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Optimal Control/Partial Differential Equations

Carleman inequality for backward stochastic parabolic equations with general coefficients

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Abstract

In this Note, we present a Carleman inequality for linear backward stochastic parabolic equations (BSPEs) with general coefficients, and its applications in the observability of BSPEs, and in the null controllability of forward stochastic parabolic equations with general coefficients. ***To cite this article: S. Tang, X. Zhang, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Résumé

Inégalité de Carleman pour les équations stochastiques paraboliques rétrogradés avec coefficients généraux. Dans cette Note, nous présentons une inégalité de Carleman pour les équations linéaires stochastiques paraboliques rétrogradés avec coefficients généraux, et ses applications à l'observabilité des équations stochastiques paraboliques rétrogradés avec coefficients généraux, et à la contrôlabilité nulle des équations stochastiques paraboliques progressives avec coefficients généraux. ***Pour citer cet article : S. Tang, X. Zhang, C. R. Acad. Sci. Paris, Ser. I 339 (2004).***

Version française abrégée

Soient $G \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) un ouvert borné et connexe à frontière ∂G suffisamment régulière, et $G_0 \neq G$ un sous-domaine non vide de G. On note par χ_{G_0} la fonction caractéristique de G_0. Soit $T > 0$. On note $Q \stackrel{\triangle}{=} (0, T) \times G$, $Q_0 \stackrel{\triangle}{=} (0, T) \times G_0$ et $\Sigma \stackrel{\triangle}{=} (0, T) \times \partial G$. Supposons que $a^{ij} \in C^{1,2}(\overline{Q})$, $a^{ij} = a^{ji}$ et pour une constante $s_0 > 0$, on a l'inégalité (1).

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Soit G_1 un sous-domaine non vide de G tel que $\overline{G_1} \subset G_0$. Supposons que $\overline{G_0} \subset G$. Nous savons (voir [3]) qu'il existe une function $\psi \in C^4(\overline{G})$ telle que $\psi > 0$ dans G, $\psi = 0$ sur ∂G, et $|\nabla\psi(x)| > 0$ pour tout $x \in \overline{G} \setminus G_1$. Pour $k \geqslant 2$, $\lambda > 1$ et $\mu > 1$, on note les identités (2).

On se donne $(\Omega, \mathcal{F}, P)$, un espace de probabilité complet, et $\{w(t)\}_{t\geqslant 0}$ un processus de Wiener sur $(\Omega, \mathcal{F}, P)$ à valeur dans $\mathbb{R}$. Soient $\mathcal{F}_t$ la σ-algèbre engendrée par $w(s)$ avec $0 \leqslant s \leqslant t$ et les ensembles entiers P-nulles dans $\mathcal{F}$. Considérons le système stochastique parabolique rétrograde donné dans l'Éq. (3). Nous avons le théorème suivant :

Théorème 0.1. *Supposons que $a^{ij} \in C^{1,2}(\overline{Q})$, $a^{ij} = a^{ji}$ et la condition* (1). *Soient $a \in L^\infty_{\mathcal{F}}(0, T; L^\infty(G; \mathbb{R}^n))$, $b, c \in L^\infty_{\mathcal{F}}(0, T; L^\infty(G))$, et $f \in L^2_{\mathcal{F}}(0, T; L^2(G))$. Alors il existe une constante $C > 0$ telle que pour tout $z_T \in L^2(\Omega, \mathcal{F}_T, P; L^2(G))$, la solution de* (3) *satisfait l'inégalité* (4).

Remarque 1. Une inégalité de Carleman a été démontrée par Barbu et al. [1, Théorème 4.1] dans le cas où $(a^{ij})_{1\leqslant i,j\leqslant n}$ est la matrice d'identité, $a \equiv 0$, et b, c sont indépendants de x. Ici, notre cas est plus général, et notre méthode est nouvelle.

Une conséquence du Théorème 0.1 est l'observabilité suivante pour les Éq. (3).

Théorème 0.2. *Supposons que $a^{ij} \in C^{1,2}(\overline{Q})$, $a^{ij} = a^{ji}$ et la condition* (1). *Soient $a \in L^\infty_{\mathcal{F}}(0, T; L^\infty(G; \mathbb{R}^n))$, $b, c \in L^\infty_{\mathcal{F}}(0, T; L^\infty(G))$, et $f \in L^2_{\mathcal{F}}(0, T; L^2(G))$. Alors il existe une constante $C > 0$, indépendante de z_T et f, telle que la solution $(z, Z) \in (L^2_{\mathcal{F}}(\Omega; C([0, T]; L^2(G))) \cap L^2_{\mathcal{F}}(0, T; H^1_0(G))) \times L^2_{\mathcal{F}}(0, T; L^2(G))$ de* (3) *satisfait l'inégalité* (5).

Une application immédiate du Théorème 0.2, est la contrôlabilité nulle des équations stochastiques paraboliques données dans l'Éq. (6).

Théorème 0.3. *Supposons que $a^{ij} \in C^{1,2}(\overline{Q})$, $a^{ij} = a^{ji}$ et la condition* (1). *Soient $a \in L^\infty_{\mathcal{F}}(0, T; W^{1,\infty}(G; \mathbb{R}^n))$, et $b, c \in L^\infty_{\mathcal{F}}(0, T; L^\infty(G))$. Alors pour chaque $y_0 \in L^2(\Omega, \mathcal{F}_0, P; L^2(G))$, il existe $\gamma_1 \in L^2_{\mathcal{F}}(0, T; L^2(G_0))$ telle que la solution $y \in L^2_{\mathcal{F}}(\Omega; C([0, T]; L^2(G))) \cap L^2_{\mathcal{F}}(0, T; H^1_0(G))$ de* (6) *satisfait $y(T) = 0$ dans G, P-p.s.*

1. Introduction and main results

Let $G \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) be a given bounded domain with a C^∞ boundary ∂G, and $G_0 \neq G$ be a given subdomain of G. Denote by χ_{G_0} the characteristic function of G_0. Let $T > 0$. Put $Q \triangleq (0, T) \times G$, $Q_0 \triangleq (0, T) \times G_0$ and $\Sigma \triangleq (0, T) \times \partial G$. Let $a^{ij} \in C^{1,2}(\overline{Q})$ satisfying $a^{ij} = a^{ji}$ and for some constant $s_0 > 0$, we have

$$\sum_{i,j=1}^{n} a^{ij}\xi^i\xi^j \geqslant s_0|\xi|^2, \quad \forall\,(t, x, \xi) \equiv (t, x, \xi^1, \ldots, \xi^n) \in \overline{Q} \times \mathbb{R}^n. \tag{1}$$

For simplicity, we shall use the notation $y_i \equiv y_i(x) = \partial y(x)/\partial x_i$, where x_i is the i-th coordinate of a generic point $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$. In a similar manner, we use the notation z_i, v_i, etc. for the partial derivatives of z and v with respect to x_i.

Let G_1 be another nonempty open subsets of G such that $\overline{G_1} \subset G_0$. Without loss of generality, in what follows we assume $\overline{G_0} \subset G$. We know from [3] that there is a $\psi \in C^4(\overline{G})$ such that $\psi > 0$ in G, $\psi = 0$ on ∂G, and $|\nabla\psi(x)| > 0$ for all $x \in \overline{G} \setminus G_1$. For any $k \geqslant 2$, and any (large) parameters $\lambda > 1$ and $\mu > 1$, we put

$$\alpha(t, x) = t^{-k}(T - t)^{-k}\big[e^{\mu\psi(x)} - e^{2\mu|\psi|_{C(\overline{G})}}\big], \quad \ell = \lambda\alpha, \quad \theta = e^{\ell}, \quad \varphi(t, x) = t^{-k}(T - t)^{-k}\, e^{\mu\psi(x)}. \tag{2}$$

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $\{w(t)\}_{t\geqslant 0}$ is defined such that $\{\mathcal{F}_t\}_{t\geqslant 0}$ is the natural filtration generated by $w(\cdot)$, augmented by all the P-null sets in $\mathcal{F}$. Let H be a Banach space. We denote by $L^2_{\mathcal{F}}(0,T;H)$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^2(0,T;H)}) < \infty$, with the canonical norm; by $L^\infty_{\mathcal{F}}(0,T;H)$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted bounded processes; and by $L^2_{\mathcal{F}}(\Omega; C([0,T];H))$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted continuous processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{C([0,T];H)}) < \infty$, with the canonical norm.

In this Note, we consider the following backward stochastic parabolic equations:

$$\begin{cases} \mathrm{d}z + \sum_{i,j=1}^{n} (a^{ij} z_i)_j \,\mathrm{d}t = \big[\langle a, \nabla z\rangle + bz + cZ + f\big]\,\mathrm{d}t + Z\,\mathrm{d}w(t) & \text{in } Q,\\ z = 0 & \text{on } \Sigma,\\ z(T) = z_T & \text{in } G. \end{cases} \tag{3}$$

We obtain the following Carleman inequality for system (3).

Theorem 1.1. *Let $a^{ij} \in C^{1,2}(\overline{Q})$, $a^{ij} = a^{ji}$ and the condition* (1) *be satisfied. Let $a \in L^\infty_{\mathcal{F}}(0,T;L^\infty(G;\mathbb{R}^n))$, $b, c \in L^\infty_{\mathcal{F}}(0,T;L^\infty(G))$, and $f \in L^2_{\mathcal{F}}(0,T;L^2(G))$. Then there is a constant $C > 0$ such that for any $z_T \in L^2(\Omega, \mathcal{F}_T, P; L^2(G))$, the solution of* (3) *satisfies the following inequality*

$$\begin{aligned} &\lambda^3\mu^4\mathbb{E}\int_Q \varphi^3\theta^2 z^2\,\mathrm{d}x\,\mathrm{d}t + \lambda\mu^2\mathbb{E}\int_Q \varphi\theta^2|\nabla z|^2\,\mathrm{d}x\,\mathrm{d}t + \lambda\mu^2\mathbb{E}\int_Q \varphi\theta^2 Z^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad\leqslant C\bigg(\lambda^3\mu^4\mathbb{E}\int_{Q_0} \varphi^3\theta^2 z^2\,\mathrm{d}x\,\mathrm{d}t + \mathbb{E}\int_Q \varphi\theta^2 f^2\,\mathrm{d}x\,\mathrm{d}t\bigg). \end{aligned} \tag{4}$$

Remark 1. A Carleman inequality has been proved by Barbu et al. [1, Theorem 4.1] for the case where $(a^{ij})_{1\leqslant i,j\leqslant n}$ is the identity matrix, $a \equiv 0$, and b and c are independent of x. Here, our case is general. We give a natural and new method.

A consequence of Theorem 1.1 is the observability for (3).

Theorem 1.2. *Let $a^{ij} \in C^{1,2}(\overline{Q})$ satisfy $a^{ij} = a^{ji}$ and the condition* (1) *be satisfied. Let $a \in L^\infty_{\mathcal{F}}(0,T;L^\infty(G;\mathbb{R}^n))$, $b, c \in L^\infty_{\mathcal{F}}(0,T;L^\infty(G))$, and $f \in L^2_{\mathcal{F}}(0,T;L^2(G))$. Then there is a constant $C > 0$, independent of z_T and f, such that the solution $(z, Z) \in (L^2_{\mathcal{F}}(\Omega; C([0,T];L^2(G))) \cap L^2_{\mathcal{F}}(0,T;H^1_0(G))) \times L^2_{\mathcal{F}}(0,T;L^2(G))$ of* (3) *satisfies the following inequality*

$$\mathbb{E}\int_G \big|z(0)\big|^2\,\mathrm{d}x \leqslant C\bigg(\mathbb{E}\int_{Q_0} z^2\,\mathrm{d}x\,\mathrm{d}t + \mathbb{E}\int_Q f^2\,\mathrm{d}x\,\mathrm{d}t\bigg). \tag{5}$$

An immediate application of Theorem 1.2 is the null controllability of the following stochastic parabolic equations

$$\begin{cases} \mathrm{d}y - \sum_{i,j=1}^{n} (a^{ij} y_i)_j \,\mathrm{d}t = \big[\langle a, \nabla y\rangle + by + \chi_{G_0}\gamma_1\big]\,\mathrm{d}t + cy\,\mathrm{d}w(t) & \text{in } Q,\\ y = 0 & \text{on } \Sigma,\\ y(0) = y_0 & \text{in } G. \end{cases} \tag{6}$$

Theorem 1.3. *Let $a^{ij} \in C^{1,2}(\overline{Q})$ satisfy $a^{ij} = a^{ji}$ and* (1)*. Let $a \in L^\infty_{\mathcal{F}}(0,T; W^{1,\infty}(G;\mathbb{R}^n))$, and $b, c \in L^\infty_{\mathcal{F}}(0,T; L^\infty(G))$. Then for any $y_0 \in L^2(\Omega, \mathcal{F}_0, P; L^2(G))$, there is $\gamma_1 \in L^2_{\mathcal{F}}(0,T; L^2(G_0))$ such that the solution $y \in L^2_{\mathcal{F}}(\Omega; C([0,T]; L^2(G))) \cap L^2_{\mathcal{F}}(0,T; H^1_0(G))$ of* (6) *satisfies $y(T) = 0$ in G, P-a.s.*

As far as we know, there are only a few papers addressed to the controllability problem of (6), see [2] for the approximate controllability, and [1] for the null controllability. Note however that the controllability results in both [1] and [2] are established under some technical assumptions on the coefficients of (6), and the main idea therein is to reduce the problem to the deterministic ones. In Theorem 1.3, we have dropped the previous undesired assumptions for the null controllability of (6). Also, thanks to Theorem 1.1, we can prove the approximate controllability of (6) under the same assumptions in Theorem 1.3.

The rest of this Note is to outline the proof of Theorem 1.1. We refer to [4] for a detailed proof of the results in this Note and other related results.

2. Sketch of the proof of Theorem 1.1

In the sequel, for $r \in \mathbb{N}$, denote by $O(\mu^r)$ a function of order μ^r for large μ (which is independent of λ). We begin with the following estimate (integrably in time and pointwisely in space and sample) for stochastic parabolic operators.

Theorem 2.1. *Let $a^{ij} \in C^{1,2}(\overline{Q})$ satisfying $a^{ij} = a^{ji}$. Assume that either $(a^{ij})_{1\leqslant i,j\leqslant n}$ or $-(a^{ij})_{1\leqslant i,j\leqslant n}$ is a strictly positive definite matrix, and $s_0(>0)$ is its smallest eigenvalue. Let u be a $H^2(G)$-valued semimartingale. Set $v = \theta u$. Then*

$$\begin{aligned}
&2\int_0^T \theta\Bigg[-\sum_{i,j=1}^n (a^{ij}v_i)_j + Av\Bigg]\Bigg[\mathrm{d}u - \sum_{i,j=1}^n (a^{ij}u_i)_j\,\mathrm{d}t\Bigg] + 2\int_0^T \sum_{i,j=1}^n (a^{ij}v_i\,\mathrm{d}v)_j\\
&\quad + 2\int_0^T \sum_{i,j=1}^n \Bigg[\sum_{i',j'=1}^n \big(2a^{ij}a^{i'j'}\ell_{i'}v_iv_{j'} - a^{ij}a^{i'j'}\ell_i v_{i'}v_{j'}\big) + \Psi a^{ij}v_iv - a^{ij}\bigg(A\ell_i + \frac{\Psi_i}{2}\bigg)v^2\Bigg]_j \mathrm{d}t\\
&\quad \geqslant 2\sum_{i,j=1}^n \int_0^T c^{ij}v_iv_j\,\mathrm{d}t + \int_0^T Bv^2\,\mathrm{d}t + \int_0^T \Bigg[-\sum_{i,j=1}^n (a^{ij}v_i)_j + Av\Bigg]^2 \mathrm{d}t\\
&\quad - \int_0^T \theta^2 \sum_{i,j=1}^n a^{ij}\,\mathrm{d}u_i\,\mathrm{d}u_j - \int_0^T \theta^2 A(\mathrm{d}u)^2,
\end{aligned}\tag{7}$$

where

$$A \triangleq -\sum_{i,j=1}^n \big(a^{ij}\ell_i\ell_j - a^{ij}_j\ell_i + a^{ij}\ell_{ij}\big), \qquad c^{ij} \triangleq \sum_{i',j'=1}^n \big[2a^{ij'}(a^{i'j}\ell_{i'})_{j'} - (a^{ij}a^{i'j'}\ell_{i'})_{j'}\big] - \frac{a^{ij}_t}{2} + \Psi a^{ij},$$

$$B \triangleq 2\Bigg[A\Psi - \sum_{i,j=1}^n (Aa^{ij}\ell_i)_j\Bigg] - A_t - \sum_{i,j=1}^n (a^{ij}\Psi_j)_i - \ell_t^2, \quad \Psi = 2\sum_{i,j=1}^n a^{ij}\ell_{ij}.$$

Moreover, for λ and μ large enough, it holds

$$A = -\lambda^2\mu^2\varphi^2 \sum_{i,j=1}^{n} a^{ij}\psi_i\psi_j + \lambda\varphi\mathrm{O}(\mu^2), \qquad \sum_{i,j=1}^{n} c^{ij} v_i v_j \geqslant \big[s_0^2\lambda\mu^2\varphi|\nabla\psi|^2 + \lambda\varphi\mathrm{O}(\mu)\big]|\nabla v|^2,$$

$$B \geqslant 2s_0^2\lambda^3\mu^4\varphi^3|\nabla\psi|^4 + \lambda^3\varphi^3\mathrm{O}(\mu^3) + \lambda^2\varphi^2\mathrm{O}(\mu^4) + \lambda\varphi\mathrm{O}(\mu^4) + \lambda^2\varphi^{2+2k^{-1}}\mathrm{O}(\mathrm{e}^{4\mu|\psi|_{C(\overline{G})}}) + \lambda^2\varphi^{2+k^{-1}}\mathrm{O}(\mu^2) + \lambda\varphi^{1+k^{-1}}\mathrm{O}(\mu^2).$$

Remark 2. A crucial point in Theorem 2.1 is that we do not put any terms like ℓ_t into A.

Now, viewing the first relation of (3) as a forward stochastic partial differential equation, and using Theorem 2.1, some straightforward arguments lead to the following estimate:

$$\begin{aligned}
&\lambda\mu^2\mathbb{E}\int_Q \varphi\theta^2\big(|\nabla z|^2 + \lambda^2\mu^2\varphi^2 z^2\big)\,\mathrm{d}t\,\mathrm{d}x \\
&\quad\leqslant C\Bigg\{\mathbb{E}\int_Q \varphi\theta^2 g^2\,\mathrm{d}x\,\mathrm{d}t - \mathbb{E}\int_Q \theta^2\sum_{i,j=1}^{n} a^{ij} Z_i Z_j\,\mathrm{d}x\,\mathrm{d}t - \mathbb{E}\int_Q \theta^2 A Z^2\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \lambda\mu^2\mathbb{E}\int_0^T\int_{G_1} \varphi\theta^2\big(|\nabla z|^2 + \lambda^2\mu^2\varphi^2 z^2\big)\,\mathrm{d}x\,\mathrm{d}t\Bigg\}.
\end{aligned} \tag{8}$$

Here, $g \stackrel{\triangle}{=} \langle a, \nabla z\rangle + bz + cZ + f$. We choose a cut-off function $\zeta \in C_0^\infty(G_0;[0,1])$ so that $\zeta \equiv 1$ on G_1. Noting that $\mathrm{d}(\varphi\theta^2 z^2) = z^2\,\mathrm{d}(\varphi\theta^2) + 2\varphi\theta^2 z\,\mathrm{d}z + \varphi\theta^2(\mathrm{d}z)^2$, from (3), we have $0 = \mathbb{E}\int_{Q_0}\zeta^2[z^2\mathrm{d}(\varphi\theta^2) + 2\varphi\theta^2 z\,\mathrm{d}z + \varphi\theta^2(\mathrm{d}z)^2]\,\mathrm{d}x = \mathbb{E}\int_{Q_0}\theta^2[\zeta^2 z^2(\varphi_t + 2\lambda\varphi\alpha_t) + 2\zeta^2\varphi\sum_{i,j=1}^{n} a^{ij} z_i z_j + 2\mu\zeta^2\varphi z\sum_{i,j=1}^{n} a^{ij} z_i\psi_j + 4\lambda\mu\zeta^2\varphi^2 z\sum_{i,j=1}^{n} a^{ij} z_i\psi_j + 4\zeta\varphi z\sum_{i,j=1}^{n} a^{ij} z_i\zeta_j + 2\zeta^2\varphi g z + \zeta^2\varphi Z^2]\,\mathrm{d}t\,\mathrm{d}x$. Therefore, by dropping the last non-negative term in the above, we conclude that for any $\varepsilon > 0$, it holds

$$\begin{aligned}
&\lambda\mathbb{E}\int_{Q_0} \zeta^2\varphi\theta^2\sum_{i,j=1}^{n} a^{ij} z_i z_j\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\leqslant \varepsilon\lambda\mathbb{E}\int_{Q_0}\zeta^2\varphi\theta^2|\nabla z|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{C}{\varepsilon}\Bigg[\mu^{-2}\mathbb{E}\int_{Q_0}\varphi\theta^2 g^2\,\mathrm{d}x\,\mathrm{d}t + \lambda^3\mu^2\mathbb{E}\int_{Q_0}\varphi^3\theta^2 z^2\,\mathrm{d}x\,\mathrm{d}t\Bigg].
\end{aligned} \tag{9}$$

From (1) and (9), we conclude that

$$\lambda\mathbb{E}\int_0^T\int_{G_1}\varphi\theta^2|\nabla z|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C\Bigg[\mu^{-2}\mathbb{E}\int_Q\varphi\theta^2 g^2\,\mathrm{d}x\,\mathrm{d}t + \lambda^3\mu^2\mathbb{E}\int_{Q_0}\varphi^3\theta^2 z^2\,\mathrm{d}x\,\mathrm{d}t\Bigg]. \tag{10}$$

Combining (8) and (10), we arrive at the following estimate:

$$\begin{aligned}
&\lambda^3\mu^4\mathbb{E}\int_Q\varphi^3\theta^2 z^2\,\mathrm{d}x\,\mathrm{d}t + \lambda\mu^2\mathbb{E}\int_Q\varphi\theta^2|\nabla z|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\leqslant C\Bigg[\mathbb{E}\int_Q\varphi\theta^2\big[\langle a,\nabla z\rangle + bz + cZ + f\big]^2\,\mathrm{d}x\,\mathrm{d}t + \lambda^3\mu^4\mathbb{E}\int_{Q_0}\varphi^3\theta^2 z^2\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

$$- \mathbb{E}\int_Q \theta^2 \sum_{i,j=1}^n a^{ij} Z_i Z_j \,\mathrm{d}x\,\mathrm{d}t - \mathbb{E}\int_Q \theta^2 A Z^2 \,\mathrm{d}x\,\mathrm{d}t\Bigg]. \tag{11}$$

Further, using integration by parts, one has

$$\begin{aligned}\mathbb{E}\int_Q \theta^2 \sum_{i,j=1}^n a^{ij} Z_i Z_j \,\mathrm{d}x\,\mathrm{d}t &= \mathbb{E}\int_Q \sum_{i,j=1}^n a^{ij}\big[(\theta Z)_i - \ell_i\theta Z\big]\big[(\theta Z)_j - \ell_j\theta Z\big]\,\mathrm{d}x\,\mathrm{d}t \\ &\geqslant \mathbb{E}\int_Q \theta^2 \sum_{i,j=1}^n a^{ij}\ell_i\ell_j Z^2 \,\mathrm{d}x\,\mathrm{d}t + \mathbb{E}\int_Q \theta^2 \sum_{i,j=1}^n a^{ij}\ell_{ij} Z^2 \,\mathrm{d}x\,\mathrm{d}t + \mathbb{E}\int_Q \theta^2 \sum_{i,j=1}^n a^{ij}_j \ell_i Z^2 \,\mathrm{d}x\,\mathrm{d}t. \end{aligned}\tag{12}$$

Therefore,

$$-\mathbb{E}\int_Q \theta^2 \sum_{i,j=1}^n a^{ij} Z_i Z_j \,\mathrm{d}x\,\mathrm{d}t - \mathbb{E}\int_Q \theta^2 A Z^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant C\lambda\mu\mathbb{E}\int_Q \varphi\theta^2 Z^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{13}$$

Combining (11) and (13), we arrive at

$$\begin{aligned}&\lambda^3\mu^4\mathbb{E}\int_Q \varphi^3\theta^2 z^2 \,\mathrm{d}x\,\mathrm{d}t + \lambda\mu^2\mathbb{E}\int_Q \varphi\theta^2|\nabla z|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad\leqslant C\Bigg[\mathbb{E}\int_Q \varphi\theta^2\big(|\nabla z|^2 + z^2 + f^2\big)\,\mathrm{d}x\,\mathrm{d}t + \lambda^3\mu^4\mathbb{E}\int_{Q_0} \varphi^3\theta^2 z^2 \,\mathrm{d}x\,\mathrm{d}t + \lambda\mu\mathbb{E}\int_Q \varphi\theta^2 Z^2 \,\mathrm{d}x\,\mathrm{d}t\Bigg]. \end{aligned}\tag{14}$$

However, from $\mathrm{d}(\varphi\theta^2 z^2) = z^2\,\mathrm{d}(\varphi\theta^2) + 2\varphi\theta^2 z\,\mathrm{d}z + \varphi\theta^2(\mathrm{d}z)^2$ and noting (3), we deduce that

$$\mathbb{E}\int_Q \varphi\theta^2 Z^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant C\bigg(\lambda^2\mu^2\mathbb{E}\int_Q \varphi^3\theta^2 z^2 \,\mathrm{d}x\,\mathrm{d}t + \lambda^{-1}\mu^{-1}\mathbb{E}\int_Q \varphi\theta^2 f^2 \,\mathrm{d}x\,\mathrm{d}t\bigg). \tag{15}$$

Therefore, combining (14) and (15), we conclude the desired estimate in Theorem 1.1.

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