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Partial Differential Equations

Asymptotic profiles of solutions to convection–diffusion equations

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Abstract

The large time behavior of zero-mass solutions to the Cauchy problem for the convection–diffusion equation $u_t - u_{xx} + (|u|^q)_x = 0$, $u(x,0) = u_0(x)$ is studied when $q > 1$ and the initial datum u_0 belongs to $L^1(\mathbb{R}, (1+|x|)\,\mathrm{d}x)$ and satisfies $\int_{\mathbb{R}} u_0(x)\,\mathrm{d}x = 0$. We provide conditions on the size and shape of the initial datum u_0 as well as on the exponent $q > 1$ such that the large time asymptotics of solutions is given either by the derivative of the Gauss–Weierstrass kernel, or by a self-similar solution of the equation, or by hyperbolic N-waves. ***To cite this article: S. Benachour et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Comportement asymptotique des solutions d'équations de convection–diffusion. Le comportement asymptotique des solutions de masse nulle du problème de Cauchy pour l'équation de convection–diffusion $u_t - u_{xx} + (|u|^q)_x = 0$, $u(x,0) = u_0(x)$ est étudié lorsque $q > 1$ et la donnée initiale u_0 appartient à $L^1(\mathbb{R}, (1+|x|)\,\mathrm{d}x)$ et satisfait $\int_{\mathbb{R}} u_0(x)\,\mathrm{d}x = 0$. Nous donnons des conditions sur l'amplitude et la forme de la donnée initiale u_0 et sur l'exposant $q > 1$ sous lesquelles le comportement asymptotique des solutions est décrit par la dérivée première du noyau de Gauss–Weierstrass, ou par une solution auto-similaire de l'équation, ou par une N-onde hyperbolique. ***Pour citer cet article : S. Benachour et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

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Nous étudions le comportement asymptotique, lorsque $t \to \infty$, des solutions de masse nulle du problème de Cauchy pour l'équation de convection–diffusion (1), (2). Il est clair que cette équation met en jeu une compétition entre le terme de diffusion u_{xx} et le terme de convection non linéaire $(|u|^q)_x$, et le but de notre étude est de déterminer lequel de ces deux termes devient prépondérant lorsque $t \to \infty$. Lorsque la masse totale $M = \int_{\mathbb{R}} u_0(x)\,\mathrm{d}x \neq 0$, ce problème a été considéré en particulier dans [8,7,12]. Lorsque $M = 0$, les résultats de [8,7,12] se traduisent seulement par la convergence de $u(t)$ vers zéro dans $L^1(\mathbb{R})$. Nous présentons, dans cette Note, des résultats plus précis pour certaines classes de données initiales. En effet, nous exhibons le premier terme dans le développement asymptotique des solutions de (1), (2) selon les valeurs du paramètre $q > 1$ et le signe (et éventuellement la taille) de la primitive de u_0.

Nous supposons que u_0 vérifie (3) et désignons par u la solution correspondante de (1), (2). Signalons dès à présent que notre approche est basée sur l'étude de l'équation de Hamilton–Jacobi diffusive (7) qui est naturellement associée à (1), (2). En effet, la primitive U de u, définie par (6), est solution de (7).

En premier lieu, nous étudions le cas où la diffusion régit le comportement en temps grands.

Théorème 0.1. *Supposons que l'une des trois conditions suivantes soit vérifiée* : (i) $U_0 \geqslant 0$ *et* $q > 3/2$, (ii) $U_0 \leqslant 0$ *et* $q \geqslant 2$, *ou* (iii) $U_0 \leqslant 0$, $q \in (3/2, 2)$ *et* u_0 *satisfait* (8). *Alors,* $u \in L^q(\mathbb{R} \times (0, \infty))$ *et*

$$\lim_{t\to\infty} t^{(1-1/p)/2+1/2} \big\| u(t) - I_\infty\, G_x(t) \big\|_p = 0, \quad p \in [1, \infty],$$

où I_∞ *est le réel non nul défini par* (9) *et* G *est la solution fondamentale de l'équation de la chaleur.*

Considérons ensuite le cas où $U_0 \geqslant 0$ et $q \in (1, 3/2)$. L'influence des effets convectifs et diffusifs s'équilibre lorsque $t \to \infty$ et on a le résultat suivant :

Théorème 0.2. *Si* $q \in (1, 3/2)$ *et* $U_0 \geqslant 0$ *vérifie la condition de croissance* (11)*, alors*

$$\lim_{t\to\infty} t^{(1-1/p)/2+a/2} \big\| u(t) - W_x(t) \big\|_p = 0, \quad p \in [1, \infty],$$

où $a := (2-q)/(q-1)$ *et* W *désigne la solution très singulière de* (7).

Enfin, nous identifions une classe de données initiales u_0 pour lesquelles le comportement en temps grands est régi par la convection.

Théorème 0.3. *Supposons que* $q \in (1, 2)$ *et que* $U_0 \leqslant 0$ *vérifie* (13)*. Alors*

$$\lim_{t\to\infty} t^{(1-1/p)/q} \big\| u(t) - N_{\sigma,\sigma}(t) \big\|_p = 0, \quad p \in [1, \infty),$$

où σ *est le réel strictement positif défini par* (14) *et* $N_{\sigma,\sigma}$ *est la* N*-onde hyperbolique définie par* (5).

Lorsque $q \in (1, 4/(1+\sqrt{3}))$, le Théorème 0.3 est vrai sans la condition de taille (13) sur u_0 et nous conjecturons qu'il reste valable sans cette condition pour tout $q \in (1, 3/2)$.

On notera que, lorsque $q \in (3/2, 2)$ et selon la taille de la donnée initiale, on peut observer aussi bien un comportement asymptotique où la diffusion domine qu'un comportement où la convection est prépondérante. De plus, il existe des solutions pour lesquelles les deux effets se compensent [6].

Les preuves des résultats présentés ici, ainsi que des extensions en dimension supérieure, font l'objet de l'article [1].

1. Introduction

The large time behavior of zero-mass solutions to the Cauchy problem

$$u_t - u_{xx} + \big(|u|^q\big)_x = 0, \quad x \in \mathbb{R},\ t > 0, \tag{1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{2}$$

is investigated when $q > 1$, under the main assumption

$$u_0 \in L^1\big(\mathbb{R}, \big(1+|x|\big)\,\mathrm{d}x\big) \cap W^{1,\infty}(\mathbb{R}), \quad u_0 \not\equiv 0 \quad \text{and} \quad \int_{\mathbb{R}} u_0(x)\,\mathrm{d}x = 0. \tag{3}$$

Eq. (1) includes two competing effects, namely the diffusion u_{xx} and the nonlinear convection $(|u|^q)_x$, and the main issue in the study of the large time asymptotics is to figure out whether one of these effects dominates for large times. For integrable initial data with non-zero mass, i.e., $u_0 \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}} u_0(x)\,\mathrm{d}x = M \neq 0$, this question has already been investigated and the outcome may be summarized as follows: if $q > 2$, the large time dynamics is dominated by the diffusion and u behaves as the fundamental (or source-type) solution MG to the linear heat equation with $G(x,t) = (4\pi t)^{-1/2}\exp(-x^2/(4t))$ for $(x,t) \in \mathbb{R} \times (0,\infty)$, see, e.g., [8]. If $q \in (1,2)$ and either $u_0 \geqslant 0$ (or $u_0 \leqslant 0$) [7] or

$$\inf_{x\in\mathbb{R}} \int_{-\infty}^{x} u_0(y)\,\mathrm{d}y \leqslant 0 \tag{4}$$

[12], the convection term is preponderant for large times and there is $\sigma \in \mathbb{R}$ such that u behaves as the N-wave solution $N_{\sigma,\sigma+M}$ to the nonlinear conservation law $z_t + (|z|^q)_x = 0$. Moreover, $\sigma = 0$ if $u_0 \geqslant 0$, $\sigma = -M$ if $u_0 \leqslant 0$ and $\sigma \geqslant 0$ under the assumption (4). Recall that the N-wave solution $N_{\alpha,\beta}$ is given explicitly by

$$N_{\alpha,\beta}(x,t) = \begin{cases} \operatorname{sign} x \cdot \left(\dfrac{|x|}{qt}\right)^{1/(q-1)}, & -q\left(\dfrac{\alpha}{q-1}\right)^{(q-1)/q} \leqslant \dfrac{x}{t^{1/q}} \leqslant q\left(\dfrac{\beta}{q-1}\right)^{1/(q-1)}, \\ 0 & \text{otherwise}, \end{cases} \tag{5}$$

for $\alpha \geqslant 0$ and $\beta \geqslant 0$ (see, e.g., [14]). When $q = 2$, there is a balance between the diffusive and convective terms and u behaves as the unique self-similar source-type solution S_M to (1), that is, S_M is the unique solution to (1) with initial datum $M\,\delta$ [8].

The previous results remain of course valid when $M = 0$. However, when $q \geqslant 2$, they reduce to the convergence of $u(t)$ to zero in $L^1(\mathbb{R})$, while if $q \in (1,2)$ and (4) is fulfilled, the possibility that $\sigma = 0$ is not excluded in [12]. Obtaining non-vanishing intermediate asymptotics in the zero-mass case thus seems to be more delicate, and is the purpose of this note. On the one hand, we identify conditions under which the constant σ defined previously does not vanish, thus leading to a truly hyperbolic large time behavior. On the other hand, we complete the study performed in [11] where conditions on the initial data u_0 are found for which the diffusion still plays a role in the large time asymptotics. Indeed, under the assumption that there is $\beta \in (0,1)$ such that $\sup_{t\geqslant 0} t^{\beta/2}\, \|\mathrm{e}^{t\Delta}u_0\|_1 < \infty$ if $q \geqslant 2$ and $\sup_{t\geqslant 0} t^{\beta/2}\, \|\mathrm{e}^{t\Delta}u_0\|_1$ is sufficiently small for $q \in (1+1/(1+\beta), 2)$, the large time asymptotics is dominated by the diffusive term and the asymptotics of solutions to (1), (2) is described by self-similar solutions to the heat equation [11]. A balance between diffusion and convection is also observed in [11] in the critical case $q = 1 + 1/(1+\beta)$ and, for suitably small initial data, the asymptotics of solutions to (1), (2) is described by a new class of self-similar solutions to (1).

From now on, we assume that u_0 fulfils (3). We denote by $u = u(x,t)$ the corresponding solution to (1), (2) and set

$$U(x,t) = \int_{-\infty}^{x} u(y,t)\,\mathrm{d}y = -\int_{x}^{\infty} u(y,t)\,\mathrm{d}y, \tag{6}$$

the second inequality being true since the integral of $u(t)$ over $\mathbb{R}$ vanishes for each $t \geqslant 0$ by (3). Owing to (3), it is easy to see that $U_0 := U(0)$ belongs to $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with

$$\int_{\mathbb{R}} U_0(x)\,\mathrm{d}x = -\int_{\mathbb{R}} x u_0(x)\,\mathrm{d}x,$$

and it readily follows from (1) that U is a solution to the viscous Hamilton–Jacobi equation

$$U_t - U_{xx} + |U_x|^q = 0, \quad x \in \mathbb{R},\ t > 0, \qquad U(x,0) = U_0(x), \quad x \in \mathbb{R}. \tag{7}$$

This observation is the basis of our approach towards the study of the large time behavior of zero-mass solutions to (1), (2), since we will actually investigate the large time dynamics of U and then deduce corresponding results for u.

2. Diffusion-dominated case

We begin with the case when the diffusive term is preponderant for large times.

Theorem 2.1. *Assume that u_0 fulfils* (3) *and that one of the following assertions hold true*: (i) $U_0 \geqslant 0$ *and* $q > 3/2$, (ii) $U_0 \leqslant 0$ *and* $q \geqslant 2$, (iii) $U_0 \leqslant 0$, $q \in (3/2, 2)$, *and*

$$\left|\int_{\mathbb{R}} x u_0(x)\,\mathrm{d}x\right| \|u_0\|_\infty^{2q-3} \quad \textit{is sufficiently small.} \tag{8}$$

Then, $u \in L^q(\mathbb{R} \times (0,\infty))$, the constant I_∞ defined by

$$I_\infty := -\lim_{t\to\infty} \int_{\mathbb{R}} x u(x,t)\,\mathrm{d}x = -\int_{\mathbb{R}} x u_0(x)\,\mathrm{d}x - \int_0^\infty \int_{\mathbb{R}} \left|u(x,s)\right|^q \mathrm{d}x\,\mathrm{d}s \tag{9}$$

is well-defined with $I_\infty > 0$ if $U_0 \geqslant 0$ and $I_\infty < 0$ if $U_0 \leqslant 0$, and

$$\lim_{t\to\infty} t^{(1-1/p)/2+1/2} \left\|u(t) - I_\infty\, G_x(t)\right\|_p = 0 \quad \textit{for every } p \in [1,\infty]. \tag{10}$$

A result similar to Theorem 2.1 is also valid for U [1, Theorems 2.1 and 2.3]. The proof of Theorem 2.1 splits in two steps: first, the fact that $u \in L^q(\mathbb{R} \times (0,\infty))$ and $I_\infty \in \mathbb{R} \setminus \{0\}$ under the assumptions of Theorem 2.1 follows from [3,5] in the case (i) and [13] in the cases (ii) and (iii). Next, the proof of (10) and its analogue for U relies on the representation of U by the Duhamel formula and L^∞- and L^q-estimates on $\partial_x U = u$ established in [3,10] and [13], respectively, from which the estimates

$$\sup_{t>0}\left\{t^{1/2}\left\|u(t)\right\|_1\right\} + \sup_{t>0}\left\{t\left\|u(t)\right\|_\infty\right\} < \infty$$

follow. Classical properties of the heat semi-group $\mathrm{e}^{t\Delta}$ then allow us to complete the proof of Theorem 2.1.

3. Convergence towards very singular solutions

We next study the case $q \in (1, 3/2)$ and $U_0 \geqslant 0$ where there is a balance between the diffusive and convective effects, and a particular self-similar solution of (1) appears in the large time asymptotics.

Theorem 3.1. *Suppose that u_0 fulfils* (3). *If $q \in (1, 3/2)$ and U_0 is nonnegative and satisfies*

$$\lim_{|x|\to\infty} |x|^a\, U_0(x) = 0 \quad \textit{with } a = \frac{2-q}{q-1}, \tag{11}$$

then

$$\lim_{t\to\infty} t^{(1-1/p)/2+a/2}\big\|u(t)-W_x(t)\big\|_p=0 \tag{12}$$

for every $p\in[1,\infty]$, *where* W *is the very singular solution to* (7) *and enjoys the self-similarity property* $W(x,t)=t^{-a/2}W(xt^{-1/2},1)$ *for* $(x,t)\in\mathbb{R}\times(0,\infty)$.

Note that $a>1$ for $q<3/2$, hence the convergence rate towards the self-similar profile in (12) is faster than the rate in (10).

We recall that the very singular solution W to (7) is a classical solution to (7) in $\mathbb{R}\times(0,\infty)$ which has a singular behavior as $t\to 0$, namely

$$\lim_{t\to 0}\int_{\{|x|\geqslant r\}} W(x,t)\,\mathrm{d}x=0 \quad\text{and}\quad \lim_{t\to 0}\int_{\{|x|\leqslant r\}} W(x,t)\,\mathrm{d}x=\infty$$

for each $r>0$. The existence and uniqueness of the very singular solution to (7) are established in [2,4]. The proof of Theorem 3.1 relies on a rescaling method together with some estimates from [4]. More precisely, introducing the sequence of rescaled functions $U_k(x,t):=k^a\,U(kx,k^2t),\ k\geqslant 1$, an easy computation shows that U_k is a solution to (7) while it follows from [4] that

$$\sup_{t>0}\big\{t^{(a-1)/2}\big\|U(t)\big\|_1+t^{a/2}\big\|U(t)\big\|_\infty+t^{(a+1)/2}\big\|u(t)\big\|_\infty\big\}<\infty.$$

Owing to these estimates, one can show that a subsequence of (U_k) converges towards W in $L^p(\mathbb{R})$ for every $p\in[1,\infty]$. The uniqueness of the very singular solution to (7) and the use of the Duhamel representation formula allow to extend the previous convergence to the whole sequence (U_k) and also to its first derivative, thus completing the proof of Theorem 3.1.

4. Hyperbolic-dominated case

We end up with the case when the large time behavior is ruled by the convection term and is given by self-similar solutions to the nonlinear conservation law $z_t+(|z|^q)_x=0$.

Theorem 4.1. *Consider* $q\in(1,2)$. *Assume that* u_0 *fulfils* (3) *and that* U_0 *is non-positive and such that*

$$\|U_0\|_\infty\|u_{0,x}\|_\infty^{1-2/q}\quad \textit{is sufficiently large.} \tag{13}$$

Then

$$\sigma:=\lim_{t\to\infty}\big\|u(t)\big\|_\infty\in(0,\infty)\quad\textit{and}\quad \lim_{t\to\infty}t^{(1-1/p)/q}\big\|u(t)-N_{\sigma,\sigma}(t)\big\|_p=0 \tag{14}$$

for every $p\in[1,\infty)$, $N_{\sigma,\sigma}$ *being the N-wave solution to* $z_t+(|z|^q)_x=0$ *defined by* (5).

In addition, if $q\in(1,4/(1+\sqrt{3}))$, *the previous result is valid without the condition* (13) *on* u_0.

As a consequence of Theorems 2.1 and 4.1, we realize that, when $q\in(3/2,2)$, both diffusion-dominated and hyperbolic-dominated large time behaviors are possible, according to the size of u_0. In addition, when $q\in(3/2,2)$, there are initial data u_0 (with $U_0\leqslant 0$) for which the large time behavior is governed by a balance between the diffusive and convective effects [6].

The main step of the proof of Theorem 4.1 is to show the positivity of σ in (14). This property follows from the unilateral estimate

$$U_{xx}(x,t)=u_x(x,t)\leqslant\min\big\{\|u_{0,x}\|_\infty,\,C(q)\|U_0\|_\infty^{(2-q)/q}t^{-2/q}\big\}. \tag{15}$$

This inequality can be seen as a weak form of the Oleinik gradient estimate for scalar conservation laws and is also derived in [12] (see also [9] for related estimates). We however establish a version of (15) in a multi-dimensional setting involving the Hessian matrix of U (cf. [1]). The proof of (15) follows from (1) by the maximum principle and the L^∞-estimate $\|u(t)\|_\infty \leqslant C(q)\ \|U_0\|_\infty^{1/q}\ t^{-1/q}$ [3,10]. Once the positivity of σ is established, the proof of (14) is performed by a classical rescaling method [1,12].

Remark 1. (a) Though the conditions (8) and (13) do not involve the same quantities, it can be shown by Gagliardo–Nirenberg inequalities that, if (8) is fulfilled, the quantity involved in (13) is also small.

(b) We conjecture that the condition (13) is not necessary when $q \in (1, 3/2)$ for Theorem 4.1 to hold true.

The proofs of the results presented in this Note, together with their extension to higher space dimensions are to be found in [1].

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