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Remarks on exact controllability for Stokes and Navier–Stokes systems

Enrique Fernández-Cara [a], Sergio Guerrero [a], Oleg Yurievich Imanuvilov [b], Jean-Pierre Puel [c]

[a] *Dpto. E.D.A.N., Universidad de Sevilla, Aptdo. 1160, 41080 Sevilla, Spain*
[b] *Department of Mathematics, Iowa State University, 400 Carver Hall, Ames, IA 50011-2064, USA*
[c] *Laboratoire de mathématiques appliquées, Université de Versailles-St-Quentin, 45, avenue des États Unis, 78035 Versailles, France*

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Abstract

This Note deals with the controllability of Stokes and Navier–Stokes systems with distributed controls with support in possibly small subdomains. We first present a new global Carleman inequality for the solutions to Stokes-like systems that leads to the null controllability at any time $T > 0$. Then, we present a local result concerning exact controllability to trajectories of the Navier–Stokes system. ***To cite this article: E. Fernández-Cara et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Remarques sur la contrôlabilité exacte des systèmes de Stokes et Navier–Stokes. Cette Note concerne la contrôlabilité des systèmes de Stokes et Navier–Stokes avec contrôles distribués pour lesquels les supports sont éventuellement petits. On présente d'abord une nouvelle inégalité globale de Carleman pour les solutions d'un problème de type Stokes. On en déduit la contrôlabilité exacte à zéro en tout temps $T > 0$. Ensuite, on présente un résultat local de contrôlabilité exacte sur les trajectoires pour le système de Navier–Stokes. ***Pour citer cet article : E. Fernández-Cara et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

On se donne un domaine borné et regulier $\Omega \subset \mathbf{R}^N$ avec $N = 2$ ou $N = 3$, un (petit) sous-ensemble ouvert non vide $\omega \subset \Omega$ et un nombre réel $T > 0$. Dans la suite, on utilisera la notation suivante : $Q = \Omega \times (0, T)$, $\Sigma = \partial\Omega \times (0, T)$ et n la normale unitaire extérieure à Ω.

Cette Note concerne quelques aspects de la contrôlabilité exacte du système de Navier–Stokes (2) et du système linéaire de type Stokes (3), où $\bar{y} = \bar{y}(x, t)$ est donné. On présentera des améliorations des résultats de [6].

E-mail addresses: cara@us.es (E. Fernández-Cara), sguerrero@us.es (S. Guerrero), vika@iastate.edu (O.Yu. Imanuvilov), jppuel@cmapx.polytechnique.fr (J.-P. Puel).

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Comme il est désormais bien connu, il convient d'analyser l'observabilité du système (4), qui peut être regardé comme l'adjoint de (3).

On aura besoin des espaces suivants

$$H = \{z \in L^2(\Omega)^N\colon\ \nabla \cdot z = 0 \text{ dans } \Omega,\ z \cdot n = 0 \text{ sur } \partial\Omega\},$$
$$V = \{z \in H_0^1(\Omega)^N\colon\ \nabla \cdot z = 0 \text{ dans } \Omega\}.$$

Le premier résultat principal de cette Note est une nouvelle inégalité globale de Carleman qui doit être vérifiée par les solutions de (4) :

Théorème 0.1. *Si la fonction $\bar{y}$ satisfait* (5)*, on peut trouver des constantes positives $\bar{s}$, $\bar{\lambda}$ et C qui dépendent seulement de Ω et ω telles que, pour chaque $g \in L^2(Q)^N$ et $\varphi^0 \in H$, la solution associée de* (4) *vérifie*

$$\iint_Q e^{-2s\alpha}\big((s\xi)^{-1}\big(|\varphi_t|^2 + |\Delta\varphi|^2\big) + s\lambda^2\xi|\nabla\varphi|^2 + s^3\lambda^4\xi^3|\varphi|^2\big)\,dx\,dt$$
$$\leqslant C\big(1+T^2\big)\Big(s^{15/2}\lambda^{20}\iint_Q e^{-4s\hat{\alpha}+2s\alpha^*}\hat{\xi}^{15/2}|g|^2\,dx\,dt + s^{16}\lambda^{40}\iint_{\omega\times(0,T)} e^{-8s\hat{\alpha}+6s\alpha^*}\hat{\xi}^{16}|\varphi|^2\,dx\,dt\Big) \tag{1}$$

pour $\lambda \geqslant \bar{\lambda}(1 + \|\bar{y}\|_\infty + e^{\bar{\lambda}T\|\bar{y}\|_\infty^2} + \|\bar{y}_t\|^2_{L^2(L^\sigma)})$ *et* $s \geqslant \bar{s}(T^4 + T^8)$.

Dans (1), on a utilisé les poids donnés dans (6). L'inégalité (1) conduit, à l'aide d'un argument désormais classique, à l'observabilité des solutions de (4) et, par conséquent, à la contrôlabilité exacte à zéro de (3) pour $f = 0$.

Le deuxième résultat principal est consacré à la contrôlabililité exacte locale sur les trajectoires de (2) :

Théorème 0.2. *On suppose que $(\bar{y}, \bar{p})$ est une solution du problème de Navier–Stokes* (9) *et que la condition* (5) *est vérifiée. Il existe $\varepsilon > 0$ tel que pour chaque donnée initiale $y^0 \in L^{2N-2}(\Omega)^N \cap H$ satisfaisant $\|\bar{y}^0 - y^0\|_{L^{2N-2}} < \varepsilon$, on peut trouver des contrôles $v \in L^2(\omega \times (0,T))^N$ et des états associés (y, p) tels que l'on ait* (2) *et*

$$y(x,T) = \bar{y}(x,T) \quad \text{dans } \Omega.$$

Les démonstrations détaillées de ces théorèmes seront données dans un travail à paraître.

1. Introduction and main results

Let $\Omega \subset \mathbf{R}^N$ be a bounded regular domain, with $N = 2$ or $N = 3$. Assume that $\omega \subset \Omega$ is a nonempty (small) open subset and $T > 0$ is given. In the sequel, we will use the following notation: $Q = \Omega \times (0,T)$, $\Sigma = \partial\Omega \times (0,T)$ and n the outward unit normal to Ω; C will stand for a generic positive constant that may depend on Ω and ω.

In this Note, we will be concerned with some controllability properties of the Navier–Stokes system

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla p = v1_\omega & \text{in } Q,\\ \nabla\cdot y = 0 & \text{in } Q,\\ y = 0 & \text{on } \Sigma,\\ y(x,0) = y^0(x) & \text{in } \Omega \end{cases} \tag{2}$$

and the similar linear Stokes-like problem

$$\begin{cases} y_t - \Delta y + \nabla\cdot(\bar y\otimes y + y\otimes\bar y) + \nabla p = f + v1_\omega & \text{in } Q,\\ \nabla\cdot y = 0 & \text{in } Q,\\ y = 0 & \text{on } \Sigma,\\ y(x,0) = y^0(x) & \text{in } \Omega, \end{cases} \tag{3}$$

where $\bar y = \bar y(x,t)$ is given and satisfies adequate regularity assumptions. In (3), the symbol $\otimes$ stands for the usual tensor product in $\mathbf{R}^N$.

Our results below improve the results of [6]. As usual, it will be convenient to analyze the observability properties of the following system, which can be viewed as the adjoint of (3):

$$\begin{cases} -\varphi_t - \Delta\varphi - (D\varphi)\bar y + \nabla\pi = g & \text{in } Q,\\ \nabla\cdot\varphi = 0 & \text{in } Q,\\ \varphi = 0 & \text{on } \Sigma,\\ \varphi(x,T) = \varphi^0(x) & \text{in } \Omega. \end{cases} \tag{4}$$

Here, $D\varphi = \nabla\varphi + \nabla\varphi^{\mathrm T}$. We will need some function spaces:

$$H = \{z\in L^2(\Omega)^N\colon\ \nabla\cdot z = 0 \text{ in } \Omega,\ z\cdot n = 0 \text{ on } \partial\Omega\},$$
$$V = \{z\in H_0^1(\Omega)^N\colon\ \nabla\cdot z = 0 \text{ in } \Omega\}.$$

Furthermore, the following hypotheses over $\bar y$ will be needed in order to have suitable Carleman estimates for the solutions to (4):

$$\bar y\in L^\infty(Q)^N,\quad \bar y_t\in L^2\big(0,T;L^\sigma(\Omega)\big)^N\quad \begin{pmatrix} \sigma > 6/5 & \text{if } N=3\\ \sigma>1 & \text{if } N=2\end{pmatrix}. \tag{5}$$

Our first main result is a new global Carleman estimate for the solutions to (4). Several weight functions will be needed:

$$\alpha(x,t) = \frac{\mathrm e^{5/4\lambda m\|\eta^0\|_\infty} - \mathrm e^{\lambda(m\|\eta^0\|_\infty+\eta^0(x))}}{t^4(T-t)^4},\qquad \xi(x,t) = \frac{\mathrm e^{\lambda(m\|\eta^0\|_\infty+\eta^0(x))}}{t^4(T-t)^4},$$
$$\hat\alpha(t) = \min_{x\in\overline\Omega}\alpha(x,t),\qquad \alpha^*(t) = \max_{x\in\overline\Omega}\alpha(x,t),\qquad \hat\xi(t) = \max_{x\in\overline\Omega}\xi(x,t). \tag{6}$$

Here, $m>4$ is a fixed real number and $\eta^0\in C^2(\overline\Omega)$ is a function satisfying

$$\eta^0>0\quad\text{in } \Omega,\qquad \eta^0 = 0\quad \text{on } \partial\Omega,\qquad |\nabla\eta^0|>0\quad\text{in } \overline{\Omega\setminus\omega'},$$

where $\omega'\Subset\omega$ is a nonempty open set. The existence of such a function η^0 is proved in [3].

Theorem 1.1. *Let us assume that* (5) *holds. There exist positive constants $\bar s$, $\bar\lambda$ and C, only depending on Ω and ω such that, for every $g\in L^2(Q)^N$ and $\varphi^0\in H$, the associated solution to* (4) *satisfies*

$$\iint_Q \mathrm e^{-2s\alpha}\big((s\xi)^{-1}\big(|\varphi_t|^2 + |\Delta\varphi|^2\big) + s\lambda^2\xi|\nabla\varphi|^2 + s^3\lambda^4\xi^3|\varphi|^2\big)\,\mathrm dx\,\mathrm dt$$
$$\leqslant C\big(1+T^2\big)\left(s^{15/2}\lambda^{20}\iint_Q \mathrm e^{-4s\hat\alpha+2s\alpha^*}\hat\xi^{15/2}|g|^2\,\mathrm dx\,\mathrm dt + s^{16}\lambda^{40}\iint_{\omega\times(0,T)} \mathrm e^{-8s\hat\alpha+6s\alpha^*}\hat\xi^{16}|\varphi|^2\,\mathrm dx\,\mathrm dt\right) \tag{7}$$

for any $\lambda\geqslant\bar\lambda(1+\|\bar y\|_\infty + \mathrm e^{\bar\lambda T\|\bar y\|_\infty^2} + \|\bar y_t\|^2_{L^2(L^\sigma)})$ and any $s\geqslant\bar s(T^4+T^8)$.

This Carleman inequality provides, in a classical way, an observability inequality for the solutions to (4), i.e.,

$$\|\varphi(\cdot\,,0)\|^2_{L^2(\Omega)} \leqslant C \iint\limits_{\omega\times(0,T)} |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t \tag{8}$$

for a positive constant C. It is now classical to prove that the inequality (8) implies the null controllability of (3) for $f = 0$.

The second main result in this Note concerns the local exact controllability to the trajectories of (2). It is the following:

Theorem 1.2. *Let* $(\bar{y}, \bar{p})$ *be a solution to the Navier–Stokes problem*

$$\begin{cases} \bar{y}_t - \Delta\bar{y} + (\bar{y}\cdot\nabla)\bar{y} + \nabla\bar{p} = 0 & \text{in } Q,\\ \nabla\cdot\bar{y} = 0 & \text{in } Q,\\ \bar{y} = 0 & \text{on } \Sigma,\\ \bar{y}(x,0) = \bar{y}^0(x) & \text{in } \Omega, \end{cases} \tag{9}$$

satisfying (5). *There exists* $\varepsilon > 0$ *such that, for any initial state* $y^0 \in L^{2N-2}(\Omega)^N \cap H$ *satisfying* $\|\bar{y}^0 - y^0\|_{L^{2N-2}} \leqslant \varepsilon$, *we can find controls* $v \in L^2(\omega\times(0,T))^N$ *and associated states* (y,p) *such that one has* (2) *and*

$$y(x,T) = \bar{y}(x,T) \quad \text{in } \Omega.$$

In the following sections, we will indicate the main ideas of the proofs of Theorems 1.1 and 1.2. The detailed proofs will be given in a forthcoming paper.

2. A new Carleman inequality

We will use the notation $I(s,\lambda;\varphi)$ to denote the left-hand side of (7). Let $g \in L^2(Q)^N$ and $\varphi^0 \in H$ be given and let (φ,π) be the associated solution to (4). We can first apply to each component of φ the usual Carleman inequality for the heat equation with right-hand side in $L^2(Q)$. After some arrangements, we get

$$I(s,\lambda;\varphi) \leqslant C\biggl(\iint\limits_{Q} \mathrm{e}^{-2s\alpha}\bigl(|g|^2 + |\nabla\pi|^2\bigr)\,\mathrm{d}x\,\mathrm{d}t + s^3\lambda^4 \iint\limits_{\omega'\times(0,T)} \mathrm{e}^{-2s\alpha}\xi^3|\varphi|^2\,\mathrm{d}x\,\mathrm{d}t\biggr), \tag{10}$$

for all $\lambda \geqslant C(1 + \|\bar{y}\|_\infty)$ and $s \geqslant C(T^7 + T^8)$. For the proof of (10), see [5] and [3]; for the explicit values of λ and s, see for instance [2].

In view of the main result in [7] and following the ideas of [6], we can estimate the pressure gradient in (10) and deduce that

$$\begin{aligned} &I(s,\lambda;\varphi)\\ &\quad\leqslant C\biggl(s^3\lambda^4 \iint\limits_{\omega'\times(0,T)} \mathrm{e}^{-2s\alpha}\xi^3|\varphi|^2\,\mathrm{d}x\,\mathrm{d}t + s^2\lambda^2 \iint\limits_{\omega_1\times(0,T)} \mathrm{e}^{-2s\alpha}\xi^2|\pi|^2\,\mathrm{d}x\,\mathrm{d}t + s\iint\limits_{Q} \mathrm{e}^{-2s\alpha}\xi|g|^2\,\mathrm{d}x\,\mathrm{d}t\biggr), \end{aligned} \tag{11}$$

for any $\lambda \geqslant C(1 + \|\bar{y}\|_\infty)$ and any $s \geqslant C(T^4 + T^8)$, where ω_1 is an open set such that $\omega' \Subset \omega_1 \Subset \omega$. The rest of the proof is oriented towards the absorption of the local pressure term in (11). Let us remark that we have only used the assumption $\bar{y} \in L^\infty(Q)^N$ until this moment, while more regularity on $\bar{y}$ will be needed to perform a local estimate of the pressure.

We can assume that the pressure has been chosen with zero mean in ω_1. Then,

$$\iint\limits_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^2|\pi|^2\,dx\,dt \leqslant C \iint\limits_{\omega_1\times(0,T)} e^{-2s\hat{\alpha}}\hat{\xi}^2|\nabla\pi|^2\,dx\,dt$$

and, using the equation satisfied by φ and π, we see that the task is to obtain local estimates of $\Delta\varphi$ and φ_t.

For the estimate of $\Delta\varphi$, we can use classical arguments for the heat equation; observe that $u = \Delta\varphi$ fulfills a heat equation where the pressure is absent. On the other hand, integrating by parts in time and using well known a priori estimates for the Stokes system (see [4]), we can find a local estimate of φ_t in terms of local integrals of φ and $\nabla\varphi$ and $I(s,\lambda;\varphi)$. More precisely, with $q = s^{15/2}\,e^{-2s\hat{\alpha}+s\alpha^*}\hat{\xi}^{15/2}$ and ω_2 an open set satisfying $\omega_1 \Subset \omega_2 \Subset \omega$, for any small $\varepsilon > 0$ we obtain

$$\begin{aligned} s^2\lambda^2 &\iint\limits_{\omega_1\times(0,T)} e^{-2s\hat{\alpha}}\hat{\xi}^2|\varphi_t|^2\,dx\,dt \\ &\leqslant \varepsilon I(s,\lambda;\varphi) + C_\varepsilon\lambda^{20}(1+T)\big(\|qg\|^2_{L^2(L^2)} + \|q\varphi\|^2_{L^2(L^2(\omega_2))} + \|q\nabla\varphi\|^2_{L^2(L^2(\omega_2))}\big) \end{aligned}$$

for $\lambda \geqslant C(1 + \|\bar{y}\|_\infty + e^{CT\|\bar{y}\|_\infty^2} + \|\bar{y}_t\|^2_{L^2(L^\sigma)})$. Let us remark that proving such a local estimate requires many technical computations and led us to assume $\bar{y}_t \in L^2(L^\sigma)$.

The local estimates of $\Delta\varphi$ and φ_t lead to the desired Carleman inequality (7).

3. Local null controllability for the Navier–Stokes system

The proof of Theorem 1.2 follows the ideas in [6]. Thus, we deduce in a first step a null controllability result for (3) with suitable right-hand side f.

More precisely, let us set $Ly = y_t - \Delta y + \nabla\cdot(\bar{y}\otimes y + y\otimes\bar{y})$ and let us introduce the spaces E_N, with

$$\begin{aligned} E_2 = \big\{(y,v):\ & e^{2s\hat{\beta}-s\beta^*}\hat{\gamma}^{-15/4}y,\, e^{4s\hat{\beta}-3s\beta^*}\hat{\gamma}^{-8}v1_\omega \in L^2(Q)^2, \\ & e^{s\beta^*/2}(\gamma^*)^{-1/4}y \in L^2(0,T;V)\cap L^\infty(0,T;H), \\ & \exists p:\ e^{s\beta^*}(\gamma^*)^{-1/2}(Ly+\nabla p - v1_\omega) \in L^2\big(0,T;H^{-1}(\Omega)^2\big)\big\} \end{aligned}$$

and

$$\begin{aligned} E_3 = \big\{& e^{2s\hat{\beta}-s\beta^*}\hat{\gamma}^{-15/4}y,\, e^{4s\hat{\beta}-3s\beta^*}\hat{\gamma}^{-8}v1_\omega \in L^2(Q)^3,\, e^{s\beta^*/2}(\gamma^*)^{-1/4}y \in L^2(0,T;V)\cap L^\infty(0,T;H), \\ & e^{s\beta^*/2}(\gamma^*)^{-1/4}y \in L^4\big(0,T;L^{12}(\Omega)^3\big), \\ & \exists p:\ e^{s\beta^*}(\gamma^*)^{-1/2}(Ly+\nabla p - v1_\omega) \in L^2\big(0,T;W^{-1,6}(\Omega)^3\big)\big\}, \end{aligned}$$

where the new weight functions β, β^*, etc. are given by

$$\beta(x,t) = \frac{e^{5/4\lambda m\|\eta^0\|_\infty} - e^{\lambda(m\|\eta^0\|_\infty+\eta^0(x))}}{\ell(t)^4},\quad \hat{\beta}(t) = \min_{x\in\overline{\Omega}}\beta(x,t),\quad \beta^*(t) = \max_{x\in\overline{\Omega}}\beta(x,t),$$

$$\gamma(x,t) = \frac{e^{\lambda(m\|\eta^0\|_\infty+\eta^0(x))}}{\ell(t)^4},\quad \hat{\gamma}(t) = \max_{x\in\overline{\Omega}}\gamma(x,t),\quad \gamma^*(t) = \min_{x\in\overline{\Omega}}\gamma(x,t).$$

Here, we have introduced

$$\ell(t) = \begin{cases} T^2/4 & \text{for } 0 < t < T/2, \\ t(T-t) & \text{for } T/2 < t < T. \end{cases}$$

We then have:

Proposition 3.1. *Assume that* $\bar{y}$ *satisfies* (5) *and*:

- $y^0 \in H$, $e^{s\beta^*}(\gamma^*)^{-1/2} f \in L^2(0,T; H^{-1}(\Omega)^2)$ *if* $N=2$,
- $y^0 \in L^4(\Omega)^3 \cap H$, $e^{s\beta^*}(\gamma^*)^{-1/2} f \in L^2(0,T; W^{-1,6}(\Omega)^3)$ *if* $N=3$.

Then, there exists a control $v \in L^2(\omega \times (0,T))^N$ *such that, if* y *is the associated solution to* (3), *we have* $(y,v) \in E_N$.

Notice that this is actually a null controllability result for (3). Indeed, if $(y,v) \in E$, we have in particular that $y(x,T)=0$ in Ω.

The rest of the proof of Theorem 1.2 relies on an appropriate *inverse mapping theorem*. More precisely, we use the following result (see [1]):

Proposition 3.2. *Let* E, F *be two Banach spaces and let* $\mathcal{A}: E \mapsto F$ *satisfy* $\mathcal{A} \in C^1(E;F)$. *Assume that* $e_0 \in E$, $\mathcal{A}(e_0) = h_0$ *and* $\mathcal{A}'(e_0): E \mapsto F$ *is an epimorphism. Then, there exists* $\delta > 0$ *such that, for every* $h \in F$ *satisfying* $\|h - h_0\|_F < \delta$, *there exists a solution of the equation*

$$\mathcal{A}(e) = h, \quad e \in E.$$

Let us consider the mapping $\mathcal{A}: E \mapsto F$, given by

$$\mathcal{A}(y,v) = \big(Ly + (y \cdot \nabla)y + \nabla p - v1_\omega, y(\cdot,0)\big) \quad \forall (y,v) \in E,$$

where $E = E_N$ and

$$F = \begin{cases} L^2\big(e^{s\beta^*}(\gamma^*)^{-1/2}; H^{-1}(\Omega)^2\big) \times H & \text{if } N=2, \\ L^2\big(e^{s\beta^*}(\gamma^*)^{-1/2}; W^{-1,6}(\Omega)^3\big) \times \big(L^4(\Omega)^3 \cap H\big) & \text{if } N=3. \end{cases}$$

From the definition of E_N, one can easily check that $\mathcal{A}$ is well defined and satisfies $\mathcal{A} \in C^1(E;F)$. Furthermore, the identity

$$\operatorname{Im}\big(\mathcal{A}'(0,0)\big) = F$$

is equivalent to the result stated in Proposition 3.1. Therefore, we can apply Proposition 3.2 to $\mathcal{A}$ with $e_0 = (0,0)$ and $h_0 = (0,0)$. This ends the proof of Theorem 1.2.

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