

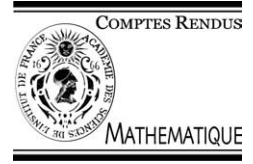


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Homological Algebra/Topology

## Hopf-cyclic homology and cohomology with coefficients

Piotr M. Hajac<sup>a,b</sup>, Masoud Khalkhali<sup>c</sup>, Bahram Rangipour<sup>c</sup>, Yorck Sommerhäuser<sup>d</sup>

<sup>a</sup> *Instytut Matematyczny, Polska Akademia Nauk, ul. Śniadeckich 8, Warszawa, 00-956 Poland*

<sup>b</sup> *Katedra Metod Matematycznych Fizyki, Uniwersytet Warszawski, ul. Hoża 74, Warszawa, 00-682 Poland*

<sup>c</sup> *Department of Mathematics, University of Western Ontario, London ON, Canada*

<sup>d</sup> *Mathematisches Institut, Universität München, Theresienstr. 39, 80333 München, Germany*

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### Abstract

Following the idea of an invariant differential complex, we construct general-type cyclic modules that provide the common denominator of known cyclic theories. The cyclicity of these modules is governed by Hopf-algebraic structures. We prove that the existence of a cyclic operator forces a modification of the Yetter–Drinfeld compatibility condition leading to the concept of a stable anti-Yetter–Drinfeld module. This module plays the role of the space of coefficients in the thus obtained cyclic cohomology of module algebras and coalgebras, and the cyclic homology and cohomology of comodule algebras. Along the lines of Connes and Moscovici, we show that there is a pairing between the cyclic cohomology of a module coalgebra acting on a module algebra and closed 0-cocycles on the latter. The pairing takes values in the usual cyclic cohomology of the algebra. Similarly, we argue that there is an analogous pairing between closed 0-cocycles of a module coalgebra and the cyclic cohomology of a module algebra. **To cite this article:** *P.M. Hajac et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).*

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### Résumé

**Homologie et cohomologie Hopf-cycliques à coefficients.** Suivant l'idée d'un complexe différentiel invariant, nous construisons des modules cycliques de type général qui fournissent un dénominateur commun aux théories cycliques connues. Le caractère cyclique de ces modules est gouverné par des structures Hopf-algébriques. Nous montrons que l'existence d'un opérateur cyclique oblige à une modification de la condition de compatibilité de Yetter–Drinfeld et mène au concept de module anti-Yetter–Drinfeld stable. Ce module joue le rôle d'espace de coefficients pour la cohomologie de modules algèbres et de modules cogèbres ainsi obtenue, ainsi que pour l'homologie et la cohomologie cycliques de comodules algèbres. Comme l'ont fait Connes et Moscovici pour leur cohomologie, nous montrons qu'il existe un appariement entre la cohomologie cyclique d'un module cogèbre agissant sur un module algèbre et les 0-cycles fermés sur ce dernier. Cet appariement prend ses valeurs dans la cohomologie cyclique usuelle de l'algèbre. De façon similaire, nous établissons un appariement analogue entre les 0-cycles fermés d'un module cogèbre et la cohomologie cyclique d'un module algèbre. **Pour citer cet article:** *P.M. Hajac et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).*

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*E-mail addresses:* [p.m.hajac@impan.gov.pl](mailto:p.m.hajac@impan.gov.pl) (P.M. Hajac), [masoud@uwo.ca](mailto:masoud@uwo.ca) (M. Khalkhali), [brangipo@uwo.ca](mailto:brangipo@uwo.ca) (B. Rangipour), [sommerh@mathematik.uni-muenchen.de](mailto:sommerh@mathematik.uni-muenchen.de) (Y. Sommerhäuser).

*URL:* <http://www.fuw.edu.pl/~pmh> (P.M. Hajac).

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### Version française abrégée

Le résultat principal de cet article est la construction de modules cycliques qui nous permettent de définir la cohomologie cyclique de  $H$ -modules cogèbres et de  $H$ -modules algèbres à coefficients dans des modules anti-Yetter–Drinfeld stables. De plus, si un  $H$ -module cogèbre  $C$  agit sur un  $H$ -module algèbre  $A$  au sens où nous avons une application linéaire  $C \otimes A \rightarrow A$  satisfaisant  $c(ab) = (c^{(1)}a)(c^{(2)}b)$ ,  $c1 = \varepsilon(c)$ ,  $h(ca) = (hc)a$ , pour tous  $h \in H$ ,  $c \in C$ , alors il existe un certain appariement entre ces cohomologies pour  $C$  et  $A$ .

**Théorème 0.1.** *Soit  $H$  une algèbre de Hopf avec une antipode bijective et soit  $M$  un module anti-Yetter–Drinfeld sur  $H$  stable. Alors les formules (2)–(5) (resp. (8)–(11)) définissent une structure de module cyclique sur  $\{M \otimes_H C^{\otimes(n+1)}\}_{n \in \mathbb{N}}$  (resp.  $\{\text{Hom}^H(M \otimes A^{\otimes(n+1)}, k)\}_{n \in \mathbb{N}}$ ) pour tout  $H$ -module cogèbre  $C$  (resp. algèbre  $A$ ). Si de plus  $C$  agit sur  $A$  et  $\mathbb{Q} \subseteq k$ , alors, pour tout  $n \in \mathbb{N}$ , les formules (12) et (13) définissent, respectivement, les homomorphismes suivants :*

$$HC_H^n(C, M) \otimes HC_H^0(A, M) \xrightarrow{\#_{n,0}} HC^n(A), \quad HC_H^0(C, M) \otimes HC_H^n(A, M) \xrightarrow{\#_{0,n}} HC^n(A). \quad (1)$$

De façon similaire, pour tout comodule algèbre, les formules (15)–(18) et (19)–(22) déterminent respectivement notre construction d’une homologie et d’une cohomologie cycliques à coefficients dans un module anti-Yetter–Drinfeld stable.

## 1. Introduction

Ever since its invention, among the main applications of cyclic cohomology was the computation of  $K$ -theoretical invariants. While enhancing the feasibility of such computations, Connes and Moscovici discovered a new type of cyclic cohomology, notably the cyclic cohomology of Hopf algebras [4]. Invariant cyclic homology introduced in [7] generalizes the Connes–Moscovici theory and its dual version [8]. It shows that passage from the cyclic homology of algebras to the cyclic cohomology of Hopf algebras is remarkably similar to passage from de Rham cohomology to cohomology of Lie algebras via invariant de Rham cohomology [2]. The idea of employing invariant complexes proved to be a key in resolving by significantly more effective means the technical challenge of showing that the  $(n + 1)$ th power of the cyclic operator  $\tau_n$  is identity [5, p. 102], and allowed the introduction of higher-dimensional coefficients.

We continue this strategy herein. Our motivation is to obtain and understand computable invariants of  $K$ -theory. The aim of this paper is to provide a general framework for cyclic theories whose cyclicity is based on Hopf-algebraic structures. We refer to such homologies and cohomologies as *Hopf-cyclic*. The definition and sources of examples of stable anti-Yetter–Drinfeld modules that play the role of coefficients for Hopf-cyclic theory are provided in the preceding article [6]. (Note that modular pairs in involution are precisely 1-dimensional stable anti-Yetter–Drinfeld modules.) Here we construct cyclic module structures on invariant complexes for module coalgebras and module algebras, respectively. It turns out that the cyclic cohomology of Hopf algebras is a special case of the former, whereas both twisted [9] and usual cyclic cohomology are special cases of the latter. As a result of this generality, we obtain now a very short proof of the Connes–Moscovici key result [5, Theorem 1]. Furthermore, as  $\delta$ -invariant  $\sigma$ -traces can be viewed as closed 0-cocycles on a module algebra, our pairing for Hopf-cyclic cohomology generalizes the Connes–Moscovici transfer map [5, Proposition 1] from the cyclic cohomology of Hopf algebras to ordinary cyclic cohomology. Finally, we end this article by deriving Hopf-cyclic homology and cohomology of comodule algebras. This extends the formalism for comodule algebras provided in [7].

The coproduct, counit and antipode of  $H$  are denoted by  $\Delta$ ,  $\varepsilon$  and  $S$ , respectively. For the coproduct we use the notation  $\Delta(h) = h^{(1)} \otimes h^{(2)}$ , for a left coaction on  $M$  we write  ${}_M\Delta(m) = m^{(-1)} \otimes m^{(0)}$ , and for a right coaction  $\Delta_M(m) = m^{(0)} \otimes m^{(1)}$ . The summation symbol is suppressed everywhere. We assume all algebras to be associative,

unital and over the same ground field  $k$ . Partly for the sake of simplicity, we also assume that the antipodes of all Hopf algebras under consideration are bijective.

## 2. Cyclic cohomology of module algebras and coalgebras

An algebra  $A$  which is a module over a Hopf algebra  $H$  such that  $h(ab) = (h^{(1)}a)(h^{(2)}b)$  and  $h1 = \varepsilon(h)$ ,  $\forall h \in H$ ,  $a, b \in A$ , is called an  $H$ -module algebra. Similarly, a coalgebra  $C$  which is a module over a Hopf algebra  $H$  such that  $\Delta(hc) = (h^{(1)}c^{(1)}) \otimes (h^{(2)}c^{(2)})$  and  $\varepsilon(hc) = \varepsilon(h)\varepsilon(c)$ ,  $\forall h \in H, c \in C$ , is called an  $H$ -module coalgebra. In this section we construct cyclic modules for both module algebras and coalgebras.

We begin with the coalgebra case. First, we take a left  $H$ -comodule  $M$ ,  $H$ -module coalgebra  $C$ , and define an auxiliary simplicial complex  $\mathcal{C}^n(C, M) := M \otimes C^{\otimes(n+1)} = (M \otimes C) \otimes C^{\otimes n}$ ,  $n \in \mathbb{N}$ , by endowing  $M \otimes C$  with the bicomodule structure  ${}_{M \otimes C} \Delta(m \otimes c) = m^{(-1)}c^{(1)} \otimes (m^{(0)} \otimes c^{(2)})$ ,  $\Delta_{M \otimes C}(m \otimes c) = (m \otimes c^{(1)}) \otimes c^{(2)}$ . Then the standard formulas for faces and degeneracies on the complex  $\{\text{bicomodule} \otimes C^{\otimes n}\}_{n \in \mathbb{N}}$  translate immediately into

$$\delta_i(m \otimes c_0 \otimes \cdots \otimes c_{n-1}) = m \otimes c_0 \otimes \cdots \otimes c_i^{(1)} \otimes c_i^{(2)} \otimes c_{n-1}, \quad 0 \leq i < n \quad (\text{faces}), \tag{2}$$

$$\delta_n(m \otimes c_0 \otimes \cdots \otimes c_{n-1}) = m^{(0)} \otimes c_0^{(2)} \otimes c_1 \otimes \cdots \otimes c_{n-1} \otimes m^{(-1)}c_0^{(1)} \quad (\text{flip-over face}), \tag{3}$$

$$\sigma_i(m \otimes c_0 \otimes \cdots \otimes c_{n+1}) = m \otimes c_0 \otimes \cdots \otimes \varepsilon(c_{i+1}) \otimes \cdots \otimes c_{n+1}, \quad 0 \leq i \leq n \quad (\text{degeneracies}). \tag{4}$$

It is straightforward to check that these morphisms together with

$$\tau_n(m \otimes c_0 \otimes \cdots \otimes c_n) = m^{(0)} \otimes c_1 \otimes \cdots \otimes c_n \otimes m^{(-1)}c_0 \tag{5}$$

form a paracyclic structure on  $\{\mathcal{C}^n(C, M)\}_{n \in \mathbb{N}}$ . Now let us assume that  $M$  is also a right  $H$ -module. We can treat  $C^{\otimes(n+1)}$  as a left  $H$ -module via the diagonal action (i.e.,  $h(c_0 \otimes \cdots \otimes c_n) = h^{(1)}c_0 \otimes \cdots \otimes h^{(n+1)}c_n$ ) and define the quotient (invariant) complex  $\mathcal{C}_H^n(C, M) := M \otimes_H C^{\otimes(n+1)}$ ,  $n \in \mathbb{N}$ . Except for  $\tau_n$  and  $\delta_n$ , it is clear that the aforementioned morphisms are well defined on the quotient complex. The key result of this paper is that  $\tau_n$  is well defined if and only if  $M$  is an anti-Yetter–Drinfeld module.

**Theorem 2.1.** *Let  $M$  be a left  $H$ -comodule and a right  $H$ -module. Then the map  $\tau_n$  given by the formula (5) is well defined on  $M \otimes_H C^{\otimes(n+1)}$  for any  $n \in \mathbb{N}$  and any  $H$ -module coalgebra  $C$  if and only if  $M$  is an anti-Yetter–Drinfeld module. If furthermore  $M$  is stable, then  $\{\mathcal{C}_H^n(C, M)\}_{n \in \mathbb{N}}$  is a cyclic module via (2)–(5).*

**Proof.** First note that  $\tau_n$  is well defined on  $M \otimes_H C^{\otimes(n+1)}$  if and only if

$$m^{(0)} \otimes_H (h^{(2)}(c_1 \otimes \cdots \otimes c_n) \otimes m^{(-1)}h^{(1)}c_0) = (hm)^{(0)} \otimes_H (c_1 \otimes \cdots \otimes c_n \otimes (hm)^{(-1)}c_0). \tag{6}$$

The equality  $m^{(0)} \otimes_H (h^{(2)} \otimes m^{(-1)}h^{(1)}) = (hm)^{(0)} \otimes_H (1 \otimes (hm)^{(-1)})$  evidently implies (6) for any  $n$  and  $C$ , and follows if we assume that (6) holds for any  $n$  and  $C$ . (E.g., take  $n = 2$ ,  $C = H$  and  $c_0 = 1 = c_1$ .) Now, observe that there is an isomorphism  $\Phi : H \otimes H \rightarrow H \otimes H$ ,  $\Phi(h \otimes k) = h^{(1)} \otimes S(h^{(2)})k$ ,  $\Phi^{-1}(h \otimes k) = h^{(1)} \otimes h^{(2)}k$ .  $\Phi$  is left  $H$ -linear if we view the domain as a left  $H$ -module via the diagonal action and the counter-domain as a left  $H$ -module via the multiplication in the first factor. Thus we have an isomorphism

$$M \otimes_H (H \otimes H) \xrightarrow{\text{id} \otimes_H \Phi} (M \otimes_H H) \otimes H \cong M \otimes H. \tag{7}$$

Applying this isomorphism to the equality below Eq. (6) yields  $m^{(0)}h^{(2)} \otimes S(h^{(3)})m^{(-1)}h^{(1)} = (hm)^{(0)} \otimes (hm)^{(-1)}$ . This is equivalent to the anti-Yetter–Drinfeld condition, as desired. Next, since  $\delta_n = \tau_n \delta_0$ , all morphisms are well defined on  $\mathcal{C}_H^n(C, M)$ , if  $M$  is an anti-Yetter–Drinfeld module. If  $M$  is also stable, then the equality  $(\tau_n)^{n+1}(m \otimes_H (c_0 \otimes \cdots \otimes c_n)) = m^{(0)}m^{(-1)} \otimes_H (c_0 \otimes \cdots \otimes c_n)$  entails that  $(\tau_n)^{n+1} = \text{id}$ , as needed.  $\square$

For  $C = H$  and  $M = {}^\sigma k_\delta$ , the complex  $\{\mathcal{C}_H^n(C, M)\}_{n \in \mathbb{N}}$  becomes the cyclic module of Connes–Moscovici [5]. As an intermediate step, one can take the Hopf cotriples of [7].

We can proceed much the same way in the algebra case. Again, we first have an auxiliary complex  $\mathcal{C}^n(A, M) := \text{Hom}(M \otimes A^{\otimes(n+1)}, k) \cong \text{Hom}(A^{\otimes n}, \text{Hom}(M \otimes A, k))$ . Here  $A$  is a left  $H$ -module algebra and  $M$  a left  $H$ -comodule. The formulas  $b(m \otimes a) = m^{(0)} \otimes (S^{-1}(m^{(-1)})b)a$ ,  $(m \otimes a)b = m \otimes ab$  turn  $M \otimes A$ , and consequently  $\text{Hom}(M \otimes A, k)$ , into a bimodule. Thus we can use the standard formulas [10, p. 37] to define a simplicial structure on  $\{\mathcal{C}^n(A, M)\}_{n \in \mathbb{N}}$ . To define an invariant subcomplex, we assume now  $M$  to be also a right  $H$ -module, view  $A^{\otimes(n+1)}$  as a left  $H$ -module via the diagonal action,  $M \otimes A^{\otimes(n+1)}$  as a right  $H$ -module via  $(m \otimes \tilde{a})h = mh^{(1)} \otimes S(h^{(2)})\tilde{a}$ , and  $k$  as a right  $H$ -module via the counit map. Then we can restrict to right  $H$ -linear holomorphisms and put  $\mathcal{C}_H^n(A, M) := \text{Hom}_H(M \otimes A^{\otimes(n+1)}, k)$ ,  $n \in \mathbb{N}$ . Also along the same lines, one can prove that if  $M$  satisfies the anti-Yetter–Drinfeld condition, then the morphisms

$$(\delta_i f)(m \otimes a_0 \otimes \cdots \otimes a_n) = f(m \otimes a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n), \quad 0 \leq i < n, \tag{8}$$

$$(\delta_n f)(m \otimes a_0 \otimes \cdots \otimes a_n) = f(m^{(0)} \otimes (S^{-1}(m^{(-1)})a_n)a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}), \tag{9}$$

$$(\sigma_i f)(m \otimes a_0 \otimes \cdots \otimes a_n) = f(m \otimes a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes \cdots \otimes a_n), \quad 0 \leq i \leq n, \tag{10}$$

$$(\tau_n f)(m \otimes a_0 \otimes \cdots \otimes a_n) = f(m^{(0)} \otimes S^{-1}(m^{(-1)})a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}), \tag{11}$$

define a paracyclic structure on  $\{\mathcal{C}_H^n(A, M)\}_{n \in \mathbb{N}}$ . (This means that all axioms of the cyclic structure are satisfied except for  $(\tau_n)^{n+1} = \text{Id}$ .) Adding the stability assumption on  $M$ , one obtains:

**Theorem 2.2.** *If  $M$  is a stable anti-Yetter–Drinfeld module and  $A$  is a left  $H$ -module algebra, then the maps (8)–(11) define a cyclic module structure on  $\{\mathcal{C}_H^n(A, M)\}_{n \in \mathbb{N}}$ .*

For  $H = k = M$  we recover the usual cyclic cohomology of algebras. For the Laurent polynomial Hopf algebra ( $H = k[\sigma, \sigma^{-1}]$ ) and  $M = {}^\sigma k_\varepsilon$  we obtain the twisted cyclic cohomology [9]. We can also take as a stable anti-Yetter–Drinfeld module a Hopf algebra  $K$  thought of as a left comodule over itself via the coproduct, and as a right module over itself via  $k \cdot h = S(h^{(2)})kh^{(1)}$ . Then we arrive at the construction considered in [1]. This is a special case of the construction in [6, Proposition 3.1]:  $P = K^{\text{cop}} = H$ .

Finally, if we take  $M = {}^\sigma k_\delta$ , we can view a  $\delta$ -invariant  $\sigma$ -trace of [5, Definition 1] as a closed 0-cocycle in our complex. On the other hand, if  $\mathbb{Q} \subseteq k$ , we can define cyclic cohomology as follows:  $\mathcal{Z}_H^n(*, M) := \text{Ker } b_n|_{\text{Ker}(\tau_n - (-1)^n)}$ ,  $\mathcal{B}_H^n(*, M) := \text{Im } b_{n-1}|_{\text{Ker}(\tau_{n-1} - (-1)^{n-1})}$ ,  $HC_H^n(*, M) := \mathcal{Z}_H^n(*, M)/\mathcal{B}_H^n(*, M)$ ,  $n \in \mathbb{N}$ . This brings us to the following generalization of [5, Proposition 1]:

**Proposition 2.3.** *Let  $C$  be a left  $H$ -module coalgebra,  $A$  a left  $H$ -module algebra, and  $M$  a stable right-left anti-Yetter–Drinfeld module over  $H$ . Assume also that  $\mathbb{Q} \subseteq k$  and  $C$  acts on  $A$  in the sense that we have a linear map  $C \otimes A \rightarrow A$  satisfying  $c(ab) = (c^{(1)}a)(c^{(2)}b)$ ,  $c1 = \varepsilon(c)$ ,  $h(ca) = (hc)a$ , for all  $h \in H$ ,  $c \in C$ ,  $a, b \in A$ . Then, for all  $n \in \mathbb{N}$ , the formulas*

$$([m \otimes_H (c_0 \otimes \cdots \otimes c_n)]\#f)(a_0 \otimes \cdots \otimes a_n) = f(m \otimes_H (c_0 a_0) \cdots (c_n a_n)), \tag{12}$$

$$((m \otimes_H c)\#[f])(a_0 \otimes \cdots \otimes a_n) = f(m \otimes_H (c^{(1)}a_0) \otimes \cdots \otimes (c^{(n+1)}a_n)), \tag{13}$$

define, respectively, the following homomorphisms:

$$HC_H^n(C, M) \otimes \mathcal{Z}_H^0(A, M) \xrightarrow{\#_{n,0}} HC^n(A), \quad \mathcal{Z}_H^0(C, M) \otimes HC_H^n(A, M) \xrightarrow{\#_{0,n}} HC^n(A). \tag{14}$$

We conjecture that one can construct along these line a cup product between the Hopf-cyclic cohomology of a module coalgebra and the Hopf-cyclic cohomology of a module algebra. This would generalize, in some sense, the cup product in [3, p. 105].

### 3. Cyclic theory for comodule algebras

In this section we define cyclic cohomology with coefficients in a right-right stable anti-Yetter–Drinfeld module, and cyclic homology with coefficients in a left-left stable anti-Yetter–Drinfeld module. The latter case generalizes theory introduced in [7]. In both cases, we assume that  $A$  is a right  $H$ -comodule algebra. Our strategy for constructing cyclic theory is the same as before: bimodule structure  $\rightsquigarrow$  simplicial structure  $\rightsquigarrow$  paracyclic structure  $\rightsquigarrow$  invariant complex  $\rightsquigarrow$  cyclic theory. In the cohomology case, we define the invariant subcomplex as  $\mathcal{C}^{n,H}(A, M) := \text{Hom}^H(A^{\otimes(n+1)}, M)$ ,  $n \in \mathbb{N}$ . Here  $M$  is a right-right stable anti-Yetter–Drinfeld module,  $A^{\otimes(n+1)}$  is viewed as a right comodule via the diagonal coaction, and  $\text{Hom}^H$  stands for right  $H$ -colinear holomorphisms. The following maps define, respectively, faces, degeneracies and cyclic operators on  $\{\mathcal{C}^{n,H}(A, M)\}_{n \in \mathbb{N}}$ .

$$(\delta_i f)(a_0 \otimes \cdots \otimes a_n) = f(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n), \quad 0 \leq i < n, \tag{15}$$

$$(\delta_n f)(a_0 \otimes \cdots \otimes a_n) = f(a_n^{(0)} a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}) a_n^{(1)}, \tag{16}$$

$$(\sigma_i f)(a_0 \otimes \cdots \otimes a_n) = f(a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes \cdots \otimes a_n), \quad 0 \leq i < n, \tag{17}$$

$$(\tau_n f)(a_0 \otimes \cdots \otimes a_n) = f(a_n^{(0)} \otimes a_0 \otimes \cdots \otimes a_{n-1}) a_n^{(1)}. \tag{18}$$

In the homology case, we take a left-left stable anti-Yetter–Drinfeld module  $M$  and define the invariant subcomplex via the cotensor product:  $\mathcal{C}_n^H(A, M) := A^{\otimes(n+1)} \square_H M$ ,  $n \in \mathbb{N}$ . (Recall that  $X \square_H Y := \text{Ker}(\Delta_X \otimes \text{id} - \text{id} \otimes \Delta_Y)$ .) Here, again, we view  $A^{\otimes(n+1)}$  as a right  $H$ -comodule via the diagonal coaction. The following homomorphisms establish, respectively, the desired faces, degeneracies and cyclic operators on  $\{\mathcal{C}_n^H(A, M)\}_{n \in \mathbb{N}}$ .

$$d_i(a_0 \otimes \cdots \otimes a_n \otimes m) = a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \otimes m, \quad 0 \leq i < n, \tag{19}$$

$$d_n(a_0 \otimes \cdots \otimes a_n \otimes m) = a_n^{(0)} a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes a_n^{(1)} m, \tag{20}$$

$$s_i(a_0 \otimes \cdots \otimes a_n \otimes m) = a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes \cdots \otimes a_n \otimes m, \quad 0 \leq i \leq n, \tag{21}$$

$$t_n(a_0 \otimes \cdots \otimes a_n \otimes m) = a_n^{(0)} \otimes a_0 \otimes \cdots \otimes a_{n-1} \otimes a_n^{(1)} m. \tag{22}$$

**Theorem 3.1.** *Let  $A$  be a right  $H$ -comodule algebra and  $M$  a right-right (resp. left-left) stable anti-Yetter–Drinfeld module over  $H$ . Then the formulas (15)–(18) (resp. (19)–(22)) define a cyclic module structure on  $\{\text{Hom}^H(A^{\otimes(n+1)}, M)\}_{n \in \mathbb{N}}$  (resp.  $\{A^{\otimes(n+1)} \square_H M\}_{n \in \mathbb{N}}$ ).*

Now one can either define the cyclic cohomology (resp. homology) of  $A$  with coefficients in  $M$  as the total cohomology (resp. homology) of an appropriate double complex [10, p. 77 and p. 72], or assume that  $\mathbb{Q} \subseteq k$  and proceed as above Proposition 2.3.

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