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Complex Analysis

Levi-flat extensions from a part of the boundary

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Abstract

Let G be a bounded domain in $\mathbb{C}\times\mathbb{R}$ such that $G\times\mathbb{R}\subset\mathbb{C}^2$ is strictly pseudoconvex and U an open subset of bG. We define an open subset Ω^U of $\overline{G}$ with the property $\Omega^U\cap bG=U$ such that the following extension theorem holds true: for every $\varphi\in C(U)$ there exist two functions $\Phi^{\pm}\in C(\Omega^U)$ such that $\Phi^{\pm}|_U=\varphi$ and the graphs $\Gamma(\Phi^{\pm})$ of $\Phi^{\pm}$ are Levi-flat over $\Omega^U\cap G$. Moreover, for each $\Phi\in C(\Omega^U)$ such that $\Phi|_U=\varphi$ and $\Gamma(\Phi)$ is Levi-flat over $\Omega^U\cap G$ one has $\Phi^-\leqslant\Phi\leqslant\Phi^+$ on Ω^U. We also show that if G is diffeomorphic to a 3-ball and U is the union of simply-connected domains each of which is contained either in the "upper" or in the "lower" part of bG (with respect to the u-direction), then Ω^U is the maximal domain of Levi-flat extensions for some function $\varphi\in C(U)$. ***To cite this article: N. Shcherbina, G. Tomassini, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Extensions Levi-plates d'une partie du domaine borné. Soient G un domaine borné dans $\mathbb{C}\times\mathbb{R}$ tel que $G\times\mathbb{R}\subset\mathbb{C}^2$ soit strictement pseudoconvexe et U un sous-ensemble ouvert de bG. On définit un sous-ensemble ouvert Ω^U de $\overline{G}$ avec la propriété $\Omega^U\cap bG=U$ tel que le résultat suivant soit valable : pour toute fonction $\varphi\in C(U)$ il existe deux fonctions $\Phi^{\pm}\in C(\Omega^U)$ telles que $\Phi^{\pm}|_U=\varphi$ et les graphes $\Gamma(\Phi^{\pm})$ de $\Phi^{\pm}$ soient Levi-plats sur $\Omega^U\cap G$. De plus, pour toute $\Phi\in C(\Omega^U)$ telle que $\Phi|_U=\varphi$ et $\Gamma(\Phi)$ soit Levi-plat sur $\Omega^U\cap G$, on a $\Phi^-\leqslant\Phi\leqslant\Phi^+$ sur Ω^U. On démontre aussi que si G est difféomorphe à la boule et U est une réunion de domaines simplement connexes, chacun d'entre eux étant contenu soit dans la partie supérieure, soit dans la partie inférieure de bG (par rapport à la direction u), alors Ω^U est le domaine maximal pour l'extension Levi-plate d'une certaine fonction $\varphi\in C(U)$. ***Pour citer cet article : N. Shcherbina, G. Tomassini, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

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Soient G un domaine dans $\mathbb{C}_z \times \mathbb{R}_u \subset \mathbb{C}^2_{z,w}$ ($w = u + \mathrm{i}v$). On dit que le graphe $\Gamma(\Phi)$ d'une fonction continue $\Phi : \overline{G} \to \mathbb{R}_v$ est *Levi-plat* s'il est feuilleté par des courbes complexes. Soit $\varphi : bG \to \mathbb{R}_v$ une fonction continue et $\Gamma(\varphi)$ son graphe. Le problème de prolonger φ par une fonction continue $\Phi : \overline{G} \to \mathbb{R}_v$ dont le graphe soit Levi-plat sur G a été étudié par plusieurs auteurs ([1,2,4–6,3] dans le cas où G est borné et [7] dans le cas où G est non borné). Dans ce papier on s'intéresse à une version *semi-locale* du problème, à savoir celle de l'extension Levi-plate du graphe $\Gamma(\varphi)$ d'une fonction continue donnée sur un ouvert U de bG. Précisément, nous définissons une notion d'enveloppe $\Omega^U \subset \overline{G}$ de l'ouvert U et montrons que toute fonction continue $\varphi : U \to \mathbb{R}_v$ a une extension Levi-plate à Ω^U.

Soit $E_1 \Subset E_2 \Subset \cdots$ une suite exhaustive de compacts dans U. Pour tout $n \in \mathbb{N}$ soient $K_n = E_n \times [-n, n]$ et $\widehat{K}_n$ la $\mathcal{A}(G \times \mathbb{R}_v)$-enveloppe de K_n. Alors le sous-ensemble $W = \bigcup_n \widehat{K}_n$ est ouvert dans $\overline{G} \times \mathbb{R}_v$ et invariant par rapport aux translations dans la direction v, i.e. $W = \Omega^U \times \mathbb{R}_v$ pour un certain domaine $\Omega^U \subset \overline{G}$.

Le résultat central du travail est contenu dans le théorème suivant :

Théorème principale. *Soit G un domaine borné dans $\mathbb{C}_z \times \mathbb{R}_u$ tel que le domaine $G \times \mathbb{R}_v \subset \mathbb{C}^2_{z,w}$ soit strictement pseudoconvexe. Soit U un sous-ensemble ouvert dans bG. Pour toute fonction continue $\varphi \in C(U)$ il existe deux extensions Φ^+, Φ^- sur Ω^U qui jouissent des propriétés suivantes : les fonctions $\Phi^\pm$ sont continues dans Ω^U, leurs graphes $\Gamma(\Phi^\pm)$ sont Levi-plats sur $\Omega^U \cap G$ et $\Phi^\pm|_U = \varphi$.*

De plus, pour toute fonction $\Phi \in C(\Omega^U)$ telle que le graphe $\Gamma(\Phi)$ soit Levi-plat sur $\Omega^U \cap G$ et que $\Phi|_U = \varphi$ aux points $(z,u) \in \Omega^U$, on a

$$\Phi^-(z,u) \leqslant \Phi(z,u) \leqslant \Phi^+(z,u).$$

En ce qui concerne la maximalité de Ω^U on a le résultat suivant

Théorème. *Soit $G \subset \mathbb{C}_z \times \mathbb{R}_u$ un domaine borné difféomorphe à la* 3*-boule et tel que le domaine $G \times \mathbb{R}_v$ soit strictement pseudoconvexe. Soit U un sous-ensemble ouvert de bG réunion disjointe de domaines simplement connexes, chacun d'entre eux étant contenu soit dans la partie supérieure, soit dans la partie inférieure de bG* (*par rapport à la direction u*)*. Alors il existe une fonction $\varphi \in C(U)$ telle que Ω^U soit le domaine maximal pour l'extension Levi-plate du graphe de φ.*

1. Introduction

Let G be a domain in $\mathbb{C}_z \times \mathbb{R}_u \subset \mathbb{C}^2_{z,w}$ ($w = u + \mathrm{i}v$). Let $\varphi : bG \to \mathbb{R}_v$ be a continuous function and $\Gamma(\varphi)$ its graph. The extendability of φ to a continuous function $\Phi : \overline{G} \to \mathbb{R}_v$ with Levi-flat (i.e., foliated by holomorphic curves) graph $\Gamma(\Phi)$ has been studied by several authors ([1,2,4–6,3] in the case G is bounded, and [7] in the unbounded case) under the assumption that bG is smooth and strictly pseudoconvex (i.e., $bG \times \mathbb{R}_v$ is a strictly pseudoconvex hypersurface of $\mathbb{C}^2_{z,w}$). In this paper we study a *semi-local* version of the extension problem, namely, when the function φ is prescribed on an open subset U of bG, where bG is smooth and strictly pseudoconvex. Namely, we define a notion of the hull $\Omega^U \subset \overline{G}$ of U and prove that every continuous function $\varphi : U \to \mathbb{R}_v$ has a Levi-flat extension to Ω^U.

2. Results

Given an open subset D of $\mathbb{C}^2_{z,w}$ we denote $\mathcal{A}(D)$ the Fréchet algebra $\mathcal{O}(D) \cap C(\overline{D})$. If H is a subset of D, let us denote by $\widehat{H}^{\mathcal{A}(D)}$ (or simply by $\widehat{H}$) the $\mathcal{A}(D)$-hull of H.

Let $E_1 \Subset E_2 \Subset \cdots$ be an exhaustion of U by compact subsets such that $E_n \Subset \mathring{E}_{n+1}$ for every $n \in \mathbb{N}$. Consider for each $n \in \mathbb{N}$ the set $K_n = E_n \times [-n, n]$, and the $\mathcal{A}(G \times \mathbb{R}_v)$-hull $\widehat{K}_n$ of K_n. Define $W = \bigcup_n \widehat{K}_n \subset \overline{G} \times \mathbb{R}_v$.

Theorem 2.1. *The set W has the following properties*:

(1) *W is open in $\overline{G} \times \mathbb{R}_v$ and $W \cap (G \times \mathbb{R}_v)$ is pseudoconvex.*
(2) *W is invariant under translation in v-direction. In particular, there is an open subset Ω^U of $\overline{G}$ such that $W = \Omega^U \times \mathbb{R}_v$.*

If, moreover, U is the union of simply-connected subdomains of bG, then

(3) *W is the CR-hull of $U \times \mathbb{R}_v$.*

In the case when G is a topological 3-ball or U has only simply-connected components, the domain Ω^U enjoies the following property.

Theorem 2.2. *Let $G \subset \mathbb{C}_z \times \mathbb{R}_u$ be a bounded domain with smooth boundary such that $G \times \mathbb{R}_v \subset \mathbb{C}^2_{z,w}$ is strictly pseudoconvex. Let U be an open subset of bG and let $U_1, U_2, U_3, \ldots$ be the connected components of U. Assume that at least one of the following conditions is satisfied*:

(i) *G is diffeomorphic to a* 3*-ball.*
(ii) *All connected components U_n of U are simply-connected.*

Then the domains Ω^{U_m} are connected and disjoint and, moreover, $\Omega^U = \bigcup_m \Omega^{U_m}$.

Remark 1. Note that the statement of Theorem 2.2 is in general false, if both conditions (i) and (ii) are not satisfied, as it is shown by the following

Example. Let G be the solid torus in $\mathbb{C}_z \times \mathbb{R}_u$ defined by the inequality $(|z| - 2)^2 + u^2 < 1$. By an easy direct computation one shows that the domain $G \times \mathbb{R}_v \subset \mathbb{C}^2_{z,w}$ is strictly pseudoconvex (see also Theorem 1 in [3]). Consider an open subset $U = U_1 \cup U_2$, where

$$U_1 = \left\{(z, u) \in bG\colon\ |z| < \frac{3}{2}\right\}, \qquad U_2 = \left\{(z, u) \in bG\colon\ |z| > \frac{5}{2}\right\}.$$

Then the connected components U_1 and U_2 of U are obviously disjoint, but the set $\Omega^U = \{(z, u) \in G\colon\ |u| < \sqrt{3}/2\}$ is connected (the set $\Omega^U \times \mathbb{R}_v$ is foliated by annuli $A_c = \{(z, w) \in G \times \mathbb{R}_v\colon\ u = \operatorname{Re} C,\ v = \operatorname{Im} C\}$ where C satisfyies the inequality $|\operatorname{Re} C| < \sqrt{3}/2\}$.

Let, as above, $G \subset \mathbb{C}_z \times \mathbb{R}_u$ be a domain such that $G \times \mathbb{R}_v \subset \mathbb{C}^2_{z,w}$ is strictly pseudoconvex, U an open subset of bG and $\varphi : U \to \mathbb{R}_v$ a continuous function. Consider a strictly increasing exhaustion $E_1 \Subset E_2 \Subset \cdots \subset U$ of U by compact subsets and then for each $n \in \mathbb{N}$ consider the sets

$$M_n^+ = \bigl\{(z, w) \in E_n \times \mathbb{R}_v\colon\ \varphi(z, u) + 1/n \leqslant v \leqslant n\bigr\}$$

and

$$M_n^- = \bigl\{(z, w) \in E_n \times \mathbb{R}_v\colon\ -n \leqslant v \leqslant \varphi(z, u) - 1/n\bigr\}.$$

Define $W^+ = \bigcup_{n=1}^{+\infty} \widehat{M}_n^+$ and $W^- = \bigcup_{n=1}^{+\infty} \widehat{M}_n^-$.

Then the following theorem holds.

Theorem 2.3. *$W^{\pm}$ are open subsets of the domain $\overline{G} \times \mathbb{R}_v$. They can be represented in the following form*:

(1) $W^+ = \{(z,w) \in \overline{G} \times \mathbb{R}_v\colon\ v > \Phi^+(z,u),\ (z,u) \in \Omega^U\}$,
(2) $W^- = \{(z,w) \in \overline{G} \times \mathbb{R}_v\colon\ v < \Phi^-(z,u),\ (z,u) \in \Omega^U\}$,

for some functions $\Phi^{\pm}$ defined on the set Ω^U.

Having defined the functions $\Phi^{\pm}$, we can formulate the central result of the paper.

Main Theorem. *Let G be a bounded domain in $\mathbb{C}_z \times \mathbb{R}_u$ such that the domain $G \times \mathbb{R}_v \subset \mathbb{C}^2_{z,w}$ is strictly pseudoconvex. Let U be an open subset of bG and Ω^U the defined above subdomain of $\overline{G}$. Then for every $\varphi \in C(U)$ there exist two continuous extensions Φ^+, Φ^- in Ω^U with the properties*: *the functions $\Phi^{\pm}$ are continuous on Ω^U, their graphs $\Gamma(\Phi^{\pm})$ are Levi-flat over $\Omega^U \cap G$ and $\Phi^{\pm}|_U = \varphi$.*

Moreover, for any function $\Phi \in C(\Omega^U)$ such that $\Gamma(\Phi)$ is Levi-flat over $\Omega^U \cap G$ and $\Phi|_U = \varphi$ one has

$$\Phi^-(z,u) \leqslant \Phi(z,u) \leqslant \Phi^+(z,u)$$

for each point $(z,u) \in \Omega^U$.

Remark 2. The functions $\Phi^{\pm}$ do not coincide in general, as it is shown by Example 8.1 in [7].

Sketch of the proof. First, we prove that there is an exhaustion $\Omega_1 \subset \Omega_2 \subset \cdots$ of Ω^U such that for each $n \in \mathbb{N}$ the domain $\Omega_n \times \mathbb{R}_v$ is piece-wise strictly pseudoconvex and, moreover, the sequence of sets $A_n = bG \cap \overline{\Omega}_n$ is an exhaustion of U. Then, for each $n \in \mathbb{N}$ and $s > 0$ we consider the set

$$B_{n,s} = \{P \in b\Omega_n\colon \operatorname{dist}(P, bG) \geqslant 1/s\}.$$

Further, we define a one-parameter family of continuous functions $\varphi_{n,t}$, $t \in \mathbb{R}$, on $b\Omega_n$, continuously depending on the parameter t and such that: $\varphi_{n,t} = \varphi$ on A_n (here φ is the given function on U), $\varphi_{n,t} = t$ on $B_{n,|t|}$, and $\varphi_{n,t_1} \leqslant \varphi_{n,t_2}$ if $t_1 \leqslant t_2$. Then, in view of the construction of Ω_n, we can apply Theorem 2 of [3] to the functions $\varphi_{n,t}$ and obtain functions $\Phi_{n,t} \in C(\overline{\Omega}_n)$ such that $\Phi_{n,t} = \varphi_{n,t}$ on $b\Omega_n$, for which the graph $\Gamma(\Phi_{n,t}) \setminus \Gamma(\varphi_{n,t})$ is (locally) foliated by 1-dimensional complex submanifolds. Define the functions $\widetilde{\Phi}^-$ and $\widetilde{\Phi}^+$ as

$$\widetilde{\Phi}^-(P) = \lim_{n\to+\infty} \lim_{t\to-\infty} \Phi_{n,t}(P) \quad \text{and} \quad \widetilde{\Phi}^+(P) = \lim_{n\to+\infty} \lim_{t\to+\infty} \Phi_{n,t}(P),$$

$P \in \Omega^U$. Then, using Levi-flatness of the graphs $\Gamma(\Phi_{n,t}) \setminus \Gamma(\varphi_{n,t})$ and the definition of W, we prove that the functions $\Phi_{n,t}$ are uniformly bounded on compact subsets of Ω^U for sufficiently big n. More precisely, we show that for each compact subset $K \subset \Omega^U$ there exist $C > 0$ and N such that $|\Phi_{n,t}(P)| \leqslant C$ for all $P \in K$, $t \in \mathbb{R}$ and $n \geqslant N$. This allows us to use the argument of Lemmas 3.2–3.5 in [6] and Lemma 6.2 and Proposition 6.3 in [7] for proving that the graphs $\Gamma(\widetilde{\Phi}^{\pm}) \setminus \Gamma(\varphi)$ are foliated by holomorphic curves. Next, we show that each holomorphic leaf contained in one of the graphs $\Gamma(\widetilde{\Phi}^{\pm}) \setminus \Gamma(\varphi)$ has an accumulation point in $\Gamma(\varphi)$. Then, using the argument of Lemmas 6.1 and 7.2 in [7], we conclude that the functions $\widetilde{\Phi}^{\pm}$ are continuous in Ω^U. Finally, we prove that continuity of $\widetilde{\Phi}^{\pm}$, Levi-flatness of $\Gamma(\widetilde{\Phi}^{\pm}) \setminus \Gamma(\varphi)$ and the definition of $\Phi^{\pm}$ imply that the functions $\widetilde{\Phi}^{\pm}$ coincide with $\Phi^{\pm}$, respectively.

The last statement of Main Theorem follows by “Kontinuitätsatz” from the definition of the sets $W^{\pm}$ and Levi-flatness of $\Gamma(\Phi) \setminus \Gamma(\varphi)$. □

As for the maximality of Ω^U we have the following

Theorem 2.4. *Let $G \subset \mathbb{C}_z \times \mathbb{R}_u$ be a bounded domain diffeomorphic to a 3-ball such that the domain $G \times \mathbb{R}_v$ is strictly pseudoconvex. Let U be an open subset of bG constituted by the disjoint union of simply-connected domains each of which is contained either in the "upper" or in the "lower" part of bG (with respect to the u-direction). Then there is a function $\varphi \in C(U)$ such that Ω^U is the maximal domain where the Levi-flat extension of the graph of φ can be defined.*

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