EIGENVALUATIONS

By Charles FAVRE and Mattias JONSSON[1]

ABSTRACT. – We study the dynamics in $\mathbf{C}^2$ of superattracting fixed point germs and of polynomial maps near infinity. In both cases we show that the asymptotic attraction rate is a quadratic integer, and construct a plurisubharmonic function with the adequate invariance property. This is done by finding an infinitely near point at which the map becomes rigid: the critical set is contained in a totally invariant set with normal crossings. We locate this infinitely near point through the induced action of the dynamics on a space of valuations. This space carries an $\mathbf{R}$-tree structure and conveniently encodes local data: an infinitely near point corresponds to an open subset of the tree. The action respects the tree structure and admits a fixed point—or eigenvaluation—which is attracting in a certain sense. A suitable basin of attraction corresponds to the desired infinitely near point.
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RÉSUMÉ. – Nous étudions la dynamique dans $\mathbf{C}^2$ des germes d'applications holomorphes superattractives et des applications polynomiales près de l'infini. Dans les deux cas, nous montrons que le taux asymptotique d'attraction (vers l'origine ou vers l'infini respectivement) est un entier quadratique, et nous construisons une fonction pluri-sous-harmonique vérifiant l'équation d'invariance correspondante. Pour cela, nous exhibons un point infiniment proche en lequel l'application devient rigide au sens où son ensemble critique est contenu dans une courbe à croisements normaux et totalement invariante. Nous localisons ce point infiniment proche en analysant l'action induite par l'application sur un espace adéquat de valuations. Cet espace est un arbre réel qui code de manière efficace des données locales liées aux singularités de courbes : un point infiniment proche correspond ainsi à un ouvert dans cet arbre. L'action respecte la structure d'arbre et admet un point fixe (appelé « valuation propre ») qui est attirant en un certain sens. Le bassin d'attraction de ce point correspond au point infiniment proche désiré.
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0. Introduction

Our objective in this paper is to study dynamics in $\mathbf{C}^2$ in two situations: holomorphic fixed point germs f at the origin and polynomial maps F at infinity. We shall refer to these two cases as *local* and *affine*[2], and we are interested in how fast orbits approach the origin and infinity, respectively.

In the affine case we can measure the rate of attraction to infinity by the algebraic degree $\deg(F)$. Indeed, for "most" points p near infinity we have $\|F(p)\| \simeq \|p\|^{\deg(F)}$. It is easy to see

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[1] Second author supported by the NSF, the Swedish Research Council and the Gustafsson foundation.
[2] The affine case refers to $\mathbf{C}^2$ with a fixed algebraic embedding into $\mathbf{P}^2$. The local case applies to a (classical) neighborhood of any smooth point on a surface.

that $d_\infty := \lim_{n\to\infty} \deg(F^n)^{1/n}$ exists [3] . However, it is not obvious in what sense orbits are actually attracted at the rate d_∞.

Similarly, in the local case, we define $c(f) \geqslant 1$ as the smallest degree of any term in the Taylor expansion of f in local coordinates. Thus $\|f(p)\| \simeq \|p\|^{c(f)}$ for "most" p near the origin. Again, the existence of the limit $c_\infty = \lim_{n\to\infty} c(f^n)^{1/n}$ is easy to establish, but its dynamical significance is less clear.

To avoid trivialities, all maps considered will be *dominant*, i.e. their Jacobian determinants do not vanish identically. We are also mainly interested in superattracting behavior, so we shall assume throughout that $d_\infty > 1$ and $c_\infty > 1$, respectively.

Our aim is then to answer two questions. First, what numbers d_∞ and c_∞ may appear, and what can be said about the convergence of $\deg(F^n)^{1/n}$ and $c(f^n)^{1/n}$ to d_∞ and c_∞, respectively? Second, in what sense are orbits actually attracted to infinity (the origin) at the rate d_∞ (c_∞)?

The next two theorems provide answers in the local case.

THEOREM A. – *For any dominant fixed point germ $f : (\mathbf{C}^2, 0) \to (\mathbf{C}^2, 0)$, the asymptotic attraction rate $c_\infty = c_\infty(f)$ is a quadratic integer, i.e. there exist integers a, b such that $c_\infty^2 + ac_\infty + b = 0$.*
Moreover, there exists $\delta \in (0, 1]$ such that $\delta c_\infty^n \leqslant c(f^n) \leqslant c_\infty^n$ for all $n \geqslant 1$.

Our proof also goes through for formal fixed points germs (defined by formal power series). The bound on $c(f^n)$ allows us to improve upon Theorem A′ from [11] (see the remark after that theorem).

THEOREM B. – *If $f : (\mathbf{C}^2, 0) \to (\mathbf{C}^2, 0)$ is a dominant fixed point germ with $c_\infty(f) > 1$, then there exist a plurisubharmonic function u defined near the origin and a constant $C > 0$ such that $C^{-1} \log \|\cdot\| \leqslant u \leqslant C \log \|\cdot\|$ and such that $c_\infty^{-n} u \circ f^n$ decreases to a plurisubharmonic function $u_\infty \not\equiv -\infty$ with $u_\infty \circ f = c_\infty u_\infty$.*

In particular, for any point p outside a pluripolar set, we have $\delta_1^{c_\infty^n} \leqslant \|f^n(p)\| \leqslant \delta_2^{c_\infty^n}$ for all $n \geqslant 1$ and some constants $\delta_i = \delta_i(p) > 0$. Of course, not every orbit must converge to the origin at the rate c_∞ and f may even contract some curves.

Hubbard and Papadopol [17] also studied superattracting fixed point germs, but their objective was different from ours: for the mappings they considered, $c(f^n) = c^n$, and the existence of u_∞ is easier to establish.

The corresponding results in the affine case are as follows.

THEOREM A′. – *For any dominant polynomial map $F : \mathbf{C}^2 \to \mathbf{C}^2$, the asymptotic attraction rate $d_\infty = d_\infty(F)$ is a quadratic integer.*
Moreover, if $d_\infty > 1$, then either there exists $D \in [1, \infty)$ such that $d_\infty^n \leqslant \deg(F^n) \leqslant Dd_\infty^n$ for all $n \geqslant 1$; or there exists a change of coordinates by a polynomial automorphism of $\mathbf{C}^2$ such that F becomes a skew product, $F(X, Y) = (P(X), Q(X, Y))$, with $d_\infty(F) = \deg(P) = \deg_Y(Q) > \deg_Y Q(X_0, Y)$ for some $X_0 \in \mathbf{C}$.

THEOREM B′. – *Suppose $F : \mathbf{C}^2 \to \mathbf{C}^2$ is a dominant polynomial map with $d_\infty(F) > 1$ and not conjugate to a skew product. Then there exist $C > 0$ and a plurisubharmonic function $U \geqslant 0$ on $\mathbf{C}^2$, such that $C^{-1} \log \|\cdot\| \leqslant U \leqslant C \log \|\cdot\|$ near infinity, and such that $d_\infty^{-n} U \circ F^n$ decreases to a plurisubharmonic function $U_\infty \not\equiv 0$ with $U_\infty \circ F = d_\infty U_\infty$. The positive closed current $T := dd^c U_\infty$ satisfies $F^* T = d_\infty T$.*

Skew products were considered in [10, Section 6]. The construction of an invariant current for rational maps is due to Sibony [29] in the "algebraically stable" case, when $\deg(F^n) = \deg(F)^n$

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[3] This number is also known as the (first) dynamical degree of F, see [27].

for all $n \geqslant 1$. This was later extended to birational surface maps in [7], and to special non-invertible rational maps in [23]. All these constructions are based on a precise understanding on the growth of degrees; the same is true here.

The set $\{U_\infty = 0\}$ above is not pluripolar in general and points in $\{U_\infty = 0\}$ may converge to infinity at a speed different from d_∞: see [8,31] for interesting (algebraically stable) examples.

Let us now explain our approach to the main results above. We first focus on the local case where the ideas are less involved.

A common approach to study dynamic behavior is to find coordinates in which the mapping has a particularly simple expression—a *normal form*. In one dimension, a very simple normal form is provided by Boettcher's theorem (see [4]): there exists a local coordinate near the origin such that the map becomes $\zeta \mapsto \zeta^c$ for some $c \geqslant 2$.

As noted by Hubbard and Papadopol, the naive generalization of Boettcher's theorem fails in higher dimensions. For instance, $f(x, y) = (x^2, y^2 - x^3)$ cannot be conjugated to $f_0(x, y) = (x^2, y^2)$. Indeed, the critical set $C_{f_0} = \{xy = 0\}$ is totally invariant for f_0, but $C_f = \{xy = 0\}$ is not totally invariant for f. For this example, the analysis by Hubbard and Papadopol (see also [2] for the affine case) provides a conjugacy between f and f_0 on a subset of $\mathbf{C}^2$, but that relies on much stronger (hyperbolicity) assumptions on f than we wish to impose.

Instead, we take inspiration from singularity theory and allow ourselves to work with birational changes of coordinates and with slightly more general normal forms.

Let us call a fixed point germ *rigid* if the critical set is contained in a totally invariant set with normal crossings singularities. Thus f_0 above is rigid while f is not. Rigid (contracting) germs were introduced and classified in [9]. There are seven classes of them, each containing a simple normal form.

The concept of a rigid germ is a dynamic version of simple normal crossings singularities for curves. The next theorem, which we shall refer to as *rigidification*, can thus be viewed as an analogue of embedded resolution of plane curve singularities. By a *modification* we shall mean a composition of point blowups above the origin. A (closed) point on the exceptional divisor is called an *infinitely near point*.

THEOREM C. – *For any dominant fixed point germ $f : (\mathbf{C}^2, 0) \to (\mathbf{C}^2, 0)$ with $c_\infty(f) > 1$, there exist a modification $\pi : X \to (\mathbf{C}^2, 0)$ and an infinitely near point $p \in \pi^{-1}(0)$ such that the lift $\hat{f} := \pi^{-1} \circ f \circ \pi$ is holomorphic at p, $\hat{f}(p) = p$, and the fixed point germ $\hat{f} : (X, p) \to (X, p)$ is rigid.*

If we allow ourselves to use *different* modifications at the source and target spaces, we can make any map holomorphic and locally *monomial* in suitable coordinates, see [5] and Proposition 1.1 below. Although this fact is not a direct consequence of Hironaka's desingularization theorem, it is considerably simpler to establish than Theorem C, and serves as an important ingredient in the proof of the latter.

We now indicate how Theorem C can be used to prove Theorems A and B. One can show that $c_\infty(f)$ is controlled by the speed at which orbits at p under the lift $\hat{f}$ approach the exceptional divisor $\pi^{-1}(0)$. In view of the classification of rigid germs, this speed is either an integer or the spectral radius of a 2×2 matrix with integer coefficients, hence a quadratic integer.

The construction of the function u with $u \circ f \leqslant c_\infty u$ in Theorem B requires a separate argument, but once we have found u, the fact that $c_\infty^{-n} u \circ f^n \not\to -\infty$ can be established by showing that at least one orbit of $\hat{f}$ cannot approach $\pi^{-1}(0)$ too fast.

So how, then, could one prove rigidification as in Theorem C? When resolving curve singularities, the modification $\pi : X \to (\mathbf{C}^2, 0)$ can be found inductively by successively blowing up points where we do not have normal crossings. The main problem then becomes proving that

the inductive procedure terminates: this is done by showing that the curves involved always become less singular in the sense that a certain discrete numerical invariant decreases.

To prove rigidification, such an approach seems difficult to carry out successfully. Instead we take a different tack: we consider a space parameterizing *all* possible modifications at the same time, study the dynamics induced by f on this space, and deduce dynamical information about f itself.

To explain this strategy, note that any irreducible component E of the exceptional divisor of a modification defines a *divisorial valuation* on the ring R of holomorphic germs at the origin: the value on a germ is the order of vanishing along E of its pullback. It is possible to define a natural induced action of f on the space $\mathcal{V}_{\mathrm{div}}$ of divisorial valuations. However, this space has a lack of completeness which makes it hard to properly analyze the induced action.

Instead, we consider a larger space $\mathcal{V}$ consisting of *all* valuations on R (centered at the origin, and suitably normalized). This space is automatically compact for the topology of the weak convergence, and we showed in [12] that $\mathcal{V}$ admits a natural *tree structure*. Roughly speaking, $\mathcal{V}$ is a collection of real line segments welded together in such a way that no cycles appear. More precisely, $\mathcal{V}$ is an $\mathbf{R}$-tree for a natural metric and possesses strong self-similarity properties. The line segments are made up of monomial valuations in suitable coordinates at infinitely near points above the origin, and are related to the Farey blowups studied in [18]. The set $\mathcal{V}_{\mathrm{div}}$ above is a subset of $\mathcal{V}$ in essentially the same way as $\mathbf{Q}$ is a subset of the (compactified) real line. In the sequel, we refer to $\mathcal{V}$ as the *valuative tree*.

The valuative tree can be used to efficiently encode local information near the origin in $\mathbf{C}^2$. We used it to study singularities of curves and ideals in [12], of plurisubharmonic functions in [13], and multiplier ideals in [14].

Here we prove that any dominant fixed point germ f induces a natural selfmap $f_\bullet : \mathcal{V} \to \mathcal{V}$, and that this selfmap respects the tree structure in a strong sense. For topological reasons, $f_\bullet$ must then admit a fixed point, or *eigenvaluation*. We show that this eigenvaluation admits a basin of attraction U, which can be chosen to be of a special type $U(p)$ corresponding to an infinitely near point p as follows. Fix a modification $\pi : X \to (\mathbf{C}^2, 0)$ such that $p \in \pi^{-1}(0) \subset X$. Then $U(p) \subset \mathcal{V}$ is the set of all valuations whose center on X is p. The invariance of $U(p)$ under $f_\bullet$ translates into the lift $\hat{f}$ of f being holomorphic at p with $\hat{f}(p) = p$. By choosing the basin carefully, we can even make $\hat{f}$ rigid at p, leading to Theorem C.

The induced selfmap $f_\bullet$ of $\mathcal{V}$ seems to have quite strong global attracting features, similar to holomorphic selfmaps of the unit disk in $\mathbf{C}$. There are also close connections to p-adic dynamics, more precisely to the study of rational maps on $\mathbf{C}_p$ (see [25]).

In the affine setting, i.e. when proving Theorems A′ and B′, the main idea is the same as in the local case, and consists of studying the action of F on the set $\mathcal{V}_0$ of (normalized) valuations on $\mathbf{C}[X, Y]$ centered at infinity. However, in the same way that the study of plane affine curves has a quite different flavor than the theory of germs of analytic curves, the affine case proves to be more delicate than the local case. Let us indicate two difficulties that we need to overcome.

First, a dominant polynomial map of $\mathbf{C}^2$ need not be proper, so the line at infinity is not invariant. In particular, the push-forward of a valuation centered at infinity may no longer be centered at infinity. As a consequence, the induced selfmap of $\mathcal{V}_0$ is not everywhere defined. Second (and more importantly), a valuation is by its very nature a local object and measures behavior at an infinitely near point. But for Theorem A′ we need to control degrees, which has a more global flavor.

As it turns out, we can address both difficulties by working with a suitable subtree $\mathcal{V}_1$ of $\mathcal{V}_0$. On this tree, the relation between local and global information is well behaved. Moreover, any dominant polynomial map acts on this tree in a way that preserves the tree structure. In particular,

there is again an eigenvaluation, and this provides us with a starting point for the proofs of Theorems A$'$ and B$'$. The case when F can be conjugated to a skew product arises exactly when the eigenvaluation is an end of the tree of a special type: it is associated to a pencil of rational curves having one place at infinity. The Line Embedding Theorem [1,30] then provides the conjugacy.

The space $\mathcal{V}_1$ is the key new idea in the affine case. Roughly speaking, a valuation is in $\mathcal{V}_1$ if it takes negative values on any polynomial as well as on the form $dX \wedge dY$ on $\mathbf{C}^2$. To our knowledge, such valuations have not previously been systematically studied. A crucial property that they enjoy is that they are dominated by affine curves with one place at infinity. While such curves have received much attention in the literature (see e.g. [1,3,15,30]), our focus on valuations made it difficult to extract the information that we needed. We therefore explore the structure on $\mathcal{V}_1$ in an appendix, using key polynomials as in [12], a technique originating with MacLane [19] and subsequently developed by Abhyankar–Moh [1] under the name approximate roots.

The paper is divided into eight sections. In the first one, we briefly recall the structure of the valuative tree as described in [12]. In the next five sections, we only deal with the local case, leaving affine problems for Section 7. We thus fix a fixed point germ f. In Section 2, we describe how f maps one valuation to another. We then investigate the action of f (without iteration) on the valuative tree in Section 3, and show that it preserves—in a strong sense—the tree structure. Section 4 is devoted to the existence of an eigenvaluation and the second part of Theorem A. In Section 5, we construct basins of attraction, deduce a stronger version of Theorem C, and exhibit normal forms of the lift f at the point where it is rigid. This enables us to complete the proof of Theorem A. Finally, Section 6 is devoted to the proof of Theorem B, which relies heavily on Theorem C. In Section 7, we turn to the affine case. The paper ends with appendix detailing the structure of the valuation spaces $\mathcal{V}_0$ and $\mathcal{V}_1$ in the same spirit as the local case in [12].

1. Background

In this section we briefly review the approach to valuation theory worked out in [12]. Except in Section 7 and Appendix A we let R denote the ring of holomorphic germs at the origin in $\mathbf{C}^2$. This is a local ring. Its maximal ideal $\mathfrak{m}$ is the set of germs vanishing at the origin, and its residue field is $\mathbf{C}$. We write $(\hat{R}, \hat{\mathfrak{m}})$ for the completion of R. It is the ring of formal power series in two complex variables.

1.1. Valuations

We consider the space $\mathcal{V}$ of centered, normalized *valuations* on R, i.e. the set of functions $\nu : R \to [0, \infty]$ satisfying:
  (i) $\nu(\psi\psi') = \nu(\psi) + \nu(\psi')$ for all ψ, ψ';
  (ii) $\nu(\psi + \psi') \geqslant \min\{\nu(\psi), \nu(\psi')\}$ for all ψ, ψ';
  (iii) $\nu(0) = \infty$, $\nu|_{\mathbf{C}^*} = 0$, $\nu(\mathfrak{m}) := \min\{\nu(\psi); \ \psi \in \mathfrak{m}\} = 1$.
Then $\mathcal{V}$ is equipped with a natural *partial ordering*: $\nu \leqslant \mu$ iff $\nu(\psi) \leqslant \mu(\psi)$ for all $\psi \in \mathfrak{m}$. The *multiplicity valuation* $\nu_{\mathfrak{m}}$ defined by $\nu_{\mathfrak{m}}(\psi) = m(\psi) = \max\{k; \ \psi \in \mathfrak{m}^k\}$ is the unique minimal element of $\mathcal{V}$.

1.2. Curve valuations

Some natural maximal elements (for the partial ordering above) are the *curve valuations* defined as follows. To each irreducible curve C we associate $\nu_C \in \mathcal{V}$ defined by $\nu_C(\psi) =$

$C \cdot \psi^{-1}\{0\}/m(C)$, where "$\cdot$" denotes intersection multiplicity and m multiplicity. We call ν_C *analytic* when C is an analytic curve; otherwise ν_C is *formal*. If $C = \phi^{-1}(0)$ for $\phi \in \hat{\mathfrak{m}}$, then we also write $\nu_C = \nu_\phi$. Given a parameterization $h \colon (\mathbf{C}, 0) \to (\mathbf{C}^2, 0)$ of C we have $\nu_C(\psi) = m(C)^{-1} \operatorname{div}_t(\psi \circ h(t))$, where div_t denotes the order of vanishing at $t = 0$. Note that $\nu_\phi(\phi) = \infty$.

The set $\mathcal{C}$ of local irreducible curves carries a natural (ultra)metric in which $\mathcal{C}$ has diameter 1. It is given by $d_{\mathcal{C}}(C, D) = m(C)m(D)/C \cdot D$.

1.3. Quasimonomial valuations and tree structure

Arguably the most important valuations in $\mathcal{V}$ are the *quasimonomial* ones. They are of the form $\nu_{C,t}$, where $C \in \mathcal{C}$, $t \in [1, \infty)$, and satisfy $\nu_{C,t}(\psi) = \min\{\nu_D(\psi); \ d_{\mathcal{C}}(C, D) \leqslant t^{-1}\}$. We have $\nu_{C,s} = \nu_{D,t}$ iff $s = t \leqslant d_{\mathcal{C}}(C, D)^{-1}$; and $\nu_{C,t} \geqslant \nu_{C,t'}$ iff $t \geqslant t'$. Thus $\mathcal{V}_{\mathrm{qm}}$, the set of all quasimonomial valuations, is naturally a quotient of $\mathcal{C} \times [1, \infty)$, and has a natural tree structure: if $\nu, \nu' \in \mathcal{V}_{\mathrm{qm}}$ and $\nu < \nu'$, then the *segment* $[\nu, \nu'] = \{\mu \in \mathcal{V}_{\mathrm{qm}}; \ \nu \leqslant \mu \leqslant \nu'\}$ is isomorphic to a compact real interval. We set $\nu_{\phi,t} := \nu_{C,t}$ when $C = \phi^{-1}(0)$. Quasimonomial valuations are of two types: *divisorial* and *irrational*, depending on whether the parameter t is rational or irrational.

The full space $\mathcal{V}$ is the completion of $\mathcal{V}_{\mathrm{qm}}$ in the sense that every element in $\mathcal{V}$ is the limit of an increasing sequence in $\mathcal{V}_{\mathrm{qm}}$. It is hence also naturally a tree, called the *valuative tree*. The *ends* (i.e. maximal elements) of $\mathcal{V}$ are exactly the elements of $\mathcal{V} \setminus \mathcal{V}_{\mathrm{qm}}$ and are either curve valuations or *infinitely singular* valuations [4].

A *subtree* of $\mathcal{V}$ is a subset $\mathcal{T}$ such that $\nu \in \mathcal{T}$ and $\mu \leqslant \nu$ implies $\mu \in \mathcal{T}$. A subtree is *finite* if it has finitely many ends.

1.4. Numerical invariants

A curve valuation $\nu = \nu_\phi$ is characterized by the fact that the prime ideal $\mathfrak{p}_\nu := \{\nu = +\infty\}$ is nontrivial, generated by ϕ.

If $\nu \in \mathcal{V}$ is not a curve valuation, then its *type* (divisorial, irrational or infinitely singular) is determined by its *rational rank* $\operatorname{rat.rk} \nu$ and *transcendence degree* $\operatorname{tr.deg} \nu$, see [12, Table C.1]. These invariants are defined as follows. The semigroup $\nu(R \setminus \{0\}) \subset \mathbf{R}$ generates a group $\Gamma_\nu \subset \mathbf{R}$, the value group of ν. One sets $\operatorname{rat.rk} \nu = \dim_{\mathbf{Q}}(\Gamma_\nu \otimes_{\mathbf{Z}} \mathbf{Q})$. The residue field of the valuation is $k_\nu := \{\phi \in R; \ \nu(\phi) \geqslant 0\}/\{\phi \in R; \ \nu(\phi) > 0\}$ and $\operatorname{tr.deg} \nu$ is its transcendence degree over $\mathbf{C}$.

1.5. Skewness and infimum

An important invariant of a valuation is its *skewness* α defined by $\alpha(\nu) = \sup\{\nu(\phi)/m(\phi); \ \phi \in \mathfrak{m}\}$. Skewness naturally *parameterizes* the trees $\mathcal{V}_{\mathrm{qm}}$ and $\mathcal{V}$ in the sense that $\alpha \colon \mathcal{V}_{\mathrm{qm}} \to [1, \infty)$ is strictly increasing and restricts to a bijection onto its image on any segment; indeed $\alpha(\nu_{\phi,t}) = t$ for any $\nu_{\phi,t} \in \mathcal{V}_{\mathrm{qm}}$. Thus divisorial (irrational) valuations have rational (irrational) skewness. Curve valuations have infinite skewness: see [12, Table C.1].

The tree structure on $\mathcal{V}$ implies that any collection $(\nu_i)_{i \in I}$ of valuations in $\mathcal{V}$ admits an *infimum* $\bigwedge_i \nu_i$. There are two useful formulas involving the infimum and skewness. First, if $\nu \in \mathcal{V}$ and $\phi \in \mathfrak{m}$ is irreducible, then $\nu(\phi) = m(\phi)\alpha(\nu \wedge \nu_\phi)$. Second, if $C, D \in \mathcal{C}$, then $C \cdot D = m(C)m(D)\alpha(\nu_C \wedge \nu_D)$.

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[4] The latter are represented by infinite Puiseux series whose exponents are rational numbers with unbounded denominators.

1.6. Tangent space and weak topology

Let ν be a valuation in $\mathcal{V}$. Declare $\mu, \mu' \in \mathcal{V} \setminus \{\nu\}$ to be equivalent if the segments $]\nu, \mu]$ and $]\nu, \mu']$ intersect. An equivalence class is called a *tangent vector* at ν and the set of tangent vectors at ν, the *tangent space*, denoted by $T\nu$. If $\vec{v}$ is a tangent vector, we denote by $U(\vec{v})$ the set of points in $\mathcal{V}$ defining the equivalence class $\vec{v}$. The points in $U(\vec{v})$ are said to *represent* $\vec{v}$. Irrational valuations are *regular points* of $\mathcal{V}$ in the sense that the tangent space has exactly two elements, whereas divisorial points are *branch points*: see [12, Table C.1]. We refer to Section 1.10 for a geometric interpretation of the (uncountable) tangent space at a divisorial valuation.

We endow $\mathcal{V}$ with the *weak topology*, generated by the sets $U(\vec{v})$ over all tangent vectors $\vec{v}$; this turns $\mathcal{V}$ into a compact (Hausdorff) space. The weak topology on $\mathcal{V}$ is characterized by $\nu_k \to \nu$ iff $\nu_k(\phi) \to \nu(\phi)$ for all $\phi \in R$.

1.7. Multiplicity and thinness

By setting $m(\nu) = \inf\{m(C); \ C \in \mathcal{C}, \ \nu_C \geqslant \nu\}$ we extend the notion of *multiplicity* from $\mathcal{C}$ to $\mathcal{V}$. The infinitely singular valuations are characterized as having infinite multiplicity.

As m is increasing and integer valued, it is piecewise constant on any segment $[\nu_{\mathfrak{m}}, \nu_C]$, where $C \in \mathcal{C}$. This naturally defines $m(\vec{v})$ for any tangent vector $\vec{v}$. If ν is nondivisorial, then $m(\vec{v}) = m(\nu)$ for any $\vec{v} \in T\nu$. When ν is divisorial, the situation is more complicated. It suffices to say that there exists an integer $b(\nu)$, divisible by $m(\nu)$, such that $m(\vec{v}) = b(\nu)$ for all but at most two tangent vectors $\vec{v}$ at ν. We call $b(\nu)$ the *generic multiplicity* of ν.

A quantity that combines skewness and multiplicity is *thinness* A, defined by $A(\nu) = 2 + \int_{\nu_{\mathfrak{m}}}^{\nu} m(\mu)\, d\alpha(\mu)$. The integral is in fact a finite or countable sum as $\mu \mapsto m(\mu)$ is increasing and integer-valued on $[\nu_{\mathfrak{m}}, \nu[$. Thinness defines an increasing parameterization of $\mathcal{V}$ with values in $[2, +\infty]$.

If ν_C is a curve valuation, and $\nu_1 < \nu_C$ is a quasimonomial valuation such that $m := m(\nu_1) = m(C)$, then the multiplicity is constant on $[\nu_1, \nu_C]$, hence $A(\nu) = m\alpha(\nu) + B$ for some rational number B.

1.8. Tree potentials

The set of *tree potentials* is by definition the smallest set of real-valued functions on $\mathcal{V}_{\mathrm{qm}}$ containing all functions of the form $\nu \mapsto \nu(\phi)$ for $\phi \in R$, and closed under positive linear combinations, minima, and pointwise limits. It is the analog for a real tree of the set of concave functions on the real line.

The tree potentials that we shall use are associated to germs $\phi \in R$ or ideals $I \subset R$. In the first case, write $\phi = \prod \phi_i^{m_i}$ with ϕ_i irreducible. Then $\nu(\phi) = \sum m_i \alpha(\nu \wedge \nu_{\phi_i})$. In particular, the tree potential $\nu \mapsto \nu(\phi)$ is locally constant outside the finite tree whose ends are the curve valuations ν_{ϕ_i}; and is increasing, concave and piecewise linear with integer slope as a function of skewness on any segment.

These properties carry over to the tree potential $\nu \mapsto \nu(I)$ associated to an ideal I, as follows from $\nu(I) = \min_j \nu(\phi_j)$, where (ϕ_j) is any finite set of generators.

1.9. Dual graphs

Every divisorial valuation ν arises as follows: there exist a *modification* (i.e. a finite composition of point blowups) $\pi: X \to (\mathbf{C}^2, 0)$ and an *exceptional component* E (i.e. an irreducible component of $\pi^{-1}(0)$) such that the center of ν on X equals E. More precisely, $\nu = \nu_E := b_E^{-1} \pi_* \operatorname{div}_E$, where $b_E = \operatorname{div}_E(\pi^* \mathfrak{m}) \in \mathbf{N}^*$ is the generic multiplicity at ν. The ideal

$\pi^*\mathfrak{m}$ is locally principal along E, of the form (z^{b_E}) if $E = \{z = 0\}$. Further, the thinness of ν is $A(\nu) = a_E/b_E$, where $a_E = 1 + \mathrm{div}_E(\pi^*\omega)$ and ω is a nonvanishing holomorphic two-form, say $\omega = dx \wedge dy$ in local coordinates at $(\mathbf{C}^2, 0)$.

To a modification $\pi: X \to (\mathbf{C}^2, 0)$ is associated a *dual graph*: its set of vertices Γ_π^* is the set of exceptional components $E \subset \pi^{-1}(0)$ and $E, F \in \Gamma_\pi^*$ are joined by an edge iff $E \cap F \neq \emptyset$. Since the dual graph has no cycles, Γ_π^* admits a natural poset structure, with unique minimal element given by the blowup of the origin. Then Γ_π^* embeds in $\mathcal{V}$ in the sense that $E \mapsto \nu_E$ is injective and $E \in [F, F']$ iff $\nu_E \in [\nu_F, \nu_{F'}]$.

More generally, for any curve C and any modification $\pi: X \to (\mathbf{C}^2, 0)$ such that $\pi^{-1}(C)$ has *simple normal crossings*, the set of irreducible components of $\pi^{-1}(C)$ has a natural poset structure and embeds in $\mathcal{V}$ as a set of divisorial and curve valuations. See [12, Fig. 9].

1.10. Infinitely near points

An *infinitely near point* is a (closed) point $p \in \pi^{-1}(0)$ where $\pi: X \to (\mathbf{C}^2, 0)$ is a modification. Such a point defines an open subset $U(p) \subset \mathcal{V}$, consisting of all valuations $\nu \in \mathcal{V}$ whose center on X is p, i.e. $\nu = \pi_*\mu$, where μ is a valuation on the local ring $(\mathcal{O}_p, \mathfrak{m}_p)$ of holomorphic germs at p with $\mu \geqslant 0$ on $\mathcal{O}_p$ and $\mu > 0$ on $\mathfrak{m}_p$.

If p is a smooth point on $\pi^{-1}(0)$, belonging to a unique exceptional component E, then $U(p) = U(\vec{v})$, where $\vec{v}$ is the tangent vector at ν_E represented by ν_F and F is the exceptional divisor obtained by blowing up p. If p is a singular point on $\pi^{-1}(0)$, say $\{p\} = E \cap E'$, then $U(p) = U(\vec{v}) \cap U(\vec{v}')$, where $\vec{v}$ is the tangent vector at ν_E represented by $\nu_{E'}$ and $\vec{v}'$ is the tangent vector at $\nu_{E'}$ represented by ν_E. See [12, Fig. 9].

It follows that if p is any point on an exceptional component $E \subset \pi^{-1}(0)$, then all valuations in $U(p)$ represent a common tangent vector $\vec{v}_p$ at ν_E.

1.11. Monomialization of valuations

A valuation ν on R is *monomial* if it is monomial in some local coordinates (x, y), i.e. if there exist $s, t \geqslant 0$ such that $\nu(\sum_{i,j} a_{ij} x^i y^j) = \min\{si + tj; a_{ij} \neq 0\}$. In particular $\nu(x) = s$, $\nu(y) = t$.

We shall repeatedly reduce questions regarding quasimonomial valuations to the monomial case using modifications.

First, consider a modification $\pi: X \to (\mathbf{C}^2, 0)$ and an infinitely near point $p \in \pi^{-1}(0)$ at the intersection of two exceptional components $E = \{z = 0\}$ and $F = \{w = 0\}$ in X. Let $\nu_E, \nu_F \in \mathcal{V}$ be the associated divisorial valuations and b_E, b_F their generic multiplicities. Denote by μ_t the monomial valuation in (z, w) with $\mu_t(z) = (1-t)/b_E$ and $\mu_t(w) = t/b_F$. The ideal $\pi^*\mathfrak{m}$ is locally principal, generated by $z^{b_E} w^{b_F}$ at p (see Section 1.9 above), so $\nu_t := \pi_*\mu_t$ is normalized by $\nu_t(\mathfrak{m}) = 1$. Moreover $[0, 1] \ni t \mapsto \nu_t$ gives a parameterization of the segment $[\nu_E, \nu_F]$ in $\mathcal{V}$.

Second, consider an irreducible curve C, a desingularization $\pi: X \to (\mathbf{C}^2, 0)$ of C, and pick local coordinates (z, w) at the intersection point p between the strict transform $\tilde{C} = \{w = 0\}$ and the unique exceptional component $E = \{z = 0\}$ intersecting $\tilde{C}$. Define a monomial valuation μ_t in (z, w) by $\mu_t(z) = 1/b_E$, $\mu_t(w) = t$. Then $\nu_t := \pi_*\mu_t$ is normalized and $[0, \infty] \ni t \mapsto \nu_t$ gives a parameterization of the segment $[\nu_E, \nu_C]$ in $\mathcal{V}$.

1.12. Monomialization of maps

A holomorphic map $f : (\mathbf{C}^2, 0) \to (\mathbf{C}^2, 0)$ is *monomial* if one can find coordinates (z, w) at the source and (z', w') at the target space such that $(z', w') = f(z, w) = (z^a w^b, z^c w^d)$ for some integers a, b, c, d. Then f is dominant iff $ad \neq bc$.

PROPOSITION 1.1. – *For any dominant holomorphic map $f : (\mathbf{C}^2, 0) \to (\mathbf{C}^2, 0)$ and for any curve C, there exist modifications $\pi : X \to (\mathbf{C}^2, 0)$ and $\pi' : X' \to (\mathbf{C}^2, 0)$ such that the total transform of C by π and of $f(C)$ by π' both have simple normal crossings, and such that the lift $\hat{f} : X \to X'$ of f is holomorphic and monomial at any point $p \in \pi^{-1}(0)$.*

A proof is given in [5, Section 3]. While the result is not known in high dimensions, its proof is elementary in dimension two.

2. Image of a single valuation

From now on and until Section 7, $f : (\mathbf{C}^2, 0) \to (\mathbf{C}^2, 0)$ denotes a holomorphic fixed point germ. We will always assume f *dominant*, that is, the Jacobian determinant Jf is not identically zero. In this section, we describe how f maps one valuation to another.

2.1. Definitions

An element $\nu \in \mathcal{V}$ is by definition a valuation $\nu : R \to \overline{\mathbf{R}}_+$ centered at the origin and normalized by $\nu(\mathfrak{m}) = 1$. We let $f_* \nu(\phi) := \nu(f^* \phi)$. This defines $f_* \nu$ as a centered valuation on R with values in $\overline{\mathbf{R}}_+$. However, $f_* \nu$ need not be normalized, and can even be identically $+\infty$ on $\mathfrak{m}$. The latter situation appears exactly when $f^* \mathfrak{m}$ is included in the prime ideal $\mathfrak{p}_\nu = \{\nu = +\infty\}$. When nontrivial, $\mathfrak{p}_\nu$ is generated by an irreducible element, $\mathfrak{p}_\nu = (\phi)$, and $\nu = \nu_\phi$ is a curve valuation. Geometrically, the inclusion $f^* \mathfrak{m} \subset (\phi)$ means that the curve $C = \{\phi = 0\}$ is contracted, $f(C) = 0$. We therefore call $\nu_\phi = \nu_C$ a *contracted curve valuation* in this case. Any contracted curve is critical, so there are most finitely many contracted curve valuations and they are all analytic. In summary:

LEMMA 2.1. – *Fix $\nu \in \mathcal{V}$. Then $f_* \nu$ defines a valuation on R with values in $\overline{\mathbf{R}}_+$, which is identically $+\infty$ on $\mathfrak{m}$ iff $\nu \in \mathfrak{C}_f$, where $\mathfrak{C}_f$ denotes the (finite) set of contracted curve valuations.*

DEFINITION 2.2. – If $\nu \in \mathcal{V}$ is not a contracted curve valuation, then $f_\bullet \nu \in \mathcal{V}$ is the (unique) valuation equivalent to $f_* \nu$ and normalized by $(f_\bullet \nu)(\mathfrak{m}) = 1$.

DEFINITION 2.3. – For $\nu \in \mathcal{V}$ we define the *attraction rate of f along ν* to be the number $c(f, \nu) := \nu(f^* \mathfrak{m}) \in [1, +\infty]$.

The quantity $c(f)$ defined in the introduction equals $c(f, \nu_\mathfrak{m})$. Clearly

$$(2.1) \qquad f_* \nu = c(f, \nu) \times f_\bullet \nu,$$

for any $\nu \in \mathcal{V} \setminus \mathfrak{C}_f$.

If f and g are dominant fixed point germs, then $(f \circ g)_* \nu = f_* g_* \nu$ for any $\nu \in \mathcal{V}$, and $(f \circ g)_\bullet \nu = f_\bullet g_\bullet \nu$ as long as $\nu \notin \mathfrak{C}_{f \circ g}$. This yields $c(f \circ g, \nu) = c(g, \nu) \times c(f, g_\bullet \nu)$. In particular, we get

$$(2.2) \qquad c(f^n) = \prod_{k=0}^{n-1} c\big(f, f_\bullet^k \nu_\mathfrak{m}\big),$$

for all $n \geqslant 1$.

2.2. Preservation of type

Next we show that $f_\bullet$ preserves the type of a valuation.

PROPOSITION 2.4. – *The map $f_\bullet : \mathcal{V} \setminus \mathfrak{C}_f \to \mathcal{V}$ preserves the set of divisorial (resp. irrational, infinitely singular, or curve) valuations. Moreover, there exists an integer $N \geqslant 1$ such that for any $\nu \in \mathcal{V}$, the set $f_\bullet^{-1}\{\nu\}$ has cardinality at most N.*

We shall see in Section 2.4 that $f_\bullet$ extends in a natural way to $\mathfrak{C}_f$. When $\nu \in \mathfrak{C}_f$, however, $f_\bullet \nu$ is a divisorial valuation.

Proof. – Pick $\nu \in \mathcal{V} \setminus \mathfrak{C}_f$. Let us show that ν and $f_\bullet \nu$ have the same type. We shall prove in Section 2.4 that if ν is a curve valuation then so is $f_\bullet \nu$. Hence we may assume ν takes the value ∞ only on $0 \in R$. Since f is dominant, $f_* \nu$ has the same property. As follows from the discussion in Section 1.4, we only have to prove that ν and $f_* \nu$ have the same rational rank and transcendence degree.

Using Proposition 1.1, we reduce to the case where f is monomial, hence rational. Let K be the field of rational functions on $\mathbf{C}^2$. As f is dominant, $f^* : K \to K$ is injective and K is a *finite* extension of $f^* K := \{\phi \circ f, \phi \in K\}$ of some degree $N \geqslant 1$.

Let us compare the two value groups Γ_ν and $\Gamma_{f_* \nu}$. Clearly $\Gamma_{f_* \nu} \subset \Gamma_\nu$. Conversely consider $\gamma = \nu(\phi) \in \Gamma_\nu$. We may find $a_i \in f^* K$ such that $\sum_{i=0}^N a_i \phi^i = 0$. Thus there exist $i < j$ such that $\nu(a_i \phi^i) = \nu(a_j \phi^j)$. In particular, $(j - i)\gamma = (j - i)\nu(\phi) = \nu(a_i) - \nu(a_j) \in \Gamma_{f_* \nu}$. We infer $\Gamma_\nu \otimes_{\mathbf{Z}} \mathbf{Q} \subset \Gamma_{f_* \nu} \otimes_{\mathbf{Z}} \mathbf{Q}$ and hence $\Gamma_\nu \otimes_{\mathbf{Z}} \mathbf{Q} = \Gamma_{f_* \nu} \otimes_{\mathbf{Z}} \mathbf{Q}$. Thus the rational ranks of ν and $f_* \nu$ coincide. The transcendence degrees must also coincide, as the extension of residue fields $k_{f_* \nu} \subset k_\nu$ is finite.

To prove the last assertion in the proposition, we may again assume f is rational. Pick a finite set of distinct valuations $\{\mu_i\}_{i \in I} \subset \mathcal{V}$ for which $f_* \mu_i$ are all proportional one to another. Suitable multiples of the μ_i thus define one and the same valuation μ on the field $f^* K$. But the field extension K over $f^* K$ is finite of degree N, so μ admits at most N extensions to K by [32, Theorem 19, p. 55]. Thus $\#I \leqslant N$. □

2.3. Image of a divisorial valuation

It is important for applications to understand geometrically what the image of a valuation is. We first treat the case of a divisorial valuation (cf. Section 1.9).

PROPOSITION 2.5. – *Let ν be a divisorial valuation and set $\nu' = f_\bullet \nu$. Then there exist modifications $\pi : X \to (\mathbf{C}^2, 0)$, $\pi' : X' \to (\mathbf{C}^2, 0)$, and exceptional components $E \subset \pi^{-1}(0)$, $E' \subset (\pi')^{-1}(0)$, such that $\nu = \nu_E$, $\nu' = \nu_{E'}$ and such that the map f lifts to a holomorphic map $\hat{f} : X \to X'$ sending E onto E'. Moreover,*

$$(2.3) \qquad\qquad c(f, \nu_E) = \frac{b_{E'}}{b_E} k.$$

Here b_E and $b_{E'}$ are the generic multiplicities of ν_E and $\nu_{E'}$, respectively and $k \geqslant 1$ is the largest integer such that $\hat{f}^* E' \geqslant kE$ as divisors.

Proof. – Proposition 2.4 implies that ν' is divisorial. Choose modifications $\pi : X \to (\mathbf{C}^2, 0)$, $\pi' : X' \to (\mathbf{C}^2, 0)$, such that the center E of ν in X and E' of ν' in X' are both one dimensional. By blowing up the source space further, we may assume that the lift $\hat{f}$ of f is holomorphic. By construction, a holomorphic function locally defined at a point $p' \in E'$ vanishes along E' iff its pull back by $\hat{f}$ vanishes along E. So f maps E onto E'. Pick $k \geqslant 1$ maximal with $\hat{f}^* E' \geqslant kE$.

Since $\pi^* f^* \mathfrak{m}$ and $\hat{f}^* (\pi')^* \mathfrak{m}$ vanish along E to order $b_E c(f, \nu_E)$ and $b_{E'} k$, respectively, we get (2.3). $\quad\square$

2.4. Image of a curve valuation

Next we give an analogue of Proposition 2.5 for analytic curve valuations. We distinguish between curves that are contracted and those that are not.

PROPOSITION 2.6. – *Let C be an analytic irreducible curve such that $f(C) \neq 0$, i.e. $\nu_C \notin \mathfrak{C}_f$. Then $C' := f(C)$ is an analytic irreducible curve and $f_\bullet \nu_C = \nu_{C'}$. Further,*

$$(2.4) \qquad c(f, \nu_C) = \frac{m(C')}{m(C)} e(f, C),$$

where $e(f, C) \in \mathbf{N}^*$ denotes the topological degree of the restriction $f : C \to C'$.

As the proof shows, the proposition is also valid for formal irreducible curves.

PROPOSITION 2.7. – *Suppose C is an irreducible curve germ such that $f(C) = 0$, i.e. $\nu_C \in \mathfrak{C}_f$. Then $c(f, \nu_C) = \infty$. Further, the limit of $f_\bullet \nu$ as ν increases to ν_C exists, and is a divisorial valuation that we denote by $f_\bullet \nu_C$. It can be interpreted geometrically as follows. There exist modifications $\pi : X \to (\mathbf{C}^2, 0)$ and $\pi' : X' \to (\mathbf{C}^2, 0)$, such that f lifts to a holomorphic map $\hat{f} : X \to X'$ sending C onto a curve germ included in an exceptional component E' of $\pi'^{-1}(0)$, for which $f_\bullet \nu_C = \nu_{E'}$.*

Proof of Proposition 2.6. – Write $C = \phi^{-1}(0)$ and $C' = (\phi')^{-1}(0)$ for $\phi, \phi' \in \mathfrak{m}$ irreducible. Then $(f_* \nu_C)(\phi') = \nu_C(\phi' \circ f) = \infty$ as ϕ divides $\phi' \circ f$. Hence $f_\bullet \nu_C = \nu_{C'}$.

Pick parameterizations $h : \Delta \to C$ and $h' : \Delta \to C'$. The composition $(h')^{-1} \circ f \circ h$ has topological degree $e := e(f, C)$ at the origin. We may assume $f \circ h(t) = h'(t^e)$. Now pick $x \in \mathfrak{m}$ of multiplicity 1, and generic so that $\nu_C(x) = \nu_{C'}(x) = 1$. Then

$$c(f, \nu_C) = (f_* \nu_C)(x) = \nu_C(x \circ f) = m(C)^{-1} \operatorname{div}_t \big(x \circ f \circ h(t) \big)$$

$$= m(C)^{-1} \operatorname{div}_t \big(x \circ h'(t^e) \big) = e \times m(C)^{-1} \operatorname{div}_t \big(x \circ h'(t) \big) = e \times \frac{m(C')}{m(C)}.$$

This concludes the proof of the proposition. $\quad\square$

Proof of Proposition 2.7. – We know from Lemma 2.1 that $c(f, \nu_C) = \infty$. Let div_C be the *divisorial* valuation on R associated to C. It is not centered at the origin, but since C is contracted, $f_* \operatorname{div}_C$ is equivalent to a divisorial valuation $\nu' \in \mathcal{V}$.

Pick modifications $\pi : X \to (\mathbf{C}^2, 0)$ and $\pi' : X' \to (\mathbf{C}^2, 0)$ such that $\pi^{-1}(C)$ has simple normal crossings, the center of ν' on X is an exceptional component $E' \subset (\pi')^{-1}(0)$ and such that the lift $\hat{f} : X \to X'$ of f is holomorphic and locally monomial at every point on $\pi^{-1}(0)$.

Write $\tilde{C}$ for the strict transform of C: it intersects the exceptional divisor at a single point p on a unique exceptional component E. Set $p' = \hat{f}(p)$. As in Proposition 2.5, $\hat{f}$ maps $\tilde{C}$ to the germ of E' at p'. In suitable coordinates (z, w) at p and (z', w') at p' we have $\hat{f}(z, w) = (z^a, z^c w^d)$ with $a, d \geqslant 1$, $b \geqslant 0$. Here $\tilde{C} = \{w = 0\}$, $E = \{z = 0\}$ and $E' = \{w' = 0\}$. The ideals $\pi^* \mathfrak{m}$ and $(\pi')^* \mathfrak{m}$ are principal at p and p', generated by z^m and $z'^k w'^l$, respectively, where $m = m(C) \geqslant 1$, $k \geqslant 0$, $l \geqslant 1$.

Let $\mu_{s,t}$ (resp. $\mu'_{s,t}$) be the monomial valuation with weights (s, t) on the coordinates (z, w) (resp. (z', w')). The valuation $\nu_t := \pi_* \mu_{1/m, t}$ is an element of $\mathcal{V}$ and increases to ν_C as $t \to \infty$. Clearly $f_* \nu_t = \pi'_* \mu'_{a/m, c/m + dt}$. As $t \to \infty$, $f_\bullet \nu_t$ converges to $\pi'_* \mu'_{0, d/l} = \nu'$. The proof is complete. $\quad\square$

3. Action on the valuative tree

Having defined the map $f_\bullet$ pointwise on $\mathcal{V} \setminus \mathfrak{C}_f$, we now investigate how it interacts with the structure on $\mathcal{V}$. First we show that it preserves the tree structure in a strong sense (Theorem 3.1). We then show how $f_\bullet$ can be used to analyze the lift of f to a (meromorphic) map between surfaces obtained by blowing up points over the origin (Propositions 3.2 and 3.3).

THEOREM 3.1. – *The map $f_\bullet: \mathcal{V} \setminus \mathfrak{C}_f \to \mathcal{V}$ extends uniquely to a map $f_\bullet: \mathcal{V} \to \mathcal{V}$ which is regular in the following sense: for any $\nu \in \mathcal{V}$, we can decompose the segment $[\nu_\mathfrak{m}, \nu]$ into finitely many subsegments $I_j = [\nu_j, \nu_{j+1}]$, $0 \leqslant j \leqslant k$ with $\nu_0 = \nu_\mathfrak{m}$, $\nu_{k+1} = \nu$ and ν_i divisorial, such that $f_\bullet$ is a monotone homeomorphism of I_j onto its image.*

For $t \in [1, \alpha(\nu)]$ denote by ν_t the unique valuation dominated by ν and of skewness t. Then on any segment $[\alpha(\nu_i), \alpha(\nu_{i+1})]$ we have $\alpha(f_\bullet \nu_t) = \frac{a+bt}{c+dt}$ for nonnegative integers a, b, c, d with $ad \neq bc$.

PROPOSITION 3.2. – *Let $\pi: X \to (\mathbf{C}^2, 0)$, $\pi': X' \to (\mathbf{C}^2, 0)$ be modifications, and let $\hat{f}: X \dashrightarrow X'$ be the lift of f. For an infinitely near point $p \in \pi^{-1}(0) \subset X$, let $U(p) \subset \mathcal{V}$ be the open set of valuations whose center on X is p (see Section 1.10). Then $\hat{f}$ is holomorphic at p iff $f_\bullet U(p)$ does not contain any divisorial valuation associated to an exceptional component of π'. When $\hat{f}$ is holomorphic at p, the point $p' = \hat{f}(p) \in X'$ is characterized by $f_\bullet U(p) \subset U(p')$.*

It follows from Theorem 3.1 that f induces a *tangent map* $Df_\bullet$ between tree tangent spaces. Indeed, if $\nu \in \mathcal{V}$ is any valuation and $\vec{v}$ is a tangent vector at ν, then $\vec{v}$ is represented by a valuation μ such that $f_\bullet$ is a homeomorphism of $[\nu, \mu]$ onto $[f_\bullet \nu, f_\bullet \mu]$. We let $Df_\bullet \vec{v}$ be the tangent vector at $f_\bullet \nu$ represented by $f_\bullet \mu$; it clearly does not depend on the choice of μ.

At a divisorial valuation, the tangent map has the following interpretation.

PROPOSITION 3.3. – *Consider a divisorial valuation ν and set $\nu' = f_\bullet \nu$. Let $\pi: X \to (\mathbf{C}^2, 0)$, $\pi': X' \to (\mathbf{C}^2, 0)$ be modifications such that ν and ν' are associated to exceptional components $E \subset \pi^{-1}(0)$, and $E' \subset (\pi')^{-1}(0)$, respectively. The lift $\hat{f}: X \dashrightarrow X'$ sends E surjectively onto E'. Let f_E be its restriction to E.*

For any point $p \in E$ let $\vec{v}_p$ be the tangent vector at ν defined by p as in Section 1.10. Use the same notation for points on E' (tangent vectors are now at ν'). Then $Df_\bullet(\vec{v}_p) = \vec{v}_{p'}$ where $p' = f_E(p)$.

When $f_\bullet \nu = \nu$ and $\pi = \pi'$, f_E is a rational selfmap of E, hence admits a noncritical fixed point. This fact will be used in the proof of Theorem C.

3.1. The critical tree

As a step towards Theorem 3.1 we first investigate how $f_\bullet$ interacts with the partial ordering on $\mathcal{V}$. Note that f_* is order preserving: if $\nu \geqslant \nu'$ in the sense that $\nu(\phi) \geqslant \nu'(\phi)$ for all ϕ, then $f_* \nu \geqslant f_* \nu'$. The situation concerning $f_\bullet$ is more complicated as the function $\nu \mapsto c(f, \nu)$ is also increasing; thus $f_\bullet \nu$ is the quotient of two increasing functions, and in general not increasing.

PROPOSITION 3.4. – *The subset $\mathcal{T}_f$ of $\mathcal{V}$ where $c(f, \cdot)$ is not locally constant is a finite closed subtree of $\mathcal{V}$. Its ends (i.e. its maximal elements) are exactly the maximal elements in the finite set $\mathcal{E}_f$ consisting of divisorial valuations ν with $f_\bullet \nu = \nu_\mathfrak{m}$ and of contracted curve valuations.*

In particular, all contracted curve valuations are ends in $\mathcal{T}_f$. We call $\mathcal{T}_f$ the *critical tree* of f. Note that $f_\bullet$ is order preserving on $\mathcal{V} \setminus \mathcal{T}_f$.

Example 3.5. – Let $f(x,y) = (x^2 y, xy^3)$. The critical tree consists of all monomial valuations, and its ends are the two contracted curve valuations ν_x and ν_y. The monomial valuation $\nu_{x,2}$ with weight 2 on x and 1 on y is the only preimage of $\nu_{\mathfrak{m}}$, so $\mathcal{E}_f = \{\nu_x, \nu_y, \nu_{x,2}\}$. Note that $f_{\bullet}$ is not surjective onto $\mathcal{V}$, as $\nu_x \notin f_{\bullet}(\mathcal{V})$.

Proof of Proposition 3.4. – The function $c(f, \cdot)$ is a tree potential (see Section 1.8). Indeed, if we write $f = (g, h)$ in coordinates, then $c(f, \nu) = \min\{\nu(g), \nu(h)\}$. It is then a general fact that the locus where $c(f, \cdot)$ is not locally constant is a subtree of $\mathcal{V}$. In our case, this locus $\mathcal{T}_f$ is a finite subtree as it is contained in the subtree $\mathcal{S}$ consisting of valuations dominated by curve valuations ν_{ϕ} such that ϕ divides gh.

Let ν be an end of $\mathcal{T}_f$. We shall prove that it is a maximal element in $\mathcal{E}_f$. Note that ν is quasimonomial or a curve valuation, as $\mathcal{S}$ does not contain infinitely singular valuations. First suppose that $\nu = \nu_{\phi}$ is a curve valuation. The tree potential $c(f, \mu) = \mu(f^*\mathfrak{m})$ is piecewise affine on any segment with *integer* slope in terms of skewness. Thus $c(f, \mu) \to \infty$ as μ increases to ν_{ϕ}, so ϕ divides any element in $f^*\mathfrak{m}$. This means ν_{ϕ} is a contracted curve valuation, hence a maximal element of $\mathcal{E}_f$.

Now assume the end ν of $\mathcal{T}_f$ is quasimonomial. Let $\mathcal{L}$ denote the set of functions $L_{a,b} = ax + by$, $(a, b) \neq (0, 0)$. Suppose there exists $L \in \mathcal{L}$ such that $\nu(f^*L) > c(f, \nu)$. Then the same inequality holds in a neighborhood U of ν. Note that $c(f, \mu) = \min\{\mu(f^*L), \mu(f^*L')\}$ for any fixed $L' \neq L$ and any $\mu \in \mathcal{V}$. Thus $c(f, \mu) = \mu(f^*L')$ for any $\mu \in U$. Now a tree potential of the type $\mu \to \mu(\phi)$, $\phi \in R$ cannot attain a local maximum at a point in $\mathcal{V}_{\mathrm{qm}}$ except if it is locally constant there. Hence there exists $\mu > \nu$ for which $c(f, \mu) > c(f, \nu)$, a contradiction. Thus $f_*\nu(L) = \nu(f^*L) = c(f, \nu)$ is a constant independent of $L \in \mathcal{L}$. This implies that $f_*\nu$ is proportional to $\nu_{\mathfrak{m}}$, so $\nu \in \mathcal{E}_f$. Now the set $\{\mu > \nu\}$ does not meet $\mathcal{E}_f$ as $c(f, \cdot)$ is constant and $f_{\bullet}$ order-preserving on it, so ν is a maximal element of $\mathcal{E}_f$.

Conversely, let ν be a maximal element of $\mathcal{E}_f$. If ν is a contracted curve valuation, then $c(f, \mu) \to \infty$ as μ increases to ν, so ν is an end in $\mathcal{T}_f$. If ν is divisorial and $f_{\bullet}\nu = \nu_{\mathfrak{m}}$, then $f_{\bullet}$ cannot be order preserving near ν, so $\nu \in \mathcal{T}_f$. Since ν is maximal in $\mathcal{E}_f$, as are all ends of $\mathcal{T}_f$, ν must be an end of $\mathcal{T}_f$. $\square$

3.2. Regularity

We are now ready to prove Theorem 3.1.

Note first that the image of a monomial valuation $\nu_{s,t}$ with $\nu_{s,t}(z) = s$, $\nu_{s,t}(w) = t$ by a monomial map $f(z, w) = (z^a w^b, z^c w^d)$ equals $\nu_{as+bt, cs+dt}$. In particular, when f is dominant the induced map f_* on the set of monomial valuations is injective. More precisely, $f_*\nu_{s,t}$ and $f_*\nu_{s',t'}$ are proportional iff the two vectors (s, t), (s', t') are.

Now consider a general fixed point germ f. We first treat the case of an analytic curve valuation $\nu = \nu_C$ and refer to the end of proof for the other cases. By Proposition 1.1, one can find modifications $\pi \colon X \to (\mathbf{C}^2, 0)$, $\pi' \colon X' \to (\mathbf{C}^2, 0)$ such that $\pi^{-1}(C)$ has simple normal crossings and the lift $\hat{f} \colon X \to X'$ of f is holomorphic and locally monomial at any point on $\pi^{-1}(0)$.

By Section 1.9, the dual graph Γ of the (reducible) curve $\pi^{-1}(C)$ embeds in $\mathcal{V}$. It contains $\nu_{\mathfrak{m}}$ and ν_C, hence the full segment $I = [\nu_{\mathfrak{m}}, \nu_C]$. Let $E_0, \ldots, E_n$ be the vertices of Γ (i.e. irreducible components of $\pi^{-1}(C)$) whose associated valuations belong to I. We order them by $\nu_{\mathfrak{m}} = \nu_{E_0} < \nu_{E_1} < \cdots < \nu_{E_n} = \nu_C$. The segment $[\nu_{E_i}, \nu_{E_{i+1}}]$ consists of valuations in $\mathcal{V}$ that are monomial in suitable coordinates at $p_i := E_i \cap E_{i+1}$: see Section 1.11. As $\hat{f}$ is monomial in these coordinates, the restriction of $f_{\bullet}$ to $[\nu_{E_i}, \nu_{E_{i+1}}]$ is injective. This proves the first part of the theorem.

To prove the second part, we need to control skewness. Pick $\nu_0, \nu_1 \in \mathcal{V}$ such that $\nu_0 < \nu_1$, $f_{\bullet}$ is monotone on $[\nu_0, \nu_1]$, and ν_1 is not infinitely singular. Then $f_{\bullet}[\nu_0, \nu_1]$ is a segment containing

no infinitely singular valuation so we can pick a curve valuation $\nu_\psi \in \mathcal{V}$ such that $\nu_\psi \geqslant f_\bullet \mu$ for all $\mu \in [\nu_0, \nu_1]$. Then

$$(3.1) \qquad \alpha(f_\bullet \mu) = \frac{f_\bullet \mu(\psi)}{m(\psi)} = \frac{f_* \mu(\psi)}{c(f,\mu) \cdot m(\psi)} = \frac{\mu(f^* \psi)}{c(f,\mu) \cdot m(\psi)}.$$

Now $\mu \mapsto \mu(f^*\psi) = \sum_i \alpha(\mu \wedge \nu_{\psi_i})$ is piecewise linear in $\alpha(\mu)$ with nonnegative integer coefficients. (Here $f^*\psi = \prod \psi_i$ with ψ_i irreducible.) The same is true for $c(f,\mu) = \min\{\mu(f^*x), \mu(f^*y)\}$. Thus $\alpha(f_\bullet \mu)$ is piecewise a Möbius function of $\alpha(\mu)$ with nonnegative integer coefficients, as desired.

Finally we consider the statement in Theorem 3.1 when ν is not an analytic curve valuation. When ν is quasimonomial it is dominated by an analytic curve valuation, so the previous argument applies. Otherwise it is either a formal curve or infinitely singular valuation. But then ν does not belong to the critical tree $\mathcal{T}_f$, so we may find a divisorial valuation $\nu_1 \leqslant \nu$ such that $c(f, \cdot)$ is constant on $[\nu_1, \nu]$, equal to $c := c(f, \nu)$. In particular, $f_\bullet$ is increasing on $[\nu_1, \nu]$, and gives a bijection onto $[f_\bullet \nu_1, f_\bullet \nu]$. We claim that $\alpha(f_\bullet \mu)$ is in fact an *affine* function of $\alpha(\mu)$ on $[\nu_1, \nu]$, at least if ν_1 is chosen large enough. To prove this, pick $\nu_2 \in [\nu_1, \nu]$. Fix $\psi \in \mathfrak{m}$ irreducible, such that $\nu \wedge \nu_\psi > \nu_2$. As $c(f, \cdot)$ is constant, (3.1) shows that $\alpha(f_\bullet \mu)$ is piecewise linear with rational coefficients on each segment $[\nu_1, \nu_2]$. But the number of points in $[\nu_1, \nu_2]$ at which $\alpha(f_\bullet \mu)$ is not smooth is bounded from above by a constant depending only on f (and not on ψ) by Proposition 2.4. We can hence decompose $[\nu_1, \nu]$ into finitely many segments, on which $\alpha(f_\bullet \mu)$ is piecewise linear in $\alpha(\mu)$.

This completes the proof of Theorem 3.1.

3.3. Geometric interpretation of indeterminacy points

We now prove Proposition 3.2. Suppose first that p is not an indeterminacy point of $\hat{f}$ and write $p' = \hat{f}(p)$. Pick a valuation $\nu \in U(p)$. This means $\nu = \pi_* \mu$, for some centered valuation μ on $\mathcal{O}_p$. As $\hat{f}$ is holomorphic at p we have $\hat{f}^* \mathcal{O}_{p'} \subset \mathcal{O}_p$ and $\hat{f}^* \mathfrak{m}_{p'} \subset \mathfrak{m}_p$. By duality, $\mu' := \hat{f}_* \mu$ is a centered valuation on $\mathcal{O}_{p'}$. Now $\pi'_* \mu'$ is proportional to $f_\bullet \nu$, so $f_\bullet \nu \in U(p')$. Thus $f_\bullet U(p) \subset U(p')$. On the other hand, the center of a divisorial valuation ν' associated to an exceptional component $E' \subset (\pi')^{-1}(0)$ is equal to $E' \neq p'$, so $\nu' \notin U(p')$.

Conversely, let p be an indeterminacy point of $\hat{f}$. There exists a modification $\varpi : Y \to X$ such that $\hat{f}$ lifts to a holomorphic map, $g : Y \to X'$ and the image of $\varpi^{-1}(p)$ under g contains a curve $E' \subset X'$. We may thus find an exceptional component $E \subset \varpi^{-1}(p)$ that is sent by g surjectively onto E'. Hence the center of ν_E on X is p, so $\nu_E \in U(p)$. By construction, $f_\bullet \nu_E = \nu_{E'}$. The proof is complete.

3.4. The tangent map

Finally we prove Proposition 3.3. The lift $\hat{f} : X \dashrightarrow X'$ needs not be holomorphic, but we may find a modification $\varpi : Y \to X$ such that $\hat{f}$ lifts to a holomorphic map $g : Y \to X'$. Pick a point $q \in \varpi^{-1}(p)$ on the strict transform of E.

If a valuation in $\mathcal{V}$ has center q on Y, then it has center p on X. Hence $U(q) \subset U(p)$. Moreover, ν lies in the closure of $U(q)$ and all valuations in $U(p)$ represent the same tangent vector $\vec{v}_p$ at ν. Similarly, $\nu' \in \overline{U(p')}$ and all valuations in $U(p')$ represent the same tangent vector $\vec{v}_{p'}$ at ν'.

Proposition 3.2 shows that $f_\bullet U(q) \subset U(p')$. By the definition of the tangent map $Df_\bullet$, we conclude that $Df_\bullet \vec{v}_p = \vec{v}_{p'}$. This concludes the proof.

4. Eigenvaluations

The goal of this section is to show that the induced map $f_\bullet : \mathcal{V} \to \mathcal{V}$ admits a fixed point and uses the latter to analyze the attraction rate of f.

DEFINITION 4.1. – The *attraction rate* of f is $c(f) := c(f, \nu_\mathfrak{m}) = \nu(f^*\mathfrak{m})$. The *asymptotic attraction rate* is the limit $c_\infty := \lim_{n \to \infty} c(f^n)^{1/n}$.

The limit defining c_∞ exists as the sequence $c(f^n)$ is supermultiplicative. The main result of this section is

THEOREM 4.2. – *Any dominant fixed point germ* $f : (\mathbf{C}^2, 0) \to (\mathbf{C}^2, 0)$ *with* $c_\infty(f) > 1$ *admits a valuation* $\nu_\star \in \mathcal{V}$ *with* $f_\bullet \nu_\star = \nu_\star$ *and* $c(\nu_\star) = c_\infty$. *The valuation* $\nu_\star$ *may be chosen quasimonomial, infinitely singular, or as a noncontracted, analytic curve valuation. When* $\nu_\star$ *is not quasimonomial, there exists* $\nu_0 < \nu_\star$ *such that* $c(f, \nu_0) = c_\infty$, $f_\bullet$ *is order-preserving on* $\{\nu \geqslant \nu_0\}$ *and* $f_\bullet \nu > \nu$ *for* $\nu \in [\nu_0, \nu_\star[$.
Finally, we may find $0 < \delta \leqslant 1$ *such that* $\delta \cdot c_\infty^n \leqslant c(f^n) \leqslant c_\infty^n$ *for all* $n \geqslant 1$.

DEFINITION 4.3. – A valuation $\nu_\star$ as above is called an *eigenvaluation* for f. Note that $\nu_\star$ may be viewed as an eigenvector for f_* with eigenvalue c_∞.

We do not claim that the eigenvaluation is unique. The proof of Theorem 4.2 will be given in Section 4.2 after we have discussed the main ingredient: a purely tree-theoretic fixed point theorem.

4.1. Fixed point theorem on trees

Let us recall some general definitions on trees taken from [12, Chapter 3]. A (rooted, nonmetric) *complete tree* is a poset $(\mathcal{T}, \leqslant)$ with a unique minimal element τ_0 (the root) such that for any $\tau, \tau' \in \mathcal{T}$ with $\tau < \tau'$ there exists an increasing bijection from $[0, 1]$ (endowed with the standard order relation) onto the *segment* $[\tau, \tau'] := \{\sigma; \ \tau \leqslant \sigma \leqslant \tau'\}$ and such that any totally ordered subset of $\mathcal{T}$ has a majorant in $\mathcal{T}$. To any segment $I = [\tau, \tau'] \subset \mathcal{T}$ with $\tau \leqslant \tau'$ is associated a *retraction* $\pi_I : \mathcal{T} \to I$: $\pi_I(\sigma)$ is the maximal element in $I \cap [\tau_0, \sigma]$ if this set is nonempty, and $\pi_I(\sigma) = \tau$ otherwise. The map π_I is continuous and surjective onto I, and $\pi_I|_I = \mathrm{id}$. We endow $\mathcal{T}$ with the weakest topology in which π_I is continuous for every segment I. It is generated by sets $U(\vec{v})$ over tangent vectors $\vec{v}$ as in Section 1.6.

DEFINITION 4.4. – A map $\mathcal{F} : \mathcal{T} \to \mathcal{T}'$ between complete trees $\mathcal{T}, \mathcal{T}'$ is a *tree map* if it is continuous on branches, i.e. if the restriction $\mathcal{F}|_I : I \to \mathcal{T}'$ is continuous for every segment I in $\mathcal{T}$.
A tree map $\mathcal{F}$ is *regular* if any segment I in $\mathcal{T}$ can be decomposed into finitely many segments on each of which $\mathcal{F}$ is a homeomorphism onto its image.

There is a natural way in which selfmaps of trees induce selfmaps of segments. Let $\mathcal{F} : \mathcal{T} \to \mathcal{T}$ be a tree map and let I be a segment in $\mathcal{T}$. Set $\mathcal{F}_I := \pi_I \circ \mathcal{F}|_I$, where $\pi_I : \mathcal{T} \to I$ is the retraction defined above. Then $\mathcal{F}_I$ is a continuous selfmap of I. Notice that $\mathcal{F} \neq \mathcal{F}_I$ on I in general: the set $\{\tau \in I; \ \mathcal{F}(\tau) \neq \mathcal{F}_I(\tau)\}$ is an open subset of I on which $\mathcal{F}_I$ is locally constant.
We define an *end* $\tau \in \mathcal{T}$, i.e. a maximal element under $\leqslant$, to be *weakly attracting* if there exists a segment $I = [\tau', \tau]$ such that τ is a globally attracting fixed point for the induced map $\mathcal{F}_I$ in the sense that $\mathcal{F}_I(\sigma) > \sigma$ for every $\sigma \in [\tau', \tau[$. The end is *strongly attracting* when in addition the segment I can be chosen $\mathcal{F}$-invariant.
We are now ready to state the following fixed point property for tree maps.

THEOREM 4.5. – *Let $\mathcal{T}$ be a complete tree and $\mathcal{F}: \mathcal{T} \to \mathcal{T}$ a tree map* (*resp. a regular tree map*). *Then one of the following two statements holds*:
- *$\mathcal{F}$ admits a fixed point τ which is not an end*;
- *$\mathcal{F}$ admits a weakly* (*resp. strongly*) *attracting end.*

Remark 4.6. – Theorem 4.5 was proved independently by Rivera [26, Section 8.3] in his work on p-adic dynamics. It may be viewed as a "non-invertible" analog of the description of isometries on metric **R**-trees (see [22,24]).

Proof of Theorem 4.5. – Suppose first $\mathcal{F}$ is a tree map. We may assume that $\mathcal{F}\tau_0 \neq \tau_0$, or else there is nothing to prove. Pick any end τ_0' such that $\tau_0 < \mathcal{F}\tau_0 \leqslant \tau_0'$ and let $I_0 = [\tau_0, \tau_0']$. We consider the induced selfmap $\mathcal{F}_0 = \mathcal{F}_{I_0}$ of I_0 defined above.

Since I_0 is homeomorphic to a compact interval, $\mathcal{F}_0$ admits a fixed point $\tau_1 \in I_0$. Further, $\tau_1 > \tau_0$ as $\mathcal{F}_0\tau_0 > \tau_0$. Notice that we may assume that τ_1 is not an end unless $\mathcal{F}_0\tau > \tau$ for every $\tau \in [\tau_0, \tau_0'[$, in which case $\tau_1 = \tau_0'$ is a weakly attracting fixed point for $\mathcal{F}$.

If $\tau_1 \in]\tau_0, \tau_0'[$, then τ_1 is either a fixed point of $\mathcal{F}$, in which case we are done, or satisfies $\mathcal{F}\tau_1 > \tau_1$. In the latter case pick an end τ_1' with $\tau_1 < \mathcal{F}\tau_1 \leqslant \tau_1'$ and let $I_1 = [\tau_1, \tau_1']$. We write $\mathcal{F}_1 := \mathcal{F}_{I_1}$ for the induced selfmap of I_1. The map $\mathcal{F}_1$, being a continuous selfmap of a compact interval, has a fixed point $\tau_2 \in I_1$. Since $\mathcal{F}\tau_1 > \tau_1$ we have $\mathcal{F}_1\tau_1 > \tau_1$, which implies $\tau_2 > \tau_1$. We may assume $\tau_1 < \tau_2 < \tau_1'$ unless τ_1' is a weakly attracting fixed point for $\mathcal{F}$. If $\tau_2 \neq \tau_1'$ is an end, then we assume $\mathcal{F}\tau_2 > \tau_2$, or else we are done. Then the inductive procedure continues.

If this procedure stops, we obtain a fixed point for $\mathcal{F}$: a non-end or a weakly attracting end. Suppose the procedure continues indefinitely. We get a sequence of points $\tau_0 < \tau_1 < \tau_2 < \cdots$ such that $\mathcal{F}\tau_n > \tau_n$ for every n. Let $\tau_\infty = \sup \tau_n$. We claim that $\mathcal{F}\tau_\infty = \tau_\infty$. To see this, consider τ_n', the inductively constructed ends. Since $\mathcal{F}_n\tau_n = \tau_n$, the segments $[\tau_n, \tau_\infty]$ and $[\tau_n, \tau_n']$ intersect only at τ_n. In particular, $\mathcal{F}\tau_n \not\geqslant \tau_\infty$. By continuity this implies that $\mathcal{F}\tau_\infty \not> \tau_\infty$. On the other hand, $\mathcal{F}\tau_n > \tau_n$, so continuity also gives $\mathcal{F}\tau_\infty \geqslant \tau_\infty$. Thus τ_∞ is a fixed point for $\mathcal{F}$.

If τ_∞ is an end, then set $I = [\tau_0, \tau_\infty]$ and $I^* = [\tau_0, \tau_\infty[$. We claim that either $\mathcal{F}$ has a fixed point $\tau \in I^*$, or $\mathcal{F}_I\tau > \tau$ for all $\tau \in I^*$. In either case the proof is complete.

Therefore assume that $\mathcal{F}$ has no fixed point in I^* and that $\mathcal{F}_I\tau \leqslant \tau$ for some $\tau \in I^*$. Since $\mathcal{F}_I\tau_n > \tau_n$ and $\tau_n \to \infty$, then there exists a fixed point $\tau' \in I$ for $\mathcal{F}_I$. By assumption, $\mathcal{F}\tau' \neq \tau'$, so we must have $\mathcal{F}\tau' > \tau'$ and $\mathcal{F}\tau' \notin I$. Thus $\mathcal{F}_I$ is locally constant at τ'. Pick $\tau'' \in I$ maximal such that $\mathcal{F}_I\tau < \tau$ on $]\tau', \tau''[$. We have $\tau'' < \tau_\infty$ since $\mathcal{F}_I\tau_n > \tau_n$ for every n. Clearly $\mathcal{F}_I\tau'' = \tau''$ and we claim that in fact $\mathcal{F}\tau'' = \tau''$. Indeed, if not, then $\mathcal{F}_I$ is locally constant at τ'', which implies that $\mathcal{F}_I\tau > \tau$ for $\tau \approx \tau''$, $\tau < \tau''$. This is a contradiction, completing the proof when $\mathcal{F}$ is a tree map.

Finally suppose $\mathcal{F}$ is regular, and τ is a weakly attracting end. Consider a segment $I = [\tau_0, \tau[$, such that $\mathcal{F}_I\tau' > \tau'$ for all $\tau' \in I$. Since $\mathcal{F}$ is regular, we may in fact assume that $\mathcal{F}$ is a homeomorphism on I. This implies that $\mathcal{F}_I = \mathcal{F}$ on I. $\square$

4.2. Proof of Theorem 4.2

We know from [12, Theorem 3.14] that $(\mathcal{V}, \leqslant)$ is a complete tree. Theorem 3.1 implies that the induced map $f_\bullet: \mathcal{V} \to \mathcal{V}$ is a regular tree map. Thus Theorem 4.5 applies, and provides a fixed point $\nu_\star \in \mathcal{V}$.

If $\nu_\star$ is not an end, then it is quasimonomial, with finite skewness $\alpha_\star := \alpha(\nu_\star)$. This implies $\nu_\mathfrak{m}(\phi) \leqslant \nu_\star(\phi) \leqslant \alpha_\star \nu_\mathfrak{m}(\phi)$ for any $\phi \in R$. But $c(f^n, \nu) = \nu(f^{n*}\mathfrak{m})$, so we get $c(f^n, \nu_\mathfrak{m}) \leqslant c(f^n, \nu_\star) \leqslant \alpha_\star c(f^n, \nu_\mathfrak{m})$. As $f_\bullet \nu_\star = \nu_\star$, we have $c(f^n, \nu_\star) = c(f, \nu_\star)^n$. This proves that $c(f^n) \leqslant c(f, \nu_\star)^n \leqslant \alpha_\star c(f^n)$. In particular, $c_\infty = c(f, \nu_\star)$ and $c(f^n)/c_\infty^n \geqslant \delta$ for all n with $\delta = \alpha_\star^{-1}$.

Now suppose $\nu_\star$ is a strongly attracting end. By definition this implies that $f_\bullet \nu > \nu$ on $[\nu_0, \nu_\star[$ for some $\nu_0 < \nu_\star$. We claim that $\nu_\star$ cannot belong to $\mathcal{T}_f$, the critical tree of f. Indeed, by Proposition 3.4 the ends of $\mathcal{T}_f$ are either quasimonomial or contracted curve valuations. But the latter are mapped by $f_\bullet$ to divisorial valuations by Proposition 2.6, hence cannot be fixed points. Thus $\nu_\star \notin \mathcal{T}_f$. Since $\mathcal{T}_f$ is closed, we may assume $\nu_0 \notin \mathcal{T}_f$ so that $c(f, \cdot)$ is constant and $f_\bullet$ order-preserving on $\{\nu \geqslant \nu_0\}$. This implies $c(f^n, \nu_0) = \prod_0^{n-1} c(f, f_\bullet^k \nu) = c_\star^n$, where $c_\star = c(f, \nu_\star)$. The skewness α_0 of ν_0 is finite and $c(f^n) \leqslant c(f^n, \nu_0) \leqslant \alpha_0 c(f^n)$. As before we conclude that $c_\star = c_\infty$ and that $c(f^n)/c_\infty^n \geqslant \alpha_0^{-1}$.

If $\nu_\star$ is a strongly attracting end, then it is infinitely singular or a curve valuation. Suppose it is a curve valuation. We saw above that it cannot be contracted. To prove that it is analytic, we rely on the following useful formula.

LEMMA 4.7. – *For any $\nu \in \mathcal{V} \setminus \mathfrak{C}_f$ we have*

$$(4.1) \qquad c(f, \nu) A(f_\bullet \nu) = A(\nu) + \nu(Jf).$$

Here A denotes thinness and Jf is the Jacobian determinant of f. To prove (4.1), we may by continuity assume ν is divisorial and use the geometric interpretation in Section 1.9. With the help of Proposition 2.5, the proof reduces to the change of variables formula. The details are left to the reader.

Continuing the proof of the theorem, if $\nu_\star = \nu_C$ is a curve valuation, pick ν_0 as above such that $m(\nu_0) = m(C)$. Apply (4.1) with $\nu = \nu_{C,t}$ for large t. Then $A(\nu) = mt + B$ for some constant B (see Section 1.7), $c(f, \nu) = c_\infty > 1$ and $A(f_\bullet \nu) > A(\nu)$, so we get $\nu(Jf) > (c_\infty - 1)(mt + B) \to \infty$ as $t \to \infty$. Thus C belongs to the critical set of f and in particular is an analytic curve.

This completes the proof of Theorem 4.2.

5. Local normal forms

In this section, we prove a quite precise version of the rigidification statement in Theorem C. As an easy consequence we will also obtain Theorem A. The key step in achieving rigidification involves finding suitable basins of attraction of the eigenvaluation: see Proposition 5.2.

Recall that a fixed point germ is called *rigid* when its critical set is contained in a totally invariant set with normal crossings.

THEOREM 5.1. – *Let f be a dominant fixed point germ with $c_\infty(f) > 1$. Then one can find a modification $\pi: X \to (\mathbf{C}^2, 0)$ and an infinitely near point $p \in \pi^{-1}(0)$ such that the lift $\hat{f}$ of f is holomorphic at p, $\hat{f}(p) = p$ and $\hat{f}: (X, p) \to (X, p)$ is rigid. Moreover, there exist local coordinates (z, w) at p in which $\hat{f}$ takes one of the following forms*:
  (i) $\hat{f}(z, w) = (z^a w^b (1 + \phi), \lambda w (1 + \psi))$ *with* $a \geqslant 2$, $b \geqslant 1$, $\lambda \in \mathbf{C}^*$ *and* $\phi(0) = \psi(0) = 0$; *then* $c_\infty(f) = a$;
  (ii) $\hat{f}(z, w) = (z^a w^b, z^c w^d)$ *with* $a, b, c, d \in \mathbf{N}$, $ad \neq bc$; *then* $c_\infty(f)$ *is the spectral radius of the 2×2 matrix with entries a, b, c and d*;
  (iii) $\hat{f}(z, w) = (z^a, \lambda z^c w^d)$ *with* $d \geqslant a \geqslant 2$, $c \geqslant 1$, $\lambda \in \mathbf{C}^*$; *then* $c_\infty(f) = a$;
  (iv) $\hat{f}(z, w) = (z^a, \lambda z^c w + P(z))$ *with* $a \geqslant 2$, $c \geqslant 1$, $\lambda \in \mathbf{C}^*$ *and* $P \not\equiv 0$ *is a polynomial*; *then* $c_\infty(f) = a$.
The exceptional divisor is included in $\{zw = 0\}$. In the first case, it equals $\{zw = 0\}$. In the other three cases, it coincides with the curves that are contracted to 0 by $\hat{f}$.

The four cases arise according to the type of the eigenvaluation $\nu_\star$. In order, they appear when $\nu_\star$ is divisorial [5] (the exceptional component is $\{z = 0\}$), irrational, a curve valuation (the curve equals $\pi\{w = 0\}$) or an infinitely singular valuation.

As we do not claim that the eigenvaluation is unique, the four cases are a priori not mutually exclusive.

Proof of Theorem A. – We know from Theorem 4.2 that the quotient $c_\infty^n / c(f^n)$ is bounded from above and below by positive constants. By inspecting the local forms in Theorem 5.1 it is clear that c_∞ is a quadratic integer. $\quad\square$

5.1. Basins of attraction

The first step in the proof of Theorem 5.1 is to find small invariant regions in $\mathcal{V}$ that serve as basins of attraction for the eigenvaluation. We need some flexibility in the choice of these basins; this explains the somewhat technical appearance of the result below.

PROPOSITION 5.2. – *Let f be a dominant fixed point germ with $c_\infty(f) > 1$. Let $\nu_\star$ be an eigenvaluation for f whose existence is guaranteed by Theorem 4.2.*

  (i) *If $\nu_\star$ is infinitely singular or an analytic curve valuation, then for any $\nu_0 \in \mathcal{V}$ with $\nu_0 < \nu_\star$ and ν_0 sufficiently close to $\nu_\star$, $f_\bullet$ maps the segment $I = [\nu_0, \nu_\star]$ strictly into itself and is order-preserving there. Moreover, set $U = U(\vec{v})$ where $\vec{v}$ is the tangent vector at ν_0 represented by $\nu_\star$. Then $f_\bullet$ also maps the open set U strictly into itself and $f_\bullet^n \to \nu_\star$ as $n \to \infty$ on U.*

  (ii) *If $\nu_\star$ is divisorial, then there exists a tangent vector $\vec{w}$ at $\nu_\star$ such that for any $\nu_0 \in \mathcal{V}$ representing $\vec{w}$ and sufficiently close to $\nu_\star$, $f_\bullet$ maps the segment $I = [\nu_\star, \nu_0]$ strictly into itself and is order-preserving there. Moreover, set $U = U(\vec{v}) \cap U(\vec{w})$, where $\vec{v}$ is the tangent vector at ν_0 represented by $\nu_\star$. Then $f_\bullet(U) \subset U$. Further $f_\bullet^n \to \nu_\star$ as $n \to \infty$ on U.*

  (iii) *If $\nu_\star$ is irrational, then there exist $\nu_0, \nu_1 \in \mathcal{V}$, arbitrarily close to $\nu_\star$, with $\nu_0 < \nu_\star < \nu_1$ such that $f_\bullet$ maps the segment $I = [\nu_0, \nu_1]$ into itself. Let $\vec{v}_1$ ($\vec{v}_2$) be the tangent vector at ν_0 (ν_1) represented by $\nu_\star$ and set $U = U(\vec{v}_1) \cap U(\vec{v}_2)$. Then $f_\bullet U \subset U$. Further, either $f_\bullet^2 = \mathrm{id}$ on I or $f_\bullet^n \to \nu_\star$ as $n \to \infty$ on U.*

The different basins U are illustrated in Figs. 1 and 2.

Remark 5.3. – In the case (iii) of an irrational eigenvaluation, $f_\bullet$ may be order-reversing near $\nu_\star$ and not all segments containing $\nu_\star$ are $f_\bullet$-invariant: an $f_\bullet$-invariant segment has to be roughly symmetric around $\nu_\star$. However, $f_\bullet^2$ is always order-preserving and it is in fact much easier to prove Theorem 5.1 for f^2 rather than f (Lemmas 5.7 and 5.8 below are not needed).

Example 5.4. – Let $f(x, y) = (x^2 y^3 + x^7, x^7)$. The segment $I = [\nu_{\mathrm{m}}, \nu_{y,3/2}]$ consists of monomial valuations of the form $\nu_{y,t}$, $1 \leqslant t \leqslant 3/2$, with $\nu_{y,t}(x) = 1$ and $\nu_{y,t}(y) = t$. We have $f_\bullet I \subset I$ and the restriction of $f_\bullet$ to I is given by $t \mapsto 7/(2 + 3t)$. One can check that the open subset $U \subset \mathcal{V}$ consisting of all valuations $\nu \in \mathcal{V}$ with $\nu(x) = 1$ and $1 < \nu(y) < 3/2$ is invariant under $f_\bullet$.

There exists an eigenvaluation $\nu_\star$ belonging to I, which corresponds to $t = t_\star = \frac{1}{3}(\sqrt{22} - 1)$. We have $c_\infty(f) = c(f, \nu_\star) = 3t_\star + 2 = 1 + \sqrt{22}$. The basin of attraction U is of the form $U = U(p)$, where p is an infinitely near point on a model obtained by three blowups. In suitable coordinates at p, the lift of f takes the form $\hat{f}(z, w) = (z^2 w^3, z^7)$.

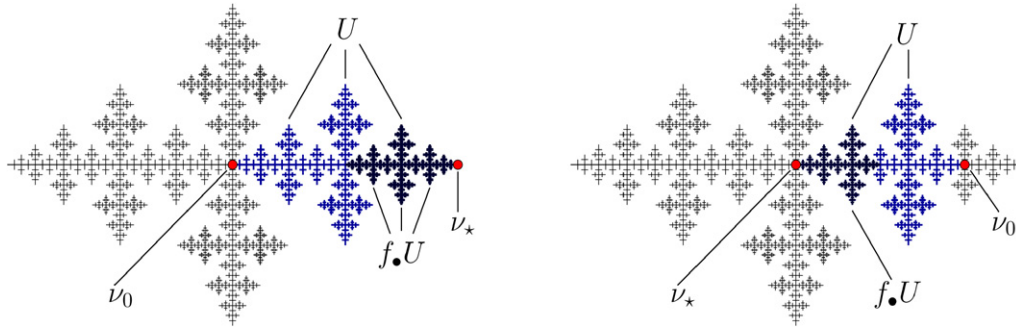[5] When $|\lambda| < 1$, one can make $\phi \equiv 0$.

Fig. 1. Basins of attraction of an infinitely singular or curve eigenvaluation (left) and a divisorial eigenvaluation (right). See cases (i) and (ii) in Proposition 5.2.
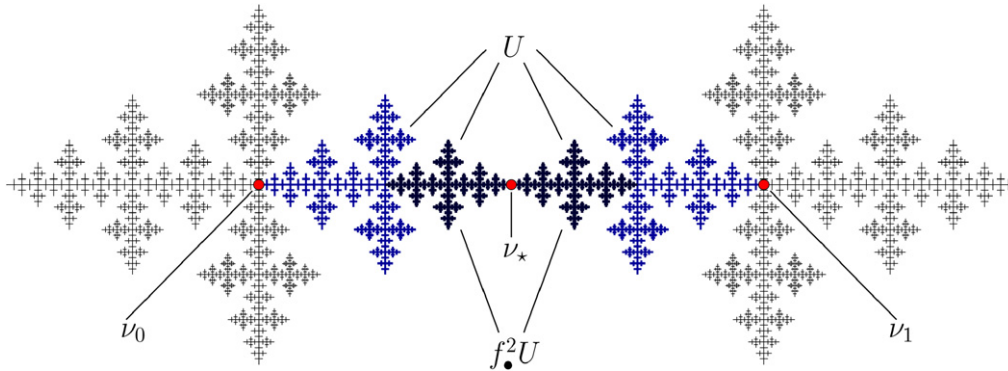

Fig. 2. A basin of attraction of an irrational eigenvaluation. See case (iii) in Proposition 5.2.

Proof of Proposition 5.2. – When $\nu_\star$ is not quasimonomial, Theorem 4.2 immediately implies all the relevant assertions in (i) above.

For the case of a quasimonomial eigenvaluation, we rely on the fact that $f_\bullet$ is piecewise Möbius in the parameterization of $\mathcal{V}$ by skewness. In doing so, we shall use the following elementary result, whose proof is left to the reader.

LEMMA 5.5. – *Set $M(t) = \frac{at+b}{ct+d}$ where $a, b, c, d \in \mathbf{N}$, $ad \neq bc$. Suppose $M(t_\star) = t_\star$ with $t_\star > 0$. Then either $M \circ M \equiv \mathrm{id}$, or $t_\star$ is locally attracting, i.e. $|M'(t_\star)| < 1$.*

Now suppose $\nu_\star$ is irrational. Pick valuations μ_0, μ_1 close to $\nu_\star$ with $\mu_0 < \nu_\star < \mu_1$. By Theorem 3.1, we may assume that $f_\bullet$ is monotone on the segment $J := [\mu_0, \mu_1]$, and $\alpha(f_\bullet \nu)$ is a Möbius function of $\alpha(\nu)$ with nonnegative integer coefficients. The segment $f_\bullet J$ contains $\nu_\star$. The intersection $J \cap f_\bullet J$ is thus a nontrivial segment I' and $f_\bullet I' \subset J$. Apply Lemma 5.5 on I'.

Suppose first $f_\bullet^2 = \mathrm{id}$ on I'. If $f_\bullet = \mathrm{id}$, we set $\nu_0 := \mu_0$, and $\nu_1 := \mu_1$. The segment $[\nu_0, \nu_1]$ is clearly invariant. Otherwise $f_\bullet^2 = \mathrm{id}$ but $f_\bullet \neq \mathrm{id}$, so that $f_\bullet$ is order-reversing. We choose $\nu_0 < \nu_\star$ in the segment I', and we set $\nu_1 = f_\bullet \nu_0$. The segment $I := [\nu_0, \nu_1]$ then contains $\nu_\star$ and is $f_\bullet$-invariant (and $f_\bullet^2 = \mathrm{id}$ on I). When instead $f_\bullet^2 \neq \mathrm{id}$, $\nu_\star$ is attracting and for any two valuations $\nu_0 < \nu_\star < \nu_1$ at the same distance to $\nu_\star$, the segment $I := [\nu_0, \nu_1]$ is invariant. Moreover, $f_\bullet^n \to \nu_\star$ on I in this case. Note also that in all cases, we may choose ν_0 and ν_1 arbitrarily close to $\nu_\star$.

Now consider the region U associated to I. Pick a curve valuation $\nu_\psi > \nu_1$. Then $U = \{\nu;\ \nu_0 < \nu \wedge \nu_\psi < \nu_1\}$. Write $f^*\psi = \prod \psi_k$ with $\psi_k \in \mathfrak{m}$ irreducible. The curve valuations ν_{ψ_k} are the preimages of ν_ψ. After shrinking I we may assume that none of them lies in U. We may also assume that the critical tree $\mathcal{T}_f$ of f intersects U only along I (or not at all). Thus $\mu \mapsto \mu(\psi_k)$ and $\mu \mapsto c(f,\mu)$ are constant on any segment $[\tilde\nu, \nu]$, where $\nu \in U$ and $\tilde\nu := \nu \wedge \nu_\psi \in I$. We infer

$$(f_\bullet \nu)(\psi) = \frac{\nu(f^*\psi)}{c(f,\nu)} = \frac{\sum \nu(\psi_k)}{c(f,\nu)} = \frac{\sum \tilde\nu(\psi_k)}{c(f,\tilde\nu)} = (f_\bullet \tilde\nu)(\psi).$$

Thus $f_\bullet \nu \wedge \nu_\psi = f_\bullet \tilde\nu \wedge \nu_\psi \in f_\bullet I \subset I$. This implies $f_\bullet U \subset U$. When $f_\bullet^2 \neq \mathrm{id}$ on I, $f_\bullet^n \to \nu_\star$ on I, hence also on U.

Suppose finally that $\nu_\star$ is divisorial. By the remark following Proposition 3.3 there exists a tree tangent vector $\vec{w}$ at $\nu_\star$ invariant and non-critical under the tangent map. Now pick $\mu_0 \in \mathcal{V}$ representing this tangent vector and close enough to $\nu_\star$ so that $f_\bullet$ is monotone on the segment $J := [\nu_\star, \mu_0]$. This is always possible by Theorem 3.1. Then $I' := f_\bullet J \cap J \neq \emptyset$. If $f_\bullet \not\equiv \mathrm{id}$ on I', then we may argue as in the irrational case to construct I and U with the desired properties.

To conclude, let us therefore assume that $f_\bullet = \mathrm{id}$ on I' and derive a contradiction, previewing arguments to appear in Section 5.2. The function $c(f,\cdot)$ must be constant on I', say $c(f,\cdot) \equiv k$ for some integer $k > 1$. (See Proposition 2.5.) We may find a valuation $\nu_0 \in I'$ near $\nu_\star$ and a modification $\pi : X \to (\mathbf{C}^2, 0)$ such that $\nu_\star$ and ν_0 are associated to exceptional components E, E_0 intersecting in some point p corresponding to $\vec{w}$. Then the lift $\hat f : X \dashrightarrow X$ of f is holomorphic at p and $\hat f(p) = p$. Moreover, $\hat f^* E = kE$ and $\hat f^* E_0 = kE_0$, forcing $\hat f|_E$ to be critical at p, a contradiction. $\quad\square$

5.2. Rigidification

We now prove Theorem 5.1. Let $\nu_\star$ be an eigenvaluation for f as in Theorem 4.2. The idea is to show that the basins of attractions found in Proposition 5.2 can be chosen as $U = U(p)$ for an infinitely near point p. This will imply that a suitable lift $\hat f$ is holomorphic at p. Choosing U small enough will allow us to control the critical set and make $\hat f$ rigid. The normal forms follow from the analysis in [9] and the formulas for c_∞ from direct computations. We split into cases depending on the type of $\nu_\star$.

First suppose $\nu_\star = \nu_C$ is a noncontracted analytic curve valuation. Pick ν_0 as in Proposition 5.2. By increasing ν_0, we may assume it is divisorial. Pick a modification $\pi : X \to (\mathbf{C}^2, 0)$ such that $\nu_0 = \nu_{E_0}$ for some exceptional component $E_0 \subset \pi^{-1}(0)$ and such that $\pi^{-1}(C)$ has simple normal crossings. There is a unique exceptional component $E \subset \pi^{-1}(0)$ that intersects the strict transform $\tilde C$ of C, say at the point $p \in E$. It follows from [12, Proposition 6.32] that the divisorial valuation ν_E associated to E satisfies $\nu_E < \nu_C$ and $\nu_F \notin {]\nu_E, \nu_C[}$ for any $F \subset \pi^{-1}(0)$. In particular $\nu_0 \leqslant \nu_E < \nu_C$. The region $U = U(p) \subset \mathcal{V}$ consisting of valuations whose center on X is p coincides with the set of valuations $\nu \in \mathcal{V}$ representing the same tangent vector as $\nu_\star$ at ν_E. By Proposition 5.2 we have $f_\bullet U \Subset U$.

It now follows from Proposition 3.2 that the lift $\hat f := \pi^{-1} \circ f \circ \pi$ is holomorphic at p and $\hat f(p) = p$. Notice that ν_0 could be chosen arbitrarily close to ν_C. Hence we may assume that U does not contain any curve valuations associated to an irreducible component of the critical set, nor any preimages of ν_C, except ν_C itself. The critical set of $\hat f$ is therefore equal to $E \cup \tilde C$, which is totally invariant and has simple normal crossings. Thus $\hat f$ is rigid at p.

To get the normal form of $\hat f$ at p, we note that $\hat f$ is superattracting at p. Indeed, E is contracted to p since $f_\bullet \nu_E > \nu_E$, and the restriction of $\hat f$ to $\tilde C$ has topological degree $c(f, \nu_C) = c_\infty > 1$ by Proposition 2.6. We may thus apply the results from [9]. The critical set of $\hat f$ has two branches:

E, which is contracted to p, and $\tilde{C}$, which is fixed. Thus $\hat{f}$ is of Type 6, see [9, Table I, p. 478], so $\hat{f}(z,w) = (z^a, z^c w^d)$ with $a \geqslant 2$, $c \geqslant 1$ and $d \geqslant 2$. In fact, since the curve valuation ν_C is attracting, a simple calculation gives $d \geqslant a$. The argument also shows that $c_\infty = 1$.

Now suppose $\nu_\star$ is infinitely singular. Again pick $\nu_0 < \nu_\star$ divisorial as in Proposition 5.2. The multiplicity function $m = m(\nu)$ is nondecreasing, integer-valued and unbounded on the segment $[\nu_\mathfrak{m}, \nu_\star[$, see Section 1.7. We may therefore find a divisorial valuation $\nu \in]\nu_0, \nu_\star[$ where the multiplicity jumps. By [12, Proposition 6.40] there exists a modification $\pi : X \to (\mathbf{C}^2, 0)$ such that $\nu = \nu_E$ is associated to an exceptional component $E \subset \pi^{-1}(0)$ and such that the center of $\nu_\star$ on X lies on E and is a *smooth* point $p \in \pi^{-1}(0)$ (in [12], such a point is called a free point). As above we may set $U = U(p)$ and conclude that $f_\bullet U \Subset U$.

Again the lift $\hat{f} := \pi^{-1} \circ f \circ \pi$ is holomorphic at p and $\hat{f}(p) = p$. We may ensure that U does not contain any curve valuations associated to irreducible components of the critical set. This implies that $\hat{f}$ is rigid at p, and that the critical set of $\hat{f}$ is equal to E, which is contracted to p. Comparing with [9] we see that $\hat{f}$ is of Type 4, i.e. $\hat{f}(z,w) = (z^a, \lambda z^c w + P(z))$ for some polynomial P in suitable coordinates. Since $f_\bullet^n \to \nu_\star$ on U, $\hat{f}$ does not leave any curve invariant, hence $P \not\equiv 0$.

A direct computation gives $c(f, \nu_E) = a$ and in fact also $c(f^n, \nu_E) = a^n$, for any $n \geqslant 1$. But $c(f^n, \nu_E)/c(f^n)$ is bounded from above and below by positive constants, so $c_\infty = \lim_{n\to\infty} c(f^n, \nu_E)^{1/n} = a$.

Next suppose $\nu_\star$ is divisorial. Pick ν_0 as in Proposition 5.2; we may assume it is divisorial. Then pick a modification $\pi : X \to (\mathbf{C}^2, 0)$ such that both $\nu_\star = \nu_E$ and $\nu_0 = \nu_{E_0}$ are associated to exceptional components $E, E_0 \subset \pi^{-1}(0)$. There is then a unique exceptional component $F \subset \pi^{-1}(0)$ intersecting E and such that $\nu_F \in]\nu_E, \nu_0]$ (see [12, Corollary 6.32]). Replace ν_0 by ν_F. The set U in Proposition 5.2 is of the form $U = U(p)$, where $E \cap F = \{p\}$.

As before, $f_\bullet U \subset U$, which implies that the lift $\hat{f}$ of f is holomorphic at p and $\hat{f}(p) = p$. By moving the original divisorial valuation ν_0 closer to $\nu_\star$, we may assume that U does not contain any curve valuations associated with irreducible components of the critical set of f. Then the critical set of $\hat{f}$ is locally included in the exceptional divisor $E \cup F$, which is totally invariant since $f_\bullet U \subset U$. Hence $\hat{f}$ is rigid at p.

As for the normal form at p, we may assume that p is a noncritical fixed point for the restriction of $\hat{f}$ on E: see the remark after Proposition 3.3. Choose local coordinates (z, w) at p such that $E = \{z = 0\}$, and $F = \{w = 0\}$. Using the fact that $\hat{f}^{-1} E \subset E \cup F$, $\hat{f}E = E$, and $\hat{f}F = p$, we easily get that $\hat{f}$ can be written in the form in (i). Proposition 2.5 with $E' = E$ then implies $c_\infty = c(f, \nu_E) = a$.

Finally suppose $\nu_\star$ is irrational. We rely on the following result, which strengthens Proposition 5.2 above.

LEMMA 5.6. – *There exist a modification $\pi : X \to (\mathbf{C}^2, 0)$ and exceptional components $E_0, E_1 \subset \pi^{-1}(0)$, intersecting in some point $p \in \pi^{-1}(0)$, such that*:
- *the divisorial valuations ν_0 and ν_1, associated to E_0 and E_1, respectively, satisfy $\nu_0 < \nu_\star < \nu_1$;*
- *the open set $U := \{\nu;\ \nu_0 < \nu \wedge \nu_1 < \nu_1\}$ is $f_\bullet$-invariant.*

Moreover, ν_0 and ν_1 can be chosen to be arbitrarily close to $\nu_\star$.

We give a proof below. The set U of the lemma is of the form $U = U(p)$. As U is $f_\bullet$-invariant, we conclude as before that the lift $\hat{f}$ of f is holomorphic and rigid at p.

To get the normal form of $\hat{f}$ at p, we proceed as follows. Pick local coordinates (z, w) such that $E_0 = \{z = 0\}$ and $E_1 = \{w = 0\}$. Since $E_0 \cup E_1$ is locally totally invariant under $\hat{f}$ we have $\hat{f}(z, w) = (z^a w^b \phi, z^c w^d \psi)$, where $a, b, c, d \in \mathbf{N}$ and $\phi, \psi \in \mathcal{O}_p \setminus \mathfrak{m}_p$. For $s, t \geqslant 0$, let $\mu_{s,t}$

be the monomial valuation on $\mathcal{O}_p$ in coordinates (z,w) with $\mu_{s,t}(z)=s$ and $\mu_{s,t}(w)=t$. Then the segment $I=[\nu_0,\nu_1]$ is parameterized by $\pi_*\mu_{s,t}$ as $b_{E_0}s+b_{E_1}t=1$. Now $\pi_*\mu_{1,0}=b_{E_0}\nu_0$ and $\pi_*\mu_{0,1}=b_{E_1}\nu_1$: see Section 1.11. Also note that $\hat{f}_*\mu_{1,0}=\mu_{a,b}$ and $\hat{f}_*\mu_{0,1}=\mu_{c,d}$. Since $f_\bullet I\subset I$, we get $a,b,c,d\geqslant 0$. But $f_\bullet$ is injective on $[\nu_0,\nu_1]$, so $\mu_{a,c}$ and $\mu_{b,d}$ are not proportional, i.e. $ad\neq bc$. We have proved that $\hat{f}$ is of Type 6 in [9], hence can be conjugated to a monomial map.

Denote by ρ the spectral radius of the 2×2 matrix M with entries a,b,c,d. If a_n,b_n,c_n,d_n are the entries of M^n, then for any $s,t>0$, the sequence $\{(sa_n+tc_n)/\rho^n\}_1^\infty$ is bounded from above and below by positive constants. We have

$$c(f^n,\nu_0)=\nu_0(f^{n*}\mathfrak{m})=b_{E_0}^{-1}\operatorname{div}_z(\pi^*f^{n*}\mathfrak{m})=b_{E_0}^{-1}\operatorname{div}_z(\hat{f}^{n*}\pi^*\mathfrak{m})$$
$$=b_{E_0}^{-1}\operatorname{div}_z(\hat{f}^{n*}(z^{b_{E_0}}w^{b_{E_1}}))=b_{E_0}^{-1}(a_nb_{E_0}+c_nb_{E_1}).$$

Thus $c_\infty=\lim c(f^n,\nu_0)^{1/n}\to\rho$.

To conclude the proof of Theorem 5.1, we now prove Lemma 5.6.

Proof of Lemma 5.6. – We first apply Proposition 5.2 and pick valuations $\mu_0<\nu_\star<\mu_1$ (arbitrarily) close to $\nu_\star$, such that $f_\bullet I\subset I$ with $I:=[\mu_0,\mu_1]$. On I the multiplicity is constant, equal to $m\geqslant 1$. We rely on the following two lemmas.

LEMMA 5.7. – *Let $\bar{a},\bar{b},\bar{c},\bar{d}\geqslant 0$ be integers. Suppose $M(t):=(\bar{a}t+\bar{b})/(\bar{c}t+\bar{d})$ has an irrational fixed point $t_+>0$. Then there exist arbitrarily large integers p,q,p',q' such that $p/q<t_+<p'/q'$, $p'q-pq'=1$ and M maps the interval $(p/q,p'/q')$ into itself.*

LEMMA 5.8. – *Let ν be a divisorial valuation of multiplicity $m\geqslant 1$, of skewness p/m^2q with $p,q\in\mathbf{N}^*$, and generic multiplicity b. Then $b\leqslant mq$.*

Lemma 5.8 is a direct consequence of [12, Lemmas 3.53, 3.54]; the details are left to the reader. A proof of Lemma 5.7 is given below.

We continue the proof of Lemma 5.6. Denote by μ_t the unique valuation in I of skewness t. By Theorem 3.1, we have $\alpha(f_\bullet\mu_t)=(at+b)/(ct+d)$ for some integers $a,b,c,d\geqslant 0$. We apply Lemma 5.7 to the 4-tuple $(\bar{a},\bar{b},\bar{c},\bar{d})=(am^2,bm^4,c,dm^2)$. Note that $t_+:=m^2\alpha(\nu_\star)>1$ is irrational and a fixed point of $M(t):=(\bar{a}t+\bar{b})/(\bar{c}t+\bar{d})$. This gives us integers p,q,p',q'. Define $\nu_0:=\mu_{p/m^2q}$ and $\nu_1:=\mu_{p'/m^2q'}$. The condition $p/q<t_+<p'/q'$ implies $p/m^2q<\alpha(\nu_\star)<p'/m^2q'$, hence $\nu_0<\nu_\star<\nu_1$. The fact that M maps $(p/q,p'/q')$ into itself shows that $f_\bullet]\nu_0,\nu_1[\subset]\nu_0,\nu_1[$. When ν_0 and ν_1 are close enough to $\nu_\star$, this yields $f_\bullet U\subset U$.

On I, the multiplicity equals m and $\alpha(\nu_1)-\alpha(\nu_0)=(p'q-pq')/(m^2qq')=1/m^2qq'$. Lemma 5.8 gives $b(\nu_0)\leqslant mq$ and $b(\nu_1)\leqslant mq'$, so that $\alpha(\nu_1)-\alpha(\nu_0)\leqslant 1/b(\nu_0)b(\nu_1)$. We now apply [12, Proposition 6.38]: there exist a modification $\pi\colon X\to(\mathbf{C}^2,0)$ and exceptional components $E_0,E_1\subset\pi^{-1}(0)$, such that ν_0 (resp. ν_1) is associated to E_0 (resp. E_1), and E_0, E_1 intersect in a point. $\square$

Proof of Lemma 5.7. – The proof was suggested to us by R.A. Mollin.

For simplicity, write a,b,c,d instead of $\bar{a},\bar{b},\bar{c},\bar{d}$. After conjugating by $t\mapsto t^{-1}$ if necessary, we may assume $t_+>1$. Write $\delta:=a-d$. Then t_+ is a root of the quadratic polynomial $ct^2-\delta t-b=0$. The other root t_- is negative, since $t_+t_-=-b/c<0$. Define the integer $n\geqslant 0$ by $t_-+n\in(-1,0)$. Then t_++n has a periodic continued fraction expansion, see [21, p. 241]: $t_++n=[a_0,a_1,\ldots]$ with $a_{j+l}=a_j$ for all j, where $l\geqslant 1$. Write $P_j/Q_j=[a_0,\ldots,a_j]$. Then $P_j/Q_j\to t_++n$ as $j\to\infty$.

Take k to be an *even* multiple of l in the sequel. As in [21, p. 240] we have $t_++n=[a_0,\ldots,a_{k-1};t_++n]=\frac{P_{k-1}(t_++n)+P_{k-2}}{Q_{k-1}(t_++n)+Q_{k-2}}$, thus there exists $\lambda_k>0$ rational such that

$\lambda_k b = P_{k-2} + n(P_{k-1} - Q_{k-2} - nQ_{k-1})$; $\lambda_k \delta = P_{k-1} - Q_{k-2} - 2nQ_{k-1}$; and $\lambda_k c = Q_{k-1}$. From [21, p. 225], we infer $P_{k-1}Q_{k-2} - P_{k-2}Q_{k-1} = (-1)^k = 1$. Set

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} P_{k-1} - nQ_{k-1} \\ Q_{k-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} P_{k-2} + nP_{k-1} - nQ_{k-2} - n^2Q_{k-1} \\ Q_{k-2} + nQ_{k-1} \end{pmatrix}.$$

A direct computation yields $q' = \lambda_k c$, $p' - q = \lambda_k \delta$, $p'q - pq' = 1$ and $p = \lambda_k b$, so

(5.1) $$q^2 + q\lambda_k \delta - \lambda_k^2 bc = 1.$$

Clearly q', p and q are positive, and $p'q = 1 + pq'$ implies $p' > 0$. Notice that $p'/q' \to t_+$ as $k \to \infty$, hence $p/q = p'/q' - 1/qq' \to t_+$ as well. From the fact $P_{k-1}/Q_{k-1} > t_+ + n$, we infer $p'/q' > t_+$, thus $p/q = \lambda_k b/(p' - \lambda_k \delta) < \lambda_k b/(\lambda_k ct_+ - \lambda_k \delta) = t_+$. To conclude, it suffices to prove that M maps $(p/q, p'/q')$ into itself, or more precisely $p'/q' > M(p'/q') > p/q$, and $p'/q' > M(p/q) > p/q$. Now

$$M(p'/q') - p/q = \frac{aq^2 + a\delta\lambda_k q - \lambda_k^2 bc(\delta + d)}{q(cp' + dq')} \overset{\text{by (5.1)}}{=} \frac{a}{q(cp' + dq')} > 0,$$

$$p'/q' - M(p'/q') = \frac{bq'^2 - cp'^2 - \delta p'q'}{q'(cp' + dq')} = \frac{q'(-b + c(p'/q')^2 - \delta(p'/q'))}{cp' + dq'}.$$

But $p'/q' > t_+$ and $t \mapsto ct^2 - \delta t - b$ is increasing near t_+, thus $p'/q' - M(p'/q') > 0$. Hence $p'/q' > M(p'/q') > p/q$. Similarly, $p'/q' - M(p/q) = \frac{d}{q'(cp+dq)} > 0$ and $M(p/q) - p/q > 0$ (for k large). Thus $M(p/q, p'/q') \subset (p/q, p'/q')$.   □

6. Proof of Theorem B

For background on plurisubharmonic (psh) functions, see e.g. [6]. In [13] we showed that any (quasimonomial) valuation ν may be naturally evaluated on any psh function u and that $\nu \mapsto \nu(u)$ defines a tree potential—the tree transform of u. Here we shall only need the case when $u = \log\max_i |\phi_i|^{t_i}$, for holomorphic germs ϕ_i, and $t_i > 0$, $i = 1, 2$, in which case $\nu(u) = \min_i t_i\nu(\phi_i)$.

We prove Theorem B in several steps, the proofs of which are given at the end of the section. The first lemma below is an improvement of [13, Proposition 4.2].

LEMMA 6.1. – *If* $g : \mathcal{V}_{qm} \to [0, \infty[$ *is a tree potential, then so is the function* $f^*g : \mathcal{V}_{qm} \to [0, \infty[$, *defined by* $(f^*g)(\nu) := c(f, \nu)g(f_\bullet\nu)$.

LEMMA 6.2. – *There exists a quasimonomial valuation* $\nu_0 \in \mathcal{V}$ *such that* $f_\bullet\nu_0 \geqslant \nu_0$ *and* $c(f, \nu_0) = c_\infty$. *In particular,* $f_*\nu_0 \geqslant c_\infty\nu_0$.

Define tree potentials g_0 and g_1 by $g_0(\nu) = \alpha(\nu \wedge \nu_0)$ and $g_1 := c_\infty^{-1}f^*g_0$.

LEMMA 6.3. – *We have* $g_1 \geqslant g_0$.

Since ν_0 is quasimonomial, we can write $\nu_0 = \nu_{\phi,t}$ with $t > 0$, $\phi \in \mathfrak{m}$ irreducible and $m(\phi) = m(\nu) =: m$. Pick a coordinate x transverse to ϕ, i.e. $x \in \mathfrak{m}$ is irreducible, $m(x) = 1$ and $\nu_x \wedge \nu_\phi = \nu_\mathfrak{m}$. Define the psh function $u_0 := \log\max\{|\phi|^{1/m}, |x|^t\}$ and set $u_1 = c_\infty^{-1}u_0 \circ f$.

LEMMA 6.4. – *The tree transforms of* u_0 *and* u_1 *are* g_0 *and* g_1, *respectively. Hence* $\nu(u_0) \leqslant \nu(u_1)$ *for every* $\nu \in \mathcal{V}_{qm}$.

LEMMA 6.5. – *We have $u_1 \leqslant u_0 + \mathrm{O}(1)$ and $\delta \log \|\cdot\| \leqslant u_0 \leqslant \log \|\cdot\| + \mathrm{O}(1)$ for some $\delta > 0$.*

By subtracting a constant from u_0 (i.e. multiplying x and ϕ by constants), we may assume $u_1 \leqslant u_0$. It is then clear that $u_n := c_\infty^{-n} u_0 \circ f^n$ defines a decreasing sequence of psh functions, hence converges to a function u_∞.

LEMMA 6.6. – *The function u_∞ is not identically $-\infty$. In particular it defines a psh function satisfying the invariance relation $u_\infty \circ f = c_\infty u_\infty$.*

This lemma completes the proof of Theorem B.

Proof of Lemma 6.1. – First, if $g(\nu) = \nu(\psi)$ for some $\psi \in R$, then $f^* g$ is a tree potential as $(f^* g)(\nu) = c(f, \nu)(f_\bullet \nu)(\psi) = \nu(f^* \psi)$. Second, if $g = g_I$ with $g_I(\nu) = \nu(I)$ for some ideal $I \subset R$, then $f^* g$ is the minimum of tree potentials, hence a tree potential. Now, a general tree potential g is the (pointwise) limit of tree potentials of the form $c\, g_I$, where $c > 0$ and I is an ideal. It follows that $f^* g$ is a limit of tree potentials, hence a tree potential. □

Proof of Lemma 6.2. – Let $\nu_\star$ be an eigenvaluation as in Theorem 4.2. When $\nu_\star$ is quasimonomial, set $\nu_0 := \nu_\star$. Otherwise, $\nu_\star$ is an end and we may pick ν_0 as in the statement of Theorem 4.2. □

Proof of Lemma 6.3. – Clearly, $g_1(\nu_0) = c_\infty^{-1} c_\infty \alpha(f_\bullet \nu_0 \wedge \nu_0) = \alpha(\nu_0) = g_0(\nu_0)$. As g_1 is a tree potential, $g_1(\nu_0) \leqslant \alpha(\nu_0) g_1(\nu_\mathrm{m})$, so $g_1(\nu_\mathrm{m}) \geqslant 1 = g_0(\nu_\mathrm{m})$. Since g_1 is concave and g_0 affine on the segment $[\nu_\mathrm{m}, \nu_0]$, we get $g_1 \geqslant g_0$ there. Outside this segment, g_0 is locally constant and g_1 is nondecreasing, so $g_1 \geqslant g_0$ everywhere. □

Proof of Lemma 6.4. – The tree transform of u_0 is g_0. Indeed, as $\nu_0 = \nu_{\phi,t}$:

$$\nu(u_0) = \min\{m^{-1}\nu(\phi), t\nu(x)\} = \min\{\alpha(\nu \wedge \nu_\phi), t\alpha(\nu \wedge \nu_x)\} = \alpha(\nu \wedge \nu_0) = g_0(\nu)$$

for any $\nu \in \mathcal{V}_\mathrm{qm}$. The special form of u_0 directly implies that the tree transform of $u_0 \circ f$ is $f^* g_0$ (see also [13, Proposition 4.2]), which completes the proof. □

Proof of Lemma 6.5. – Notice that $u_1 = \log \max_i |\phi_i|^{t_i}$ for holomorphic germs ϕ_1, ϕ_2 and $t_1, t_2 > 0$. Pick a modification $\pi: X \to (\mathbf{C}^2, 0)$ such that the total transform of the curve $\{\phi \phi_1 \phi_2 = 0\}$ has simple normal crossings. For any point $p \in \pi^{-1}(0)$, we can find local coordinates (z, w) at p such that $\phi(z, w) = z^k w^l$ for some $k, l \in \mathbf{N}$. This implies

$$u_0 \circ \pi(z, w) = c_0 \log |z| + d_0 \log |w| + \log \max\{|z|^{r_0}, |w|^{s_0}\} + \mathrm{O}(1)$$

for some constants $c_0, d_0, r_0, s_0 \geqslant 0$. These constants can be computed in terms of the tree transform of u_0. For instance, if p is the intersection point between two exceptional components $E = \{z = 0\}$ and $F = \{w = 0\}$ with associated divisorial valuations ν_E, ν_F and generic multiplicities b_E, b_F, then $c_0 = b_E \nu_E(u_0)$, $d_0 = b_F \nu_F(u_0)$. Further, r_0 and s_0 may be computed in terms of the restriction of the tree transform to $[\nu_E, \nu_F]$, using the monomialization procedure in Section 1.11.

Similarly, we have

$$u_1 \circ \pi(z, w) = c_1 \log |z| + d_1 \log |w| + \log \max\{|z|^{r_1}, |w|^{s_1}\} + \mathrm{O}(1)$$

for some constants $c_1, d_1, r_1, s_1 \geqslant 0$. The fact that $\nu(u_0) \leqslant \nu(u_1)$ for all quasimonomial ν implies that $c_0 \leqslant c_1$, $d_0 \leqslant d_1$, $r_0 \leqslant r_1$ and $s_0 \leqslant s_1$. We conclude that $u_1 \circ \pi \leqslant u_0 \circ \pi + \mathrm{O}(1)$ in a neighborhood of p. As this is true for any p on the exceptional divisor of π, this inequality can be pushed down to a neighborhood of the origin in $\mathbf{C}^2$, in which we get $u_1 \leqslant u_0 + \mathrm{O}(1)$.

The same proof applied to u_0 and $u = \log \| \cdot \|$ gives $\alpha(\nu_0)^{-1} u + \mathrm{O}(1) \leqslant u_0 \leqslant u + \mathrm{O}(1)$. Indeed, $1 = \nu(u) \leqslant \nu(u_0) \leqslant \alpha(\nu_0)$ for all $\nu \in \mathcal{V}_{\mathrm{qm}}$. $\square$

Proof of Lemma 6.6. – This is the key point in the proof of Theorem B. Let $\nu_\star$ be an eigenvaluation as in Theorem 4.2. We use the normal forms in Theorem 5.1.

When $\nu_\star = \nu_C$ is an analytic curve valuation, the normal form is $\hat{f}(z, w) = (z^a, \lambda z^c w^d)$. Here $E := \{z = 0\}$ is an exceptional component, whereas $\{w = 0\}$ is the strict transform of C. Write $\hat{f}^n(z, w) = (z_n, w_n)$. We have $c_\infty = a$ so it is clear that $c_\infty^{-n} \log |z_n| = \log |z|$ for any point in a neighborhood Ω of p. Moreover, $\log \|\pi(z, w)\| = b_E \log |z| + \mathrm{O}(1)$ and $u_0 \geqslant \delta \log \| \cdot \| + \mathrm{O}(1)$ so $u_\infty = \lim c_\infty^{-n} u_0 \circ f^n$ is finite in $\pi(\Omega \setminus E)$.

The case when $\nu_\star$ is infinitely singular is completely analogous and left to the reader. When $\nu_\star$ is irrational, the normal form is $\hat{f}(z, w) = (z^a w^b, z^c w^d)$, and one checks that $\|\hat{f}^n(z, w)\| \geqslant c(z, w) \rho^n$ when $zw \neq 0$, for some $c(z, w) > 0$, where $\rho = c_\infty(f)$ is the spectral radius of the 2×2 matrix with entries a, b, c, d.

The final—and most delicate—case is when $\nu_\star$ is divisorial. The problem is that we cannot apply Theorem 5.1 directly: for the normal form (i) there need not exist any invariant open region attracted to the origin.

By Proposition 3.3, the tangent map $Df_\bullet$ at $\nu_\star$ may be viewed as a surjective rational selfmap of $\mathbf{P}^1$. We consider three different cases.

In the first case, the tangent map has topological degree at least two. It then has infinitely many *repelling* periodic orbits [4]. Pick one of them, such that for any tangent vector $\vec{v}$ in this orbit, the open set $U(\vec{v})$ does not intersect the critical tree of f. Replacing f by a suitable iterate we may assume that $\vec{v}$ is a fixed point for the tangent map. Then $f_\bullet$ maps $U(\vec{v})$ into itself. We may assume that $\vec{v}$ has multiplicity $m(\vec{v}) = b(\nu_\star)$ (this is true for all but at most two $\vec{v}$). By [12, Proposition 6.40] there exists a modification such that $\nu_\star$ is associated with an exceptional component E of π, $\vec{v}$ is associated with a point $p \in E$ that is a smooth point on $\pi^{-1}(0)$. Then $U(\vec{v}) = U(p)$. Hence $\hat{f}$ is holomorphic at p. Now $\hat{f}(E) = E$, and p is a repelling fixed point for the restriction $\hat{f}|_E$. On the other hand, $\hat{f}^* E = c_\infty E$ and $c_\infty > 1$, so E is superattracting. Thus there exists a stable manifold through p, i.e. a holomorphic curve V which is smooth and transverse to E and invariant under $\hat{f}$, see [28, Example III.2, p. 68]. Notice that V is *not* contracted to p. It is now easy to put $\hat{f}$ in the form $\hat{f}(z, w) = (z^a(1 + \phi), \lambda w(1 + \psi))$, where $\phi(0) = \psi(0) = 0$, $a = c_\infty$, $|\lambda| > 1$, $E = \{z = 0\}$ and $V = \{w = 0\}$. Arguing as before we see that $u_\infty \neq -\infty$ on $\pi(V \setminus E)$.

In the second case, the tangent map is a Möbius map with two distinct fixed points. At least one of these must be non-repelling. Use this tangent vector in the proof of Theorem 5.1 to get the normal form

$$\hat{f}(z, w) = \left(z^a w^b\big(1 + \phi(z, w)\big), \lambda w\big(1 + z\psi(z, w)\big)\right),$$

with $a = c_\infty \geqslant 2$, $b \geqslant 1$, $0 < |\lambda| \leqslant 1$ and $\phi(0) = 0$. Here we have used the fact that a Möbius map of $\mathbf{P}^1$ with two fixed points is *linear* in some affine coordinate. Fix $\varepsilon > 0$ small and pick $\varepsilon_0 > 0$ such that $|\phi(z, w)| \leqslant \varepsilon$, $|\psi(z, w)| \leqslant \varepsilon/4$ when $|z|, |w| \leqslant \varepsilon_0$. A simple induction shows that if $0 < |z_0|, |w_0| \leqslant \varepsilon$, then $|z_n| \leqslant 2^{-n} \varepsilon$ and $|2\lambda|^{-n} |w_0| \leqslant |w_n| \leqslant 2|w_0|$. It then follows that $a^{-(n+1)} \log |z_{n+1}| = a^{-n} \log |z_n| + \mathrm{O}(na^{-n})$. Thus $c_\infty^{-n} \log |z_n| \nrightarrow -\infty$ and $c_\infty^{-n} \log |w_n| \to 0$. This implies $u_\infty(\pi(z_0, w_0)) \neq -\infty$.

In the third and final case, the tangent map is a Möbius map with a single (parabolic) fixed point. We can then obtain the normal form

$$\hat{f}(z,w) = \left(z^a w^b \big(1 + \phi(z,w) \big), \frac{w}{1+w} \big(1 + z\psi(z,w) \big) \right),$$

with $a = c_\infty > 1$, $b \geqslant 1$ and $\phi(0) = 0$. A direct computation shows that for small ε, the region defined by $0 \leqslant |\operatorname{Im} w| < |\operatorname{Re} w| < \varepsilon$ and $0 < |z| < |w|^3$ is invariant and that $|w_n| \geqslant 2^{-n}$ in this region. As in the previous case, this gives $a^{-(n+1)} \log|z_{n+1}| = a^{-n} \log|z_n| + \mathrm{O}(na^{-n})$, which allows us to conclude. $\square$

7. The affine case

Now we turn to polynomial maps and prove Theorems A′ and B′. The approach mimics the one in the local case and is based on the action on the set $\mathcal{V}_0$ of normalized valuations centered at infinity. This set has a tree structure similar to that of the valuative tree $\mathcal{V}$, but the induced map on $\mathcal{V}_0$ is not everywhere defined. We therefore also work with an invariant subtree $\mathcal{V}_1$ of $\mathcal{V}_0$: this is the main difference to the local case. The tree structure of the spaces $\mathcal{V}_0$ and $\mathcal{V}_1$ is detailed in the appendix.

7.1. Induced action: domain of definition

For the rest of the paper, R denotes the ring of polynomials in two complex variables. The role of the maximal ideal $\mathfrak{m}$ in the local case is played by the vector space $\mathcal{L}$ of affine functions on $\mathbf{C}^2$. We refer to $X, Y \in \mathcal{L}$ as *coordinates* if $R = \mathbf{C}[X, Y]$.

The valuation space $\mathcal{V}_0$ is by definition the set of valuations $\nu \colon R \to (-\infty, \infty]$ normalized by $\nu(\mathcal{L}) := \min\{\nu(L);\ L \in \mathcal{L}\} = -1$. We refer to Section 7.5 for a description of its tree structure. It suffices to say that $\mathcal{V}_0$ comes equipped with a partial ordering $\leqslant$, a multiplicity function m, a decreasing parameterization $\alpha \colon \mathcal{V}_0 \to [1, -\infty]$ by skewness and an increasing parameterization $A \colon \mathcal{V}_0 \to [-2, \infty]$ by thinness. The valuation $-\deg$ is the root of $\mathcal{V}_0$.

If F is a dominant polynomial map of $\mathbf{C}^2$ and ν is a nonconstant valuation on R, then $(F_*\nu)(P) := \nu(F^*P)$, $P \in R$ defines a nonconstant valuation $F_*\nu$. The proof of Proposition 2.4 applies also in this context, so that F_* sends divisorial (resp. irrational, curve or infinitely singular) valuations to valuations of the same type.

For $\nu \in \mathcal{V}_0$ set

$$(7.1) \qquad\qquad d(F, \nu) := -\nu(F^*\mathcal{L}) := -\min\{\nu(F^*L);\ L \in \mathcal{L}\}.$$

Thus $d(F, \nu) \geqslant 0$ and $F_*\nu$ is centered at infinity exactly when $d(F, \nu) > 0$. We then write $F_*\nu = d(F, \nu) F_\bullet \nu$ with $F_\bullet \nu \in \mathcal{V}_0$. Clearly $d(F, -\deg) = \deg(F)$.

DEFINITION 7.1. – We denote by $\mathcal{D}_F^o$ the set of valuations $\nu \in \mathcal{V}_0$ such that $F_*\nu$ is centered at infinity, and let $\mathcal{D}_F$ be its closure in $\mathcal{V}_0$.

Thus $\mathcal{D}_F^o = \{d(F, \cdot) > 0\}$. We summarize in the following proposition the structure of the set $\mathcal{D}_F$ and the behavior of the function $d(F, \cdot)$ on $\mathcal{V}_0$. Compare with Proposition 3.4 in the local case.

PROPOSITION 7.2. – *The function $d(F, \cdot)$ is continuous and decreasing on $\mathcal{V}_0$. The locus $\mathcal{T}_F$ where $d(F, \cdot)$ is not locally constant is a finite subtree of $\mathcal{V}_0$. The ends of $\mathcal{T}_F$ are the maximal elements of the finite set $\mathcal{E}_F$ consisting of divisorial valuations $\nu \in \mathcal{V}_0$ satisfying one of the following properties*:

(i) $d(F, \nu) > 0$ *and* $F_\bullet \nu = -\deg$;

(ii) $d(F, \nu) = 0$, $F_* \nu = c \operatorname{div}_C$, *where* $c > 0$ *and* $C \subset \mathbf{C}^2$ *is a rational curve with one place at infinity.*

The valuations in (ii) *are always maximal in* $\mathcal{E}_F$*. Moreover,* $\mathcal{D}_F^o$ *and* $\mathcal{D}_F$ *are subtrees of* $\mathcal{V}_0$*, and* $\mathcal{D}_F \setminus \mathcal{D}_F^o$ *is a finite set consisting of all divisorial valuations satisfying* (ii).

Recall that a curve $C \subset \mathbf{C}^2$ has *one place at infinity* when its closure in $\mathbf{P}^2$ intersects the line at infinity at a single point p, and admits a single branch there.

Proof of Proposition 7.2. – We have $d(F, \nu) = -\min\{\nu(F^* X), \nu(F^* Y), 0\}$ in coordinates (X, Y). Thus the function $d(F, \cdot)$ is clearly decreasing and continuous. It is moreover a tree potential in the sense of Appendix A.4; this implies that $\mathcal{T}_F$ is a tree.

The elements in $\mathcal{E}_F$ are divisorial by Proposition 2.4; let us prove that they are finite in number. Pick a modification $\pi: S \to \mathbf{P}^2$ such that F lifts to a holomorphic map $\hat{F}: S \to \mathbf{P}^2$. The image of a divisorial valuation ν centered at a (closed) point $p \in S$ is a divisorial valuation centered at the point $\hat{F}(p)$. In particular, $F_* \nu$ is proportional to neither $-\deg$ nor div_C, where $C \subset \mathbf{C}^2$. So any valuation in $\mathcal{E}_F$ is associated to an irreducible component of $\pi^{-1}(L_\infty)$, where L_∞ denotes the line at infinity. Thus $\mathcal{E}_F$ is finite.

We now show that the ends of $\mathcal{T}_F$ are exactly the maximal elements of $\mathcal{E}_F$. Pick an end ν of $\mathcal{T}_f$. If $d(F, \nu) > 0$, then $F_* \nu$ is centered at infinity and we may argue as in the proof of Proposition 3.4 to show that $F_\bullet \nu = -\deg$ and that ν is a maximal element of $\mathcal{E}_F$. Hence assume $d(F, \nu) = 0$. Then $F_* \nu$ is not centered at infinity, so its center is either a point $p \in \mathbf{C}^2$ or an irreducible curve $C \subset \mathbf{C}^2$.

In the first case, pick affine coordinates (X, Y) such that p is the origin. Then $\nu(F^* X)$, $\nu(F^* Y) > 0$. The same inequality must hold in a neighborhood of ν. Thus $d(F, \cdot) \equiv 0$ on this neighborhood, contradicting $\nu \in \mathcal{T}_F$.

If instead $F_* \nu$ is centered at an irreducible curve $C \subset \mathbf{C}^2$, then $F_* \nu$ is proportional to div_C. It remains to be shown that C is a rational curve with one place at infinity. Pick a modification $\pi: S \to \mathbf{P}^2$ such that ν is associated to an exceptional component E of π and F lifts to a holomorphic map $\hat{F}: S \to \mathbf{P}^2$. Then $\hat{F}(E)$ equals $\bar{C}$, the closure of C in $\mathbf{P}^2$. As E is a rational curve, so is C. If $\bar{C}$ has several branches at infinity, one can find two distinct points $p, q \in E$ such that $\hat{F}(p), \hat{F}(q) \in L_\infty$. Denote by $U(p)$ (resp. $U(q)$) the set of valuations in $\mathcal{V}_0$ whose center is p (resp. q). These are disjoint open subsets of $\mathcal{V}_0$ containing ν in their closures, so at least one of them contains a valuation $\mu > \nu$. Then $F_* \mu$ is still centered at infinity but $d(F, \mu) \leqslant d(F, \nu) = 0$, so $d(F, \mu) = 0$, a contradiction. We conclude that C has one place at infinity. In particular, $\nu \in \mathcal{E}_F$.

Next we show that all valuations satisfying (ii) are maximal elements of $\mathcal{E}_F$ as well as ends in $\mathcal{T}_f$. This will in particular complete the proof that all ends in $\mathcal{T}_F$ are maximal elements in $\mathcal{E}_F$. Hence pick ν satisfying (ii). Clearly $d(F, \nu) = 0$. It suffices to show that $d(F, \cdot) \not\equiv 0$ at ν. In the notation above, let p' be the unique intersection point of $\bar{C} = \hat{F}(E)$ with the line at infinity and pick $p \in \hat{F}^{-1}(p') \cap E$. We may approximate ν by valuations μ whose center on S is the point p. Since $\hat{F}$ is holomorphic, $F_* \mu$ is centered at $p' \in L_\infty$, so $d(F, \mu) > 0$. Hence $d(F, \cdot) \not\equiv 0$ at ν.

Next consider a divisorial valuation ν satisfying (i) and maximal in $\mathcal{E}_F$. As $F_\bullet \nu = -\deg$, $F_\bullet$ cannot be order preserving near ν, thus $\nu \in \mathcal{T}_F$. It is an end of $\mathcal{T}_F$, since we already proved that all ends of $\mathcal{T}_F$ belong to $\mathcal{E}_F$.

That $\mathcal{D}_F^o$, and hence $\mathcal{D}_F$, is a subtree of $\mathcal{V}_0$ follows from the fact that $d(F, \cdot)$ is decreasing. If $\nu \in \mathcal{D}_F \setminus \mathcal{D}_F^o$, then $d(F, \nu) = 0$, but $d(F, \cdot)$ is not locally constant at ν. This implies that ν is an end of $\mathcal{T}_F$. $\square$

7.2. Induced action: regularity

Having described its domain of definition, we now show that the induced map on the valuation space exhibits many of the features present in the local case. The following two results together generalize Theorem 3.1.

THEOREM 7.3. – *The map $F_\bullet : \mathcal{D}_F^o \to \mathcal{V}_0$ extends uniquely to a surjective, regular tree map $F_\bullet : \mathcal{D}_F \to \mathcal{V}_0$. The valuations in $\mathcal{D}_F \setminus \mathcal{D}_F^o$ are mapped to curve valuations associated with rational curves having one place at infinity.*

The set $\mathcal{V}_1$ is the closure of the set $\mathcal{V}_1^o$ of valuations $\nu \in \mathcal{V}_0$ such that $A(\nu) < 0$ and $\nu(P) < 0$ for all nonconstant P. See Section A.3 for details.

THEOREM 7.4. – *The subtree $\mathcal{V}_1 \subset \mathcal{V}_0$ is included in $\mathcal{D}_F$, and $F_\bullet \mathcal{V}_1 \subset \mathcal{V}_1$. As a consequence, $F_\bullet : \mathcal{V}_1 \to \mathcal{V}_1$ is a regular tree map.*
Moreover, for any $\nu \in \mathcal{V}_1$, we can decompose the segment $[-\deg, \nu]$ into finitely many subsegments $I_j = [\nu_j, \nu_{j+1}]$, $0 \leqslant j \leqslant k$, with $\nu_0 = -\deg$, $\nu_{k+1} = \nu$ and ν_i divisorial, such that $F_\bullet$ is a monotone homeomorphism of I_j onto its image, and $\alpha(F_\bullet \cdot)$ is a piecewise Möbius function of $\alpha(\cdot)$ with nonnegative integer coefficients.

Remark 7.5. – As follows from the proof of Theorem 7.3, the map $F_\bullet : \mathcal{D}_F \to \mathcal{V}_0$ is also piecewise Möbius along segments, but the coefficients may be negative in general. Indeed, let $F(X, Y) = (X, Y^2(XY - 1))$ and denote by ν_t the minimal valuation with $\nu_t(X) = -1$, $\nu_t(Y) = 1$ and $\nu_t(XY - 1) = t > 0$. It has skewness $-t - 1$. Now $F_\bullet \nu_t$ is monomial, with values -1 and $2 + t$ on X and Y respectively. Its skewness is $-t - 2$. Hence $\alpha(F_\bullet \nu_t) = \alpha(\nu_t) - 1$.

Proof of Theorem 7.3. – The proof is similar to the local case, so we shall skip some of the details and focus on the new features in the affine situation.

Let ν be an end in $\mathcal{D}_F$. We need to study the behavior of $F_\bullet$ on the closed segment $[-\deg, \nu]$ when $\nu \in \mathcal{D}_F^o$, and on the half-open segment $[-\deg, \nu[$ when $\nu \in \mathcal{D}_F \setminus \mathcal{D}_F^o$. In both cases we need to decompose the segment into finitely many subsegments on each of which $F_\bullet$ is a monotone homeomorphism.

First suppose $\nu \in \mathcal{D}_F \setminus \mathcal{D}_F^o$. We know from Proposition 7.2 that ν is divisorial and that $F_* \nu$ is proportional to div_C, where $C \subset \mathbf{C}^2$ is a rational curve with one place at infinity. Let $\bar{C}$ denote the closure of C in $\mathbf{P}^2$. We may pick modifications $\pi : S \to \mathbf{P}^2$ and $\pi' : S' \to \mathbf{P}^2$ such that $\pi^{-1}(C \cup L_\infty)$ has simple normal crossings and such that the lift $\hat{F} : S \to S'$ of F is holomorphic and locally monomial at any point on $\pi^{-1}(L_\infty)$. We may also assume that ν is associated to an irreducible component E of $\pi^{-1}(L_\infty)$ that is mapped by $\hat{F}$ onto $\tilde{C}$, the strict transform of $\bar{C}$ by π'.

Let $-\deg < \nu_1 < \cdots < \nu_n = \nu$ be the divisorial valuations in $[-\deg, \nu]$ whose centers on S are one-dimensional. As in the proof of Theorem 3.1, $F_\bullet$ is a monotone homeomorphism of $[\nu_i, \nu_{i+1}[$ onto its image for $0 \leqslant i < n$. Moreover, $F_\bullet \mu \to \nu_C$ as μ increases to ν.

Next consider the case when ν is an analytic curve valuation in $\mathcal{D}_F$. The proof just given still applies in exactly the same way, the only difference being that ν is mapped to a curve valuation of the same type.

If instead ν is a formal curve valuation or an infinitely singular valuation, then ν does not belong to the critical tree $\mathcal{T}_F$ of F. Thus there exists a divisorial valuation $\nu_0 < \nu$ such that $F_\bullet$ is increasing and continuous, hence a homeomorphism, on $[\nu_0, \nu]$. This completes the proof that $F_\bullet$ is a regular tree map, as ν_0 is dominated by an analytic curve valuation in $\mathcal{D}_F$.

To see that $F_\bullet$ is surjective, it suffices to show that any divisorial valuation $\nu' \in \mathcal{V}_0$ has a preimage. Pick modifications $\pi : S \to \mathbf{P}^2$, $\pi' : S' \to \mathbf{P}^2$ such that ν' is associated to some exceptional component $E' \subset S'$, and F lifts to a holomorphic map $\hat{F} : S \to S'$. As $\mathbf{P}^2$ is

compact, $\hat{F}$ is surjective, so one can find an irreducible curve $E \subset S$ such that $\hat{F}(E) = E'$. Since F is a polynomial map of $\mathbf{C}^2$, the curve E is mapped by π into the line at infinity in $\mathbf{P}^2$, hence the divisorial valuation ν associated to E is centered at infinity. We can normalize it by $\nu(F^*\mathcal{L}) = -1$, so that $\nu \in \mathcal{V}_0$. Then $F_\bullet \nu = \nu'$. This finishes the proof of the theorem. $\square$

Proof of Theorem 7.4. – If $\nu \in \mathcal{V}_1^o$, then $\nu(P) < 0$ for any nonconstant P. This immediately implies $d(F, \nu) > 0$ so $\nu \in D_F^o$. Taking closures yields $\mathcal{V}_1 \subset \mathcal{D}_F$.

The fact that $F_\bullet$ leaves $\mathcal{V}_1$ invariant relies on the following affine version of the Jacobian formula (4.1). As in the local case, it ultimately reduces to the change of variables formula.

LEMMA 7.6. – *For any valuation $\nu \in \mathcal{D}_F$, we have*

$$(7.2) \qquad \nu(JF) + A(\nu) = d(F, \nu)\, A(F_\bullet \nu),$$

where JF denotes the Jacobian determinant of F in $\mathbf{C}^2$.

If $\nu \in \mathcal{V}_1^0$, then $\nu(JF) < 0$, $A(\nu) < 0$ and $d(F, \nu) > 0$, hence $A(F_\bullet \nu) < 0$. On the other hand, it is clear that $(F_\bullet \nu)(P) = \nu(F^*P)/d(F, \nu) < 0$ for all nonconstant polynomials P. Thus $F_\bullet \nu \in \mathcal{V}_1^o$, so $F_\bullet \mathcal{V}_1^o \subset \mathcal{V}_1^o$. By continuity, $F_\bullet \mathcal{V}_1 \subset \mathcal{V}_1$.

To control skewness, we proceed as in the local case. First assume that $\nu \in \mathcal{V}_1$ is not infinitely singular. By Theorem 7.3, $[-\deg, \nu]$ can be decomposed into finitely many segments $[\nu_i, \nu_{i+1}]$ on which $F_\bullet$ is monotone. All the ν_i can be chosen divisorial. The key remark is that since $F_\bullet[\nu_i, \nu_{i+1}]$ is a segment containing no infinitely singular valuation, we can pick an affine curve $C = \{Q = 0\}$ with one place at infinity such that the associated curve valuation ν_C dominates $F_\bullet \mu$ for all $\mu \in [\nu_i, \nu_{i+1}]$ (see Theorem A.7). Then

$$(7.3) \qquad \alpha(F_\bullet \mu) = \frac{-F_\bullet \mu(Q)}{\deg(Q)} = \frac{-F_* \mu(Q)}{d(F, \mu) \cdot \deg(Q)} = \frac{-\mu(F^*Q)}{d(F, \mu) \cdot \deg(Q)}.$$

The function $\mu \mapsto -\mu(F^*Q)$ is piecewise linear in $\alpha(\mu)$ with nonnegative integer coefficients, see Lemma A.8. The same is true for the function $d(F, \mu) = -\min\{\mu(F^*X), \mu(F^*Y), 0\}$. Thus $\alpha(F_\bullet \mu)$ is a Möbius function of $\alpha(\mu)$ with nonnegative integer coefficients, as claimed.

When ν is infinitely singular, $F_\bullet$ is monotone on $[\nu_0, \nu]$ for ν_0 divisorial close enough to ν. For $\mu \in [\nu_0, \nu]$ divisorial, we have seen that the segment $[-\deg, \mu]$ can be decomposed into finitely many segments on which the function $\alpha(F_\bullet \cdot)$ is Möbius with nonnegative integer coefficients. The number of such segments is bounded by the number of irreducible components of F^*C (with C as above), which via Bezout's Theorem can be bounded by $\deg(F)^2$. As this bound is uniform in μ, the segment $[-\deg, \nu]$ can also be decomposed into finitely many segments on which $\alpha(F_\bullet \cdot)$ is Möbius. This completes the proof. $\square$

7.3. Normal forms

We now work towards the proof of Theorems A$'$ and B$'$. Let $F : \mathbf{C}^2 \to \mathbf{C}^2$ be a dominant polynomial map. The analogue of (2.2) is

$$(7.4) \qquad \deg(F^n) = \prod_{k=0}^{n-1} d\big(F, F_\bullet^k(-\deg)\big).$$

Since $d(F, \cdot)$ is decreasing, the sequence $\deg(F^n)$ is submultiplicative (as can also be verified directly), so the limit $d_\infty := \lim \deg(F^n)^{1/n}$ exists.

The quantity d_∞ is invariant under conjugacy: if $F_2 = G \circ F_1 \circ G^{-1}$ for a polynomial automorphism G, then $D^{-1} \leqslant \deg(F_1^n)/\deg(F_2^n) \leqslant D$ for all n, where $D = \deg(G)\deg(G^{-1})$, hence $d_\infty(F_1) = d_\infty(F_2)$.

In order to control the degrees $\deg(F^n)$ we show that F can be made rigid.

THEOREM 7.7. – *Let $F: \mathbf{C}^2 \to \mathbf{C}^2$ be a dominant polynomial map with $d_\infty(F) > 1$. Assume F is not conjugate to a skew product. Then one can find a modification $\pi: S \to \mathbf{P}^2$, and $p \in \pi^{-1}(L_\infty)$ such that the lift $\hat{F}$ of F is holomorphic at p, $\hat{F}(p) = p$ and $\hat{F}: (S,p) \to (S,p)$ is rigid. Further, there exist local coordinates (z,w) at p in which $\hat{F}$ takes one of the following forms*:*
  (i) $\hat{F}(z,w) = (z^a w^b(1+\phi), \lambda w^d(1+\psi))$ with $b \geqslant 1$, $a > d \geqslant 1$, $\lambda \in \mathbf{C}^*$ and $\phi(0) = \psi(0) = 0$; then $d_\infty = a$;*
  (ii) $\hat{F}(z,w) = (z^a w^b, z^c w^d)$ with $a,b,c,d \in \mathbf{N}$, $ad \neq bc$; then d_∞ is the spectral radius of the 2×2 matrix with entries a, b, c, d;*
  (iii) $\hat{F}(z,w) = (z^a, \lambda z^c w + P(z))$ with $a \geqslant 2$, $c \geqslant 1$, $\lambda \in \mathbf{C}^*$ and $P \not\equiv 0$ is a polynomial; then $d_\infty = a$.*

Remark 7.8. – As in the local case, the proof is based on the construction of an eigenvaluation $\nu_\star \in \mathcal{V}_1$ (not necessarily unique) and the normal form above depends on the type of $\nu_\star$, namely divisorial, irrational, or infinitely singular. In the normal form (i) we can assume $d = 1$ and $|\lambda| \geqslant 1$ except if $\nu_\star$ is divisorial with $\alpha(\nu_\star) > 0$ and $A(\nu_\star) = 0$ (a case which we suspect may in fact never appear).

Proof. – By Theorem 7.4, $F_\bullet: \mathcal{D}_F \to \mathcal{V}_0$ restricts to a regular tree map $F_\bullet: \mathcal{V}_1 \to \mathcal{V}_1$. Theorem 4.5 implies the existence of a fixed point (eigenvaluation) $\nu_\star$, which is either an interior point of $\mathcal{V}_1$ or a strongly attracting end on $\mathcal{V}_1$ in the sense of Section 4.1. We now proceed according to the nature of $\nu_\star$: it can be divisorial, irrational, or infinitely singular, but not a curve valuation. We further subdivide the divisorial case into the subcases $\alpha(\nu_\star) > 0$ and $\alpha(\nu_\star) = 0$. The latter will not be discussed here, as we will show later that F is then conjugate to a skew product.

First assume that $\nu_\star \in \mathcal{V}_1$ is divisorial, with skewness $\alpha_\star := \alpha(\nu_\star) > 0$. Then

$$\deg(F^n) \leqslant d(F^n, \nu_\star) = d(F, \nu_\star)^n = -\nu_\star(F^{n*}\mathcal{L}) \leqslant \alpha_\star^{-1} \deg(F^n)$$

by Theorem A.7. Thus $d(F, \nu_\star) = d_\infty$ and $\deg(F^n)/d_\infty^n \in [1, \alpha_\star^{-1}]$ for all n.

We may now prove that a suitable lift $\hat{F}$ of F may be put in the form (i) by arguing as in the local case. First suppose $\nu_\star$ is not an end of $\mathcal{V}_1$. Then $F_\bullet$ induces an action on the tree tangent space at $\nu_\star$, and this action admits a noncritical fixed point: see the remark after Proposition 3.3. By the Möbius property in Theorem 7.4, we may construct basins of attraction U of $\nu_\star$ as in Proposition 5.2. Notice that these basins are subsets of $\mathcal{V}_0$, and not $\mathcal{V}_1$. They may be chosen as $U = U(p)$ for some infinitely near point p, and by choosing them small enough, the lift $\hat{F}$ of F will be holomorphic and rigid at p. The normal form (i) with $d = 1$ is an easy consequence.

If instead $\nu_\star$ is an end of $\mathcal{V}_1$, then the argument above breaks down as the tangent vector constructed may not be represented by any valuation in $\mathcal{V}_1$, so the Möbius property may fail. However, the tangent vector represented by the valuation $-\deg$ is invariant since $\mathcal{V}_1$ is invariant. Proceeding as above using this tangent vector yields the normal form in (i). We have $a > d$ since $\nu_\star$ is an attracting end in $\mathcal{V}_1$.

That $a = d_\infty$ (in both cases) is a simple computation. Indeed, $\hat{F}^*\pi^*\mathcal{L}$ and $\pi^*F^*\mathcal{L}$ have poles of order $ab_\star$ and $d(F, \nu_\star)b_\star$, respectively, where $b_\star$ is the generic multiplicity of $\nu_\star$. Hence $d_\infty = d(F, \nu_\star) = a$.

Next suppose $\nu_\star$ is infinitely singular. It is by construction a strongly attracting end, so there exists $\nu_0 < \nu_\star$ such that $d(F, \cdot)$ is constant on $[\nu_0, \nu_\star]$. Arguing as in the local case we see that $d_\infty = d(F, \nu_\star)$ and $\deg(F^n)/d_\infty^n \in [1, \alpha(\nu_0)^{-1}]$ for all n. The construction of the basin $U = U(p)$ and the local normal form (ii) are also proved as in the local case. We get $d_\infty = a$ for the same reason.

Finally assume $\nu_\star$ is irrational. Then $\alpha_\star := \alpha(\nu_\star) > 0$ and the argument in the divisorial case shows that $d_\infty = d(F, \nu_\star)$ and $\deg(F^n)/d_\infty^n \in [1, \alpha_\star^{-1}]$ for all n. Rigidification can be proved as in the local case: the most delicate situation is when $F_\bullet$ is order-reversing at $\nu_\star$. The proof relies on the analogue of Lemma 5.6 with details being left to the reader. The monomial form of $\hat{F}$ follows from [9] and the expression for $d(F, \nu_\star) = d_\infty$ is proved as in the local case, by pulling back $\mathcal{L}$. $\quad\square$

7.4. Proof of Theorem A′

We have already done most of the work by finding the normal forms in Theorem 7.7. Indeed, assuming that the eigenvaluation $\nu_\star$ was not divisorial with $\alpha(\nu_\star) = 0$ we saw that $\deg(F^n)/d_\infty^n \in [1, D]$ for all n for some $D \geqslant 1$. Moreover, d_∞ is an integer except when $\nu_\star$ is irrational. In the latter case, it is the spectral radius of a 2×2 matrix with integer coefficients, hence a quadratic integer.

Let us therefore assume $\nu_\star$ is a divisorial valuation with $\alpha(\nu_\star) = 0$. It is then an end of $\mathcal{V}_1$, and, by Theorem A.7, associated to a rational pencil of curves with one place at infinity. The Line Embedding Theorem [1,30] shows that there exists a change of coordinates by a polynomial automorphism such that the pencil becomes $\{X = \text{const}\}$. Writing $F = (P, Q)$, we infer

$$0 = d(F, \nu_\star)\nu_\star(X) = \nu_\star(F^*X) = \nu_\star(P) = \min_{c \in \mathbf{C}} - \deg_Y\big(P(c, Y)\big).$$

Hence P depends only on X, and $F(X, Y) = (P(X), Q(X, Y))$ is a skew product. As $\nu_\star$ was chosen to be a strongly attracting end, we get $d_\infty(F) = \deg(P) \geqslant \deg_Y(Q)$. It is easy to see that $\deg(F^n)/d_\infty^n$ is unbounded iff $\deg_Y(Q) = \deg(P) > \deg_Y(Q(X_0, Y))$ for some $X_0 \in \mathbf{C}$. This completes the proof of Theorem A′.

7.5. Proof of Theorem B′

The proof follows the same steps as in Theorem B.

By adapting the proof of Lemma 6.2 and using the assumption that F is not conjugate to a skew product we find a quasimonomial valuation $\nu_1 \in \mathcal{V}_1$ with $F_\bullet \nu_1 \geqslant \nu_1$, $\alpha(\nu_1) > 0$ and $d(F, \nu_1) = d_\infty$. Indeed, either the eigenvaluation $\nu_\star$ has positive skewness, in which case we take $\nu_1 = \nu_\star$, or $\nu_\star$ is an infinitely singular valuation $\nu_\star$, in which case we take $\nu_1 < \nu_\star$ sufficiently close to $\nu_\star$.

The function $g_0 : \mathcal{V}_0 \to \mathbf{R}$ given by $g_0(\nu) = -\alpha(\nu \wedge \nu_1)$ is a tree potential. We then prove that $g_1(\nu) := d_\infty^{-1} d(F, \nu) g_0(F_\bullet \nu)$ defines a tree potential and that $g_1 \geqslant g_0$.

We know that ν_1 is dominated by a pencil valuation $\nu_{|P|}$ for some polynomial P of degree m. Set $U_0 = \log^+ \max\{|P|^{1/m}, |X|^t\}$, where $t = \alpha(\nu_1)$ and X is a generic affine function. By Corollary A.9, U_0 is a psh function whose tree transform is g_0. Now $U_1 := d_\infty^{-1} F^* U_0$ is also psh, with tree transform g_1. Since $g_1 \geqslant g_0$ we get $U_1 \leqslant U_0$, after subtracting a constant from U_0, if necessary. We also obtain $\delta \log^+ \|\cdot\| \leqslant U_0 \leqslant \log^+ \|\cdot\| + \mathrm{O}(1)$ for some $\delta > 0$. Thus $U_n := d_\infty^{-n} U_0 \circ F^n$ defines a decreasing sequence of psh functions which are uniformly bounded from below. Hence U_n converges to a nonnegative psh function U_∞. The proof is completed by finding one point $q \in \mathbf{C}^2$ such that $U_\infty(q) > 0$, or, equivalently, $\log \|F^n(q)\| \geqslant \varepsilon d_\infty^n$ for some $\varepsilon > 0$, and this is done exactly as in the proof of Theorem B.

Acknowledgement

Appendix A. Valuations centered at infinity

In this appendix we define and analyze valuation spaces adapted to the analysis of polynomial maps of $\mathbf{C}^2$ at infinity.

A.1. The full valuation space

Let R be the polynomial ring in two complex variables. Denote by $\mathcal{L}$ the vector space of affine functions on $\mathbf{C}^2$. We refer to $X, Y \in \mathcal{L}$ as *coordinates* if $R = \mathbf{C}[X, Y]$. By a *valuation* on R we mean a function $\nu : R \to (-\infty, \infty]$ satisfying $\nu(PQ) = \nu(P) + \nu(Q)$ and $\nu(P + Q) \geqslant \min\{\nu(P), \nu(Q)\}$ for all $P, Q \in R$; and such that $\nu(0) = \infty$, $\nu|_{\mathbf{C}^*} = 0$.

Any valuation extends to the fraction field of R, and then restricts to a valuation on the local ring at any point in the compactification $\mathbf{P}^2$ of $\mathbf{C}^2$. Its *center* is the unique (not necessarily closed) point such that the valuation is nonnegative on the corresponding local ring and (strictly) positive on its maximal ideal.

In particular, a valuation is *centered at infinity* if there exists a polynomial $P \in R$ with $\nu(P) < 0$. Then $\nu(\mathcal{L}) := \min\{\nu(L); \ L \in \mathcal{L}\} < 0$ and $\nu(\mathcal{L}) = \min\{\nu(X), \nu(Y)\}$ for any coordinates (X, Y).

DEFINITION A.1. – We let $\mathcal{V}_0$ be the set of valuations $\nu : R \to (-\infty, \infty]$ centered at infinity, normalized by $\nu(\mathcal{L}) = -1$.

Our first goal is to show that $\mathcal{V}_0$ is a complete tree which can be endowed with a multiplicity function as well as two natural parameterizations (called skewness and thinness as in the local case), see Theorem A.3 below.

We equip $\mathcal{V}_0$ with the topology of pointwise convergence (in which it is compact), and with the partial ordering defined by $\mu \leqslant \nu$ iff $\mu(P) \leqslant \nu(P)$ for all polynomials P. The unique minimal element is the valuation $-\deg$ whose value on P is $-\deg(P)$. It is divisorial by definition and its center on $\mathbf{P}^2$ is L_∞, the line at infinity.

Any other valuation is centered at a (closed) point p on L_∞ and is classified as a divisorial, quasimonomial, irrational, curve, or infinitely singular valuation according to the analysis in Section 1. Pick local coordinates (z, w) at p such that $(1/z, w/z)$ are affine coordinates on $\mathbf{C}^2$. Note that $L_\infty = \{z = 0\}$. The set of valuations $\nu \in \mathcal{V}_0$ centered at p can be identified with the set $\tilde{\mathcal{V}}_p$ of centered valuations on $\mathcal{O}_p$, normalized by $\nu(z) = 1$. The union of $\tilde{\mathcal{V}}_p$ and $\mathrm{div}_z = -\deg$ is referred to as *the relative valuation space* in [12, Section 3.9]. It has a natural tree structure induced by the natural partial ordering, with root div_z. Since this partial ordering is compatible to the partial ordering on $\mathcal{V}_0$ we see that $\mathcal{V}_0$ is naturally a complete tree, rooted in $-\deg$.

The relative valuation space $\tilde{\mathcal{V}}_p$ comes equipped with a multiplicity function and two natural parameterizations that we now recall. First, the *relative multiplicity* $\tilde{m}(C)$ of a (formal) curve C is the intersection multiplicity at 0 of C with $\{z = 0\}$. For $P \in \hat{\mathcal{O}}_p$ set $\tilde{m}(P) = \tilde{m}(P^{-1}(0))$; this is the order of vanishing of $P(0, w)$ at 0. The relative multiplicity of a (quasimonomial) valuation ν is the infimum of $\tilde{m}(C)$ over all irreducible formal curves C such that the curve valuation ν_C

associated to C dominates ν. The *relative skewness* is defined by $\tilde{\alpha}(\nu) = \sup \nu(P)/\tilde{m}(P)$, and the *relative thinness* by $\tilde{A}(\nu) = 1 + \int_{\mathrm{div}_z}^{\nu} \tilde{m}(\mu)\, d\tilde{\alpha}(\mu)$.

Thus we have defined $\tilde{m}(\nu)$, $\tilde{\alpha}(\nu)$ and $\tilde{A}(\nu)$ for any $\nu \in \mathcal{V}_0$, $\nu \neq -\deg$. It turns out to be more natural to modify these quantities slightly.

DEFINITION A.2. – We define the *multiplicity* $m(\nu)$, *skewness* $\alpha(\nu)$ and *thinness* $A(\nu)$ of a valuation $\nu \in \mathcal{V}_0$ as follows. If $\nu \neq -\deg$, then

$$m(\nu) := \tilde{m}(\nu), \quad \alpha(\nu) := 1 - \tilde{\alpha}(\nu) \quad \text{and} \quad A(\nu) := \tilde{A}(\nu) - 3,$$

and for $\nu = -\deg$ we set $m = 1$, $\alpha = 1$, $A = -2$.

The definition of these invariants may seem arbitrary, but are justified by Theorem A.7; see also formula (A.1).

THEOREM A.3. – *The valuation space $(\mathcal{V}_0, \leqslant)$ is a complete nonmetric tree rooted in $-\deg$. Moreover, $\alpha : \mathcal{V}_0 \to [-\infty, +1]$ (resp. $A : \mathcal{V}_0 \to [-2, +\infty]$) gives a decreasing (resp. increasing) parameterization of $\mathcal{V}_0$; and $m : \mathcal{V}_0 \to \overline{\mathbf{N}}$ is nondecreasing. Further, for any $\nu \in \mathcal{V}_0$ we have $A(\nu) = -2 - \int_{-\deg}^{\nu} m(\mu)\, d\alpha(\mu)$.*

Proof. – These statements follow from the properties of the relative valuation space: see [12, Proposition 3.61, Corollary 3.66]. The integral formula reduces to the formula for $\tilde{A}$ in terms of $\tilde{\alpha}$ and $\tilde{m}$. $\square$

We conclude this section by giving a geometric interpretation of thinness in the case of a divisorial valuation. Any divisorial valuation $\nu \in \mathcal{V}_0$ is obtained as follows. First, $-\deg(P)$ is the order of vanishing of the (rational) function P along the line L_∞ at infinity. Second, if $\nu \neq -\deg$, there exist a modification $\pi : S \to \mathbf{P}^2$ and an exceptional component $E \subset \pi^{-1}(L_\infty)$ such that $\nu(P) = b^{-1} \mathrm{div}_E(\pi^* P)$. Here div_E denotes the order of vanishing along E and $b = b(\nu) = -\mathrm{div}_E(\pi^* \mathcal{L})$. Set $a(\nu) := 1 + \mathrm{div}_E(\pi^* \Omega)$, where $\Omega = dX \wedge dY$. Then

$$(\mathrm{A.1}) \qquad\qquad\qquad A(\nu) = a(\nu)/b(\nu).$$

Compare this with the corresponding local result in Section 1.9. Formula (A.1) can easily be reduced to the local case, using [12, Theorem 6.50].

A.2. Pencils

An affine curve C is said to have *one place at infinity* if its closure $\mathbf{P}^2$ intersects the line at infinity in a single point and is analytically irreducible there. Such curves have received considerable attention: see e.g. [1,3,30], and play a crucial role in our study. The branch of C at infinity defines a curve valuation $\nu_C \in \mathcal{V}_0$. If $C = P^{-1}(0)$ for $P \in R$, then we also say that P has one place at infinity and write $\nu_C = \nu_P$. Notice that $\nu_P(P) = \infty$.

By a theorem of Moh [20] (see also [3, p. 565]), the curve $P^{-1}(\lambda)$ has one place at infinity for any $\lambda \in \mathbf{C}$. Denote by $|C|$ or $|P|$ the corresponding pencil of curves in $\mathbf{C}^2$ and define the function $\nu_{|C|} = \nu_{|P|}$ by

$$(\mathrm{A.2}) \qquad\qquad\qquad \nu_{|C|}(Q) = \min_{\lambda \in \mathbf{C}} \nu_{P+\lambda}(Q).$$

PROPOSITION A.4. – *The function $\nu = \nu_{|C|}$ defines a divisorial valuation centered at infinity. We have $\nu_{|P|}(Q) \leqslant 0$ for all $Q \in R$, and $\nu_{|P|}(Q) = 0$ iff $Q = c(P + \lambda)$ for some $c \in \mathbf{C}^*$, $\lambda \in \mathbf{C}$. Moreover, $A(\nu) = (2g - 1)/b$, where $b = b(\nu)$ is the generic multiplicity of ν and g is the genus*

of a generic element in the pencil $|C|$. *In particular,* $A(\nu) \leqslant 0$ *iff* $|C|$ *is a pencil of rational curves.*

We shall call any such valuation a *pencil valuation* and, when the generic element of the associated pencil is rational, a *rational pencil valuation*.

In the proof we shall use the following lemma, which can be proved using Bezout's theorem and the interpretation of a curve valuation as a local intersection number.

LEMMA A.5. – *If C is a curve with one place at infinity and $\nu_C \in \mathcal{V}_0$ is the associated curve valuation, then*

$$(A.3) \qquad \nu_C(Q) = -\frac{(C \cdot D)_{\mathbf{C}^2}}{\deg(C)},$$

where $D = Q^{-1}(0)$ and $(C \cdot D)_{\mathbf{C}^2}$ denotes the total number of intersection points in $\mathbf{C}^2$, counting multiplicities.

Proof of Proposition A.4. – Pick a modification $\pi : S \to \mathbf{P}^2$ such that the pencil has no base point on S. The strict transform $\tilde{C}_\lambda$ of a generic member C_λ of the pencil is then smooth, and intersects the exceptional divisor transversely. But C_λ has one place at infinity, so $\tilde{C}_\lambda$ intersects the exceptional divisor at a *unique* smooth point, say on an exceptional component E, independent of the choice of (generic) element of the pencil. From this geometric representation, it is easy to conclude that $\nu = \nu_{|C|}$ is the divisorial valuation associated to E. The fact that ν takes only nonpositive values follows from (A.3). To compute the thinness of ν, write $a = 1 + \operatorname{div}_E(\pi^*\Omega)$, and $b = \operatorname{div}_E(\pi^*\mathcal{L})$ as in the previous section. By the genus formula [16, p. 471], the genus g of $\tilde{C}_\lambda$ satisfies $2g - 2 = K_S \cdot \tilde{C}_\lambda + \tilde{C}_\lambda \cdot \tilde{C}_\lambda$, where K_S is a canonical divisor of S. Now $\tilde{C}_\lambda \cdot \tilde{C}_\lambda = 0$ as $\tilde{C}_\lambda$ is an element of a pencil on S without base points. Moreover, K_S can be chosen as the divisor of the 2-form $\pi^*\Omega$ on S. This form has a zero of order $a - 1$ along E. Thus $K_S \cdot \tilde{C}_\lambda = a - 1$, so (A.1) gives $2g - 1 = a = A(\nu)b$. □

A.3. A smaller valuation space

We shall now define a subtree $\mathcal{V}_1$ of $\mathcal{V}_0$ which is crucial to our study of (possibly nonproper) polynomial maps of $\mathbf{C}^2$.

DEFINITION A.6. – Let $\mathcal{V}_1^o$ be the set of quasimonomial valuations $\nu \in \mathcal{V}_0$ with negative thinness $A(\nu) < 0$ and such that $\nu(P) < 0$ for all nonconstant polynomials P. Let $\mathcal{V}_1$ be the closure of $\mathcal{V}_1^o$ in $\mathcal{V}_0$.

THEOREM A.7. – *The space $\mathcal{V}_1$ is a complete subtree of $\mathcal{V}_0$. Moreover:*
  (i) *every quasimonomial valuation $\nu \in \mathcal{V}_1$ is dominated by a pencil valuation; the multiplicity $m(\nu)$ is the smallest degree of any such pencil;*
  (ii) *skewness $\alpha : \mathcal{V}_1 \to [0, 1]$ defines a decreasing parameterization of $\mathcal{V}_1$; further, for any $\nu \in \mathcal{V}_1$, we have $\alpha(\nu) = -\sup \frac{\nu(P)}{\deg(P)}$, taken over nonconstant polynomials P; when ν is quasimonomial, the supremum is attained by any polynomial with one place at infinity whose pencil valuation dominates ν;*
  (iii) *thinness $A : \mathcal{V}_1 \to [-2, 0]$ defines an increasing parameterization of $\mathcal{V}_1$; further, for any $\nu \in \mathcal{V}_1$, we have $A(\nu) = -2 - \int_{-\deg}^{\nu} m(\mu) \, d\alpha(\mu)$;*
  (iv) *the ends of $\mathcal{V}_1$, none of which is contained in $\mathcal{V}_1^o$, are of the form:*
    (a) *rational pencil valuations; these have $\alpha = 0$ and $A < 0$;*
    (b) *divisorial valuations with $\alpha > 0$ and $A = 0$;*
    (c) *infinitely singular valuations with $\alpha \geqslant 0$ and $A \leqslant 0$.*

We postpone the proof of Theorem A.7 to Section A.5, but point out that the main difficulty is to prove (i). Specifically we need to construct affine curves with one place at infinity.

Notice that the description above of the tree structure (partial ordering, multiplicity, skewness, thinness) on $\mathcal{V}_1$ is purely in terms of polynomials on $\mathbf{C}^2$.

A.4. Potential theory

We can develop a potential theory on $\mathcal{V}_0$ similar to the one on the valuative tree $\mathcal{V}$ as described in Section 1.8. In fact, doing so is a purely tree-theoretic endeavor, and we refer to [12, Chapter 7] for the precise construction of a potential theory on an arbitrary tree. The main feature distinguishing $\mathcal{V}_0$ from $\mathcal{V}$ is that the parameterization $\alpha : \mathcal{V}_0 \to [-\infty, 1]$ by skewness is *decreasing* in the partial ordering, and that a general (irreducible) polynomial does not determine a unique valuation in $\mathcal{V}_0$. This accounts for the different properties of tree potentials in these two contexts.

The set of tree potentials is the smallest set of functions on $\mathcal{V}_0$ that is closed under sums, multiplication by positive real constants, under minima and pointwise limits, and contains all functions of the form $\nu \mapsto -\alpha(\mu \wedge \nu)$ for any valuation $\mu \in \mathcal{V}_0$. With this definition any tree potential is nonpositive on $\mathcal{V}_1$, but may be positive elsewhere on $\mathcal{V}_0$. Alternatively, tree potentials can be characterized as increasing functions on $\mathcal{V}_0$ with certain (strong) concavity properties. In particular, a tree potential restricts to an increasing, concave function on any segment $[-\deg, \nu]$ parameterized by skewness.

An important property that we shall use is that, for any polynomial $P \in R$, the function $\nu \mapsto \nu(P)$ is a tree potential. More precisely, we have

LEMMA A.8. – *Let $P \in R$ be a polynomial and let $\nu_i \in \mathcal{V}_0$, $i = 1, \ldots, d$, be the curve valuations associated to the irreducible branches of $P^{-1}(0)$ at infinity. Then there exist integers $m_i \geqslant 1$ such that $\sum_1^d m_i = \deg(P)$ and*

$$(A.4) \qquad \nu(P) = -\sum_1^d m_i \, \alpha(\nu \wedge \nu_i) \quad \text{for all } \nu \in \mathcal{V}_0.$$

The proof is given below. In particular, on any segment, the function $\nu \mapsto \nu(P)$ is piecewise linear with integer coefficients as a function of skewness. Moreover, it does not attain a local maximum at a quasimonomial valuation unless it is locally constant there.

For the proof of Theorem B$'$, we need

COROLLARY A.9. – *Suppose a polynomial $P \in R$ has one place at infinity. For $t \in [0, 1]$, let μ_t be the unique valuation in $[-\deg, \nu_P]$ with skewness t. Then*

$$(A.5) \qquad \min\{-\nu(P)/\deg(P), t\} = -\alpha(\nu \wedge \mu_t) \quad \text{for any } \nu \in \mathcal{V}_0.$$

Proof. – Lemma A.8 shows that $\nu(P) = -\deg(P)\alpha(\nu \wedge \nu_P)$ for all $\nu \in \mathcal{V}_0$. The proof reduces to showing that $\alpha(\nu \wedge \mu_t) = \min\{\alpha(\nu \wedge \nu_P), t\}$. Both sides of this equation are locally constant off the segment $[-\deg, \nu_P]$, and in fact also off the subsegment $I = [-\deg, \mu_t]$. On I they both coincide with $\alpha(\nu)$, which completes the proof. □

Proof of Lemma A.8. – By linearity, we may assume P to be irreducible and reduced. Denote by $C_1, \ldots, C_d$ the (analytic) branches of the curve $C := P^{-1}(0) \subset \mathbf{P}^2$ at infinity, set $m_i := C_i \cdot L_\infty$, and let $\nu_i \in \mathcal{V}_0$ be the curve valuation associated to C_i.

Then $\sum_1^d m_i = \deg(P)$ by Bezout's theorem. In particular (A.4) holds for $\nu = -\deg$. Now consider $\nu \neq -\deg$ and let $p \in L_\infty$ be the center of ν on $\mathbf{P}^2$, so that $\nu \in \tilde{\mathcal{V}}_p$. As above, choose coordinates (z, w) at p such that $(1/z, w/z)$ are affine coordinates in $\mathbf{C}^2$. Thus $L_\infty = \{z = 0\}$. Set $\tilde{P}(z, w) = z^{\deg(P)} P(1/z, w/z)$. Then $\nu(P) = -\deg(P) + \nu(\tilde{P})$. Suppose $C_1, \ldots, C_k$ are exactly those branches of C that contain p. Pick a defining equation (possibly transcendental) $\phi_i \in \hat{\mathcal{O}}_p$ for each branch C_i. As P is reduced, we have, locally at p, $\tilde{P} = \psi \prod_1^k \phi_i$ where $\psi(0) \neq 0$. Thus $\nu(\tilde{P}) = \sum_1^k \nu(\phi_i) = \sum m_i \tilde{\alpha}(\nu \wedge \nu_{\phi_i})$ by [12, Propositions 3.25, 3.65]. On the other hand, for any $i > k$, we have $\nu \wedge \nu_i = -\deg$ so $\alpha(\nu \wedge \nu_i) = 1$. This yields

$$\nu(P) = -\deg(P) + \nu(\tilde{P}) = -\sum_1^d m_i + \sum_1^k m_i\, \tilde{\alpha}(\nu \wedge \nu_{\phi_i})$$

$$= -\sum_{k+1}^d m_i\, \alpha(\nu \wedge \nu_i) + \sum_1^k m_i\big(\tilde{\alpha}(\nu \wedge \nu_{\phi_i}) - 1\big)$$

$$= -\sum_1^d m_i \alpha(\nu \wedge \nu_i),$$

and completes the proof. $\square$

A.5. Key polynomials

The rest of the appendix is devoted to the (rather technical) proof of Theorem A.7 on the structure of the valuation space $\mathcal{V}_1$. If $\nu \in \mathcal{V}_0$ is any valuation centered at some point p at infinity, then ν can be completely described in terms of its values on special polynomials: the *key polynomials* of ν, in local coordinates at p. When ν belongs to $\mathcal{V}_1$, the key polynomials define curves with one place at infinity, see Lemma A.12(ii). This fact holds the key to Theorem A.7.

A.5.1. Definitions

Let us first review some material taken from [12].

Roughly speaking, an SKP is a sequence of polynomials $(\tilde{U}_j)_0^k$, $1 \leqslant k \leqslant \infty$ (in some coordinates (z, w)) together with a sequence $(\tilde{\beta}_j)_0^k$ of positive real numbers, satisfying certain combinatorial relations. To any SKP is attached a valuation ν with $\nu(\tilde{U}_j) = \tilde{\beta}_j$ for all j. Conversely, to any valuation ν is attached a unique SKP.

More precisely, an SKP is a sequence $[(\tilde{U}_j)_0^k; (\tilde{\beta}_j)_0^k]$ such that:

(i) $\tilde{U}_0 = z, \tilde{U}_1 = w, \tilde{\beta}_0 = 1, \tilde{\beta}_1 > 0$;

(ii) if $k > 1$, then $\tilde{\beta}_j \in \mathbf{Q}$ for $1 \leqslant j < k$, and $\tilde{\beta}_{j+1} > n_j \tilde{\beta}_j = \sum_{l=0}^{j-1} m_{j,l} \tilde{\beta}_l$, where $n_j = \min\{l; \, l\tilde{\beta}_j \in \sum_0^{j-1} \mathbf{Z}\tilde{\beta}_i\}$ and $0 \leqslant m_{j,l} < n_l$ for $1 \leqslant l < j$;

(iii) for $1 \leqslant j < k$: $\tilde{U}_{j+1} = \tilde{U}_j^{n_j} - \theta_j \cdot \tilde{U}_0^{m_{j,0}} \ldots \tilde{U}_{j-1}^{m_{j,j-1}}$, where $\theta_j \in \mathbf{C}^*$.

LEMMA A.10. – *The polynomial $\tilde{U}_j$ is irreducible both in $\mathbf{C}[z, w]$ and in $\mathbf{C}\{z, w\}$. In particular, the curve $\tilde{U}_j^{-1}(0)$ is globally irreducible in $\mathbf{P}^2$, and analytically irreducible at $(z, w) = (0, 0)$. Moreover, if $j \geqslant 1$, $\tilde{U}_j$ monic in w, and its intersection multiplicity with $\{z = 0\}$ is given by $d_j := n_1 \ldots n_{j-1}$, with the convention $d_1 = 1$.*

We refer to [12, Lemma 2.4] for a proof.

Let us indicate how an SKP determines a valuation. We construct by induction on k a valuation ν_k associated to the data $[(\tilde{U}_j)_0^k; (\tilde{\beta}_j)_0^k]$. First, ν_1 is the monomial valuation with $\nu_1(z) = \tilde{\beta}_0 = 1, \nu_1(w) = \tilde{\beta}_1$. Now assume $j > 1$, that the SKP $[(\tilde{U}_j)_0^k; (\tilde{\beta}_j)_0^k]$ is given and that

$\nu_1, \ldots, \nu_{k-1}$ have been defined. Consider a polynomial $\phi \in \mathbf{C}[z, w]$. As $\tilde{U}_k$ is monic in w, we can divide ϕ by U_k in $\mathbf{C}(z)[w]$: $\phi = \phi_0 + \tilde{U}_k \psi$ with $\deg_z(\phi_0) < d_k = \deg_w(\tilde{U}_k)$ and $\psi \in \mathbf{C}[z, w]$. Iterating the procedure we get a unique decomposition $\phi = \sum_j \phi_j \tilde{U}_k^j$ with $\phi_j \in \mathbf{C}[z, w]$ and $\deg_z(\phi_j) < d_k$. Define

$$\text{(A.6)} \qquad \nu_k(\phi) = \min_j \nu_k\big(\phi_j \tilde{U}_k^j\big) = \min_j\big\{\nu_{k-1}(\phi_j) + j\tilde{\beta}_k\big\}.$$

It is clear that $\nu_k(\tilde{U}_k) = \tilde{\beta}_k$, so $\nu_k(\tilde{U}_j) = \tilde{\beta}_j$ for all $j \leqslant k$. One can prove that ν_k is indeed a valuation, see [12, Theorem 2.8].

For the converse construction, suppose we are given a valuation ν with $\nu(z) = 1$, $\nu(w) > 0$. Set $\tilde{\beta}_0 = 1$, $\tilde{\beta}_1 := \nu(w)$, and suppose $\tilde{U}_j$, $\tilde{\beta}_j$ have been defined for $j = 1, \ldots, k$. If ν coincides with the valuation ν_k attached to $[(\tilde{U}_j)_0^k; (\tilde{\beta}_j)_0^k]$, then we are done. Otherwise, we still have $\nu(\tilde{U}_j) = \nu_k(\tilde{U}_j) = \tilde{\beta}_j$, and it follows from the definition of ν_k that $\nu \geqslant \nu_k$. In particular the set $\mathcal{D} := \{\phi;\ \nu(\phi) > \nu_k(\phi)\}$ is an ideal of the graded ring of ν_k (over $\mathbf{C}(z)[w]$). This graded ring can be shown to be Euclidean, and its irreducible elements are all of the form $\tilde{U}_k^{n_k} - \theta_k \cdot \tilde{U}_0^{m_{k,0}} \ldots \tilde{U}_{k-1}^{m_{k,k-1}}$, where $\theta_k \in \mathbf{C}^*$, $n_k = \min\{l;\ l\tilde{\beta}_k \in \sum_0^{k-1} \mathbf{Z}\tilde{\beta}_j\}$, and $n_k\tilde{\beta}_k = \sum m_{k,j}\tilde{\beta}_j$, see [12, Theorem 2.29]. We choose $\tilde{U}_{k+1}$ to be the generator of $\mathcal{D}$, and set $\tilde{\beta}_{k+1} := \nu(\tilde{U}_{k+1})$.

A.5.2. Global properties

Let $[(\tilde{U}_j)_0^k; (\tilde{\beta}_j)_0^k]$ be an SKP in coordinates (z, w). Write D_j for the degree of $\tilde{U}_j$ as a polynomial in (z, w). By Lemma A.10 and Bezout's theorem, we have $d_j \leqslant D_j$ for $1 \leqslant j < k + 1$. Moreover, if we identify $\{z = 0\}$ with the line at infinity in $\mathbf{P}^2$, then the curve $\tilde{U}_j^{-1}(0)$ has one place at infinity iff $D_j = d_j$. The following technical result exhibits one important case when this occurs.

LEMMA A.11. – *Consider a finite SKP $[(\tilde{U}_j)_0^{k+1};\ (\tilde{\beta}_j)_0^{k+1}]$. Assume $\tilde{\beta}_j \leqslant d_j$ for $1 \leqslant j \leqslant k$ and $\tilde{\beta}_1 + \sum_1^{k-1}(\tilde{\beta}_{j+1} - n_j\tilde{\beta}_j) < 2$. Then $D_j = d_j$ for $1 \leqslant j \leqslant k + 1$.*

It is important to notice that the assumptions in the lemma involve only the indices $j \leqslant k$ but the conclusions apply also for $j = k + 1$.

Proof. – Recall that $\tilde{U}_1 = w$, so that $d_1 = D_1 = 1$. By induction it suffices to prove $d_{k+1} = D_{k+1}$ under the assumption $d_j = D_j$ for $1 \leqslant j < k + 1$. Now

$$\tilde{U}_{k+1} = \tilde{U}_k^{n_k} - \theta_k \cdot \tilde{U}_0^{m_{k,0}} \ldots \tilde{U}_{k-1}^{m_{k,k-1}}.$$

The second term in the right-hand side has degree $D' := \sum_0^{k-1} m_{k,j} D_j$. The first term has degree $n_k D_k = n_k d_k = d_{k+1} = \prod_1^k n_j$, and $\tilde{U}_k(0, w)^{n_k}$ has degree d_{k+1} by Lemma A.10. It hence suffices to prove the inequalities $D' \leqslant d_{k+1}$, and $m_{k,0} \geqslant 1$. For the first, we use $m_{k,j} < n_j$ for $0 < j < k$ together with the inductive assumptions:

$$D' = m_{k,0} + \sum_1^{k-1} m_{k,j} d_j = \sum_0^{k-1} m_{k,j}\tilde{\beta}_j + \sum_1^{k-1} m_{k,j}\big(d_j - \tilde{\beta}_j\big)$$

$$\leqslant n_k\tilde{\beta}_k + \sum_1^{k-1}(n_j - 1)\big(d_j - \tilde{\beta}_j\big) = (n_k - 1)\tilde{\beta}_k + d_k - 1 + \tilde{\beta}_1 + \sum_1^{k-1}\big(\tilde{\beta}_{j+1} - n_j\tilde{\beta}_j\big)$$

$$< (n_k - 1)\tilde{\beta}_k + d_k + 1 \leqslant (n_k - 1)d_k + d_k + 1 = d_{k+1} + 1.$$

Hence $D' \leqslant d_{k+1}$. The fact that $m_{k,0} \geqslant 1$ is proved in a similar way. Namely:

$$\sum_1^{k-1} m_{k,j}\tilde{\beta}_j \leqslant \sum_1^{k-1}(n_j - 1)\tilde{\beta}_j \leqslant n_{k-1}\tilde{\beta}_{k-1} - \tilde{\beta}_1 < \tilde{\beta}_k,$$

so we get $m_{k,0} = n_k\tilde{\beta}_k - \sum_1^{k-1} m_{k,j}\tilde{\beta}_j > 0$. $\square$

A.5.3. The SKP in affine coordinates

Consider a valuation $\nu \in \mathcal{V}_0$ centered at $p \in L_\infty$ so that $\nu \in \tilde{\mathcal{V}}_p$. Pick local coordinates (z, w) at p such that $(X, Y) = (1/z, w/z)$ are affine coordinates on $\mathbf{C}^2$. Suppose ν is given by the SKP $[(\tilde{U}_j)_0^k; (\tilde{\beta}_j)_0^k]$, $1 \leqslant k \leqslant \infty$. Define $U_j(X, Y) = X^{D_j}\tilde{U}_j(1/X, Y/X)$ for $1 \leqslant j < k + 1$, with $D_j = \deg(\tilde{U}_j)$.

LEMMA A.12. – *If $A(\nu) \leqslant 0$ and $\nu(P) \leqslant 0$ for all polynomials P, then*:
  (i) *$\tilde{\beta}_j \leqslant d_j$ for $1 \leqslant j < k + 1$; and $\alpha(\nu) = 1 - \tilde{\alpha}(\nu) = 1 - \sup_j \frac{\tilde{\beta}_j}{d_j} \geqslant 0$;*
  (ii) *$d_j = D_j$, i.e. $U_j^{-1}(0)$ has one place at infinity for $1 \leqslant j < k + 1$.*
If in addition ν is quasimonomial, then:
  (iii) *the SKP of ν is finite, i.e. $k < \infty$, the curve valuation $\nu_{U_k} \in \mathcal{V}_0$ dominates ν, and $m(\nu) = D_k = \deg(U_k)$.*

Proof. – Since $\nu(Y) \leqslant 0$ we have $\tilde{\beta}_1 = \nu(w) \leqslant 1 = d_1 = D_1$. Suppose the inequality in (i) does not always hold and pick j minimal, $1 \leqslant j < k$ such that $\tilde{\beta}_{j+1} > d_{j+1}$. Let $\mu \in \tilde{\mathcal{V}}_p$ be the valuation defined by the truncated SKP $[(\tilde{U}_l)_0^j; (\tilde{\beta}_l)_0^j]$. Then $\mu < \nu$ so $\tilde{A}(\mu) < \tilde{A}(\nu) = A(\nu) + 3 \leqslant 3$. Now $\tilde{A}(\mu) = 1 + \tilde{\beta}_1 + \sum_1^{j-1}(\tilde{\beta}_{l+1} - n_l\tilde{\beta}_l)$, so we may apply Lemma A.11 to the SKP $[(\tilde{U}_l)_0^{j+1}; (\tilde{\beta}_l)_0^{j+1}]$ and conclude that $D_l = d_l$ for $1 \leqslant l \leqslant j + 1$. But then

$$\nu(U_{j+1}) = \nu(\tilde{U}_{j+1}) - D_{j+1} = \tilde{\beta}_{j+1} - d_{j+1} > 0,$$

contradicting the assumption $\nu(P) \leqslant 0$ for all polynomials P. Thus $\tilde{\beta}_j \leqslant d_j$ for all j. By [12, Lemma 3.32], the relative skewness of ν is given by $\tilde{\alpha}(\nu) = \sup_j \tilde{\beta}_j/d_j$. This shows that $\tilde{\alpha}(\nu) \leqslant 1$, so that $\alpha(\nu) = 1 - \tilde{\alpha}(\nu) \geqslant 0$, completing the proof of (i).

Applying Lemma A.11 again we conclude that $d_j = D_j$ for all j. Now $d_j = D_j$ is equivalent to the curve $U_j^{-1}(0)$ having one place at infinity. Thus (ii) holds as well. Finally, if ν is quasimonomial, then its SKP is finite: see [12, Definition 2.23]. Moreover, the curve valuation $\nu_{\tilde{U}_k} \in \tilde{\mathcal{V}}_p$ dominates ν and $\tilde{m}(\nu) = \tilde{m}(\tilde{U}_k) = d_k = D_k$: see [12, Lemma 3.42]. By the definition of the multiplicity on $\mathcal{V}_0$, this yields (iii). $\square$

A.5.4. Proof of Theorem A.7

That $\mathcal{V}_1$ is a complete subtree of $\mathcal{V}_0$ is clear by definition. By continuity, $A(\nu) \leqslant 0$ and $\nu(P) \leqslant 0$ for $\nu \in \mathcal{V}_1$ and $P \in R$.

For (i), pick $\nu \in \mathcal{V}_1$ quasimonomial. If $\nu = -\deg$, then $m(\nu) = 1$ and ν is dominated by the pencil valuation $\nu_{|X|}$ for any affine function X, so suppose $\nu \neq -\deg$. The key fact is that there exists a polynomial P with one place at infinity such that the curve valuation ν_P dominates ν and $m(\nu) = \deg(P)$. The existence of P is guaranteed by Lemma A.12(iii). We claim ν is also dominated by the pencil valuation $\nu_{|P|}$. Indeed, both ν and $\nu_{|P|}$ are dominated by ν_P, hence they are comparable. If $\nu \in \mathcal{V}_1^o$, then $\nu < 0$ on all nonconstant polynomials, but $\nu_{|P|}(P) = 0$ so $\nu < \nu_{|P|}$. If $\nu \in \mathcal{V}_1 \setminus \mathcal{V}_1^o$ we obtain $\nu \leqslant \nu_{|P|}$ by approximation.

Thus ν is dominated by the valuation associated to the pencil $|P|$, whose degree is $m(\nu)$. On the other hand, if C is any curve with one place at infinity and $\nu \leqslant \nu_{|C|}$, then $\nu < \nu_C$. Now the multiplicity of the curve valuation $\nu_C \in \mathcal{V}_0$ equals the intersection multiplicity of C with the line at infinity, hence equals the degree of C by Bezout's theorem. Thus $m(\nu) \leqslant m(\nu_C) = \deg(C)$, completing the proof of (i).

Next we consider (ii). By definition, $\alpha(-\deg) = 1$. For $\nu \neq -\deg$, Lemma A.12 applies and shows that $\alpha(\nu) \geqslant 0$. By Theorem A.3 it is then clear that skewness restricts to decreasing parameterization of $\mathcal{V}_1$ with values in $[0, 1]$.

If P is any nonconstant polynomial, then (A.4) shows that

$$\nu(P) = -\sum_1^d m_i \alpha(\nu \wedge \nu_i) \leqslant -\sum_1^d m_i \alpha(\nu) = -\alpha(\nu) \deg(P)$$

for any $\nu \in \mathcal{V}_1$. Equality holds iff all the curve valuations ν_i associated to irreducible local branches of P dominate ν. In particular, this holds when P has one place at infinity and $\nu_P > \nu$. By (i), such a polynomial P exists when $\nu \in \mathcal{V}_1$ is quasimonomial. When ν is not quasimonomial, we may find $\nu_n \in \mathcal{V}_1$ quasimonomial increasing to ν. For each n there exists a polynomial P_n with $\nu_n(P_n) = -\alpha(\nu_n) \deg(P_n)$. Thus

$$\frac{\nu(P_n)}{\deg(P_n)} \geqslant \frac{\nu_n(P_n)}{\deg(P_n)} = -\alpha(\nu_n) \to -\alpha(\nu),$$

which completes the proof of (ii).

All the statements in (iii) are consequences of Theorem A.3 so we now turn to (iv). Consider an end ν of $\mathcal{V}_1$. We have seen that $A(\nu) \leqslant 0$, which implies that ν cannot be a curve valuation. Hence ν is either infinitely singular or quasimonomial. When ν is infinitely singular, we are in case (c). Otherwise, ν is quasimonomial, hence $\alpha(\nu) = -\sup \nu(P)/\deg(P)$ is attained for some polynomial P having one place at infinity, by (ii). It thus defines a pencil valuation $\nu_{|P|}$, and we have seen above that $\nu_{|P|} \geqslant \nu$. If $\nu < \nu_{|P|}$, then $\mu(Q) \leqslant 0$ for all polynomials Q and all $\mu \in [\nu, \nu_{|P|}]$. By assumption, ν is an end of $\mathcal{V}_1$ thus $A(\nu) = 0$. We are in case (b). The remaining case is when $\nu = \nu_{|P|}$, i.e. ν is a pencil valuation. But $A(\nu) \leqslant 0$, so Proposition A.4 implies that ν is a rational pencil valuation. We are then in case (a). Finally, one checks that any valuation of the type (a), (b), or (c) is an end of $\mathcal{V}_1$ and does not belong to $\mathcal{V}_1^o$. This completes the proof of (iv) and hence the proof of Theorem A.7.

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Charles Favre
Université Paris VII,
Institut de Mathématiques, CNRS,
Équipe Géométrie et Dynamique,
Boîte 7012,
2 place Jussieu,
75251 Paris Cedex 05, France

Unidade Mista CNRS-IMPA,
Estrada Dona Castorina 110,
Rio de Janeiro, Brazil 22460-320
E-mail: favre@math.jussieu.fr

Mattias Jonsson
Department of Mathematics,
University of Michigan,
Ann Arbor, MI 48109, USA

KTH,
Department of Mathematics,
100 44 Stockholm, Sweden
E-mail: mattiasj@umich.edu