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GENERALIZED RING OF NORMS AND GENERALIZED (φ, Γ)-MODULES

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ABSTRACT. – We construct a generalization of the field of norms functor, due to J.-M. Fontaine and J.-P. Wintenberger for local fields, in the case of a ring R which is p-adically formally étale over the Tate algebra of convergent power series $V\{T_1, T_1^{-1}, \ldots, T_d, T_d^{-1}\}$ over a complete discrete valuation ring V of characteristic 0 and with perfect residue field of positive characteristic p. We use this to show that the category of p-adic representations of the fundamental group of $R[\frac{1}{p}]$ is equivalent, as a tensor abelian category, to the category of so-called étale (φ, Γ_R)-modules.

RÉSUMÉ. – On généralise la théorie du corps des normes, due à J.-M. Fontaine et J.-P. Wintenberger dans le cas des corps locaux, au cas d'un anneau R, p-adiquement étale sur l'algèbre de Tate de séries formelles convergentes $V\{T_1, T_1^{-1}, \ldots, T_d, T_d^{-1}\}$ sur un anneau de valuation discrète complet V de caractéristique 0 à corps résiduel parfait de caractéristique p. On en déduit une équivalence de catégories abéliennes tensorielles entre celle des représentations p-adiques du groupe fondamental de $R[\frac{1}{p}]$ et celle des (φ, Γ_R)-modules étales.

Contents

1. Introduction

Let V be a complete discrete valuation ring of characteristic 0 and with perfect residue field k of positive characteristic p. Fix a "very ramified" extension $V \subset V_\infty$ (see 2.1). For example, one may take V_∞ to be the ring of integers of the cyclotomic extension of $V[p^{-1}]$ obtained by adjoining all p^n-th roots of unity for every $n \in \mathbf{N}$. The classical theory of the field of norms, due to J.-M. Fontaine and J.-P. Wintenberger [8,9,17], associates to V_∞ a complete, equicharacteristic p discrete valuation ring $\mathbf{E}_V^+$ and provides an equivalence of categories between the category of finite extensions of V_∞, which are normal as rings, and the category of finite extensions of $\mathbf{E}_V^+$, which are generically separable and are normal as rings.

The main goal of this paper is to generalize such construction. Let R be a p-adically complete and separated, noetherian algebra over the ring of convergent power series $V\{T_1,T_1^{-1},\ldots,T_d,T_d^{-1}\}$, flat as V-algebra such that (I) $k\subset R\otimes_V k$ is geometrically integral and (II) the image of $T_1,\ldots,T_d$ in $R\otimes_V k$ is an absolute p-basis for $R\otimes_V k$. Fix a "very ramified" extension $R\otimes_V V_\infty\subset R_\infty$ as in 2.2. For example, one may consider the "cyclotomic case" in which $V\subset V_\infty$ is given by the cyclotomic tower and $R\otimes_V V_\infty\subset R_\infty$ is given by taking p^n-th roots of the variables $T_1,\ldots,T_d$ for every $n\in\mathbf{N}$. Let $\overline{\pi}_K$ be a uniformizer of $\mathbf{E}_V^+$. We associate to R_∞ a ring $\mathbf{E}_R^+$, which is a $\overline{\pi}_K$-adically complete and separated, noetherian algebra over the ring of convergent power series $\mathbf{E}_V^+\{x_1,x_1^{-1},\ldots,x_d,x_d^{-1}\}$, flat as $\mathbf{E}_V^+$-algebra and having $\overline{\pi}_K,x_1,\ldots,x_d$ as absolute p-basis. More generally, we construct a functor $\mathbf{E}_-^+$ of "generalized ring of norms" associating to an extension S_∞ of R_∞, which is finite and étale after inverting p and is normal as a ring, a $\mathbf{E}_R^+$-algebra $\mathbf{E}_S^+$. Then,

THEOREM 6.3(I). – *For every extension S_∞ of R_∞, which is finite and étale after inverting p and is normal as a ring, $\mathbf{E}_S^+$ is a finite extension of $\mathbf{E}_R^+$, which is étale after inverting $\overline{\pi}_K$ and is normal as a ring.*

Since $R_\infty\subset S_\infty$ is finite and étale after inverting p there exists a canonical diagonal idempotent $\mathfrak{e}_\infty\in(S_\infty\otimes_{R_\infty}S_\infty)[p^{-1}]$. Following [6, Def. 2.1] we say that $R_\infty\subset S_\infty$ is *almost étale* if, letting m_∞ be the maximal ideal of V_∞, we have that $m_\infty\mathfrak{e}_\infty$ is in the image of $S_\infty\otimes_{R_\infty}S_\infty$. Then,

THEOREM 6.3(II). – *Assume that every extension S_∞ of R_∞, which is finite and étale after inverting p and is normal as a ring, is almost étale. Then, the functor $\mathbf{E}_-^+$ defines an equivalence of categories from the category of extensions of R_∞, which are finite and étale after inverting p and are normal rings, to the category of extensions of $\mathbf{E}_R^+$, which are finite and étale over $\mathbf{E}_R^+[\overline{\pi}_K^{-1}]$ and which are normal rings.*

The assumption in the theorem is easily proven to be satisfied if R is of Krull dimension $\leqslant 2$. In general, it holds whenever G. Faltings' "almost purity theorem" [7, Thm. 4] applies. For example, it is the case if R is the completion, with respect to an ideal containing p, of the localization with respect to a multiplicative system of an algebra étale over the polynomial ring $V[T_1,T_1^{-1},\ldots,T_d,T_d^{-1}]$; see 5.12.

Let R be the completion, with respect to an ideal containing p, of the localization with respect to a multiplicative system of an étale extension of $V[T_1,T_1^{-1},\ldots,T_d,T_d^{-1}]$ and assume also that we are in the cyclotomic case. Let S be a finite extension of R, which is étale after inverting p and is normal as a ring. Denote by S_∞ the normalization of $S\otimes_R R_\infty$. Assume it is an integral domain and denote by Γ_S the automorphism group of S_∞ as S-algebra. It is a finite index subgroup of the Galois group of $R\subset R_\infty$; the latter is isomorphic to the semidirect product of the Galois group Γ_V of $V\subset V_\infty$ ($\cong\mathbf{Z}_p$ up to a finite group) and of the Galois group $\widetilde{\Gamma}_R\cong\mathbf{Z}_p^d$ of $R\otimes_V V_\infty\subset R_\infty$. We construct a noetherian regular integral domain $\mathbf{A}_S$ of characteristic 0 such that $\mathbf{A}_S/p\mathbf{A}_S=\mathbf{E}_S^+[\overline{\pi}_K^{-1}]$, it is p-adically complete and separated, it is endowed with a continuous action of Γ_S and with a continuous operator φ commuting with Γ_S and reducing to Frobenius modulo p. A (φ,Γ_S)-module D is a finitely generated $\mathbf{A}_S$-module, endowed with semilinear, commuting actions of Γ_S and φ. It is called étale if $\varphi\otimes 1\colon D\otimes_{\mathbf{A}_S}^{\varphi}\mathbf{A}_S\to D$ is an isomorphism of $\mathbf{A}_S$-modules. We let $(\varphi,\Gamma_S)-\mathrm{Mod}_{\mathbf{A}_S}^{\mathrm{et}}$ be the category of étale (φ,Γ_S)-modules. It is an abelian tensor category. Let $\mathcal{G}_S:=\pi_1(S[\frac{1}{p}],\eta)$ be the fundamental group of $\mathrm{Spec}(S[p^{-1}])$, where the base point η is chosen to be the generic one. Let $\mathrm{Rep}(\mathcal{G}_S)$ be the category of p-adic representations of $\mathcal{G}_S$ i.e., of finitely generated $\mathbf{Z}_p$-modules endowed with a continuous action of $\mathcal{G}_S$. We then deduce from Theorem 6.3 the following analogue of the classical theory of (φ,Γ_V)-modules due to J.-M. Fontaine [10] in the case $S=V$:

THEOREM 7.11. – *There is an equivalence of abelian tensor categories between the category* $\mathrm{Rep}(\mathcal{G}_S)$ *and the category* $(\varphi,\Gamma_S)-\mathrm{Mod}^{\mathrm{et}}_{\mathbf{A}_S}$.

Here, our work is a generalization of that of Fontaine: the statement of the main theorem and the strategy of the proof are, mutatis mutandis, as in loc. cit. The key ingredient of the proof is the relation between the category of p-adic representations of the fundamental group $\mathbf{E}_S^+[\bar\pi_K]$ and the category of unit root φ-crystals over $\mathbf{E}_S^+[\bar\pi_K^{-1}]$ established by N. Katz in [12].

In [4] a generalization of the theory of Fontaine has already been given. It is different from ours: in loc. cit. the object of study is the category of representations of the Galois group of $\mathrm{Frac}(V)$ with more general coefficients (not only $\mathbf{Z}_p$ as in [10], but any complete noetherian ring with finite residue field). In the case that R is the discrete valuation ring associated to an higher-dimensional local field an independent construction of the field of norms functor can be found in [1]. If R is a discrete valuation ring with imperfect residue field admitting a finite p-basis, a theory of the field of norms and of (φ,Γ)-modules has been independently developed in [14].

The paper is organized as follows. Section 2 contains preliminary definitions and constructions. We write R_∞ as the union of subrings $R=R_0\subset R_1\subset\cdots\subset R_n\subset\cdots$. In the cyclotomic case we take V_n to be the normalization of $V(\zeta_{p^n})$, where ζ_{p^n} is a primitive p^n-th root of unity, and R_n to be $R\otimes_{V[T_1,\ldots,T_d]}V_n[T_1^{\frac{1}{p^n}},\ldots,T_d^{\frac{1}{p^n}}]$. Let $R\subset S$ be a finite extension, étale after inverting p. Let S_n be the normalization of $S\otimes_R R_n$.

In Section 3 we study the behavior of the ramification of the extensions $R_n\subset S_n$ as n increases. We rely on [6]. Contrary to what is needed in almost étale theory, though, for our purposes it is not sufficient that differents get smaller as n increases tending to 0. One needs a better control on how fast they decrease; see 3.8. In the case of local fields this follows from the original approach of J. Tate [16] via class field theory.

Section 4 is devoted to the construction and the study of the first properties of the generalized ring of norms associated to the extension $R\subset S$ above. We denote it by $\mathbf{E}_S^+$ following what now seems the standard notation at least for $S=V$; cf. [3]. Then, $\mathbf{E}_S^+$ is a normal $\mathbf{E}_R^+$-algebra and $\mathbf{E}_R^+\subset\mathbf{E}_S^+$ is finite and étale over $\mathbf{E}_R^+[\bar\pi_K^{-1}]$; see 4.9.

In Section 5 we state a condition on the ramification of $\{R_n\subset S_n\}_n$ called (RAE) (= refined almost étaleness). It implies the key property that $\mathbf{E}_R^+\subset\mathbf{E}_S^+$ is generically of the right degree i.e., $[\mathrm{Frac}(S):\mathrm{Frac}(R)]$; see 5.3. In 5.1 we prove that (RAE) is in fact equivalent to Faltings' almost étaleness (AE). We also give other equivalent formulations in terms of properties of $\mathbf{E}_S^+$.

In Section 6 we prove Theorem 6.3. The key point is to construct an inverse to the functor $\mathbf{E}_{\_}^+$ (under the assumption that (RAE) holds). We deduce from 6.3 that the fundamental groups $\pi_1(R_\infty[p^{-1}],\eta)$ and $\pi_1(\mathbf{E}_R^+[\bar\pi_K^{-1}],\nu)$, η and ν being corresponding base points, are canonically isomorphic; see 6.4 and 6.6.

In Section 7 we prove Theorem 7.11. Thanks to Corollary 6.6 and due to the properties of $\mathbf{A}_S$, it is a consequence of [12].

The rings $\mathbf{E}_S^+$, $\mathbf{A}_S$ and the theory of (φ,Γ_S)-modules depend on the structure of R as $V\{T_1,T_1^{-1},\ldots,T_d,T_d^{-1}\}$-algebra. In 5.6 we present some mild functoriality properties of the ring $\mathbf{E}_R^+$ relatively to the ring R. Unfortunately, one cannot and does not expect to sheafify these constructions in order to get a global version of 6.3 or 7.11 for (formal) schemes over V. Still, our results seem a valuable tool for the study of the various comparison theorems of Fontaine's theory, for computing p-adic étale cohomology of schemes over K and for providing new methods of constructing and studying classical (φ,Γ_V)-modules. We will come back to these topics soon.

2. The basic ring R

2.1. Notation

Let V be a ring complete with respect to a discrete valuation $\mathbf{v}$ satisfying $\mathbf{v}(p)=1$, with perfect residue field k of characteristic p and with fraction field $K=\mathrm{Frac}(V)$ of characteristic 0. Let $K\subset K_\infty$ be a Galois extension with group Γ_V such that it contains $\mathbf{Z}_p\subset\Gamma_V$ as a finite index subgroup. Define the tower

$$K_0:=K\subset K_1=K_\infty^{\mathbf{Z}_p}\subset K_2=K_\infty^{p\mathbf{Z}_p}\subset\cdots\subset K_n:=K_\infty^{p^{n+1}\mathbf{Z}_p}\subset\cdots.$$

We *assume* that $K_n\subset K_\infty$ is totally ramified for some $n\gg 0$. Let V_n be the normalization of V in K_n. It is complete with respect to the unique discrete valuation extending $\mathbf{v}$. Let k_n be its residue field. Let

$$V_\infty=\bigcup_n V_n,\qquad \widehat{V_\infty}=\lim_{\infty\leftarrow n}V_\infty/p^nV_\infty.$$

Let $\overline{V}$ be the normalization of V in an algebraic closure $\overline{K}$ of K. The valuation $\mathbf{v}$ extends to a unique valuation $\mathbf{v}$ on $\overline{K}$ with associated valuation ring $\overline{V}$. For every $\delta\in\mathbf{v}(\overline{K}^*)$ denote by p^δ a (any) element of $\overline{K}$ whose valuation is δ.

As in [11, 0.21.1.4] if A is a ring of characteristic p we denote by $A^{(p)}$ the A-algebra defined by the ring A with structural morphism as A-algebra given by the Frobenius homomorphism $\varphi_A:A\to A$ sending $a\mapsto a^p$. We denote by A^p the image of φ_A. Recall from [11, 0.21.1.9] that a set $\{a_1,\ldots,a_d\}$ of elements of A is called an *absolute p-basis* if the monomials $\{a_1^{i_1}\cdots a_d^{i_d}\mid 0\leqslant i_j<p\forall j\}$ is a basis of A as A^p-module.

2.2. DEFINITION. – Let R be a p-adically complete and separated, noetherian, flat V-algebra such that

I. R is an algebra over $R^0:=V\{T_1,T_1^{-1},\ldots,T_d,T_d^{-1}\}$ the Tate algebra of p-adically convergent power series in the variables $T_1,T_1^{-1},\ldots,T_d,T_d^{-1}$ and coefficients in V;

II. $T_1,\ldots,T_d$ form an absolute p-basis for $R\otimes_V k$ and $k\subset R\otimes_V k$ is geometrically integral.

Let $R^0:=R_0^{0\prime}\subset\cdots\subset R_n^{0\prime}\subset R_{n+1}^{0\prime}\subset\cdots$ be a tower of rings such that

III. $R_{n+1}^{0\prime}$ is finite and flat over $R_n^{0\prime}$ for every n and the tower $\{R_n^{0\prime}\}_n$ reduces to the Frobenius tower of $R^0\otimes_V k$ i.e.,

$$R_n^{0\prime}\otimes_V k\cong\left(R^0\otimes_V k\right)^{(p^n)}\cong\left(R^0\otimes_V k\right)\left[T_1^{\frac{1}{p^n}},\ldots,T_d^{\frac{1}{p^n}}\right].$$

Define $R_n^0:=R_n^{0\prime}\otimes_V V_n$, $R_n':=R\otimes_{R^0}R_n^{0\prime}$, $R_n:=R_n'\otimes_V V_n$ and $R_\infty:=\bigcup_n R_n$. Let $\widehat{R_\infty}$ be the p-adic completion of R_∞. The hypotheses imply that the maximal ideal of V_n generates the unique prime ideal $\mathfrak{P}_n$ of R_n over p so that $\mathfrak{P}_n$ is principal. Let $\mathfrak{P}_\infty:=\bigcup_n\mathfrak{P}_n$. Finally, let

$$L_n:=\mathrm{Frac}(R_n)\quad\text{and}\quad L_\infty:=\bigcup_n L_n=\mathrm{Frac}(R_\infty).$$

2.3. PROPOSITION. – *The following properties hold*:

(1) *the natural map* $(R\otimes_V k)\otimes_{R^0}(R^0\otimes_V k)^{(p)}\to(R\otimes_V k)^{(p)}$ *defined by Frobenius is an isomorphism. In particular, the extension* $R^0\otimes_V k\to R\otimes_V k$ *is formally étale* (*for the discrete topologies*);

(2) *for every* $n\in\mathbf{N}$ *the rings* $R_n/\mathfrak{P}_n$ *and* $R'_n\otimes_V k$ *are noetherian, geometrically regular domains*;

(3) *for every* $n\in\mathbf{N}$ *the rings* R_n *and* R'_n *are noetherian regular domains,* p*-adically complete and separated. In particular,* R_n *and* R'_n *are normal.*

Proof. – (1) The first claim follows from 2.2(II). By [11, 0.21.2.7] it implies that the ring $R\otimes_V k$ is formally smooth over $R^0\otimes_V k$. It is formally unramified due to the fact that the relative differentials $\Omega^1_{R\otimes_V k/R^0\otimes_V k}$ are zero by [11, 0.21.2.5].

(2) We have $R'_n\otimes_V k\cong(R\otimes_V k)^{(p^n)}$ and $R_n/\mathfrak{P}_n\cong R'_n\otimes_V k_n$. In particular, $R'_n\otimes_V k$ is isomorphic to $R\otimes_V k$ as a ring and $R_n/\mathfrak{P}_n$ is isomorphic to $R\otimes_V k'$ where $k\subset k'$ is a finite extension. By 2.2(II) they are then domains. Since $R^0\otimes_V k$ is smooth over k, it follows from [11, 0.19.3.5] that $R\otimes_V k'$, and thus the localization $(R\otimes_V k')_{\mathfrak{m}}$ at every maximal ideal $\mathfrak{m}$ of $R\otimes_V k'$, is formally smooth over k' for every finite extension $k\subset k'$. By [11, 0.22.5.8] this implies that $(R\otimes_V k')_{\mathfrak{m}}$ is a geometrically regular local ring. We conclude that the localization of $R_n/\mathfrak{P}_n$ and of $R'_n\otimes_V k$ at every maximal ideal is geometrically regular. Hence, $R_n/\mathfrak{P}_n$ and $R'_n\otimes_V k$ are geometrically regular by [11, 0.17.3.2]. In particular, they are regular rings, thus locally factorial and, hence, normal.

(3) By 2.2(III) the rings R_n and R'_n are finite as R-modules and, in particular, they are noetherian and p-adically complete and separated. This implies that every maximal ideal contains p. Since $\mathfrak{P}_n$ is principal, we conclude from (2) that the localization of R_n and of R'_n at every maximal ideal is regular. Hence, R_n and R'_n are regular by [11, 0.17.3.2]. In particular, they are locally factorial and, thus, normal. Then, R_n (resp. R'_n) are product of normal domains which, being quotients of R_n (resp. R'_n), are p-adically complete and separated. We have proven that $R'_n\otimes_V k$ and $R_n/\mathfrak{P}_n$ are domains. We conclude that R_n and R'_n are integral domains. □

2.4. The cyclotomic case

The main example to keep in mind is given by the cyclotomic tower

$$K=K_0\subset\cdots\subset K_n=K(\zeta_{p^n})\subset\cdots$$

and by the rings

$$R'_n:=R\big[T_1^{\frac{1}{p^n}},T_1^{\frac{-1}{p^n}},\dots,T_d^{\frac{1}{p^n}},T_d^{\frac{-1}{p^n}}\big].$$

In this case $R\subset R_\infty$ is Galois, after inverting p, with Galois group Γ_R which is the semidirect product of the Galois group $\Gamma_V=\mathrm{Gal}(K_\infty/K)$ of the tower $K\subset K_\infty$ and of the Galois group $\widetilde{\Gamma}_R=\mathrm{Gal}(R_\infty/R\otimes_V V_\infty)$. Remark that $\Gamma_V\subset\mathrm{Gal}(\mathbf{Q}_p(\zeta_{p^\infty})/\mathbf{Q}_p)\cong\mathbf{Z}_p^*$ and that $\widetilde{\Gamma}_R=\mathbf{Z}_p\gamma_1\oplus\cdots\oplus\mathbf{Z}_p\gamma_d\cong\mathbf{Z}_p^d$, where $\gamma_i\in\widetilde{\Gamma}_R$ is the element acting trivially on V_∞ and such that

$$\gamma_i\big(T_j^{\frac{1}{p^n}}\big)=\begin{cases}\zeta_{p^n}T_j^{\frac{1}{p^n}} & \text{if } i=j,\\ T_j^{\frac{1}{p^n}} & \text{if } i\neq j.\end{cases}$$

2.5. DEFINITION. – Let S be an R-algebra such that

– S is normal and V-flat;

– it is finite as R-module;
– $R[\frac{1}{p}] \subset S[\frac{1}{p}]$ is étale.

Define S_n to be the normalization of $S \otimes_R R_n$. Let M_n be the total field of fractions of S_n. Let

$$S_\infty := \bigcup_n S_n, \quad M_\infty := \bigcup_n M_n \quad \text{and} \quad \widehat{S_\infty} := \lim_{\infty \leftarrow n} S_\infty / p^n S_\infty.$$

2.6. DEFINITION. – Let R^{sep} be the direct limit of a maximal chain of normal R_∞-algebras, which are domains and, after inverting p, are finite and étale extensions of $R_\infty[\frac{1}{p}]$. If S_∞ is a normal R_∞-subalgebra of R^{sep} and $R_\infty[\frac{1}{p}] \subset S_\infty[\frac{1}{p}]$ is finite and étale, define

$$\mathcal{H}_S := \mathrm{Gal}\left(R^{\mathrm{sep}}\left[\frac{1}{p}\right] / S_\infty\left[\frac{1}{p}\right]\right).$$

If S is an R-algebra as in 2.5 contained in R^{sep}, define

$$\mathcal{G}_S := \mathrm{Gal}\left(R^{\mathrm{sep}}\left[\frac{1}{p}\right] / S\left[\frac{1}{p}\right]\right).$$

3. Ramification theory in towers

In this section we study the ramification of the extensions $\{R_n \subset S_n\}_n$. Let $\mathfrak{e}_n \in (S_n \otimes_{R_n} S_n)[p^{-1}]$ be the diagonal idempotent; see 3.2. As in [6] we measure the ramification of the extension $R_n \subset S_n$ by the smallest non-negative rational number δ_n for which there exists an element $p^{\delta_n} \in V_n$ of valuation $\mathrm{v}(p^{\delta_n}) = \delta_n$ such that $p_n^{\delta}\mathfrak{e}_n$ lies in the intersection of the localizations of $S_n \otimes_{R_n} S_n$ at all height one prime ideals. Due to work of Faltings, one knows that $\delta_n \to 0$ as $n \to \infty$; see 3.6. Our main result is a refinement of this statement, see 3.8, stating that $p^n \delta_n$ is bounded from above. This is proven by Tate [16] in the case of local fields and inspired Faltings' theory of almost étale extensions. Unfortunately, the control one has on δ_n in codimension 1 is not enough a priori to guarantee that $p^{\delta_n}\mathfrak{e}_n$ lies in the image of $S_n \otimes_{R_n} S_n$. We will prove that, under some hypotheses, this indeed holds; see 5.11.

3.1. The trace map and idempotents

Let $W \subset Z$ be a finite extension of normal rings such that the extension of fraction fields $L \subset M$ is separable. Consider the W-linear homomorphism

$$\mathrm{Tr} : Z \to \mathrm{Hom}_W(Z, W)$$

defined by $x \mapsto \mathrm{Tr}_{M/L}(x \cdot \_)$. Note that Tr is well defined since $W \subset Z$ is finite and W is normal.

Let $m : M \otimes_L M \to M$ be the multiplication map. Since M is a separable extension of L, there exists

$$\mathfrak{e} \in M \otimes_L M$$

such that $m(x) = (\mathrm{Tr}_{M/L} \otimes \mathrm{Id})(\mathfrak{e} \cdot x)$ for all $x \in M \otimes_L M$. The idempotent $\mathfrak{e}$ defines a section to the multiplication map m (as L-algebras).

Assume that there exist $\tau \in W$ and an element of $Z \otimes_W Z$ whose image in $M \otimes_L M$ is $\tau\mathfrak{e}$. By abuse of notation we denote such element by $\tau\mathfrak{e}$. Let $b_1, \ldots, b_h$ be generators of Z as W-module.

Write

$$\tau\mathfrak{e}=\sum_{i=1}^{h}a_i\otimes b_i$$

with $a_1,\dots,a_h$ in Z. We have W-linear maps

$$g\colon Z\to W^h,\qquad f\colon W^h\to Z$$

given by $g(z):=(\mathrm{Tr}(za_i))_i$ and $f((w_i)_i):=\sum_i w_ib_i$. Since for any $z\in Z$ one has $\tau z=\sum_i \mathrm{Tr}(za_i)b_i$, we deduce that $f\circ g$ is multiplication by τ. This implies that $Z[\tau^{-1}]$ is a direct summand in a free $W[\tau^{-1}]$-module of finite rank. In particular, it is a projective $W[\tau^{-1}]$-module.

Let I be the kernel of the multiplication map $m\colon Z\otimes_W Z\to Z$. The differentials $\Omega^1_{Z/W}$ are defined by I/I^2. Let $x\in I$. By assumption $\tau\mathfrak{e}-\tau$ lies in I. On the other hand, since $L\subset M$ is étale, the kernel of multiplication by $\mathfrak{e}$ on $M\otimes_L M$ is $I\otimes_Z M$. Thus, $\mathfrak{e}x=0$ and $(\tau\mathfrak{e}-\tau)x=-\tau x\in I^2$. Therefore, τ kills $\Omega^1_{Z/W}$. Thus, $Z[\tau^{-1}]$ is unramified as $W[\tau^{-1}]$-algebra. Hence, $W[\tau^{-1}]\subset Z[\tau^{-1}]$ is étale.

3.2. Notation

With the notation of 2.5, we denote by

$$\mathrm{Tr}_n\colon S_n\to \mathrm{Hom}_{R_n}(S_n,R_n)$$

the trace map $x\mapsto \mathrm{Tr}_{M_n/L_n}(x\cdot\_)$ and by $\mathfrak{e}_n\in M_n\otimes_{L_n}M_n$ the canonical idempotent associated to the separable extension $L_n\subset M_n$. Since $S_n[p^{-1}]$ is finite and étale as $R_n[p^{-1}]$-algebra, we have

$$\mathfrak{e}_n\in (S_n\otimes_{R_n}S_n)\big[p^{-1}\big].$$

3.3. Differents and discriminants

Assume that $W\subset Z$ is a finite and flat extension of products of Dedekind domains. Following [15, SIII.3] define the inverse different and the discriminant of $W\subset Z$ as

$$\mathfrak{D}^{-1}_{Z/W}:=\big\{y\in M\mid \mathrm{Tr}_{M/L}(xy)\in W\ \forall x\in Z\big\},\qquad \mathbf{D}_{Z/W}:=\mathrm{Norm}_{Z/W}(\mathfrak{D}_{Z/W}).$$

Then, $\mathfrak{D}^{-1}_{Z/W}$ contains Z because Z is integral over W and it is a locally free Z-module of rank 1. In particular, $\mathfrak{D}_{Z/W}$ is an ideal of Z. Furthermore, $\mathbf{D}_{Z/W}$ is an ideal of W.

The map $\mathfrak{D}^{-1}_{Z/W}\to \mathrm{Hom}_W(Z,W)$ defined by $x\mapsto \mathrm{Tr}_{M/L}(x\cdot\_)$ is an isomorphism of Z-modules. Let $\tau\in Z$ be an element which annihilates $Z/\mathfrak{D}_{Z/W}$. Then, τ kills the cokernel of Tr and the cokernel of

$$\mathrm{Tr}\otimes\mathrm{Id}\colon Z\otimes_W Z\to \mathrm{Hom}_Z(Z\otimes_W Z,Z)$$

as well. We conclude that $\tau\mathfrak{e}\in Z\otimes_W Z$; the notation is as in 3.1.

3.4. The definition of $\delta_n(S)$

The discriminant ideal of the extension of Dedekind rings $R_{n,\mathfrak{P}_n}\subset S_n\otimes_{R_n}R_{n,\mathfrak{P}_n}$ is a power of $\mathfrak{P}_nR_{n,\mathfrak{P}_n}$. By 2.2 we may then assume that it is generated by an element $p^{\delta_n(S)}$ of V_n dividing a power of p.

Let $\mathfrak{P}_{S_\infty}$ be a prime of S_∞ over p. Let $\mathfrak{P}_{S_n} := \mathfrak{P}_{S_\infty} \cap S_n$. Then, $\mathfrak{P}_\infty = \mathfrak{P}_{S_\infty} \cap R_\infty$ and $\mathfrak{P}_n = \mathfrak{P}_{S_\infty} \cap R_n$. The discriminant ideal of the extension of discrete valuation rings $\widehat{R_{n,\mathfrak{P}_n}} \subset \widehat{S_{n,\mathfrak{P}_{S_n}}}$ is generated by an element $p^{\delta_n(\mathfrak{P}_{S_\infty})}$ of V_n dividing a power of p. Furthermore, $\delta_n(S) = \sum_{\mathfrak{P}_{S_\infty}/\mathfrak{P}_\infty} \delta_n(\mathfrak{P}_{S_\infty})$.

3.5. LEMMA. – *There are only finitely many prime ideals of* S_∞ *over* $\mathfrak{P}_\infty$.

Proof. – Fix $n \in \mathbf{N}$. The extension $\mathrm{Frac}(R_n) \subset \mathrm{Frac}(S_n)$ is separable of degree $h := [\mathrm{Frac}(S) : \mathrm{Frac}(R)]$. Hence, the ring $S_n \otimes_{R_n} R_{n,\mathfrak{P}_n}$ is finite as $R_{n,\mathfrak{P}_n}$-module; see the discussion in 3.3. In particular, it is a Dedekind domain, free as $R_{n,\mathfrak{P}_n}$-module of rank h. We conclude that the number of prime ideals of S_n over $\mathfrak{P}_n$ is at most h. □

3.6. PROPOSITION (cf. [6, Thm. 1.2]). – *The sequence* $\{\delta_n(S)\}_{n\in\mathbf{N}}$ *is a decreasing sequence of non-negative rational numbers converging to* 0. *Furthermore, for every* $n \in \mathbf{N}$ *the cokernel of the trace map* $\mathrm{Tr}_n : S_n \to \mathrm{Hom}_{R_n}(S_n, R_n)$, *see* 3.2, *is annihilated by* $p^{\delta_n(S)}$ *and* $p^{\delta_n(S)} S_{n+1} \subset S_n \otimes_{R_n} R_{n+1}$.

Proof. – By construction $p^{\delta_n(S)}\mathrm{Hom}_{R_n}(S_n, R_n)$ is an R_n-submodule of $S_{n,\mathfrak{P}_{S_n}}$ for every height one prime ideal of S_n containing p. Since S_n is normal, S_n coincides with the intersection of the localization of S_n at all its height one prime ideals. Hence, $p^{\delta_n(S)}$ annihilates the cokernel of Tr_n as claimed. Consider the diagram

$$\begin{array}{ccc} S_n \otimes_{R_n} R_{n,\mathfrak{P}_n} =: Z_n & \longrightarrow & Z_{n+1} := S_{n+1} \otimes_{R_{n+1}} R_{n+1,\mathfrak{P}_{n+1}} \\ \uparrow & & \uparrow \\ R_{n,\mathfrak{P}_n} =: W_n & \longrightarrow & W_{n+1} := R_{n+1,\mathfrak{P}_{n+1}} \end{array}$$

We have

$$\begin{array}{ccccc} Z_n \otimes_{W_n} W_{n+1} \hookrightarrow Z_{n+1} \xrightarrow{\mathrm{Tr}} & \mathrm{Hom}_{W_n}(Z_{n+1}, W_n) & \longrightarrow & \mathrm{Hom}_{W_n}(Z_n \otimes_{W_n} W_{n+1}, W_n) \\ & \downarrow \wr & & \downarrow \wr \\ & \mathfrak{D}^{-1}_{Z_{n+1}/W_n} & \hookrightarrow & \mathfrak{D}^{-1}_{Z_n/W_n} \otimes_{W_n} \mathfrak{D}^{-1}_{W_{n+1}/W_n} \end{array}$$

Thus,

$$\mathfrak{D}_{Z_n/W_n}\mathfrak{D}_{W_{n+1}/W_n}\mathfrak{D}^{-1}_{Z_{n+1}/W_n} = \mathfrak{D}_{Z_n/W_n}\mathfrak{D}^{-1}_{Z_{n+1}/W_{n+1}} \subset Z_n \otimes_{W_n} W_{n+1} \subset Z_{n+1}. \tag{3.6.1}$$

In particular,

$$p^{\delta_n(S)-\delta_{n+1}(S)} Z_{n+1} \subset Z_n \otimes_{W_n} W_{n+1} \subset Z_{n+1}$$

so that $\delta_n(S) \geqslant \delta_{n+1}(S)$. Furthermore,

$$p^{\delta_n(S)} S_{n+1} \subset S_n \otimes_{R_n} R_{n+1,\mathfrak{P}_{n+1}}.$$

Since R_{n+1} is a free R_n-module, $R_{n+1,\mathfrak{P}_{n+1}} = R_{n+1} \otimes_{R_n} R_{n,\mathfrak{P}_n}$ and S_n is normal, the last statement of the proposition follows as well.

Due to 3.4 and 3.5 to conclude the proof of the first statement it is enough to show that for every prime $\mathfrak{P}_{S_\infty}$ the sequence $\mathbf{v}(\mathfrak{D}_{S_{n,\mathfrak{P}_{S_n}}/R_{n,\mathfrak{P}_n}})$ converges to 0. Let p^ξ be a generator of the maximal ideal of V. By construction $\Omega^1_{R'_{n+1}/R'_n}$ has $\Omega^1_{R'_{n+1}\otimes_V k/R'_n} \cong (R'_{n+1}/p^\xi R'_{n+1})^d$ as quotient. Take $N \in \mathbf{N}$ so that for every $n \geqslant N$ the extension $V_n \subset V_{n+1}$ is totally ramified. In particular, $\Omega^1_{V_{n+1}/V_n}$ is generated by one element and it is isomorphic to $V_{n+1}/\mathfrak{D}_{V_{n+1}/V_n}$.

Furthermore, $\mathbf{v}(\mathfrak{D}_{V_{n+1}/V_n})\geqslant \rho$, where $0<\rho\leqslant\xi$ is a constant independent of n; see [16, Cor. 1, §3.1]. Since by construction $\Omega^1_{R_{n+1}/R_n}\cong\Omega^1_{R'_{n+1}/R'_n}\otimes_{R'_{n+1}}R_{n+1}\oplus\Omega^1_{V_{n+1}/V_n}\otimes_{V_{n+1}}R_{n+1}$, the module of differentials $\Omega^1_{R_{n+1}/R_n}$ is generated by $d+1$ elements by Nakayama's lemma and it has $(R_{n+1}/p^\rho R_{n+1})^{d+1}$ as a quotient for every $n\geqslant N$.

Write $W_n:=R_{n,\mathfrak{P}_n}$ and, abusing the notation, $Z_n:=S_{n,\mathfrak{P}_{S_n}}$. Due to 3.5 we may choose N so that $\mathfrak{P}_{S_{n+1}}$ is the only prime ideal of S_{n+1} over $\mathfrak{P}_{S_n}$ for $n\geqslant N$. In particular, Z_{n+1} is the normalization of $Z_n\otimes_{W_n}W_{n+1}$. Consider the sequence of Z_{n+1}-modules

$$\gamma:\Omega^1_{Z_n/W_n}\otimes_{Z_n}Z_{n+1}\xrightarrow{\alpha}\Omega^1_{Z_{n+1}/W_n}\xrightarrow{\beta}\Omega^1_{Z_{n+1}/W_{n+1}}.$$

Let z_m be the residue field of Z_m with $m=n$ or $n+1$. It is a finite extension of the residue field w_n of W_n. Since $[w_n^{(p)}:w_n]=p^d$ by 2.2(II), we have $[z_m^{(p)}:z_m]=p^d$ so that $\Omega^1_{z_m/w_n}$ is generated by $\leqslant d$ elements as z_m-module. The maximal ideal I_m of Z_m is a principal ideal so that I_m/I_m^2 is a free z_m-module of rank 1. The kernel of $\Omega^1_{Z_m/W_n}\to\Omega^1_{z_m/w_n}$ is the image of I_m/I_m^2. The conclusion is that $\Omega^1_{Z_m/W_n}$ is generated by $\leqslant d+1$ elements as Z_m-module.

The kernel of β contains $\Omega^1_{W_{n+1}/W_n}\otimes_{W_{n+1}}Z_{n+1}$ as submodule by [6, Lemma 1.1]. As explained above, this is generated by $d+1$ elements and it has $(Z_{n+1}/p^\rho Z_{n+1})^{d+1}$ as quotient. Since $\Omega^1_{Z_{n+1}/W_n}$ is generated by $\leqslant d+1$ elements, the elementary divisors theorem implies that $\mathrm{Ker}(\beta)$ contains the kernel of multiplication by p^ρ on $\Omega^1_{Z_{n+1}/W_n}$. Hence, $\mathrm{Ker}(\gamma)$ contains the kernel H of multiplication by p^ρ on $\Omega^1_{Z_n/W_n}\otimes_{Z_n}Z_{n+1}$. Since $\Omega^1_{Z_n/W_n}$ is generated by $\leqslant d+1$ elements and it has the same length as $Z_n/\mathfrak{D}_{Z_n/W_n}$ as Z_n-module by [6, Lemma 1.1], we get that H has at least the length of Z_{n+1}/p^tZ_{n+1} as Z_{n+1}-module with $t=\min\{\rho,(\mathbf{v}(\mathfrak{D}_{Z_n/W_n})/d+1)\}$. It follows from (3.6.1) that the cokernel C of γ is annihilated by $\mathfrak{D}_{Z_n/W_n}\mathfrak{D}^{-1}_{Z_{n+1}/W_{n+1}}$. Since $\Omega^1_{Z_{n+1}/W_{n+1}}$ is generated by $\leqslant d+1$ elements, we conclude that the length of C is less or equal to the length of $(Z_{n+1}/\mathfrak{D}_{Z_n/W_n}\mathfrak{D}^{-1}_{Z_{n+1}/W_{n+1}})^{d+1}$. The length of $\Omega^1_{Z_n/W_n}\otimes_{Z_n}Z_{n+1}$ minus the length of $\Omega^1_{Z_{n+1}/W_{n+1}}$ is the same as the length of $\mathrm{Ker}(\gamma)$ minus the length of C. Thus,

$$\mathbf{v}(\mathfrak{D}_{Z_n/W_n})-\mathbf{v}(\mathfrak{D}_{Z_{n+1}/W_{n+1}})\geqslant t-(d+1)\mathbf{v}\big(\mathfrak{D}_{Z_n/W_n}\mathfrak{D}^{-1}_{Z_{n+1}/W_{n+1}}\big).$$

Hence, $(d+2)\mathbf{v}(\mathfrak{D}_{Z_n/W_n}\mathfrak{D}^{-1}_{Z_{n+1}/W_{n+1}})\geqslant\min\{\rho,(\mathbf{v}(\mathfrak{D}_{Z_n/W_n})/(d+1))\}$ for every $n\geqslant N$. One concludes that $\mathbf{v}(\mathfrak{D}_{Z_n/W_n})\to 0$. □

3.7. COROLLARY. – *There exists $\varepsilon'\in\mathbf{Q}$ with $1>\varepsilon'>0$ depending on $\{R_n\}_n$ and there exists $N'\in\mathbf{N}$ such that there is $p^{\varepsilon'}\in V_{N'}$ of valuation $\mathbf{v}(p^{\varepsilon'})=\varepsilon'$ and for every $n\geqslant N'$ we have*

$$R^p_{n+1}+p^{\varepsilon'}R_{n+1}=R_n+p^{\varepsilon'}R_{n+1}.$$

There exist $\varepsilon'>\varepsilon>0$ and $N=N(S,\varepsilon)\in\mathbf{N}$ such that there is $p^\varepsilon\in V_N$ of valuation ε and for all $n\geqslant N$ we have

$$S^p_{n+1}\subset S_n+p^\varepsilon S_{n+1}.$$

Proof. – Suppose that the first statement holds. Using 3.6, we have for $n\geqslant N'$

$$\begin{aligned}p^{p\delta_n(S)}S^p_{n+1}&\subset(S_n\otimes_{R_n}R_{n+1})^p\subset S_n\otimes_{R_n}R^p_{n+1}+p(S_n\otimes_{R_n}R_{n+1})\\&\subset S_n+p^{\varepsilon'}(S_n\otimes_{R_n}R_{n+1})\subset S_n+p^{\varepsilon'}S_{n+1}.\end{aligned}$$

Hence,

$$S_{n+1}^p \subset \frac{1}{p^{p\delta_n(S)}} S_n + p^{\varepsilon' - p\delta_n(S)} S_{n+1}.$$

Recall that $p^{\delta_n(S)} \in V_n$. Since the elements of S_{n+1} are integral over S_n and $\varepsilon' - p\delta_n(S) \to \varepsilon'$ for $n \to \infty$ by 3.6, we conclude that the corollary holds for the tower $\{S_n\}_n$.

We are left to prove the first statement. By construction of R_n and since $(R'_{n+1})^p + p^\xi R'_{n+1} = R'_n + p^\xi R'_{n+1}$, where p^ξ is a generator of the maximal ideal of V, it is enough to prove it for the tower $\{V_n\}_n$. In this case we may rely on the classical theory of the field of norms. Indeed, by assumption the extension $V_m \subset V_\infty$ is totally ramified for $m \gg 0$. In particular, for every $n \geqslant m$ the Norm of a uniformizer π_{n+1} of V_{n+1} is a uniformizer of V_n and V_n and V_{n+1} have the same residue field. Thus it suffices to prove that there exist $1 > \varepsilon' > 0$ and $N' \in \mathbf{N}$ such that $p^{\varepsilon'} \in V_{N'}$ and $\mathrm{Norm}(\pi_{n+1}) \equiv \pi_{n+1}^p$ modulo $p^{\varepsilon'} V_n$ for every $n \geqslant N'$. This follows from [17, Lemme 4.2.2.1 & Prop. 1.2.3]. This concludes the proof. □

3.8. THEOREM. – *Let $\mathfrak{P}_{S_\infty}$ be a prime ideal of S_∞ over p. There exists $c(\mathfrak{P}_{S_\infty})$ such that $p^n \delta_n(\mathfrak{P}_{S_\infty}) \leqslant c(\mathfrak{P}_{S_\infty})$.*

Proof. – For every $n \in \mathbf{N}$ we denote by W_n the p-adic completion of $R_{n,\mathfrak{P}_n}$. Let Z_n the p-adic completion of $S_{n,\mathfrak{P}_{S_n}}$ and let $\mathbf{v}$ be the unique valuation on Z_n extending the one on V_n. Let w_n (resp. z_n) be the residue fields of W_n (resp. Z_n).

Since $\mathbf{D}_{Z_n/W_n} Z_n = \mathfrak{D}_{Z_n/W_n}^{[Z_n:W_n]}$ and $\mathbf{D}_{Z_n/W_n} = p^{\delta_n(\mathfrak{P}_{S_\infty})} W_n$, it is enough to prove that the sequence $\{p^n \mathbf{v}(\mathfrak{D}_{Z_n/W_n})\}_n$ is bounded from above. If $Z \subset Z' \subset Z''$ are finite and generically separable extensions of discrete valuation rings, we have $\mathfrak{D}_{Z''/Z} = \mathfrak{D}_{Z''/Z'} \mathfrak{D}_{Z'/Z}$. Hence, passing to Galois closures it suffices to prove the statement assuming that $R \subset S$ is Galois with group G. □

3.8.1. LEMMA. – *There exist $1 > \varepsilon > 0$ and $N \in \mathbf{N}$ such that for every $n \geqslant N$ we have $Z_{n+1}^p + p^\varepsilon Z_{n+1} = Z_n + p^\varepsilon Z_{n+1}$ (in Z_{n+1}). In particular, Frobenius defines an isomorphism $z_{n+1} \xrightarrow{\sim} z_n \subset z_{n+1}$ and the ramification index of $Z_n \subset Z_{n+1}$ is p.*

Proof. – Due to 3.5 we may take $N \in \mathbf{N}$ so that $\mathfrak{P}_{S_n}$ is the unique prime of S_n over $\mathfrak{P}_{S_N}$. In particular, the rank of Z_n as W_n-module is constant and independent of n (for $n \geqslant N$) and the degree of $Z_n \subset Z_{n+1}$ is p^{d+1}. If we prove the first statement, then clearly Frobenius defines an isomorphism $z_{n+1} \xrightarrow{\sim} z_n$. Since this is true for $\{w_n\}$ by 3.7, we deduce that the degrees $[z_n : w_n]$ are constant and, hence, $[z_{n+1} : z_n] = [w_{n+1} : w_n]$ for every n. The latter is p^d by 2.2. In particular, the ramification index of $Z_n \subset Z_{n+1}$ is p. We are left to prove the first statement.

Fix $1 > \varepsilon' > \varepsilon$ and $N \in \mathbf{N}$ as in 3.7 so that $\varepsilon' - p\delta_N(S) \geqslant \varepsilon$. It suffices to show that for $n \geqslant N$ the map

$$Z_{n+1}/p^\varepsilon Z_{n+1} \to Z_n/p^\varepsilon Z_n,$$

defined by raising to the p-th power, is surjective. We know from 3.6 that $p^{\delta_m(Z)}$ annihilates the cokernel of the trace map $\mathrm{Tr}_m : Z_m \to \mathrm{Hom}_{W_m}(Z_m, W_m)$ and, since Z_m is a free W_m-module, also the cokernel of $\mathrm{Tr}_m \otimes 1 : Z_m \otimes_{W_m} Z_m \to \mathrm{Hom}_{Z_m}(Z_m \otimes_{W_m} Z_m, Z_m)$. By 3.1 the idempotent $\mathfrak{e}_m$ associated to the extension $\mathrm{Frac}(W_m) \subset \mathrm{Frac}(Z_m)$ satisfies $p^{\delta_m(S)} \mathfrak{e}_m \in Z_m \otimes_{W_m} Z_m$. Fix $n \geqslant N$. Write $p^{\delta_{n+1}(S)} \mathfrak{e}_{n+1} = \sum_i a_{n+1,i} \otimes b_{n+1,i}$ for suitable elements $a_{n+1,i}$ and $b_{n+1,i}$ of Z_{n+1}. Since $\mathfrak{e}_{n+1}$ is an idempotent, it follows that for $x \in Z_n$ we have

$$p^{p\delta_{n+1}(S)} x = (\mathrm{Tr}_{n+1} \otimes 1)\big(p^{p\delta_{n+1}(S)} \mathfrak{e}_{n+1}^p (x \otimes 1)\big) = \left(\sum_i \mathrm{Tr}_{n+1}\big(a_{n+1,i}^p x\big) b_{n+1,i}^p + O(p) \right).$$

Here, $O(p)$ means up to an element in pZ_{n+1}. Since $a_{n+1,i}^p$ lies in $Z_n + p^{\varepsilon'} Z_{n+1}$, the element $\mathrm{Tr}_{n+1}(a_{n+1,i}^p x)$ lies in $W_n + p^{\varepsilon'} W_{n+1}$. Since the map $W_{n+1}/p^{\varepsilon'}W_{n+1} \to W_n/p^{\varepsilon'}W_n$ defined by $z \mapsto z^p$ is surjective, there exists $\alpha \in Z_{n+1}$ such that

$$p^{p\delta_{n+1}(S)} x = \alpha^p + O\big(p^{\varepsilon'}\big).$$

Since Z_{n+1} is normal, $y := p^{-\delta_{n+1}(S)}\alpha$ lies in Z_{n+1}. Therefore, $x = y^p + p^{\varepsilon' - p\delta_{n+1}(S)}\beta$ for some $\beta \in Z_{n+1}$ as wanted. □

For every $n \in \mathbf{N}$ define W_n' as the p-adic completion of the localization of R_n' at the prime ideal $\mathfrak{P}_n \cap R_n'$ (the intersection being taken in R_n). By 2.3 the ring R_n' is regular and $R_n = R_n' \otimes_V V_n$ so that the ring W_n' is a discrete valuation ring and $W_n \cong W_n' \otimes_V V_n$. Let w_n' be the residue field of W_n'. Then, $w_n = w_n' \otimes_k k_n$ which is a separable extension of w_n'. Furthermore, $W_n' \subset W_m'$ is inert for every $m \geqslant n$. Define Z_n' as the normalization of $W_n' \otimes_{W_0} Z_0$ in Z_n. It is a discrete valuation ring. Let z_n' be its residue field.

3.8.2. LEMMA. – *There exists an integer $M \geqslant N$ such that for every $n \geqslant m \geqslant M$*
(a) $w_n' \subset z_n'$ *is separable*;
(b) *Frobenius to the p^{n-m}-th power induces isomorphisms $w_n' \xrightarrow{\sim} w_m'$ and $z_n' \xrightarrow{\sim} z_m'$*;
(c) $Z_n' \cong Z_m' \otimes_{W_m'} W_n'$;
(d) *$W_m' \subset W_n'$ and $Z_m' \subset Z_n'$ are inert.*

Proof. – Since $W_0 \subset W_n'$ is inert, the sequence $\{\mathbf{v}(\mathbf{D}_{Z_n'/W_n'})\}_n$ is discrete and bounded from below. Hence, there exists $M \in \mathbf{N}$ so that it is constant for every $n \geqslant M$. In particular, for every $n \geqslant m \geqslant M$ the map $Z_m' \otimes_{W_m'} W_n' \to Z_n'$ must be an isomorphism. Then, z_n' is a quotient of $z_m' \otimes_{w_m'} w_n'$. Thus, $[z_n' : w_n'] \leqslant [z_m' : w_m']$. We have strict inequality if and only if $w_m' \subset z_m'$ is not separable, since, by definition, Frobenius to the p^{n-m}-th power induces isomorphisms $w_n' \xrightarrow{\sim} w_m'$. Enlarging M if necessary we may assume that the sequence $\{[z_n' : w_n']\}_{n \geqslant M}$ is constant. In particular, $w_n' \subset z_n'$ is separable as claimed. Since (b) holds for $\{w_n'\}_n$ by assumption, the conclusion follows. □

Recall that $R \subset S$ is assumed to be Galois with group G. Consider a p-Sylow subgroup H of G and the associated intermediate extensions $R = T_0 \subset \cdots \subset T_i \subset T_{i+1} \subset \cdots \subset T_l = S$ such that $T_0 \subset T_1 = S^H$ has degree prime to p, $T_1 \subset T_2$ is étale and, using that a p-group is solvable, $T_i \subset T_{i+1}$ is Galois, not étale of degree p for $i \geqslant 2$. Let $T_{i,n}$ be the normalization of $T_i \otimes_R R_n$. Take N and M big enough so that the conclusions of 3.8.1 and of 3.8.2 hold for the p-adic completion of the localization of each $T_{i,n}$ at $T_{i,n} \cap \mathfrak{P}_{S_n}$ for every $n \geqslant M$. We are left to deal with the following two cases. The first one is the extension arising from $R \subset T_{2,M}$ which is tamely ramified. The second one arises from the extensions $T_{i,M} \subset T_{i+1,M}$ for $i \geqslant 2$. To ease the notation, we replace the extension $R \subset S$ with $T_{i,M} \subset T_{i+1,M}$ and for $n \geqslant M$ we denote by W_n (resp. Z_n) the p-adic completion of the localization of $T_{i,n+M}$ (resp. $T_{i+1,n+M}$) at its unique prime ideal above p. In each case, we want to show that the sequence $\{p^n \mathbf{v}(\mathfrak{D}_{Z_n/W_n})\}_n$ is bounded from above.

3.8.3. *Case I: the extension $W_n \subset Z_n$ is tamely ramified for every n*

Let

$$W_\infty' := \bigcup_n W_n', \qquad Z_\infty' := \bigcup_n Z_n' = Z_n' \otimes_{W_n'} W_\infty'.$$

They are both discrete valuation rings with perfect residue fields w'_∞ and z'_∞ respectively. Tensoring the diagram

$$\begin{array}{ccc} Z'_n & \hookrightarrow & Z_n \\ \uparrow & & \uparrow \\ W'_n & \hookrightarrow & W_n \end{array}$$

with $\bigotimes_{W'_n} W'_\infty$, we get the diagram

$$\begin{array}{ccc} Z'_\infty & \hookrightarrow & Z_n \otimes_{W'_n} W'_\infty \\ \uparrow & & \uparrow \\ W'_\infty & \hookrightarrow & W_n \otimes_{W'_n} W'_\infty \end{array}$$

Note that $W_n \otimes_{W'_n} W'_\infty$ (resp. $Z_n \otimes_{W'_n} W'_\infty$) is flat as W_n-module (resp. as Z_n-module) and $W'_n \subset W'_\infty$ (resp. $Z'_n \subset Z'_\infty$) is inert. The extensions of residue fields

$$\begin{array}{ccc} z'_n & \hookrightarrow & z_n \\ \uparrow & & \uparrow \\ w'_n & \hookrightarrow & w_n \end{array}$$

are separable since we know that $w'_n \subset w_n$ is separable and since $w_n \subset z_n$ is separable by assumption. Since $W'_n \subset W'_\infty$ and $Z'_n \subset Z'_\infty$ are inert, the quotient of $W_n \otimes_{W'_n} W'_\infty$ (resp. $Z_n \otimes_{W'_n} W'_\infty$) modulo the maximal ideal of W_n (resp. Z_n) is $w_n \otimes_{w'_n} w'_\infty$ (resp. $z_n \otimes_{z'_n} z'_\infty$) which is a perfect field. We conclude that $W_n \otimes_{W'_n} W'_\infty$ and $Z_n \otimes_{W'_n} W'_\infty$ are still discrete valuation rings and that the different of $W_n \otimes_{W'_n} W'_\infty \subset Z_n \otimes_{W'_n} W'_\infty$ is the same as the different of $W_n \subset Z_n$. Hence, to compute the behavior of the sequence $\{p^n \mathbf{v}(\mathfrak{D}_{Z_n/W_n})\}_n$ we may as well assume that the residue fields of W_n and Z_n are perfect for every n. Under this assumption it follows from [16, Prop. 5 & Pf. Prop. 9] using local class field theory that $\{p^n \mathbf{v}(\mathfrak{D}_{Z_n/W_n})\}_n$ is bounded from above for $n \gg 0$.

3.8.4. *Case II: the map $W_n \subset Z_n$ is Galois, not étale, of degree p for every n*

By 3.6 we have that $\mathbf{v}(\mathfrak{D}_{Z_n/W_n}) < \varepsilon$ for $n \gg 0$. Replacing S with S_n we may assume that this holds for every n. Let σ be a generator of the Galois group G.

Assume first that $w_n \subset z_n$ is of degree p, and hence inseparable, for some n. By 3.8.1 this holds for every n. Let $\overline{\alpha}$ be a generator of z_n as w_n-algebra. Let $\alpha \in Z_n$ be a lift of $\overline{\alpha}$. Since $W_n \subset Z_n$ is Galois of degree p, the element α is a generator of Z_n as W_n-algebra and the minimal polynomial of α over W_n is $f(X) := \prod_i (X - \sigma^i(\alpha))$. By hypothesis and 3.8.1 there exists $\beta \in Z_{n+1}$ such that $\alpha = \beta^p + p^\varepsilon \gamma$ with $\gamma \in Z_{n+1}$. Furthermore, β generates z_{n+1} over w_{n+1} and, hence, Z_{n+1} as W_{n+1}-algebra. Then, $\mathbf{v}(\mathfrak{D}_{Z_n/W_n}) = \mathbf{v}(f'(\alpha)) = \sum_{i=1}^{p-1} \mathbf{v}(\alpha - \sigma^i(\alpha))$ and, analogously, $\mathbf{v}(\mathfrak{D}_{Z_{n+1}/W_{n+1}}) = \sum_{i=1}^{p-1} \mathbf{v}(\beta - \sigma^i(\beta))$. Note that $\mathbf{v}(\sigma^i(\alpha) - \alpha) \leqslant \mathbf{v}(\mathfrak{D}_{Z_n/W_n}) < \varepsilon$. Thus, $\mathbf{v}(\beta^p - \sigma^i(\beta^p)) = \mathbf{v}(\alpha - \sigma^i(\alpha))$. Since $\varepsilon < 1 = \mathbf{v}(p)$, we have $p\mathbf{v}(\beta - \sigma^i(\beta)) = \mathbf{v}(\beta^p - \sigma^i(\beta^p))$. Thus, $\mathbf{v}(\mathfrak{D}_{Z_{n+1}/W_{n+1}}) = \frac{\mathbf{v}(\mathfrak{D}_{Z_n/W_n})}{p}$ and the conclusion follows in this case.

We are left to consider the case that $w_n = z_n$ for some (every) $n \in \mathbf{N}$. Let α be a uniformizer of Z_n. It generates Z_n as W_n-algebra. By 3.8.1 there exists $\beta \in Z_{n+1}$ such that $\beta^p \equiv \alpha$ modulo $p^\varepsilon Z_{n+1}$. In particular, since $\mathbf{v}(\mathfrak{D}_{Z_n/W_n}) < \varepsilon$ by assumption, we have $\mathbf{v}(\alpha) < \varepsilon$ so that β is a uniformizer of Z_{n+1}. Then, $\mathbf{v}(\mathfrak{D}_{Z_n/W_n}) = \sum_{i=1}^{p-1} \mathbf{v}(\alpha - \sigma^i(\alpha))$ and, analogously, $\mathbf{v}(\mathfrak{D}_{Z_{n+1}/W_{n+1}}) = \sum_{i=1}^{p-1} \mathbf{v}(\beta - \sigma^i(\beta))$. The conclusion follows as before.

3.9. DEFINITION. – With the notation of 3.8 define $c(S) := \sum c(\mathfrak{P}_{S_\infty})$, the sum being taken over all the (finitely many) prime ideals in S_∞ over p.

3.10. COROLLARY. – *The cokernel of the trace* $\mathrm{Tr}_n : S_n \to \mathrm{Hom}_{R_n}(S_n, R_n)$*, see* 3.2*, is annihilated by* $p^{\frac{c(S)}{p^n}}$*. Furthermore,* $p^{\frac{c(S)}{p^n}} S_{n+1} \subset S_n \otimes_{R_n} R_{n+1}$*. In particular,* $p^{\frac{c(S)p}{(p-1)p^n}} S_\infty \subset S_n \otimes_{R_n} R_\infty$ *for every* $n \in \mathbf{N}$.

Proof. – The assertions follow from 3.6 and 3.8. □

3.11. COROLLARY. – *We have* $S_\infty \otimes_{R_\infty} \widehat{R_\infty} \cong \widehat{S_\infty}$*. Furthermore,* $\widehat{S_\infty}$ *is* $\widehat{R_\infty}$*-torsion free as a module and is normal as a ring.*

Proof. – Since $\widehat{R_\infty}$ is a flat R_∞-module by A.7, we get from 3.10 the inclusions

$$p^{\frac{pc(S)}{p-1}} S \otimes_R \widehat{R_\infty} \hookrightarrow p^{\frac{pc(S)}{p-1}} S_\infty \otimes_{R_\infty} \widehat{R_\infty} \hookrightarrow S \otimes_R \widehat{R_\infty}.$$

Since S is a finite R-module, we conclude from A.7 that the homomorphism $S \otimes_R \widehat{R_\infty} \to \lim_{\infty \leftarrow n}(S \otimes_R R_\infty / p^n R_\infty)$ is an isomorphism. In particular, $S \otimes_R \widehat{R_\infty}$ is p-adically complete and separated. Hence, $S_\infty \otimes_{R_\infty} \widehat{R_\infty}$ is p-adically complete and separated and the natural map $S_\infty \otimes_{R_\infty} \widehat{R_\infty} \to \widehat{S_\infty}$ is an isomorphism.

Using A.9 and the first statement, we conclude that $\widehat{S_\infty}$ is p-torsion free and is a normal ring. In particular, $\widehat{S_\infty}$ injects in $\widehat{S_\infty}[p^{-1}] = S[p^{-1}] \otimes_R \widehat{R_\infty}$. Since $S[p^{-1}]$ is flat over $R[p^{-1}]$, we deduce that $\widehat{S_\infty}[p^{-1}]$ is flat, and thus torsion free, over $\widehat{R_\infty}$. We conclude that $\widehat{S_\infty}$ is $\widehat{R_\infty}$-torsion free as a module as claimed. □

4. The generalized ring of norms

The aim of this section is to introduce and study a generalization of (the ring of integers of) the field of norms of Fontaine–Wintenberger [8,9,17] for the rings R (2.2) and S (2.5). One knows from loc. cit. that $\mathbf{E}_V^+$ is a complete discrete valuation ring of characteristic p with residue field k_∞. We reprove this result in 4.6. We show that $\mathbf{E}_R^+$ is a noetherian, regular, domain formally étale over the Tate algebra $\mathbf{E}_V^+\{x_1, \ldots, x_d, \frac{1}{x_1}, \ldots, \frac{1}{x_d}\}$; see 4.7. We prove that $\mathbf{E}_S^+$ is normal and $\mathbf{E}_R^+ \subset \mathbf{E}_S^+$ is generically finite and étale; see 4.9 for a precise statement. We refer the reader to 4.5 for further properties of $\mathbf{E}_S^+$.

Let $\varepsilon \in \mathbf{Q}$ with $1 > \varepsilon > 0$ and let $N \in \mathbf{N}$ be such that 3.7 holds for the case $R = S = V$. Since $V_n \subset V_{n+1}$ is totally ramified for $n \gg 0$ by assumption, it follows from loc. cit. that there exists a system of uniformizers $\tau_i \in V_i$, for $i \geqslant N$, satisfying $\tau_{i+1}^p \equiv \tau_i \bmod p^\varepsilon$.

4.1. DEFINITION (*cf.* [10, §A.3.1.2]). – Let $\varepsilon \in \mathbf{Q}$ with $1 > \varepsilon > 0$. For S_∞ as in 2.5, define

$$\widetilde{\mathbf{E}}_{S_\infty}^+ := \lim_{\infty \leftarrow n} S_\infty / p^\varepsilon S_\infty, \qquad \widetilde{\mathbf{E}}_{S_\infty} := \widetilde{\mathbf{E}}_{S_\infty}^+ \left[\frac{1}{\overline{\pi}_K}\right]$$

where the transition morphisms in the limit are defined by raising to the p-th power and where $\overline{\pi}_K$ is the element $(\ldots, \tau_n, \tau_{n+1}, \ldots) \in \mathbf{E}_V^+$.

Possibly taking a smaller ε and a bigger $N \in \mathbf{N}$, we may assume that the conclusions of 3.7 hold for S. For every $n \geqslant N$, since $p^\varepsilon S_\infty \cap S_n = p^\varepsilon S_n$, we have that $S_n / p^\varepsilon S_n$ is a subring of $S_\infty / p^\varepsilon S_\infty$ and for every $x \in S_{n+1}/p^\varepsilon S_{n+1}$, we get from 3.7 that $x^p \in S_n / p^\varepsilon S_n$. Thus, the following makes sense:

4.2. DEFINITION. – Let

$$\mathbf{E}_S^+ := \lim_{\infty \leftarrow n} S_n/p^\varepsilon S_n$$

be the subring of $\widetilde{\mathbf{E}}_{S_\infty}^+$ consisting of elements $(x_0, x_1, \ldots, x_n, \ldots) \in \widetilde{\mathbf{E}}_{S_\infty}^+$ such that $x_n \in S_n/p^\varepsilon S_n$ for $n \geqslant N$. For every $m \in \mathbf{N}$ let

$$\mathbf{E}_S^+(m) := \lim_{\infty \leftarrow n} S_{n+m}/p^\varepsilon S_{n+m}$$

be the subring consisting of elements $(x_0, \ldots, x_n, \ldots) \in \widetilde{\mathbf{E}}_{S_\infty}^+$ with $x_n \in S_{n+m}/p^\varepsilon S_{n+m}$ for $n \geqslant N$. Define $\mathbf{E}_S := \mathbf{E}_S^+[\bar{\pi}_K^{-1}]$ and $\mathbf{E}_S(m) := \mathbf{E}_S^+(m)[\bar{\pi}_K^{-1}]$.

4.3. Convention

Let $\alpha \in \mathbf{Q}$. Following the conventions of 2.1 we denote by $\bar{\pi}_0^\alpha$ a (any) element $a = (a_0, a_1, \ldots, a_n, \ldots)$ in $\bigcup_m \mathbf{E}_V^+(m)$, if it exists, such that $\mathbf{v}(a_i) = \frac{\alpha}{p^i}$ for $i \gg 0$.

One knows that there exists ρ such that $\mathbf{v}(\tau_n) = \frac{\rho}{p^n}$ for $n \geqslant N$. Since we are assuming that there is an element $p^\varepsilon \in V_n$ of valuation ε and since τ_n is a uniformizer of V_n for $n \geqslant N$ by assumption, see 3.7, then $\beta_n := \frac{\varepsilon p^n}{\rho}$ is a non-negative integer. We can, and will, then take $\bar{\pi}_0^{p^n \varepsilon}$ to be the element $\bar{\pi}_K^{\beta_n}$ of $\mathbf{E}_V^+$.

4.4. PROPOSITION. – *The ring $\widetilde{\mathbf{E}}_{S_\infty}^+$ has the following properties*:

(1) *its idempotents are in one-to-one correspondence with those of $\widehat{S_\infty}$. It is a domain if and only if $\widehat{S_\infty}$ is a domain*;

(2) *the map $\widetilde{\mathbf{E}}_{S_\infty}^+/\bar{\pi}_0^{p^n\varepsilon}\widetilde{\mathbf{E}}_{S_\infty}^+ \to S_\infty/p^\varepsilon S_\infty$, sending $x = (x_0, x_1, \ldots, x_m, \ldots)$ to x_n is injective. It is an isomorphism if $S = R$*;

(3) *it is $\widetilde{\mathbf{E}}_{R_\infty}^+$-torsion free as a module, $\bar{\pi}_0^\varepsilon$-adically complete and separated and reduced as a ring. It is endowed with a $\bar{\pi}_0^\varepsilon$-adically continuous action of $\mathrm{Aut}(S_\infty/R)$*;

(4) *it is a perfect ring of characteristic p.*

4.5. PROPOSITION. – *The ring $\mathbf{E}_S^+$ has the following properties*:

(1) *there exists $N \in \mathbf{N}$ depending on S such that the map $\mathbf{E}_S^+/\bar{\pi}_0^{p^n\varepsilon}\mathbf{E}_S^+ \to S_n/p^\varepsilon S_n$, sending $x = (x_0, x_1, \ldots, x_m, \ldots)$ to x_n is well defined and injective for every $n \geqslant N$. It is an isomorphism if $S = R$*;

(2) *it is $\mathbf{E}_R^+$-torsion free as a module, reduced as a ring and $\bar{\pi}_K$-adically complete and separated*;

(3) *it is endowed with a $\bar{\pi}_K$-adically continuous action of $\mathrm{Aut}(S_\infty/R)$ compatible with the one on $\widetilde{\mathbf{E}}_{S_\infty}^+$*;

(4) *its idempotents are the same as those of $\widetilde{\mathbf{E}}_{S_\infty}^+$ (resp. of S_∞). It is a domain if and only if S_∞ (or equivalently $\widehat{S_\infty}$ or $\widetilde{\mathbf{E}}_{S_\infty}^+$) is a domain*;

(5) *the ring $\mathbf{E}_R^+$ is noetherian and $\mathbf{E}_S^+$ is finite as $\mathbf{E}_R^+$-module. In particular, $\mathbf{E}_S^+$ is noetherian.*

To have a good theory of the generalized field of norms, it is crucial that the maps $\widetilde{\mathbf{E}}_{S_\infty}^+/\bar{\pi}_0^{p^n\varepsilon}\widetilde{\mathbf{E}}_{S_\infty}^+ \to S_\infty/p^\varepsilon S_\infty$ and $\mathbf{E}_S^+/\bar{\pi}_0^{p^n\varepsilon}\mathbf{E}_S^+ \to S_n/p^\varepsilon S_n$ (for $n \gg 0$), defined in 4.4(2) and 4.5(1) respectively, are isomorphisms. In the next section we will show that, in fact, if one is an isomorphism the other is as well and that this is equivalent to the almost étaleness of the extension $R_\infty \subset S_\infty$; see 5.1. We refer the reader to 5.10, 5.11 and 5.12 for examples of situations when this applies. In the rest of this section we study the properties of the generalized ring of norms which are independent from almost étaleness.

4.6. COROLLARY (cf. [17, Thm. 2.1.3 & Prop. 4.2.1]). – *We have* $\mathbf{E}_V^+ \cong k_\infty[\![\bar\pi_K]\!]$, *where* k_∞ *is the residue field of* V_∞ *and* $\bar\pi_K$ *is as in* 4.2. *In particular,* $\mathbf{E}_V^+$ *is a complete dvr of characteristic* p.

Proof. – Note that k_∞ is a finite extension of k so that it is a perfect field. We have a ring homomorphism $k_\infty \mapsto \mathbf{E}_V^+$ given by $\alpha \mapsto (\alpha, \alpha^{\frac{1}{p}}, \alpha^{\frac{1}{p^2}}, \ldots)$. By 4.5 the ring $\mathbf{E}_V^+$ is noetherian and $\bar\pi_K$-adically complete and separated. We then get a homomorphism $\rho: k_\infty[\![\bar\pi_K]\!] \to \mathbf{E}_V^+ \cong \lim_n(V_n/p^\varepsilon V_n)$. By 4.5(1) it is an isomorphism modulo $\bar\pi_0^{p^n\varepsilon}$ for $n \gg 0$. Hence, it is an isomorphism. □

Let $R^0 := V\{T_1^{\pm 1}, \ldots, T_d^{\pm 1}\}$. For every $i = 1, \ldots, d$ let $x_i \in \mathbf{E}_{R^0}^+$ be the element $x_i := (T_i, T_i^{\frac{1}{p}}, T_i^{\frac{1}{p^2}}, \ldots)$ (viewed in $\lim_n R_n'/p^\rho R_n'$ with $p^\rho \in V$ generator of the maximal ideal).

4.7. COROLLARY. – *We have* $\mathbf{E}_{R^0}^+ \cong \mathbf{E}_V^+\{x_1, \ldots, x_d, \frac{1}{x_1}, \ldots, \frac{1}{x_d}\}$; *the convergence is relative to the* $\bar\pi_K$*-adic topology on* $\mathbf{E}_V^+$. *Furthermore, with the notation of* 2.1, *we have*

$$\mathbf{E}_R^+ \otimes_{\mathbf{E}_{R^0}^+} \left(\mathbf{E}_{R^0}^+\right)^{(p)} \cong \left(\mathbf{E}_R^+\right)^{(p)}. \tag{4.7.1}$$

In particular,

(i) *the extension* $\mathbf{E}_{R^0}^+ \subset \mathbf{E}_R^+$ *is formally étale* (*for the discrete topology*);

(ii) $\mathbf{E}_R^+$ *is a noetherian regular domain and* $\bar\pi_K, x_1, \ldots, x_d$ *form an absolute* p*-basis for* $\mathbf{E}_R^+$;

(iii) $\mathbf{E}_R^+(m) = \mathbf{E}_R^+ \otimes_{\mathbf{E}_{R^0}^+} \mathbf{E}_{R^0}^+(m) = \mathbf{E}_R^+[\bar\pi_K^{\frac{1}{p^m}}, x_1^{\frac{1}{p^m}}, \ldots, x_d^{\frac{1}{p^m}}]$.

4.8. An example: $\mathbf{E}_R^+$ in the cyclotomic case

Suppose we are in the case 2.4. Then, we can be more explicit. Let $\epsilon := (1, \zeta_p, \zeta_p, \ldots)$, considered as element of $\widetilde{\mathbf{E}}_{V_\infty}^+$. Note that $(\epsilon - 1) = (\bar\pi_0^{\frac{p}{(p-1)}})$ as ideals of $\widetilde{\mathbf{E}}_{V_\infty}^+$. By 4.6 we have

$$\mathbf{E}_{\mathbf{W}(k)}^+ \xrightarrow{\sim} k[\![\epsilon - 1]\!].$$

Finally, $\Gamma_V := \mathrm{Gal}(K_\infty/K)$ acts continuously on $\mathbf{E}_{\mathbf{W}(k)}^+$. Let $\chi: \Gamma_V \to \mathbf{Z}_p^*$ be the cyclotomic character determined by the choice of the roots of unity $\{\zeta_{p^n}\}$. Then,

$$\gamma(\epsilon) = \epsilon^{\chi(\gamma)} := \left(1, \zeta_p^{\chi(\gamma)}, \zeta_{p^2}^{\chi(\gamma)}, \ldots\right).$$

The action of Γ_R on $\mathbf{E}_{R^0}^+$, and hence on $\mathbf{E}_R^+$, is characterized as follows. The group Γ_V acts on $\mathbf{E}_V^+$ and acts trivially on $x_1, \ldots, x_d$. For $i = 1, \ldots, d$ the element $\gamma_i \in \widetilde\Gamma_R$ acts trivially on $\mathbf{E}_V^+$ and satisfies

$$\gamma_i(x_j) = \begin{cases} \epsilon x_j & \text{if } i = j, \\ x_j & \text{if } i \neq j. \end{cases}$$

4.9. THEOREM. – *The rings* $\mathbf{E}_S^+$ *and* $\widetilde{\mathbf{E}}_{S_\infty}^+$ *are normal. Furthermore, the extensions*

$$\mathbf{E}_R \hookrightarrow \mathbf{E}_S \quad \textit{and} \quad \widetilde{\mathbf{E}}_{R_\infty} \hookrightarrow \widetilde{\mathbf{E}}_{S_\infty}$$

are finite and étale of the same degree.

4.10. LEMMA (cf. [10, Section A.3.1.2]). – *Let $\widehat{S_\infty}$ be the p-adic completion of S_∞. Consider the ring*

$$\lim_{\infty\leftarrow n} \widehat{S_\infty} := \big\{ \big(x^{(0)}, x^{(1)}, \ldots, x^{(m)}, \ldots\big) \mid x^{(m)} \in \widehat{S_\infty},\ \big(x^{(m+1)}\big)^p = x^{(m)} \big\},$$

where the transition maps are defined by raising to the p-th power, the multiplicative structure is induced by the one on $\widehat{S_\infty}$ and the additive structure is defined by

$$\big(\ldots, x^{(m)}, \ldots\big) + \big(\ldots, y^{(m)}, \ldots\big) = \Big(\ldots, \lim_{n\to\infty} \big(x^{(m+n)} + y^{(m+n)}\big)^{p^n}, \ldots\Big).$$

Then,

(a) *the ring structure is well defined, commutative, associative with* 1;

(b) *the natural map $\lim_{\infty\leftarrow n} \widehat{S_\infty} \to \widetilde{\mathbf{E}}^+_{S_\infty}$ is an isomorphism.*

Proof. – See loc. cit. We exhibit the inverse of the map in (b). Take $\underline{x} := (x_0, \ldots, x_m, \ldots)$ in $\widetilde{\mathbf{E}}^+_{S_\infty}$. For every i, $j \in \mathbf{N}$ we have that $x_{i+j}^{p^j}$ gives a well defined element of $S_\infty/p^{j\varepsilon} S_\infty$ and $\{x_{i+j}^{p^j}\}_j$ converges to a unique $x^{(i)} \in \widehat{S_\infty}$ for $j \to \infty$. Then, $(x^{(0)}, \ldots, x^{(m)}, \ldots)$ lies in $\lim_{\infty\leftarrow n} \widehat{S_\infty}$ and we define it to be the image of $\underline{x}$. □

4.11. Proof of Proposition 4.4

(1) Let $\mathfrak{e}$ be an idempotent of $\widehat{S_\infty}$. Then $(\mathfrak{e}, \mathfrak{e}, \ldots, \mathfrak{e}, \ldots)$ is an idempotent of $\widetilde{\mathbf{E}}^+_{\widehat{S_\infty}} = \widetilde{\mathbf{E}}^+_{S_\infty}$. Vice versa, let $(x^{(0)}, \ldots, x^{(m)}, \ldots)$ be an idempotent of $\widetilde{\mathbf{E}}^+_{S_\infty}$; the notation is as in 4.10. Then, $x^{(m)} = (x^{(m+1)})^p = x^{(m+1)}$ i.e., $x^{(i)} = x^{(j)}$ for every i and $j \in \mathbf{N}$. Furthermore, $(x^{(i)})^2 = x^{(i)}$ i.e., $x^{(i)}$ is an idempotent of $\widehat{S_\infty}$.

Assume that $\widehat{S_\infty}$ is a domain. Since we have a natural inclusion $\widetilde{\mathbf{E}}^+_{S_\infty} \subset \widehat{S_\infty}^{\mathbf{N}}$ respecting the multiplicative structures, see 4.10, we conclude that $\widetilde{\mathbf{E}}^+_{S_\infty}$ is a domain. Vice versa, suppose that $\widehat{S_\infty}$ is not a domain. Since $\widehat{S_\infty}$ is normal and has as total field of fractions the finite product of fields $M_0 \otimes_{L_0} \operatorname{Frac}(\widehat{R_\infty})$ by 3.11, we conclude that $\widehat{S_\infty}$ contains a non-trivial idempotent $\mathfrak{e}$. Therefore, $\widetilde{\mathbf{E}}^+_{S_\infty}$ contains a non-trivial idempotent and, hence, it is not a domain.

(2) It suffices to show that the map $\widetilde{\mathbf{E}}^+_{S_\infty}/\bar{\pi}_0^{p^n\varepsilon}\widetilde{\mathbf{E}}^+_{S_\infty} \to S_\infty/p^\varepsilon S_\infty$, sending $x = (x^{(0)}, x^{(1)}, \ldots, x^{(m)}, \ldots)$ to the class of $x^{(n)}$ modulo p^ε, is injective. If $x^{(n)} \equiv 0 \bmod p^\varepsilon$, then $x^{(m)} \equiv 0$ modulo $p^{\frac{\varepsilon}{p^{m-n}}}$ for every $m \geqslant n$ since S_∞ is normal. Hence, $x \in \bar{\pi}_0^{p^n\varepsilon}\widetilde{\mathbf{E}}^+_{S_\infty}$. By 3.7 Frobenius on $R_\infty/p^\varepsilon R_\infty$ is surjective. The last claim follows.

(3) We deduce that $\lim_n \widetilde{\mathbf{E}}^+_{S_\infty}/\bar{\pi}_0^{p^n\varepsilon}\widetilde{\mathbf{E}}^+_{S_\infty} \xrightarrow{\sim} \lim_n S_\infty/p^\varepsilon S_\infty = \widetilde{\mathbf{E}}^+_{S_\infty}$ as claimed. The continuity of the action of $\operatorname{Aut}(S_\infty/R)$ is obvious. We have a map $\widetilde{\mathbf{E}}^+_{S_\infty} \subset \widehat{S_\infty}^{\mathbf{N}}$ (resp. $\widetilde{\mathbf{E}}^+_{R_\infty} \subset \widehat{R_\infty}^{\mathbf{N}}$) respecting the multiplication, see 4.10. It follows from 3.11 that $\widehat{S_\infty}$ is $\widehat{R_\infty}$-torsion free as a module and is reduced as a ring. Then, $\widetilde{\mathbf{E}}^+_{S_\infty}$ is $\widetilde{\mathbf{E}}^+_{R_\infty}$-torsion free and is reduced as well.

(4) If $x = (x_0, x_1, \ldots, x_m, \ldots) \in \widetilde{\mathbf{E}}^+_{S_\infty}$, then $y = (x_1, \ldots, x_m, \ldots)$ satisfies $y^p = x$. □

4.12. Proof of Proposition 4.5

The fact that $\mathbf{E}^+_S$ is $\mathbf{E}^+_R$-torsion free and reduced follows from the analogous statements for $\widetilde{\mathbf{E}}^+_{S_\infty}$ and the inclusions $\mathbf{E}^+_S \subset \widetilde{\mathbf{E}}^+_{S_\infty}$ and $\mathbf{E}^+_R \subset \widetilde{\mathbf{E}}^+_{R_\infty}$. The map $\mathbf{E}^+_S/\bar{\pi}_0^{p^n\varepsilon}\mathbf{E}^+_S \to$

$\widetilde{\mathbf{E}}^+_{S_\infty}/\overline{\pi}_0^{p^n\varepsilon}\widetilde{\mathbf{E}}^+_{S_\infty}\to S_\infty/p^\varepsilon S_\infty$, sending $(x^{(0)},x^{(1)},\ldots,x^{(m)},\ldots)$ to the class of $x^{(n)}$ modulo p^ε, factors via $S_n/p^\varepsilon S_n$ for $n\gg 0$ by construction. Using that S_n is normal for every $n\in\mathbf{N}$, one proves as in 4.4 that it is injective. By 3.7 it is also surjective if $S=R$. Moreover, we conclude that $\lim_n \mathbf{E}^+_S/\overline{\pi}_0^{p^n\varepsilon}\mathbf{E}^+_S$ coincides with $\lim_n S_n/p^\varepsilon S_n=\mathbf{E}^+_S$. Claims (1) and (2) follow.

Take $N\gg 0$ so that $\mathrm{Frac}(S_N)$ and $\mathrm{Frac}(R_\infty)$ are linearly disjoint over $\mathrm{Frac}(R_N)$ and $\mathrm{Aut}(S_\infty/R)$ is the product of $\mathrm{Aut}(S_N/R_N)$ and $\mathrm{Aut}(R_\infty/R)$. Such product acts on S_n and, hence, on $S_n/p^\varepsilon S_n$ for every $n\geqslant N$. Claim (3) follows.

We prove (4). Let $\mathfrak{e}$ be an idempotent of $\widehat{S_\infty}$. Let $\overline{\mathfrak{e}}$ be its image in $\widehat{S_\infty}/\mathfrak{P}_\infty\widehat{S_\infty}=\bigcup_n(S_n/\mathfrak{P}_nS_n)$. Then, $\overline{\mathfrak{e}}$ is an idempotent and lies in $S_n/\mathfrak{P}_nS_n$ for some $n\in\mathbf{N}$. Since S_n (resp. $\widehat{S_\infty}$) is p-adically complete and using Hensel's lemma, $\overline{\mathfrak{e}}$ lifts uniquely to an idempotent of S_n (resp. $\widehat{S_\infty}$). Thus, $\mathfrak{e}\in S_n$ and $(\mathfrak{e},\mathfrak{e},\ldots,\mathfrak{e},\ldots)$ is an idempotent of $\mathbf{E}^+_S$. Thus, the first part of (4) follows from 4.4(1).

Since S_∞ is normal, it is a domain if and only if it contains no non-trivial idempotent. By the above, this holds if and only if $\widehat{S_\infty}$ does not contain any non-trivial idempotent i.e., since it is normal, if and only if $\widehat{S_\infty}$ is a domain. By 4.4(1) this is equivalent to require that $\widetilde{\mathbf{E}}^+_{S_\infty}$ is a domain. Since $\mathbf{E}^+_S$ is a subring of $\widetilde{\mathbf{E}}^+_{S_\infty}$, if the latter is a domain also $\mathbf{E}^+_S$ is a domain. Suppose that S_∞ is not a domain. Being normal it contains a non-trivial idempotent. By the above, $\mathbf{E}^+_S$ contains a non-trivial idempotent as well and, hence, it is not a domain.

Next we prove (5). By 3.7 for every $n\geqslant N$ the map $R_{n+1}/p^\varepsilon R_{n+1}\to R_n/p^\varepsilon R_n$ (resp. $S_{n+1}/p^\varepsilon S_{n+1}\to S_n/p^\varepsilon S_n$) defined by raising to the p-th power is well defined; in the case of R_n it is also surjective. To easy the notation we assume that $N=0$ i.e., that $R_N=R$ and $S_N=S$. In particular, $\mathbf{E}^+_R/\overline{\pi}_0^\varepsilon\mathbf{E}^+_R=R/p^\varepsilon R$ is noetherian. The $\mathbf{E}^+_R$-algebra $\mathbf{E}^+_S/\overline{\pi}_0^\varepsilon\mathbf{E}^+_S$ is a $\mathbf{E}^+_R/\overline{\pi}_0^\varepsilon\mathbf{E}^+_R=R/p^\varepsilon R$-subalgebra of $S/p^\varepsilon S$. Since $S/p^\varepsilon S$ is finite as $R/p^\varepsilon R$-module and $R/p^\varepsilon R$ is noetherian, we get that $\mathbf{E}^+_S/\overline{\pi}_0^\varepsilon\mathbf{E}^+_S$ is a finite $R/p^\varepsilon R$-module. Consider the graded rings

$$\mathbf{gr}_{\overline{\pi}_0^\varepsilon}\mathbf{E}^+_R:=\bigoplus_n\big(\overline{\pi}_0^{n\varepsilon}\mathbf{E}^+_R\big)/\big(\overline{\pi}_0^{(n+1)\varepsilon}\mathbf{E}^+_R\big)\quad\text{and}\quad \mathbf{gr}_{\overline{\pi}_0^\varepsilon}\mathbf{E}^+_S:=\bigoplus_n\big(\overline{\pi}_0^{n\varepsilon}\mathbf{E}^+_S\big)/\big(\overline{\pi}_0^{(n+1)\varepsilon}\mathbf{E}^+_S\big).$$

By [2, Prop. 10.24] the ring $\mathbf{E}^+_R$ is noetherian if $\mathbf{gr}_{\overline{\pi}_0^\varepsilon}\mathbf{E}^+_R$ is. By [2, Cor. 10.25] the ring $\mathbf{E}^+_S$ is finitely generated as $\mathbf{E}^+_R$-module if $\mathbf{gr}_{\overline{\pi}_0^\varepsilon}\mathbf{E}^+_S$ is finitely generated as $\mathbf{gr}_{\overline{\pi}_0^\varepsilon}\mathbf{E}^+_R$-module. The ring $\mathbf{gr}_{\overline{\pi}_0^\varepsilon}\mathbf{E}^+_R$ (resp. $\mathbf{gr}_{\overline{\pi}_0^\varepsilon}\mathbf{E}^+_S$) is generated by $\mathbf{E}^+_R/\overline{\pi}_0^\varepsilon\mathbf{E}^+_R$ (resp. $\mathbf{E}^+_S/\overline{\pi}_0^\varepsilon\mathbf{E}^+_S$) in degree 0 and by $\overline{\pi}_0^\varepsilon$ in degree 1. Thus, by Hilbert's basis theorem [2, Cor. 7.6], both conclusions, and hence claim (5), follow remarking that $\mathbf{E}^+_R/\overline{\pi}_0^\varepsilon\mathbf{E}^+_R$ is noetherian and $\mathbf{E}^+_S/\overline{\pi}_0^\varepsilon\mathbf{E}^+_S$ is finitely generated as $\mathbf{E}^+_R/\overline{\pi}_0^\varepsilon\mathbf{E}^+_R$-module. □

4.13. Proof of Corollary 4.7

Consider the map $\varphi\otimes 1\colon\mathbf{E}^+_R\otimes_{\mathbf{E}^+_{R^0}}(\mathbf{E}^+_{R^0})^{(p)}\to(\mathbf{E}^+_R)^{(p)}$. It follows from 4.5(1) and 2.2 that, choosing ε small enough so that $V\cap p^\varepsilon V_\infty$ is the maximal ideal of V, then $\varphi\otimes 1$ modulo $(\overline{\pi}_0^{p^n\varepsilon})$ coincides with the base change via $k\to(V_n/p^\varepsilon V_n)^{(p)}$ of the homomorphism $(R'_n\otimes_V k)\otimes_{R^{0\prime}_n}(R^{0\prime}_n\otimes_V k)^{(p)}\to(R'_n\otimes_V k)^{(p)}$. This is an isomorphism by 2.2 and 2.3. Since $(\mathbf{E}^+_{R^0})^{(p)}$ is finite and free as $\mathbf{E}^+_{R^0}$-module, it is $\overline{\pi}_0^\varepsilon$-adically complete and separated. Hence, passing to the inverse limits modulo $\overline{\pi}_0^{p^n\varepsilon}$ for $n\to\infty$, we get that $\varphi\otimes 1$ is an isomorphism as claimed.

(i) It follows from (4.7.1) that $\mathbf{E}^+_{R^0}\subset\mathbf{E}^+_R$ is formally smooth (for the discrete topology) by [11, 0.21.2.7] and it is also formally unramified by [11, 0.21.2.5]. This implies the formal étaleness.

(ii) By 4.6 and 4.5(1) we have $(\mathbf{E}_R^+/\bar{\pi}_K\mathbf{E}_R^+)\cong R_n\otimes_{V_n}k_\infty$. By 2.2 the latter is isomorphic to $R'_n\otimes_k k_\infty$ which is isomorphic as a ring to $R\otimes_V k_\infty$. In particular, it is a geometrically regular domain by 2.3. Since $\mathbf{E}_R^+$ is $\bar{\pi}_K$-adically complete, every maximal ideal contains $\bar{\pi}_K$. We conclude that the localization of $\mathbf{E}_R^+$ at every maximal ideal is regular which implies that $\mathbf{E}_R^+$ is regular. In particular, it is the product of integral normal domains each of which is $\bar{\pi}_K$-adically complete and separated since it is a quotient of $\mathbf{E}_R^+$. Hence, the fact that $\mathbf{E}_R^+/\bar{\pi}_K\mathbf{E}_R^+$ is an integral domain implies that $\mathbf{E}_R^+$ is itself an integral domain. The second statement is equivalent to the isomorphism (4.7.1).

(iii) Follows from (4.7.1). □

We are left with the proof of Theorem 4.9. We start with a few lemmas.

4.14. LEMMA. – *For every n, with the notation of* 4.2*, we have that Frobenius on $\mathbf{E}_S^+(n+1)$ defines an isomorphism $\mathbf{E}_S^+(n+1)\xrightarrow{\sim}\mathbf{E}_S^+(n)\hookrightarrow\mathbf{E}_S^+(n+1)$.*

Proof. – Frobenius on $\mathbf{E}_S^+(n+1)$ is the map $(\ldots,x_m,\ldots)\mapsto(\ldots,x_m^p,\ldots)$, where $x_m\in S_{m+n+1}/p^\varepsilon S_{m+n+1}$ and $x_{m-1}=x_m^p$. Thus, its image lies in $\mathbf{E}_S^+(n)$. Vice versa, if $y=(\ldots,y_m,\ldots)$ is an element of $\mathbf{E}_S^+(n)$, then $x=(\ldots,x_m,\ldots)$, with $x_i:=y_{i+1}$, lies in $\mathbf{E}_S^+(n+1)$ and its image via Frobenius is x. Since $\mathbf{E}_S^+(n+1)$ is reduced by 4.5, Frobenius on $\mathbf{E}_S^+(n+1)$ is injective and the lemma follows. □

4.15. LEMMA. – *For every $n\in\mathbf{N}$*

(1) *we have* $\bar{\pi}_0^{\frac{c(S)}{p^n}}\mathbf{E}_S^+(n)\otimes_{\mathbf{E}_R^+(n)}\mathbf{E}_R^+(n+1)\subset\bar{\pi}_0^{\frac{c(S)}{p^n}}\mathbf{E}_S^+(n+1)\subset\mathbf{E}_S^+(n)\otimes_{\mathbf{E}_R^+(n)}\mathbf{E}_R^+(n+1)\subset\mathbf{E}_S^+(n+1)$. *In particular, these maps are isomorphisms after inverting* $\bar{\pi}_0^\varepsilon$;

(2) $\widetilde{\mathbf{E}}_{R_\infty}^+$ *the $\bar{\pi}_0^\varepsilon$-adic completion of the perfect closure of $\mathbf{E}_R^+$, the extension $\mathbf{E}_R^+(n)\subset\widetilde{\mathbf{E}}_{R_\infty}^+$ is faithfully flat and $\mathbf{E}_S^+(n)\otimes_{\mathbf{E}_R^+(n)}\widetilde{\mathbf{E}}_{R_\infty}^+$ is $\bar{\pi}_0^\varepsilon$-adically complete and separated*;

(3) *we have* $\bar{\pi}_0^{\frac{c(S)p}{p^n(p-1)}}\mathbf{E}_S^+(n)\otimes_{\mathbf{E}_R^+(n)}\widetilde{\mathbf{E}}_{R_\infty}^+\subset\bar{\pi}_0^{\frac{c(S)p}{p^n(p-1)}}\widetilde{\mathbf{E}}_{S_\infty}^+\subset\mathbf{E}_S^+(n)\otimes_{\mathbf{E}_R^+(n)}\widetilde{\mathbf{E}}_{R_\infty}^+\subset\widetilde{\mathbf{E}}_{S_\infty}^+$. *In particular,* $\mathbf{E}_S(n)\otimes_{\mathbf{E}_R(n)}\widetilde{\mathbf{E}}_{R_\infty}\xrightarrow{\sim}\widetilde{\mathbf{E}}_{S_\infty}$.

Proof. – Let ε and N be as in 3.7. Let $c(S)$ be as in 3.9. By 3.6, for every $n\geqslant N$ we have

$$(4.15.1)\qquad p^{\frac{c(S)}{p^{m+n}}}S_{m+n}\otimes_{R_{m+n}}R_{m+n+1}\subset p^{\frac{c(S)}{p^{m+n}}}S_{m+n+1}\subset S_{m+n}\otimes_{R_{m+n}}R_{m+n+1}\subset S_{m+n+1}.$$

$$(4.15.2)\qquad p^{\frac{pc(S)}{p^n(p-1)}}S_n\otimes_{R_n}R_\infty\subset p^{\frac{p\,c(S)}{p^n(p-1)}}S_\infty\subset S_n\otimes_{R_n}R_\infty\subset S_\infty.$$

(1) By construction R_{m+n+1} is free as R_{m+n}-module with a basis given by the elements $\tau_{m+n+1}^tT_1^{\frac{j_1}{p^{m+n+1}}}\cdots T_d^{\frac{j_d}{p^{m+n+1}}}$ with $0\leqslant t\leqslant p-1$ and $0\leqslant j_i\leqslant p-1$ for $n\gg 0$; see 4.2 for the notation. By 4.5(1) we have $\mathbf{E}_R^+(n)/\bar{\pi}_0^{p^m\varepsilon}\mathbf{E}_R^+(n)\xrightarrow{\sim}R_{m+n}/p^\varepsilon R_{m+n}$. The ring $\mathbf{E}_S^+(n)\otimes_{\mathbf{E}_R^+(n)}\mathbf{E}_R^+(n+1)$ is $\bar{\pi}_K$-adically complete and separated by 4.5. Its quotient modulo $\bar{\pi}_0^{p^m\varepsilon}\mathbf{E}_R^+(n+1)$ is

$$\big(\mathbf{E}_S^+(n)/\bar{\pi}_0^{p^m\varepsilon}\big)\otimes_{\mathbf{E}_R^+(n)}\big(\mathbf{E}_R^+(n+1)/\bar{\pi}_0^{p^m\varepsilon}\big)\cong\big(\mathbf{E}_S^+(n)/\bar{\pi}_0^{p^m\varepsilon}\big)\otimes_{R_{m+n}}\big(R_{m+n+1}/p^\varepsilon R_{m+n+1}\big).$$

The latter injects in $S_{m+n}\otimes_{R_{m+n}}(R_{m+n+1}/p^\varepsilon R_{m+n+1})$ since R_{m+n+1} is free as R_{m+n}-module so that the map $\mathbf{E}_S^+(n)\otimes_{\mathbf{E}_R^+(n)}\mathbf{E}_R^+(n+1)\to\lim_m S_{m+n}\otimes_{R_{m+n}}(R_{m+n+1}/p^\varepsilon)$, where the inverse limit is taken with respect to Frobenius, is injective. Using the given basis of R_{m+n+1} as R_{m+n}-module for varying m we also conclude that it is surjective. In conclusion, reducing modulo p^ε and taking inverse limits over m of (4.15.1), we get the claim.

(2) By 4.14 the perfect closure $\mathbf{E}_R^{+,\mathrm{perf}}$ of $\mathbf{E}_R^+$ coincides with the ring $\bigcup_m \mathbf{E}_R^+(m)$. Hence, $\mathbf{E}_R^{+,\mathrm{perf}}/\overline{\pi}_0^{p^n\varepsilon}\mathbf{E}_R^{+,\mathrm{perf}} \cong R_\infty/p^\varepsilon R_\infty$ by 4.5(1). We conclude from 4.4 that $\widetilde{\mathbf{E}}_{R_\infty}^+$ is the $\overline{\pi}_0^\varepsilon$-adic completion of $\mathbf{E}_R^{+,\mathrm{perf}}$. By 4.7 each $\mathbf{E}_R^+(m)$ is free as $\mathbf{E}_R^+$-module. Then, the claims follow from A.7.

(3) To get the homomorphisms claimed in (3), one reduces modulo p^ε and takes inverse limits over m with respect to Frobenius of (4.15.2). One is then left to prove that map $\mathbf{E}_S^+(n)\otimes_{\mathbf{E}_R^+(n)}\widetilde{\mathbf{E}}_{R_\infty}^+ \to \lim_m(S_{n+m}\otimes_{R_{n+m}} R_\infty/p^\varepsilon R_\infty)$ is an isomorphism. One knows from (2) that $\mathbf{E}_S^+(n)\otimes_{\mathbf{E}_R^+(n)}\widetilde{\mathbf{E}}_{R_\infty}^+$ is $\overline{\pi}_0^\varepsilon$-adically complete and separated. One then argues as in the proof of (1) that the map $\mathbf{E}_S^+(n)\otimes_{\mathbf{E}_R^+(n)}\widetilde{\mathbf{E}}_{R_\infty}^+/\overline{\pi}_0^{p^m\varepsilon}\widetilde{\mathbf{E}}_{R_\infty}^+ \to S_{n+m}\otimes_{R_{n+m}} R_\infty/p^\varepsilon R_\infty$ is an isomorphism for every $m\gg 0$ since by 4.4(2) the homomorphism $\widetilde{\mathbf{E}}_{R_\infty}^+/\overline{\pi}_0^{p^m\varepsilon}\widetilde{\mathbf{E}}_{R_\infty}^+ \to R_\infty/p^\varepsilon R_\infty$, given by $(x_0,\ldots,x_n,\ldots)\mapsto x_n$, is an isomorphism and since $R_\infty/p^\varepsilon R_\infty$ is a free $R_n/p^\varepsilon R_n$-module with a basis given by the elements $\tau_{m+n}^t T_1^{\frac{j_1}{p^{m+n}}}\cdots T_d^{\frac{j_d}{p^{m+n}}}$ with $m\geqslant 1$, $1\leqslant t\leqslant p^m$ and $1\leqslant j_i\leqslant p^m$ for $m\gg 0$. □

4.16. The valuation $\mathbf{z}'$ on $\widetilde{\mathbf{E}}_{S_\infty}^+$

Let $\mathbf{z}$ be a valuation on $\widehat{S_\infty}$ associated to a prime over p. Let

$$\mathbf{z}' : \mathrm{Frac}\big(\widetilde{\mathbf{E}}_{S_\infty}^+\big)\to \mathbf{Q}\cup\{\infty\},\quad \mathbf{z}'(x) := \mathbf{z}\big(x^{(0)}\big).$$

Let $x := (x^{(0)},x^{(1)},\ldots)\in\mathrm{Frac}(\widetilde{\mathbf{E}}_{S_\infty}^+)$. Then, $(x^{(n)})^{p^m} = x^{(n-m)}$ for all integers $0\leqslant m\leqslant n$. Since $\widehat{S_\infty}$ is reduced, $x=0$ iff $x^{(n)}=0$ for every $n\in\mathbf{N}$ iff $x^{(0)}=0$. Hence, $\mathbf{z}'(x)=\infty$ if and only if $x=0$. By the definition of the product structure on $\widetilde{\mathbf{E}}_{\widehat{S}_\infty}^+$ in 4.10, we have that $\mathbf{z}'$ is multiplicative.

Let $x=a+b$. In particular, $\{x_m := (a^{(m)}+b^{(m)})^{p^m}\}_m$ is a Cauchy sequence for the p-adic topology on $\widehat{S_\infty}$ converging to $x^{(0)}$. Since the image of $\mathbf{z}$ is contained in $\mathbf{Q}$ and $\mathbf{z}(p)>0$, we have that $\mathbf{z}(x_m)\to\mathbf{z}(x^{(0)})$. Moreover,

$$\begin{aligned}\mathbf{z}(x_m) &= p^m\mathbf{z}\big(a^{(m)}+b^{(m)}\big)\\ &\geqslant p^m\min\big\{\mathbf{z}\big(a^{(m)}\big),\mathbf{z}\big(b^{(m)}\big)\big\}\\ &= \min\big\{\mathbf{z}\big(a^{(0)}\big),\mathbf{z}\big(b^{(0)}\big)\big\}.\end{aligned}$$

It follows that $\mathbf{z}'(a+b)\geqslant\min\{\mathbf{z}'(a),\mathbf{z}'(b)\}$. Hence, $\mathbf{z}'$ defines a valuation on $\widetilde{\mathbf{E}}_{S_\infty}^+$.

4.17. Proof of Theorem 4.9

By 4.14 and 4.15(1), the map $\mathbf{E}_S\otimes_{\mathbf{E}_R}\mathbf{E}_R^{(p)}\to\mathbf{E}_S^{(p)}$ induced by Frobenius is an isomorphism. By [11, 0.21.2.7] this implies that the ring $\mathbf{E}_S$ is formally smooth over $\mathbf{E}_R$ (for the discrete topology). In particular, the extension $\mathbf{E}_R\to\mathbf{E}_S$ is smooth being of finite type by [11, IV.6.8.6]. Furthermore, it is also unramified due to the fact that the relative differentials $\Omega^1_{\mathbf{E}_S/\mathbf{E}_R}$ are zero by [11, 0.21.2.7] and [11, 0.21.2.5]. Hence, the extension $\mathbf{E}_R\subset\mathbf{E}_S$ is étale as claimed. We already know it is finite by 4.5(5). By 4.15(3) the extension $\widetilde{\mathbf{E}}_{R_\infty}\hookrightarrow\widetilde{\mathbf{E}}_{S_\infty}$ is finite and étale since $\mathbf{E}_R\hookrightarrow\mathbf{E}_S$ is finite and étale.

The ring $\widetilde{\mathbf{E}}_{S_\infty}$ is normal since it is a finite étale extension of $\widetilde{\mathbf{E}}_{R_\infty}$ and $\widetilde{\mathbf{E}}^+_{R_\infty}$ is normal due to A.6 and 4.15(2). Let $x = a/\bar{\pi}_0^\delta$ be an element of $\widetilde{\mathbf{E}}_{S_\infty}$ integral over $\widetilde{\mathbf{E}}^+_{S_\infty}$ and such that $a \in \widetilde{\mathbf{E}}^+_{S_\infty}$ and $\delta \in \mathbf{Q}_{\geqslant 0}$. Write $a = (\ldots, a^{(m)}, \ldots)$ as in 4.10. Let $\mathbf{z}$ be a valuation on $\widehat{S_\infty}$ associated to a prime over p. Let $\mathbf{z}'$ be the associated valuation on $\operatorname{Frac}(\widetilde{\mathbf{E}}^+_{S_\infty})$; see Section 4.16. Since x is integral over $\widetilde{\mathbf{E}}^+_{S_\infty}$ we have that $\mathbf{z}'(x) = \mathbf{z}(x^{(0)}) = \mathbf{z}(a^{(0)}) - \mathbf{z}(p^\delta) \geqslant 0$. Hence, $\mathbf{z}(a^{(m)} p^{\frac{-\delta}{p^m}}) \geqslant 0$ for every m and $\mathbf{z}$. Since $\widehat{S_\infty}$ is normal by 3.11, we conclude that $a^{(m)} p^{\frac{-\delta}{p^m}} \in \widehat{S_\infty}$ for every m. Hence, $x \in \widetilde{\mathbf{E}}^+_{S_\infty}$. This proves that $\widetilde{\mathbf{E}}^+_{S_\infty}$ is normal.

We know that $\mathbf{E}_S$ is normal since it is an étale extension of $\mathbf{E}_R$ which is regular by 4.7. Let $0 < \varepsilon' < \varepsilon$. Let x be an element of $\mathbf{E}_S$, which is integral over $\mathbf{E}_S^+$. Write it as $x = a/\bar{\pi}_0^\delta$ with $a \in \mathbf{E}_S^+$ and $\delta \in \mathbf{Q}_{\geqslant 0}$. Write $a = (\ldots, a^{(m)}, \ldots)$ as in 4.10. Let $\mathbf{z}$ be a valuation on $\widehat{S_\infty}$ associated to a prime over p. Let $\mathbf{z}'$ be the valuation on $\operatorname{Frac}(\widetilde{\mathbf{E}}^+_{S_\infty})$ defined by $\mathbf{z}'((x^{(0)}, x^{(1)}, \ldots)) := \mathbf{z}(x^{(0)})$; see Section 4.16. Since x is integral over $\widetilde{\mathbf{E}}^+_{S_\infty}$ we have that $\mathbf{z}'(x) \geqslant 0$ i.e., $\mathbf{z}(x^{(0)}) = \mathbf{z}(a^{(0)}) - \mathbf{z}(p^\delta) \geqslant 0$. Hence, $\mathbf{z}(a^{(m)} p^{\frac{-\delta}{p^m}}) \geqslant 0$ for every m and $\mathbf{z}$. Write $a^{(m)} = a_m + p^\varepsilon s$ with $a_m \in S_m$ and $s \in \widehat{S_\infty}$. Take m satisfying $\varepsilon' < \varepsilon - \frac{\delta}{p^m}$. Since $a^{(m)} p^{\frac{-\delta}{p^m}} = a_m p^{\frac{-\delta}{p^m}} + p^{\varepsilon - \frac{\delta}{p^m}} s$ and S_m is normal, we get that $a_m p^{\frac{-\delta}{m}}$ lies in S_m. Then, $x = (\ldots, a^{(m)} p^{\frac{-\delta}{m}}, \ldots) \in \lim_m (S_m/p^{\varepsilon'} S_m) = \mathbf{E}_S^+$ as wanted. This proves that $\mathbf{E}_S^+$ is normal. □

5. Refined almost étaleness

In this section we introduce a condition, denoted by (RAE) for refined almost étaleness, on the ramification of the extensions $\{R_n \subset S_n\}_n$. See 5.1 for equivalent formulations of (RAE) among which there is Faltings' condition (AE) on almost étaleness of the extension $R_\infty \subset S_\infty$. Assuming that (RAE) holds we prove that the generic degrees of $\mathbf{E}_R^+ \subset \mathbf{E}_S^+$ and $\widetilde{\mathbf{E}}^+_{R_\infty} \subset \widetilde{\mathbf{E}}^+_{S_\infty}$ are the same as the generic degree of $R \subset S$, see 5.3. We can also prove that $\widetilde{\mathbf{E}}^+_{S_\infty}$ is the $\bar{\pi}_0^\varepsilon$-adic completion of the perfection of $\mathbf{E}_S^+$ as in classical case ($R = V$), see 5.4. Eventually, we can establish the functoriality of our construction with respect to R; see 5.6. It is quite easy to show that (RAE) holds if R is of Krull dimension $\leqslant 2$; see 5.10. In more generality it follows from Faltings' purity theorem [7, Thm. 4]; see 5.11. The notation is as in 3.2. Consider the conditions

(AE) *For every $n \in \mathbf{N}$ the element $p^{\frac{1}{p^n}} \mathfrak{e}_\infty$ is in the image of $S_\infty \otimes_{R_\infty} S_\infty$.*

(RAE) *There exists ℓ in $\mathbf{N}$ so that $p^{\frac{\ell}{p^n}} \mathfrak{e}_n$ is in the image of $S_n \otimes_{R_n} S_n$ for all $n \in \mathbf{N}$.*

RAE stands for *refined almost étaleness*: this condition implies, thus refines, the notion of almost étaleness (AE) of the extension $R_\infty \subset S_\infty$. The latter is due to [6, Def. 2.1].

5.1. THEOREM. – *The following are equivalent*:

(a) (RAE) *holds for the tower* $\{R_n \subset S_n\}$;

(b) (AE) *holds for the extension* $R_\infty \subset S_\infty$;

(c) *for every $n \in \mathbf{N}$ the map $\widetilde{\mathbf{E}}^+_{S_\infty}/\bar{\pi}_0^{p^n \varepsilon} \widetilde{\mathbf{E}}^+_{S_\infty} \to S_\infty/p^\varepsilon S_\infty$, given by $(x_0, \ldots, x_n, \ldots) \mapsto x_n$, is an isomorphism*;

(d) *there exists $N \in \mathbf{N}$ such that for every $n \geqslant N$ the map $\mathbf{E}_S^+/\bar{\pi}_0^{p^n \varepsilon} \mathbf{E}_S^+ \to S_n/p^\varepsilon S_n$, given by $(x_0, \ldots, x_n, \ldots) \mapsto x_n$, is an isomorphism.*

5.2. LEMMA. – *Let A be a noetherian, regular domain. Let B be an A-algebra, finite and torsion free as A-module and normal as a ring. Let Q be a prime ideal of A of codimension* 1 *or* 2*. Then, $B \otimes_A A_Q$ is a free A_Q-module.*

Proof. – If Q is of codimension 1 the statement is clear. Assume that it has codimension 2. It follows from the Auslander–Buchsbaum formula, cf. [5, Thm. 19.9] that

$\mathrm{pd}(B\otimes_A A_Q)=\mathrm{dp}(Q,A_Q)-\mathrm{dp}(Q,B\otimes_A A_Q)$ where pd stands for projective dimension and dp is the depth. Since A is regular and B is normal, we have $\mathrm{dp}(Q,A_Q)=2=\mathrm{dp}(Q,B\otimes_A A_Q)$. Hence, $\mathrm{pd}(B\otimes_A A_Q)=0$ and the lemma follows. □

5.3. COROLLARY. – *Assume that* 5.1(d) *holds. Then, the degree of the extension* $\mathbf{E}_R\subset\mathbf{E}_S$ *is equal to the degree of the extension* $R[p^{-1}]\subset S[p^{-1}]$.

Proof. – Let Q be a prime ideal of $\mathbf{E}_R^+$ containing $\overline{\pi}_K$ of height $\leqslant 2$. Let $\{\mathfrak{Q}_m\}_{m\gg 0}$ be the tower of prime ideals of $\{R_m\}$ containing p defined by $\mathbf{E}_R^+/(\overline{\pi}_0^{p^m\varepsilon},Q)\mathbf{E}_R^+\cong R_m/\mathfrak{Q}_mR_m$. We deduce from 5.2 that $S_{n,\mathfrak{Q}_n}:=S_n\otimes_{R_n}R_{n,\mathfrak{Q}_n}$ is a free $R_{n,\mathfrak{Q}_n}$-module for every $n\in\mathbf{N}$ of degree equal to $[\mathrm{Frac}(S):\mathrm{Frac}(R)]$. By 4.7 the ring $\widehat{\mathbf{E}_{R,Q}^+}$ is local and regular. Since $\mathbf{E}_S^+$ is normal by 4.9, it follows from 5.2 that $\mathbf{E}_S^+\otimes_{\mathbf{E}_R^+}\widehat{\mathbf{E}_{R,Q}^+}$ is free as $\widehat{\mathbf{E}_{R,Q}^+}$-module of rank equal to $[\mathrm{Frac}(\mathbf{E}_S^+):\mathrm{Frac}(\mathbf{E}_R^+)]$. By 5.1(d) we have $\mathbf{E}_S^+\otimes_{\mathbf{E}_R^+}(\mathbf{E}_{R,Q}^+/\overline{\pi}_0^{p^n\varepsilon}\mathbf{E}_{R,Q}^+)\cong(S_{n,\mathfrak{Q}_n}/p^\varepsilon S_{n,\mathfrak{Q}_n})$ and $\mathbf{E}_{R,Q}^+/\overline{\pi}_0^{p^n\varepsilon}\mathbf{E}_{R,Q}^+\cong R_{n,\mathfrak{Q}_n}/p^\varepsilon R_{n,\mathfrak{Q}_n}$ for $n\gg 0$. The corollary follows. □

5.4. COROLLARY. – *The following hold*:

(i) *the ring* $\widetilde{\mathbf{E}}_{S_\infty}^+$ *contains the perfect closure of* $\mathbf{E}_S^+$;

(ii) *if* 5.1(d) *holds,* $\widetilde{\mathbf{E}}_{S_\infty}^+$ *coincides with the completion of the perfect closure of* $\mathbf{E}_S^+$ *with respect to the ideal* $(\overline{\pi}_0^\varepsilon)$.

Proof. – (i) Let $x=(x_0,x_1,\ldots)\in\mathbf{E}_S^+$. Let $N\in\mathbf{N}$. Then, the element $(x_N,x_{N+1},\ldots)$ of $\widetilde{\mathbf{E}}_{S_\infty}^+$ is a p^N-th root of x.

(ii) Let ε and N be as in 3.7. By 4.14 we have $\mathbf{E}_S^{+,\mathrm{perf}}=\bigcup_m\mathbf{E}_S^+(m)$. By 5.1(d) the map $\mathbf{E}_S^+(m)\to S_{m+N}/p^\varepsilon S_{N+m}$, given by $(y_0,\ldots,y_n,\ldots)\mapsto y_N$, is surjective. Since $S_\infty=\bigcup_n S_n$, we conclude that the natural map $\mathbf{E}_S^{+,\mathrm{perf}}\to\widetilde{\mathbf{E}}_{S_\infty}^+/\overline{\pi}_0^{p^N\varepsilon}\widetilde{\mathbf{E}}_{S_\infty}^+=S_\infty/p^\varepsilon S_\infty$ is surjective. The conclusion follows since $\widetilde{\mathbf{E}}_{S_\infty}^+$ is $\overline{\pi}_0^\varepsilon$-adically complete and separated by 4.4(3). □

5.5. Functoriality in R

Let R and T be rings satisfying the hypotheses of 2.2 (possibly for different d's and different V's). Denote by $\{T_n\}_n$, with $T_0=T$, a tower with the properties of 2.2. Assume that there exist $N\in\mathbf{N}$ and compatible morphisms $f_n:R_n\to T_n$ for $n\geqslant N$. It follows from the definition of $\mathbf{E}_-^+$, see 4.2, that they give rise to a ring homomorphism $f_R:\mathbf{E}_R^+\to\mathbf{E}_T^+$.

Let $R\subset S$ be a finite extension, étale after inverting p with S normal. Let U be the normalization of $S\otimes_R T$ and let U_n be the normalization of $U\otimes_T T_n$. Note that for ever $n\geqslant N$ we have compatible homomorphisms $S_n\to U_n$. We then obtain a morphism of $\mathbf{E}_R^+$-algebras $f_S:\mathbf{E}_S^+\to\mathbf{E}_U^+$. Then,

5.6. COROLLARY. – *Assume that the equivalent conditions of* 5.1 *hold for the tower* $R_n\subset S_n$ *with* $\ell=\ell(S)$. *The kernel and the cokernel of the map* $f_S\otimes 1:\mathbf{E}_S^+\otimes_{\mathbf{E}_R^+}\mathbf{E}_T^+\to\mathbf{E}_U^+$ *are annihilated by* π_0^ℓ. *In particular,* $f_S\otimes 1$ *is an isomorphism after inverting* $\overline{\pi}_K$.

Proof. – Denote by $\mathfrak{e}_n$ the idempotent associated to the extension $R_n[p^{-1}]\subset S_n[p^{-1}]$ and by $\mathfrak{e}_n'$ the idempotent associated to the extension $T_n[p^{-1}]\subset U_n[p^{-1}]$. Note that $U_n[p^{-1}]=S_n[p^{-1}]\otimes_{R_n}T_n$. Hence, $\mathfrak{e}_n'$ is the image of $\mathfrak{e}_n$. Then, $p^{\frac{\ell}{p^n}}\mathfrak{e}_n'$ is in the image of $S_n\otimes_{R_n}S_n\otimes_{R_n}T_n$. Thus, we have $p^{\frac{\ell}{p^n}}S_n\otimes_{R_n}T_n\to p^{\frac{\ell}{p^n}}U_n\to S_n\otimes_{R_n}T_n\to U_n$ by 3.1. Reducing modulo p^ε and taking inverse limits for $n\gg 0$ with respect to Frobenius, we deduce from 4.5(2) and 5.1(d) homomorphisms $\overline{\pi}_0^\ell\mathbf{E}_S^+\otimes_{\mathbf{E}_R^+}\mathbf{E}_T^+\to\overline{\pi}_0^\ell\mathbf{E}_U^+\to\mathbf{E}_S^+\otimes_{\mathbf{E}_R^+}\mathbf{E}_T^+\to\mathbf{E}_U^+$ of $\mathbf{E}_T^+$-modules.

Since $\mathbf{E}_R \subset \mathbf{E}_S$ and $\mathbf{E}_T \subset \mathbf{E}_U$ are finite and étale of the same degree by 5.3, the conclusion follows. $\square$

We now come to the proof of 5.1. We start with some lemmas.

5.7. LEMMA. – *Let $0 < \varepsilon < 1$ be as in 3.7 and let $0 \leqslant \delta < \varepsilon/p$ be such that there exists an element $p^\delta \in V_\infty$ of valuation δ. Assume that $p^\delta \mathfrak{e}_\infty$ lies in the image of $S_\infty \otimes_{R_\infty} S_\infty$. Then, the map $S_\infty/p^{\varepsilon - p\delta} S_\infty \to S_\infty/p^{\varepsilon - p\delta} S_\infty$, defined by raising to the p-th power, is surjective. In particular, the natural projection*

$$\widetilde{\mathbf{E}}^+_{S_\infty}/\overline{\pi}_0^{p^n(\varepsilon - p\delta)} \widetilde{\mathbf{E}}^+_{S_\infty} \to S_\infty/p^{\varepsilon - p\delta} S_\infty,$$

given by $(x_0, \dots, x_n, \dots) \mapsto x_n$, is an isomorphism.

Proof. – The injectivity in the displayed formula follows from 4.4. Clearly the first claim implies the second. Write $p^\delta \mathfrak{e}_\infty = \sum_i a_i \otimes b_i$ for suitable elements a_i and b_i of S_∞. Since $\mathfrak{e}_\infty$ is an idempotent, it follows that for $x \in S_\infty$ we have

$$p^{p\delta} x = (\mathrm{Tr} \otimes 1)\big(p^{p\delta} \mathfrak{e}_\infty^p (x \otimes 1)\big) = \left(\sum_i \mathrm{Tr}\big(a_i^p x\big) b_i^p + O(p)\right).$$

Here, $\mathrm{Tr} : S_\infty \to R_\infty$ is the trace map and $O(p)$ means up to an element in pS_∞. Since the map $R_\infty/p^\varepsilon R_\infty \to R_\infty/p^\varepsilon R_\infty$ defined by $z \mapsto z^p$ is surjective by 3.7, there exists $\alpha \in S_\infty$ such that $p^{p\delta} x = \alpha^p + O(p^\varepsilon)$. Since S_∞ is normal, $y := p^{-\delta}\alpha$ lies in S_∞. Therefore, $x = y^p + p^{\varepsilon - p\delta}\beta$ for some $\beta \in S_\infty$ as wanted. $\square$

5.8. LEMMA. – *Let $\mathbf{E}^+_R(n) \subset \mathbf{B}(n)$ be a finite extension such that $\mathbf{E}^+_R(n)[\overline{\pi}_K^{-1}] \to \mathbf{B}(n)[\overline{\pi}_K^{-1}]$ is finite and étale. Let $\overline{\mathfrak{e}}_n$ be the associated idempotent. Assume that there exists $\ell \in \mathbf{N}$ such that $\overline{\pi}_0^{\frac{\ell}{p^n}}$ annihilates the submodule of $\mathbf{B}(n) \otimes_{\mathbf{E}^+_R(n)} \mathbf{B}(n)$ consisting of $\overline{\pi}_K$-torsion elements and such that $\overline{\pi}_0^{\frac{\ell}{p^n}} \overline{\mathfrak{e}}_n$ lies in the image of $\mathbf{B}(n) \otimes_{\mathbf{E}^+_R(n)} \mathbf{B}(n)$. Let $R_n \subset B_n$ be a finite extension such that $R_n[p^{-1}] \to B_n[p^{-1}]$ is finite and étale. Let $\mathfrak{e}_n$ be the idempotent associated as in* 3.1. *Assume that there exist $0 < \frac{4\ell}{p^n} < \varepsilon < 1$ and an isomorphism $j : \mathbf{B}(n)/\overline{\pi}_0^\varepsilon \mathbf{B}(n) \xrightarrow{\sim} B_n/p^\varepsilon B_n$ as algebras over $\mathbf{E}^+_R(n)/\overline{\pi}_0^\varepsilon \mathbf{E}^+_R(n) = R_n/p^\varepsilon R_n$ (by* 4.5(1)*). Then, $p^{\frac{2\ell}{p^n}} \mathfrak{e}_n$ lies in the image of $B_n \otimes_{R_n} B_n$.*

Proof. – Write $\tau_n := p^{\frac{2\ell}{p^n}}$. Consider the polynomial $f(X) := X^2 - \tau_n X$. Then, by assumption $\tau_n \overline{\mathfrak{e}}_n$ is a well defined element of

$$\big(\mathbf{B}(n)/\overline{\pi}_0^\varepsilon \mathbf{B}(n)\big) \otimes_{\mathbf{E}^+_R(n)} \big(\mathbf{B}(n)/\overline{\pi}_0^\varepsilon \mathbf{B}(n)\big) \xrightarrow{j \otimes j} \big(B_n/p^\varepsilon B_n\big) \otimes_{R_n} \big(B_n/p^\varepsilon B_n\big)$$

satisfying $f(\tau_n \overline{\mathfrak{e}}_n) = 0$. Take $x_0 \in B_n \otimes_{R_n} B_n$ to be any lift of $\tau_n \overline{\mathfrak{e}}_n$ modulo p^ε. Then, $f'(x_0) = 2x_0 - \tau_n$ and $f'(x_0) \equiv \tau_n(2\overline{\mathfrak{e}}_n - 1)$ modulo p^ε. Since $(2\overline{\mathfrak{e}}_n - 1)^2 = 1$, we conclude that $f'(x_0) = \tau_n u$ where u is a unit in $B_n \otimes_{R_n} B_n$. Note that the latter ring is p-adically complete and separated since it is finite as R_n-module. By Hensel's lemma [5, Thm. 7.3], there exists a root $x \in B_n \otimes_{R_n} B_n$ of $f(X)$ congruent to x_0, and hence equal to $\tau_n \overline{\mathfrak{e}}_n$, modulo $p^{\varepsilon - \frac{2\ell}{p^n}}$.

In what follows we consider $B_n \otimes_{R_n} B_n$ as a left B_n-module. Let $m_n : B_n \otimes_{R_n} B_n \to B_n$ be the multiplication map. Since the image of $\tau_n \overline{\mathfrak{e}}_n$ via the multiplication on $\mathbf{B}(n)$ is τ_n, we have $m_n(x) \equiv \tau_n$ modulo $p^{\varepsilon - \frac{2\ell}{p^n}}$. Hence, $m_n(x) = \tau_n v$ with v a unit satisfying $v^2 = v$ and $v \equiv 1$ modulo $p^{\varepsilon - \frac{4\ell}{p^n}}$. By Hensel's lemma $v = 1$, i.e. $m_n(x) = \tau_n$. It follows from the assumptions that the image of multiplication by $\tau_n \overline{e}_n$ on $\mathbf{B}(n) \otimes_{\mathbf{E}^+_R(n)} \mathbf{B}(n)$ is $\mathbf{B}(n) \cdot \tau_n \overline{\mathfrak{e}}_n$. Thus,

$$x(B_n \otimes_{R_n} B_n) \subset B_n x + p^{\varepsilon - \frac{2\ell}{p^n}} (B_n \otimes_{R_n} B_n).$$

Since $(x/\tau_n)^M = (x/\tau_n)$ for every $M \in \mathbf{N}$ we have

$$\left(\frac{x}{\tau_n}\right)(B_n \otimes_{R_n} B_n) = \left(\frac{x}{\tau_n}\right)^M (B_n \otimes_{R_n} B_n) \subset B_n \left(\frac{x}{\tau_n}\right) + p^{M(\varepsilon - \frac{4\ell}{p^n})} (B_n \otimes_{R_n} B_n)$$

(as submodules of $B_n \otimes_{R_n} B_n[p^{-1}]$). Since $B_n \otimes_{R_n} B_n$ is p-adically separated, we conclude that $\frac{x}{\tau_n}(B_n \otimes_{R_n} B_n) = B_n \frac{x}{\tau_n}$. Hence, $\frac{x}{\tau_n} = \mathfrak{e}_n$ as claimed. □

5.9. Proof of Theorem 5.1

(a) ⇒ (b) is clear.

(b) ⇒ (c): the injectivity follows from 4.4(2). To prove the surjectivity we may shrink ε. Then, the implication follows from 5.7.

(c) ⇒ (d): the fact that there exists $N \in \mathbf{N}$ such that the map in (d) is well defined and injective for $n \geqslant N$ follows from 4.5(1). We claim that $p^{\frac{c(S)}{p^n}} S_n / p^\varepsilon S_n$ is in the image of $\mathbf{E}^+_S$ for $n \geqslant N$. It suffices to prove this after base change via the faithfully flat morphism $\mathbf{E}^+_R \subset \widetilde{\mathbf{E}}^+_{R_\infty}$ (see 4.15(2)). By 4.15(3) and 3.10 we have

$$\overline{\pi}_0^{c(S)} \widetilde{\mathbf{E}}^+_{S_\infty} \subset \mathbf{E}^+_S \otimes_{\mathbf{E}^+_R} \widetilde{\mathbf{E}}^+_{R_\infty} \subset \widetilde{\mathbf{E}}^+_{S_\infty}, \qquad p^{\frac{c(S)}{p^n}} S_n \otimes_{R_n} R_\infty \subset p^{\frac{c(S)}{p^n}} S_\infty \subset S_n \otimes_{R_n} R_\infty.$$

Since $\widetilde{\mathbf{E}}^+_{R_\infty} / \overline{\pi}_0^{p^n \varepsilon} \widetilde{\mathbf{E}}^+_{R_\infty} \cong R_\infty / p^\varepsilon R_\infty$ by 4.4(2), (c) implies that $(p^{\frac{c(S)}{p^n}} S_n / p^\varepsilon S_n) \otimes_{\mathbf{E}^+_R} \widetilde{\mathbf{E}}^+_{R_\infty}$ i.e., $(p^{\frac{c(S)}{p^n}} S_n / p^\varepsilon S_n) \otimes_{R_n} R_\infty$, is in the image of $\mathbf{E}^+_S \otimes_{\mathbf{E}^+_R} \widetilde{\mathbf{E}}^+_{R_\infty}$ as claimed. If $x_n \in p^{\frac{c(S)}{p^n}} S_n / p^\varepsilon S_n$ is the image of $x = (x_0, \ldots, x_m, \ldots) \in \mathbf{E}^+_S$, from the normality of S_m we get that $x_m \in p^{\frac{c(S)}{p^m}} S_m / p^\varepsilon S_m$ for every $m \geqslant n$. Hence, $y = x / \overline{\pi}_0^{c(S)}$ lies in $\mathbf{E}^+_S$. We conclude that $S_n / p^{\varepsilon - \frac{c(S)}{p^n}} S_n$ is in the image of $\mathbf{E}^+_S$. This implies that also $\mathbf{E}^+_S \to S_n / p^\varepsilon S_n$ is surjective.

(d) ⇒ (a): by 4.9 the extension $\mathbf{E}_R(m) \subset \mathbf{E}_S(m)$ is finite and étale. Let $\overline{\mathfrak{e}}_m$ be the associated idempotent as in 3.1. Then there exists $\ell \in \mathbf{N}$ such that $\overline{\pi}_0^\ell \overline{\mathfrak{e}}_0$ lies in the image of $\mathbf{E}^+_S \otimes_{\mathbf{E}^+_R} \mathbf{E}^+_S$ and $\overline{\pi}_0^\ell$ annihilates the submodule of $\overline{\pi}_K$-torsion elements of $\mathbf{E}^+_S \otimes_{\mathbf{E}^+_R} \mathbf{E}^+_S$. We deduce from 4.14 that $\overline{\pi}_0^{\frac{\ell}{p^m}} \overline{\mathfrak{e}}_m$ is in the image of $\mathbf{E}^+_S(m) \otimes_{\mathbf{E}^+_R(m)} \mathbf{E}^+_S(m)$ and $\overline{\pi}_0^{\frac{\ell}{p^m}}$ kills the $\overline{\pi}_K$-torsion of the latter for every $m \in \mathbf{N}$. The implication follows then from 5.8 with $B_n = S_n$ and $\mathbf{B}(n) = \mathbf{E}^+_S(n)$.

5.10. PROPOSITION. – *If R is of Krull dimension $\leqslant 2$, then condition* (RAE) *holds with $\ell = c(S)$.*

Proof. – By 3.10, $p^{\frac{c(S)}{p^n}}$ kills the cokernel of the trace map $\mathrm{Tr}_n : S_n \to \mathrm{Hom}_{R_n}(S_n, R_n)$, defined in 3.2. Since S_n is projective as R_n-module by 5.2, we get that in

$$S_n \otimes_{R_n} S_n \xrightarrow{\mathrm{Tr}_n \otimes \mathrm{Id}} \mathrm{Hom}_{R_n}(S_n, R_n) \otimes_{R_n} S_n \to \mathrm{Hom}_{S_n}(S_n \otimes_{R_n} S_n, S_n)$$

the right hand map is an isomorphism. We conclude that the cokernel of the composite is killed by $p^{\frac{c(S)}{p^n}}$. Hence, reasoning as in 3.1, it follows that $p^{\frac{c(S)}{p^n}} \mathfrak{e}_n$ lies in $S_n \otimes_{R_n} S_n$. □

5.11. THEOREM (Faltings' almost purity theorem). – *Assume that the composite of the extensions* $V[T_1^{\pm 1},\dots,T_d^{\pm 1}] \to R^0 \to R$ *is flat and has geometrically regular fibers and that we are in the cyclotomic case, see* 2.4. *Then, condition* (AE) *holds.*

Proof. – Due to 2.3 the extension $k[T_1^{\pm 1},\dots,T_d^{\pm 1}] \to R \otimes_V k$ is formally étale. Hence, R satisfies the assumptions of [7, Thm. 4]. Let $\widetilde{R} := \bigcup_{m\in\mathbf{N}} R[T_1^{\frac{1}{m}},\dots,T_1^{\frac{1}{m}}]$. It is naturally a R_∞-algebra. Define $\widetilde{S} := S_\infty \otimes_{R_\infty} \widetilde{R}$. Due to [7, Thm. 4] the extension $\widetilde{R} \subset \widetilde{S}$ is almost étale. Since $\widetilde{R}$ is the union of finite and étale extensions of R_∞, the conclusion follows. □

The following proposition provides a good source of examples when 5.11 applies. Let B^0 be an excellent, noetherian, flat $V[T_1^{\pm 1},\dots,T_d^{\pm 1}]$-algebra such that $V[T_1^{\pm 1},\dots,T_d^{\pm 1}] \to B^0$ has geometrically regular fibers and the natural map $(B \otimes_V k) \otimes_{R^0} (R^0 \otimes_V k)^{(p)} \to (B \otimes_V k)^{(p)}$, induced by Frobenius, is an isomorphism. For example, one may take $B^0 = V[T_1^{\pm 1},\dots,T_d^{\pm 1}]$.

5.12. PROPOSITION. – *Suppose that* A *is the localization* $S^{-1}B$ *with respect to a multiplicative system* S *of an étale extension* B *of* B^0. *Let* $J \subset A$ *be an ideal containing* p *and let* R *be the* J*-adic completion of* A. *Assume that* $1 \neq 0$ *in* R. *Then,*

(i) R *is noetherian and* p*-adically complete and separated*;
(ii) R *is flat as* $V[T_1^{\pm 1},\dots,T_d^{\pm 1}]$*-algebra*;
(iii) $V[T_1^{\pm 1},\dots,T_d^{\pm 1}] \to R$ *has geometrically regular fibers*;
(iv) *the natural map* $(R \otimes_V k) \otimes_{R^0} (R^0 \otimes_V k)^{(p)} \to (R \otimes_V k)^{(p)}$ *is an isomorphism. Thus,* $\{T_1,\dots,T_d\}$ *is an absolute* p*-basis for* $R \otimes_V k$.

In particular, the theory developed so far applies if $k \subset R \otimes_V k$ *is geometrically integral.*

Proof. – (i) and (ii) are clear. (iii) Since B^0 is excellent, also A is excellent by [11, 7.8.3(ii)]. The claim follows from [11, 6.6.1(i)] since both $V[T_1^{\pm 1},\dots,T_d^{\pm 1}] \to A$ and $A \to R$ have geometric regular fibers (for the second map use [11, 7.8.3(v)]).

(iv) Since $\varphi \otimes 1 : (B \otimes_V k) \otimes_{B^0} (B^0 \otimes_V k)^{(p)} \to (B \otimes_V k)^{(p)}$ is an isomorphism, the same applies to the localization with respect to S and to the reduction modulo J^n for every n. Note that $(R^0 \otimes_V k)^{(p)}$ is finite and free as $R^0 \otimes_V k$-module. Hence, the J-adic completion of $(A \otimes_V k) \otimes_{R^0} (R^0 \otimes_V k)^{(p)}$ coincides with $(R \otimes_V k) \otimes_{R^0} (R^0 \otimes_V k)^{(p)}$. On the other hand, the J-adic completion of $(A \otimes_V k)^{(p)}$ is $(R \otimes_V k)^{(p)}$. The claim follows. □

6. The inverse of the functor of generalized ring of norms

Assuming that the equivalent conditions of 5.1 hold, we prove in Theorem 6.3 and its Corollary 6.4 that the formation of the generalized ring of norms, see 6.2, defines an equivalence of categories between the category of normal extensions $R_\infty \subset S_\infty$, finite and étale after inverting p, and the category of normal extensions $\mathbf{E}_R^+ \subset \mathbf{B}$, finite and étale after inverting $\overline{\pi}_K$. The essential tool is the construction of an inverse to the functor of the generalized ring of norms, see 6.2, based on Appendix B.

6.1. DEFINITION. – Consider

R_∞**-AE**: the category of normal R_∞-algebras S_∞ such that $S_\infty[\frac{1}{p}]$ is finite and étale as $R_\infty[\frac{1}{p}]$-algebra;

$\mathbf{E}_R^+$**-AE**: the category of normal $\mathbf{E}_R^+$-algebras $\mathbf{B}$ such that $\mathbf{B}[\frac{1}{\overline{\pi}_K}]$ is finite and étale as $\mathbf{E}_R^+[\frac{1}{\overline{\pi}_K}]$-algebra.

6.2. DEFINITION. – Define the functor

$$\mathbf{E}^+ : R_\infty\textbf{-AE} \to \mathbf{E}_R^+\textbf{-AE}$$

as follows. Let $R_\infty \subset S_\infty$ be an object of R_∞-**AE**. Let N be any integer for which there exists a normal R_N-subalgebra S_N of S_∞ such that $S_\infty[p^{-1}] = S_N \otimes_{R_N} R_\infty[p^{-1}]$. For every $n \geqslant N$, let S_n be the normalization of $S_N \otimes_{R_N} R_n$. Define $\mathbf{E}_S^+$ to be the $\mathbf{E}_R^+$-subalgebra of $\widetilde{\mathbf{E}}_{S_\infty}^+$ consisting of elements $(a_0,\dots,a_n,\dots)$ such that $a_n \in S_n/p^\varepsilon S_n$ for $n \geqslant N$. It is clear how to define $\mathbf{E}^+$ on morphisms.

6.3. THEOREM. – (I) *The functor* $\mathbf{E}_-^+$ *is well defined.*

(II) *Assume that the equivalent conditions of* 5.1 *hold for every object of* R_∞-**AE**. *Then, the functor* $\mathbf{E}_-^+$ *is an equivalence of categories. Furthermore, if* S_∞ *and* $\mathbf{E}_S^+$ *are two corresponding objects, then*

(1) *the generic degrees of* $R_\infty \subset S_\infty$ *and* $\mathbf{E}_R^+ \subset \mathbf{E}_S^+$ *are the same*;

(2) *the sets of idempotents of* S_∞ *and* $\mathbf{E}_S^+$ *are in natural one-to-one correspondence*;

(3) $R_\infty[p^{-1}] \subset S_\infty[p^{-1}]$ *is Galois if and only if* $\mathbf{E}_R \subset \mathbf{E}_S$ *is Galois and the two associated Galois groups are naturally identified.*

Proof. – (I) Let $R_\infty \subset S_\infty$ be an object of R_∞-**AE**. Let S_N be as in 6.2. Since $R_N \subset R_\infty$ is faithfully flat, $R_N[p^{-1}] \to S_N[p^{-1}]$ is finite and étale. By 4.9 it is normal and $\mathbf{E}_R^+ \subset \mathbf{E}_S^+$ is finite and étale after inverting $\overline{\pi}_K$. In particular, $\mathbf{E}_S^+$ is an object of $\mathbf{E}_R^+$-**AE** as claimed.

(II) We suppose for the moment that $\mathbf{E}_-^+$ is an equivalence of categories and we prove the other claims. Claim (1) follows from 5.3. Claim (2) follows from 4.5(4). Assume that $R_\infty[p^{-1}] \subset S_\infty[p^{-1}]$ is Galois with group H. Then, H acts on $\widehat{S_\infty}$ and by 3.11 the invariants $\widehat{S_\infty}^H$ coincide with $\widehat{R_\infty}$. The group H also acts on $\widetilde{\mathbf{E}}_{S_\infty}^+$ by 4.4 and, using 4.10, the invariants coincide with $\widetilde{\mathbf{E}}_{R_\infty}^+$. Eventually, H acts on $\mathbf{E}_S^+$ by 4.5(3). It follows from 4.15(2) and (3) that the invariants are given by $\mathbf{E}_R^+$. This proves Claim (3).

To prove that $\mathbf{E}_-^+$ defines an equivalence of categories we construct an inverse. This is one of the goals of the rest of this section; see 6.10. □

6.4. COROLLARY (cf. [10, Thm. A.3.1.6]). – *Under the assumptions of* 6.3 *we have an isomorphism of topological groups*

$$\mathrm{Gal}\big(R^{\mathrm{sep}}[p^{-1}]/R_\infty[p^{-1}]\big) \xrightarrow{\sim} \mathrm{Gal}\big(\mathbf{E}_R^{\mathrm{sep}}/\mathbf{E}_R\big),$$

where R^{sep} *is as in* 2.6 *and* $\mathbf{E}_R^{\mathrm{sep}}$ *is the union of the maximal chain of finite and étale extensions of* $\mathbf{E}_R$ *arising from the maximal chain of finite and étale extensions of* $R_\infty[p^{-1}]$ *chosen in* 2.6.

6.5. Base points

Assume we have a discrete valuation ring T_M, for some $M \in \mathbf{N}$, satisfying the hypotheses of 2.2 (possibly for different d's and different V's from those for R). Let $\{T_n\}_{n\geqslant M}$ be a tower with the properties of 2.2. Denote $T_\infty := \bigcup_n T_n$. Assume that we have compatible morphisms $f_n : R_n \to T_n$ for $n \geqslant M$. For example, one can take T_n to be the localization $R_{n,\mathfrak{P}_n}$ of R_n at the unique prime ideal $\mathfrak{P}_n$ over p.

As in 2.6 define T^{sep} as the direct limit of a maximal chain of normal domains which are T_∞-algebras, finite and étale over $T_\infty[p^{-1}]$. Then, $\eta_T : \mathrm{Spec}(T^{\mathrm{sep}}[p^{-1}]) \to \mathrm{Spec}(R_\infty[p^{-1}])$ is a geometric point.

On the other hand, define $\mathbf{E}_T^+ := \lim_{m\geqslant M} T_m/p^\varepsilon T_m$ as the subring of $\widetilde{\mathbf{E}}_{T_\infty}^+ = \lim T_\infty/p^\varepsilon T_\infty$ consisting of elements $(a_0,\dots,a_n,\dots)$ such that $a_n \in T_n/p^\varepsilon T_n$ for $n \gg 0$. Then, $\mathbf{E}_T^+$ is a complete discrete valuation ring and, applying the functor $\mathbf{E}_-^+$ of 6.2 to the chain of T_∞-algebras involved in the definition of T^{sep}, we obtain by 6.4 a separably closed extension $\mathbf{E}_T^{\mathrm{sep}}$ of the field $\mathbf{E}_T$. Let $\nu_T : \mathrm{Spec}(\mathbf{E}_T^{\mathrm{sep}}) \to \mathrm{Spec}(\mathbf{E}_R)$.

6.6. COROLLARY. – *Under the assumptions of* 6.3 *the functor* $\mathbf{E}^+_-$ *provides an equivalence of Galois categories between the category of finite and étale covers of* $\operatorname{Spec}(R_\infty[p^{-1}])$ *with base point* η_T *and the category of finite and étale covers of* $\operatorname{Spec}(\mathbf{E}_R)$ *with base point* ν_T. *In particular, we have a topological isomorphism of fundamental groups* $\pi_1(\operatorname{Spec}(R_\infty[p^{-1}]),\eta_T)\cong\pi_1(\operatorname{Spec}(\mathbf{E}_R),\nu_T)$.

Proof. – In the following if A is a ring and B and C are A-algebras we write $\operatorname{Hom}_A(B,C)$ for the set of homomorphisms as A-algebras. Let S_∞ be an object of R_∞**-AE**. Denote by U_∞ the normalization of T_∞ in $S_\infty\otimes_{R_\infty}T_\infty[p^{-1}]$. Then,

$$\operatorname{Hom}_{R_\infty}\big(S_\infty[p^{-1}],T^{\mathrm{sep}}[p^{-1}]\big)=\operatorname{Hom}_{R_\infty}\big(S_\infty,T^{\mathrm{sep}}\big)=\operatorname{Hom}_{T_\infty}\big(U_\infty,T^{\mathrm{sep}}\big).$$

The latter is identified with $\operatorname{Hom}_{\mathbf{E}_T}(\mathbf{E}_U,\mathbf{E}_T^{\mathrm{sep}})$ by 6.3. By 5.6, we have $\mathbf{E}_S\otimes_{\mathbf{E}_R}\mathbf{E}_T=\mathbf{E}_U$. Hence, $\operatorname{Hom}_{\mathbf{E}_T}(\mathbf{E}_U,\mathbf{E}_T^{\mathrm{sep}})=\operatorname{Hom}_{\mathbf{E}_R}(\mathbf{E}_S,\mathbf{E}_T^{\mathrm{sep}})$. Thus, we get the identification

$$\operatorname{Hom}_{R_\infty}\big(S_\infty[p^{-1}],T^{\mathrm{sep}}[p^{-1}]\big)=\operatorname{Hom}_{\mathbf{E}_R}\big(\mathbf{E}_S,\mathbf{E}_T^{\mathrm{sep}}\big)$$

as sets with actions of $\operatorname{Aut}(S_\infty/R_\infty)$, functorially in S_∞. The corollary follows. □

6.7. DEFINITION. – Let $\mathbf{B}$ be an object of $\mathbf{E}_R^+$**-AE**. Consider the direct system $\{\mathbf{B}(n)\}_{n\in\mathbf{N}}$ (resp. $\{\mathbf{E}_R^+(n)\}_{n\in\mathbf{N}}$), where for every $n\in\mathbf{N}$ we define $\mathbf{B}(n):=\mathbf{B}$ (resp. $\mathbf{E}_R^+(n):=\mathbf{E}_R^+$) and the transition map $\mathbf{B}(n)\to\mathbf{B}(n+1)$ (resp. $\mathbf{E}_R^+(n)\to\mathbf{E}_R^+(n+1)$) is Frobenius. Define the direct limits

$$\mathbf{B}^{\mathrm{perf}}:=\lim_{n\to\infty}\mathbf{B}(n)\quad(\text{resp. } \mathbf{E}_R^{+,\mathrm{perf}}:=\lim_{n\to\infty}\mathbf{E}_R^+(n)).$$

For every $n\in\mathbf{N}$, let $\bar{\mathfrak{e}}_n\in(\mathbf{B}(n)\otimes_{\mathbf{E}_R^+(n)}\mathbf{B}(n))[\bar\pi_K^{-1}]$ be the idempotent defined as in 3.1. Let $\ell\in\mathbf{N}$ be such that $\bar\pi_0^{\frac{\ell}{p^n}}\bar{\mathfrak{e}}_n$ is in the image of $\mathbf{B}(n)\otimes_{\mathbf{E}_R^+(n)}\mathbf{B}(n)$ and $\bar\pi_0^{\frac{\ell}{p^n}}$ annihilates the submodule of $\mathbf{B}(n)\otimes_{\mathbf{E}_R^+(n)}\mathbf{B}(n)$ of $\bar\pi_K$-torsion elements.

6.8. *Remark*. – Due to 4.9 if $R\subset S$ is as in 2.5, then the extension $\mathbf{E}_R^+\subset\mathbf{E}_S^+$ is an object of $\mathbf{E}_R^+$**-AE**. In 4.2 we have already introduced a ring denoted by $\mathbf{E}_S^+(n)$. By 4.14 the two notations $\mathbf{E}_S^+(n)$ given in 4.2 and in 6.7 agree.

Assume that $\bar\pi_0^\ell\bar{\mathfrak{e}}_0$ is in the image of $\mathbf{B}(0)\otimes_{\mathbf{E}_R^+(0)}\mathbf{B}(0)$, then $(\bar\pi_0^\ell\bar{\mathfrak{e}}_0)^{\frac{1}{p^n}}=\bar\pi_0^{\frac{\ell}{p^n}}\bar{\mathfrak{e}}_n$ lies in the image of $\mathbf{B}(n)\otimes_{\mathbf{E}_R^+(n)}\mathbf{B}(n)$. Analogously, if $\bar\pi_0^\ell$ annihilates the $\bar\pi_K$-torsion of $\mathbf{B}(0)\otimes_{\mathbf{E}_R^+(0)}\mathbf{B}(0)$, then $\bar\pi_0^{\frac{\ell}{p^n}}$ kills the $\bar\pi_K$-torsion of $\mathbf{B}(n)\otimes_{\mathbf{E}_R^+(n)}\mathbf{B}(n)$. Hence, the positive integer ℓ, whose existence is claimed in 6.7, does indeed exist.

6.9. THEOREM. – *For every object* $\mathbf{B}$ *of* $\mathbf{E}_R^+$**-AE** *there exists* $N=N(\mathbf{B})\in\mathbf{N}$ *and for every* $n\geqslant N$ *there is a* R_n*-algebra* B_n *with the following properties*:

(1) B_n *is normal as a ring, finite as* R_n*-module and* $R_n\subset B_n$ *is étale after inverting* p *of degree equal to the generic degree of* $\mathbf{E}_R^+\subset\mathbf{B}$;

(2) *for every* $m\geqslant n\geqslant N$ *the* R_m*-algebra* B_m *is the normalization of* $B_n\otimes_{R_n}R_m$ *in its total field of fractions*;

(3) *for every* $n\geqslant N$ *we have an isomorphism* $\beta_{\mathbf{B}(n)}:\mathbf{E}_{B_n}^+\xrightarrow{\sim}\mathbf{B}(n)$ *as* $\mathbf{E}_R^+(n)$*-algebras such that for* $m\geqslant n$ *the homomorphism* $\beta_{\mathbf{B}(m)}\circ r_{m,n}\circ\beta_{\mathbf{B}(n)}^{-1}:\mathbf{B}(n)\to\mathbf{B}(m)$ *is the natural inclusion defined in* 6.7. *Here,* $r_{m,n}:\mathbf{E}_{B_n}^+\to\mathbf{E}_{B_m}^+$ *is induced by the inclusion* $B_n\subset B_m$ *obtained from* (2) *and the functoriality of* $\mathbf{E}^+_-$;

(4) *let* $j:\mathbf{B}\to\mathbf{B}'$ *be a homomorphism of objects of* $\mathbf{E}_R^+$**-AE**. *There exists* $N(\mathbf{B},\mathbf{B}')\in\mathbf{N}$ *such that* $N(\mathbf{B},\mathbf{B}')\geqslant N(\mathbf{B})$, $N(\mathbf{B}')$ *and for every* $n\geqslant N(\mathbf{B},\mathbf{B}')$ *there is a unique homomorphism of* R_n*-algebras*

$$v_n:B_n\to B_n'$$

such that the induced homomorphism $\beta_{\mathbf{B}'(n)}\circ r_n\circ\beta_{\mathbf{B}(n)}^{-1}:\mathbf{B}(n)\to\mathbf{B}'(n)$ *is the one defined by* j. *Here,* β_{-} *are the isomorphisms defined in* (3) *and* $r_n:\mathbf{E}_{B_n}^+\to\mathbf{E}_{B_n'}^+$ *is the homomorphism associated to* v_n *applying the functor* $\mathbf{E}_{-}^+$. *Furthermore, for* $m\geqslant n$ *the map* v_m *is induced by* v_n *via* (2);

(5) *if* $\mathbf{B}=\mathbf{E}_S^+$ *with* $R\subset S$ *as in* 2.5 *and if the equivalent conditions of* 5.1 *hold, then* $B_n=S_n$.

6.10. COROLLARY. – *Consider the map*

$$\mathbf{E}_R^+\text{-}\mathbf{AE}\to R_\infty\text{-}\mathbf{AE}$$

associating to $\mathbf{E}_R^+\subset\mathbf{B}$ *the extension* $R_\infty\subset B_\infty:=\bigcup_n B_n$ *defined in* 6.9. *It is a well defined functor and the functor* $\mathbf{E}_{-}^+$*, defined in* 6.2*, provides a left inverse. Under the assumptions of* 6.3*, it is the inverse of* $\mathbf{E}_{-}^+$.

Proof of Corollary 6.10. – Due to (1) and (2) of 6.9 the R_∞-algebra B_∞ is an object of R_∞**-AE**. By 6.9(4) the formation of B_∞ is functorial. The other claims follow from (3) and (5) of 6.9.

Proof of Theorem 6.9. – By B.3 there is $N\in\mathbf{N}$ and for every $n\geqslant N$ there exist a R_n-algebra B_n and an isomorphism $\overline{\beta}_n:B_n/p^\varepsilon B_n\xrightarrow{\sim}\mathbf{B}(n)/\overline{\pi}_0^\varepsilon\mathbf{B}(n)$. Furthermore, B_n has no non-trivial p-torsion and it is finite as R_n-module. Then,

6.9.1. LEMMA. – *The extension* $R_n[p^{-1}]\to B_n[p^{-1}]$ *is finite and étale of degree equal to the generic degree of* $\mathbf{E}_R^+\subset\mathbf{B}$.

Proof. – The finiteness is clear. We prove that $\Omega^1_{B_n/R_n}$ is killed by p. Note that

$$\Omega^1_{(B_n/p^\varepsilon B_n)/R_n}=\Omega^1_{B_n/R_n}/p^\varepsilon\Omega^1_{B_n/R_n}$$

and

$$\Omega^1_{(\mathbf{B}(n)/\overline{\pi}_0^\varepsilon\mathbf{B}(n))/\mathbf{E}_R^+(n)}=\Omega^1_{\mathbf{B}(n)/\mathbf{E}_R^+(n)}/\overline{\pi}_0^\varepsilon\Omega^1_{\mathbf{B}(n)/\mathbf{E}_R^+(n)}.$$

From the isomorphism of R_n-algebras $\overline{\beta}_n:B_n/p^\varepsilon B_n\to\mathbf{B}(n)/\overline{\pi}_0^\varepsilon\mathbf{B}(n)$, we get that $\Omega^1_{B_n/R_n}/p^\varepsilon\Omega^1_{B_n/R_n}=\Omega^1_{\mathbf{B}(n)/\mathbf{E}_R^+(n)}/\overline{\pi}_0^\varepsilon\Omega^1_{\mathbf{B}(n)/\mathbf{E}_R^+(n)}$. The latter is killed by $\overline{\pi}_0^{\frac{\ell}{p^n}}$; see 3.1. Hence, $p^{\frac{\ell}{p^n}}\Omega^1_{B_n/R_n}=p^\varepsilon\Omega^1_{B_n/R_n}$ which is then equal to $p^m\Omega^1_{B_n/R_n}$ for every $m\in\mathbf{N}$. Thus, $p^{\frac{\ell}{p^n}}$ annihilates $\Omega^1_{B_n/R_n}/p^m\Omega^1_{B_n/R_n}$ for every $m\in\mathbf{N}$. But $\Omega^1_{B_n/R_n}$ is finite as R_n-module and, consequently, it is p-adically complete and separated and coincides with the inverse limit $\lim_m(\Omega^1_{B_n/R_n}/p^m\Omega^1_{B_n/R_n})$. We conclude that $p^{\frac{\ell}{p^n}}$ annihilates $\Omega^1_{B_n/R_n}$.

We prove that $R_n\to B_n$ is flat after inverting p. Let M be a finite R_n-module. Then, $\mathrm{Tor}^1(M,B_n/p^\varepsilon B_n)=\mathrm{Tor}^1(M,\mathbf{B}(n)/\overline{\pi}_0^\varepsilon\mathbf{B}(n))$ and the latter is killed by $\overline{\pi}_0^{\frac{\ell}{p^n}}$; see 3.1. Using the exact sequence

$$0\to B_n\xrightarrow{p^\varepsilon}B_n\to B_n/p^\varepsilon B_n\to 0,$$

we get that $p^{\frac{\ell}{p^n}}\mathrm{Tor}^1(M,B_n)=p^\varepsilon\mathrm{Tor}^1(M,B_n)=p^m\mathrm{Tor}^1(M,B_n)$ for every $m\in\mathbf{N}$. But $\mathrm{Tor}^1(M,B_n)$ is finite as R_n-module and, hence, it is p-adically complete and separated. Thus, $\mathrm{Tor}^1(M,B_n)$ is annihilated by $p^{\frac{\ell}{p^n}}$.

Denote by $\mathfrak{P}_n$ the unique prime ideal of R_n over p. Since B_n has no non-trivial p-torsion, $B_n\otimes_{R_n}R_{n,\mathfrak{P}_n}$ is a free $R_{n,\mathfrak{P}_n}$-module of rank equal to the generic degree d of the extension $R_n\subset B_n$. By 4.4(2) and 2.3(2) there is a unique prime ideal P_n of $\mathbf{E}_R^+(n)$ over $\bar\pi_K$. Since $\mathbf{B}(n)$ is normal by assumption, the generic degree f of $\mathbf{E}_R^+(n)\to\mathbf{B}(n)$ is equal to the rank of the free $\mathbf{E}_R^+(n)_{P_n}$-module $\mathbf{B}(n)\otimes_{\mathbf{E}_R^+(n)}\mathbf{E}_R^+(n)_{P_n}$. Note that f coincides also with the generic degree of $\mathbf{E}_R^+\to\mathbf{B}$. Since $B_n/p^\varepsilon B_n\cong\mathbf{B}(n)/\bar\pi_0^\varepsilon\mathbf{B}(n)$ and $R_{n,\mathfrak{P}_n}/p^\varepsilon R_{n,\mathfrak{P}_n}\cong\mathbf{E}_R^+(n)_{P_n}/\bar\pi_0^\varepsilon\mathbf{E}_R^+(n)_{P_n}$ by 4.4(2), we conclude that $d=f$ as claimed. This concludes the proof of the lemma. □

By B.10 for every $m\geqslant n\geqslant N$ there is a unique homomorphism $u_{m,n}:B_n\to B_m$ of R_n-algebras such that the diagram

$$\begin{array}{ccc} B_n/p^\varepsilon B_n & \xrightarrow{\overline{\beta}_n} & \mathbf{B}(n)/\bar\pi_0^\varepsilon\mathbf{B}(n)\\ {\scriptstyle u_{m,n}}\downarrow & & \downarrow\\ B_m/p^\varepsilon B_m & \xrightarrow{\overline{\beta}_m} & \mathbf{B}(m)/\bar\pi_0^\varepsilon\mathbf{B}(m)\end{array}$$

commutes. Then,

6.9.2. LEMMA. – *For every $m\geqslant n\geqslant N$ the map*

$$u_{m,n}\otimes 1:B_n\otimes_{R_n}R_m\to B_m$$

is an isomorphism after inverting p.

Proof. – Let C be the cokernel of $u_{m,n}\otimes 1$. Then, $C/p^\varepsilon C$ is identified with the cokernel of

$$\big(\mathbf{B}(n)/\bar\pi_0^\varepsilon\mathbf{B}(n)\big)\otimes_{\mathbf{E}_R^+(n)}\mathbf{E}_R^+(m)\to\mathbf{B}(m)/\bar\pi_0^\varepsilon\mathbf{B}(m).$$

If $x\in\mathbf{B}(m)$ then $\bar\pi_0^{\frac{\ell}{p^n}}x=\sum_i\mathrm{Tr}_m(\overline{a}_i^{\frac{1}{p^n}}x)\overline{b}_i^{\frac{1}{p^n}}$ lies in the image of $\mathbf{B}(n)\otimes_{\mathbf{E}_R^+(n)}\mathbf{E}_R^+(m)\to\mathbf{B}(m)$ (see B.1 for the notation). Hence, $C/p^\varepsilon C$ is annihilated by $p^{\frac{\ell}{p^n}}$ i.e., $p^{\frac{\ell}{p^n}}C=p^\varepsilon C=p^MC$ for every $M\in\mathbf{N}$. Since C is finite as R_n-module, it is p-adically complete and separated i.e., $C=\lim_M(C/p^MC)$. Thus, C is annihilated by $p^{\frac{\ell}{p^n}}$. In particular, $u_{m,n}\otimes 1:B_n\otimes_{R_n}R_m\to B_m$ is surjective after inverting p. By 6.9.1 the generic degree of $R_m\to B_m$ coincides with the generic degree of $R_n\to B_n$. Since these extensions are étale after inverting p by 6.9.1, we deduce that $u_{m,n}\otimes 1$ is an isomorphism after inverting p as claimed. This concludes the proof of the lemma. □

Let $j:\mathbf{B}\to\mathbf{B}'$ as in 6.9(4). By B.10 there exists $N\in\mathbf{N}$ and for every $n\geqslant N$ there is a unique homomorphism of R_n-algebras $v_n:B_n\to B'_n$ such that the diagram

$$\begin{array}{ccc} B_n/p^\varepsilon B_n & \xrightarrow{\overline{\beta}_n} & \mathbf{B}(n)/\bar\pi_0^\varepsilon\mathbf{B}(n)\\ {\scriptstyle v_n}\downarrow & & {\scriptstyle j}\downarrow\\ B'_n/p^\varepsilon B'_n & \xrightarrow{\overline{\beta}'_n} & \mathbf{B}'(n)/\bar\pi_0^\varepsilon\mathbf{B}'(n)\end{array}$$

commutes. Furthermore, the homomorphisms $\{v_n\}_n$ are compatible for varying n with the maps $u_{n,m}$ defined for $\{B_n\}_n$ and for $\{B'_n\}_n$. Claims (1)–(4) follow from the following:

6.9.3. LEMMA. – *The ring B_n is the normalization of $B_N \otimes_{R_N} R_n$. Furthermore, we have isomorphisms $\beta_{\mathbf{B}(n)}: \mathbf{E}^+_{B_n} \cong \mathbf{B}(n)$ such that*

(i) *they are compatible for $m \geqslant n$ with the map $u_{m,n}: B_n \to B_m$ and the inclusion $\mathbf{B}(n) \subset \mathbf{B}(m)$ of* 6.7;

(ii) *given a homomorphism $j: \mathbf{B} \to \mathbf{B}'$ as in* 6.9(4)*, they are compatible with the homomorphisms $v_n: B_n \to B'_n$.*

Proof. – By 6.9.2 the map $u_{n,N} \otimes 1: B_N \otimes_{R_N} R_n \to B_n$ is an isomorphism after inverting p. Hence, if B_n is normal the first statement follows. Let S_n be the normalization of B_n. Let $\mathfrak{e}_n$ be the idempotent associated to $R_n[p^{-1}] \subset B_n[p^{-1}] = S_n[p^{-1}]$; see 3.2. By 5.8 we have that $p^{\frac{2\ell}{p^{n+h}}} \mathfrak{e}_n$ lies in the image of $B_n \otimes_{R_n} B_n$. We deduce from 3.1 that $p^{\frac{2\ell}{p^{n+h}}} S_{n+h} \subset B_{n+h}$ for every $h \in \mathbf{N}$. We have $\mathbf{E}^+_R(n+h)$-linear homomorphisms

$$(6.9.1) \qquad p^{\frac{2\ell}{p^{n+h}}} \left(S_{n+h}/p^{\varepsilon} S_{n+h}\right) \to B_{n+h}/p^{\varepsilon} B_{n+h} \cong \mathbf{B}(n+h)/\overline{\pi}_0^{\varepsilon}\mathbf{B}(n+h)$$
$$\to S_{n+h}/p^{\varepsilon} S_{n+h}.$$

Taking inverse limits with respect to $h \in \mathbf{N}$ of the factors in S and $\mathbf{B}$ with the transition maps defined by raising to the p-th power, we get $\mathbf{E}^+_R(n)$-linear homomorphisms

$$\overline{\pi}_0^{\frac{2\ell}{p^n}} \mathbf{E}^+_S(n) \to \mathbf{B}(n) \to \mathbf{E}^+_S(n).$$

The generic degree of $R_n \to S_n$ is the same as the generic degree of $R_n \to B_n$ and coincides with the generic degree m of $\mathbf{E}^+_R \to \mathbf{B}$ by 6.9.1. By 4.9 the ring $\mathbf{E}^+_S(n)$ is normal and defines a finite and étale extension of degree m, after inverting $\overline{\pi}_K$, of $\mathbf{E}^+_R(n)$. Hence, $\mathbf{B}(n) \cong \mathbf{E}^+_S(n)$.

By 5.8 condition (RAE) holds for the tower $\{B_n\}_n$ and, hence, for the tower $\{S_n\}_n$ as well. In particular, we have $\mathbf{E}^+_S(n)/\overline{\pi}_0^{\varepsilon}\mathbf{E}^+_S(n) \xrightarrow{\sim} S_n/p^{\varepsilon} S_n$ by 5.1. We then obtain the isomorphisms

$$B_n/p^{\varepsilon} B_n \xrightarrow{\sim} \mathbf{B}(n)/\overline{\pi}_0^{\varepsilon}\mathbf{B}(n) \xrightarrow{\sim} \mathbf{E}^+_S(n)/\overline{\pi}_0^{\varepsilon}\mathbf{E}^+_S(n) \xrightarrow{\sim} S_n/p^{\varepsilon} S_n.$$

We conclude from Nakayama's lemma that the inclusion $B_n \subset S_n$ is an isomorphism. Taking inverse limits of (6.9.1) with respect to h we conclude that $\mathbf{B}(n) \cong \mathbf{E}^+_{B_n}$. Claims (i) and (ii) follow. This concludes the proof of the lemma. □

We now prove Claim (5) of 6.9. By 4.9 the extension $\mathbf{E}_R \subset \mathbf{E}_S$ is finite and étale. Consider the tower $\{R_n \subset S_n\}$. By 5.1(d) there exists $N \in \mathbf{N}$ such that $\mathbf{E}^+_S/\overline{\pi}_0^{p^n \varepsilon}\mathbf{E}^+_S \cong S_n/p^{\varepsilon} S_n$ for every $n \geqslant N$. Due to 5.1(a) and 3.1 we also conclude that there exist homomorphisms $f_n: R_n^h \to S_n$ and $g_n: S_n \to R_n^h$ such that f_n is surjective and $f_n \circ g_n$ is multiplication by $p^{\frac{\ell}{p^n}}$. Hence, S_n is a good lift of $\mathbf{E}^+_S(n)$ modulo $\overline{\pi}_0^{\varepsilon}$, in the sense of B.2, if $n \geqslant N$ and $\frac{\ell}{p^n} < \varepsilon$. But a good lift is unique by B.10. The conclusion follows. □

6.11. Other functors

Consider the following categories:

$\mathbf{E}^{+,\mathrm{perf}}_R$**-AE**: the category of normal $\mathbf{E}^{+,\mathrm{perf}}_R$-algebras, which are finite and étale as $\mathbf{E}^{+,\mathrm{perf}}_R[\overline{\pi}_0^{-\varepsilon}]$-algebras.

$\widetilde{\mathbf{E}}^+_{R_\infty}$**-AE**: the category of normal $\widetilde{\mathbf{E}}^+_{R_\infty}$-algebras, finite and étale as $\widetilde{\mathbf{E}}^+_{R_\infty}[\overline{\pi}_0^{-\varepsilon}]$-algebras.

We have the following diagram of functors

$$\begin{array}{ccc} R_\infty\text{-}\mathbf{AE} & \xrightarrow{\widetilde{\mathbf{E}}^{\pm}} & \widetilde{\mathbf{E}}^{+}_{R_\infty}\text{-}\mathbf{AE} \\ {\scriptstyle \mathbf{E}^{\pm}}\downarrow & & \uparrow{\scriptstyle \widehat{\ }} \\ \mathbf{E}^{+}_{R}\text{-}\mathbf{AE} & \xrightarrow{\text{perf}} & \mathbf{E}^{+,\text{perf}}_{R}\text{-}\mathbf{AE} \end{array}$$

Here, $\mathbf{E}^{\pm}$ is the functor defined in 6.2 and $\widetilde{\mathbf{E}}^{\pm}$ is the functor given in 4.1. The latter is well defined due to 4.9. The bottom horizontal arrow $\_^{\text{perf}}$ stands for taking the perfection and it is clearly well defined. The right vertical arrow $\widehat{\ }$ stands for the $\overline{\pi}_0^{\varepsilon}$-adic completion. It is well defined due to A.9 since $\widetilde{\mathbf{E}}^{+}_{R_\infty}$ is the $\overline{\pi}_0^{\varepsilon}$-adic completion of the perfect closure of $\mathbf{E}^{+}_{R}$ by 4.15. If the equivalent conditions of 5.1 hold for every object of R_∞-**AE**, the square in the diagram commutes by 5.4.

We conclude this section strengthening 6.4.

6.12. DEFINITION. – Let $\{S_\infty\}$ be a maximal chain of elements in R_∞-**AE** which are also integral domains. Let $\widetilde{\mathbf{E}}^{+}$ (resp. $\mathbf{E}^{+}$) be the $\overline{\pi}_0^{\varepsilon}$-adic completion of the direct limit $\lim_{S_\infty}\widetilde{\mathbf{E}}^{+}_{S_\infty}$ (resp. $\mathbf{E}^{+} := \lim_{S_\infty}\mathbf{E}^{+}_{S}$). Let $\widetilde{\mathbf{E}} := \widetilde{\mathbf{E}}^{+}[\overline{\pi}_0^{-1}]$ and $\mathbf{E} := \mathbf{E}^{+}[\overline{\pi}_K^{-1}]$ (see 4.2).

6.13. LEMMA. – *For every extension $R_\infty \subset S_\infty$ in R_∞-**ED** we have $\widehat{R^{\text{sep}}}^{\mathcal{H}_S} = \widehat{S_\infty}$.*

Proof (see [16, §3.2 Prop. 10]). – Every extension $S_\infty \subset T_\infty$, which is finite, étale and Galois after inverting p and is normal as a ring, is almost étale by assumption. It then follows from [6, Thm. 2.4(ii)] that for every $\varepsilon > 0$ there exists an element in $\overline{K}$ in the image of the trace map $\operatorname{Tr}_{T_\infty/S_\infty}: T_\infty \to S_\infty$ of valuation $\leqslant \varepsilon$. Let $f \in \widehat{R^{\text{sep}}}^{\mathcal{H}_S}$. Let $\{f_n\}_n$ be a sequence of elements in R^{sep} such that $f_n \equiv f \bmod p^n$. Fix $\varepsilon > 0$. For each n let $S_\infty \subset T_\infty^{(n)}$ be a finite and étale Galois extension after inverting p and normal as a ring containing f_n and let $y_n \in T_\infty^{(n)}$ be an element such that $x_n := \operatorname{Tr}_{T_\infty^{(n)}/S_\infty}(y_n)$ lies in $\overline{K}$ and has positive valuation $\leqslant \varepsilon$. Then, $x_n f_n - \operatorname{Tr}_{T_\infty^{(n)}/S_\infty}(y_n f_n) \equiv 0$ modulo p^n. Hence, $f_n \equiv x_n^{-1}\operatorname{Tr}_{T_\infty^{(n)}/S_\infty}(y_n f_n) \in S_\infty + p^{n-\varepsilon}T_\infty^{(n)}$. In particular, $\{x_n^{-1}\operatorname{Tr}_{T_\infty^{(n)}/S_\infty}(y_n f_n)\}_n$ is a sequence of elements of S_∞ converging p-adically to f. This implies that $f \in \widehat{S_\infty}$ as claimed. □

6.14. PROPOSITION. – *The group $\mathcal{G}_R$ acts continuously on $\widetilde{\mathbf{E}}$ and on $\mathbf{E}$, it preserves $\widetilde{\mathbf{E}}^{+}$ and $\mathbf{E}^{+}$. For every extension $R_\infty \subset S_\infty$ in R_∞-**ED** the rings $\widetilde{\mathbf{E}}^{+}_{S_\infty}$, $\widetilde{\mathbf{E}}_{S_\infty}$, $\mathbf{E}^{+}_{S}$ and $\mathbf{E}_S$ are preserved by the subgroup $\mathcal{H}_S \subset \mathcal{G}_R$, see* 2.6*, and*

$$\left(\widetilde{\mathbf{E}}^{+}\right)^{\mathcal{H}_S} = \widetilde{\mathbf{E}}^{+}_{S_\infty}, \qquad \widetilde{\mathbf{E}}^{\mathcal{H}_S} = \widetilde{\mathbf{E}}_{S_\infty}, \qquad \left(\mathbf{E}^{+}\right)^{\mathcal{H}_S} = \mathbf{E}^{+}_{S}, \qquad \mathbf{E}^{\mathcal{H}_S} = \mathbf{E}_S.$$

Proof. – Note that $\mathbf{E} \subset \widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{E}}^{+} = \widetilde{\mathbf{E}}^{+}_{R^{\text{sep}}}$ so that $\widetilde{\mathbf{E}}^{+}$ is endowed with a continuous action of $\mathcal{G}_R$. It is clearly compatible with the action of $\operatorname{Aut}(S_\infty/R)$ on $\widetilde{\mathbf{E}}^{+}_{S_\infty}$ defined in 4.4. It follows from 4.10 that $\widetilde{\mathbf{E}}^{+} \subset (\widehat{R^{\text{sep}}})^{\mathbf{N}}$ (hat meaning p-adic completion). The fact that $(\widetilde{\mathbf{E}}^{+})^{\mathcal{H}_S} = \widetilde{\mathbf{E}}^{+}_{S_\infty}$ follows then since $\widehat{R^{\text{sep}}}^{\mathcal{H}_S} = \widehat{S_\infty}$ by 6.13. Since $\widetilde{\mathbf{E}} = \widetilde{\mathbf{E}}^{+}[\overline{\pi}_0^{-1}]$ and $\widetilde{\mathbf{E}}_{S_\infty} = \widetilde{\mathbf{E}}^{+}_{S_\infty}[\overline{\pi}_0^{-\varepsilon}]$, we conclude that $\mathcal{G}_R$ acts on $\widetilde{\mathbf{E}}$, that $\mathcal{H}_S$ preserves $\widetilde{\mathbf{E}}_{S_\infty}$ and that $\widetilde{\mathbf{E}}^{\mathcal{H}_S} = \widetilde{\mathbf{E}}_{S_\infty}$.

By 4.5, $\mathbf{E}^{+}_{S}$ is endowed with an action of $\operatorname{Aut}(S_\infty/R)$ compatible with that on $\widetilde{\mathbf{E}}^{+}_{S_\infty}$ for every $R_\infty \subset S_\infty$. Thus, by definition of $\mathbf{E}^{+}$, the group $\mathcal{G}_R$ acts on $\mathbf{E}^{+}$ and on $\mathbf{E} = \mathbf{E}^{+}[\overline{\pi}_K^{-1}]$, compatibly with the action on $\widetilde{\mathbf{E}}$, and $\mathcal{H}_S$ preserves $\mathbf{E}^{+}_{S}$ and $\mathbf{E}_S = \mathbf{E}^{+}_{S}[\overline{\pi}_K^{-1}]$. Since $\mathbf{E}_R \subset \widetilde{\mathbf{E}}_{R_\infty}$

is faithfully flat, to prove that the inclusion $\mathbf{E}_S \subset \mathbf{E}^{\mathcal{H}_S}$ is an equality we may base change via $\otimes_{\mathbf{E}_R}\widetilde{\mathbf{E}}_{R_\infty}$. Then, equality holds since $\widetilde{\mathbf{E}}_{S_\infty} = \mathbf{E}_S \otimes_{\mathbf{E}_R} \widetilde{\mathbf{E}}_{R_\infty} = \widetilde{\mathbf{E}}^{\mathcal{H}_S}$, by 4.15 and the first part of the proposition, and since $\mathbf{E}^{\mathcal{H}_S} \otimes_{\mathbf{E}_R} \widetilde{\mathbf{E}}_{R_\infty} \subset \widetilde{\mathbf{E}}^{\mathcal{H}_S}$. Since $\mathbf{E}_S^+$ is normal by 4.9, we get that $\mathbf{E}_S^+ = (\mathbf{E}^+)^{\mathcal{H}_S}$. □

7. The correspondence

In this section we prove our main result concerning the p-adic representations of $\mathcal{G}_R$; see 7.11. Due to 6.6, the proof is a formal consequence of work of N. Katz [12] who constructs an equivalence between unit root φ-crystals over a normal, reduced and irreducible affine scheme of characteristic p and p-adic representations of its algebraic fundamental group.

7.1. Definition. – For any S_∞ as in 2.5 let $\widetilde{\mathbf{A}}_{S_\infty}^+ := \mathbf{W}(\widetilde{\mathbf{E}}_{S_\infty}^+)$ (resp. $\widetilde{\mathbf{A}}_{S_\infty} := \mathbf{W}(\widetilde{\mathbf{E}}_{S_\infty})$) be the Witt vectors of $\widetilde{\mathbf{E}}_{S_\infty}^+$ (resp. $\widetilde{\mathbf{E}}_{S_\infty}$). Define $\widetilde{\mathbf{A}}^+ := \mathbf{W}(\widetilde{\mathbf{E}}^+)$ and $\widetilde{\mathbf{A}} := \mathbf{W}(\widetilde{\mathbf{E}})$.

We define on $\widetilde{\mathbf{A}}$ two topologies called the *strong topology* and the *weak topology*. The strong topology is the usual p-adic topology on Witt vectors. To define the weak topology consider on $\widetilde{\mathbf{E}}$ the topology having $\{\overline{\pi}_0^n\widetilde{\mathbf{E}}^+\}_n$ as fundamental system of neighborhoods of 0. On the truncated Witt vectors $\mathbf{W}_m(\widetilde{\mathbf{E}})$ we consider the product topology via the isomorphism $\mathbf{W}_m(\widetilde{\mathbf{E}}) \cong (\widetilde{\mathbf{E}})^m$. Eventually, the weak topology is defined as the projective limit topology $\mathbf{W}(\widetilde{\mathbf{E}}) = \lim_m \mathbf{W}_m(\widetilde{\mathbf{E}})$.

7.2. Proposition. – *The following hold*:

(i) $\widetilde{\mathbf{A}}$ *is a complete and separated topological algebra for the weak topology and* $\widetilde{\mathbf{A}}_{S_\infty}^+$ *and* $\widetilde{\mathbf{A}}_{S_\infty}$ *are complete topological subalgebras*;

(ii) $\widetilde{\mathbf{A}}$ *is endowed with a continuous action of* $\mathcal{G}_R$ *and* $\widetilde{\mathbf{A}}_{S_\infty}^+ = (\widetilde{\mathbf{A}}^+)^{\mathcal{H}_S}$ *and* $\widetilde{\mathbf{A}}_{S_\infty} = \widetilde{\mathbf{A}}^{\mathcal{H}_S}$;

(iii) $\widetilde{\mathbf{A}}_{R_\infty} \subset \widetilde{\mathbf{A}}_{S_\infty}$ *lifts the finite étale extension* $\widetilde{\mathbf{E}}_{R_\infty} \subset \widetilde{\mathbf{E}}_{S_\infty}$;

(iv) $\widetilde{\mathbf{A}}$ *is endowed with a continuous map* φ*, the Frobenius on Witt vectors. It commutes with the action of* $\mathcal{G}_R$ *on* $\widetilde{\mathbf{A}}$ *and it preserves* $\widetilde{\mathbf{A}}_{S_\infty}^+$ *and* $\widetilde{\mathbf{A}}_{S_\infty}$.

Proof. – The result on the invariants in (ii) follows from 6.14. The other claims are left to the reader. □

7.3. Definition. – With the notation and assumptions of 2.2 assume furthermore that

(IV) $R\otimes_V V_n \subset R_n$ is Galois with group

$$\mathrm{Gal}(R_n/R\otimes_V V_n) \xrightarrow{\sim} \left(\mathbf{Z}/p^n\mathbf{Z}\right)^d.$$

If S is an R-algebra as in 2.5 contained in R^{sep} such that S_∞ is an integral domain, define

$$\Gamma_S := \mathrm{Gal}\left(S_\infty\left[\frac{1}{p}\right]/S\left[\frac{1}{p}\right]\right).$$

Note that Γ_S is the quotient $\mathcal{G}_S/\mathcal{H}_S$; see 2.6. It is a finite index subgroup of Γ_R. Due to assumption (IV) the latter is isomorphic to the semidirect product of the Galois group $\Gamma_V = \mathrm{Gal}(K_\infty/K)$ of the tower $K\subset K_\infty$ and of the Galois group $\widetilde{\Gamma}_R = \mathrm{Gal}(R_\infty/R\otimes_V V_\infty) \cong \mathbf{Z}_p^d$.

7.4. *Assumption*. – There exists a subring $\mathbf{A}_R$ of $\widetilde{\mathbf{A}}_{R_\infty}$ which is complete and separated for the weak topology and is stable under the actions of Γ_R and of Frobenius φ and such that $\mathbf{A}_R/p\mathbf{A}_R = \mathbf{E}_R$ (as subring of $\widetilde{\mathbf{E}}_{R_\infty} = \widetilde{\mathbf{A}}_{R_\infty}/p\widetilde{\mathbf{A}}_{R_\infty}$).

7.5. LEMMA. – *The ring $\mathbf{A}_R$ is a p-adically complete and separated, noetherian and regular domain.*

Proof. – By construction the p-adic convergence in $\widetilde{\mathbf{A}}_{R_\infty}$ implies convergence for the weak topology. In particular, $\mathbf{A}_R$ is p-adically complete and separated. Since $\mathbf{E}_R$ is noetherian and regular by 4.7, also $\mathbf{A}_R$ is noetherian and regular. We conclude that $\mathbf{A}_R$ is normal. To prove that it is a domain it thus suffices to show that it does not contain non-trivial idempotents. Since it is p-adically complete and separated it suffices by Hensel's lemma to prove this for the reduction modulo p i.e., for $\mathbf{E}_R$. By 4.7 the latter is a domain. The conclusion follows. □

7.6. THEOREM. – *Assume that R is obtained from $R^0 = V\{T_1^{\pm 1}, \ldots, T_d^{\pm 1}\}$ iterating finitely many times the following operations*:

(ét) *the p-adic completion of an étale extension*;

(loc) *the p-adic completion of the localization with respect to a multiplicative system*;

(comp) *the completion with respect to an ideal containing p.*

Assume also that we are in the cyclotomic case; *see* 2.4. *Then, Assumption* 7.4 *holds.*

Proof. – This is proven in Appendix C. □

From now on fix $\mathbf{A}_R$ as in the Assumption 7.4.

7.7. DEFINITION. – For any S_∞ as in 2.5 define $\mathbf{A}_S$ as the unique $\mathbf{A}_R$-algebra lifting the finite and étale extension $\mathbf{E}_R \subset \mathbf{E}_S$.

7.8. PROPOSITION. – *For every R_∞-algebra S_∞ such that S_∞ is a normal domain and $R_\infty \subset S_\infty$ is finite and étale after inverting p we have*

(i) *$\mathbf{A}_S$ is a subring of $\widetilde{\mathbf{A}}_{S_\infty}$. It is complete for the topology induced from $\widetilde{\mathbf{A}}$ and it is stable under the actions of* $\mathrm{Aut}(S_\infty/R)$ *and* φ;

(ii) *$\mathbf{A}_S$ is a regular domain of characteristic* 0.

Let $\mathbf{A}$ be the closure of the subring $\bigcup_{S_\infty} \mathbf{A}_S$ of $\widetilde{\mathbf{A}}$ for the p-adic topology. Then,

(iii) $\mathbf{A}_S = \mathbf{A}^{\mathcal{H}_S}$.

Proof. – Statements (i) and (ii) hold for $\mathbf{A}_R$ by assumption and 7.5. Since $\mathbf{A}_R \subset \mathbf{A}_S$ lifts the finite and étale extension $\mathbf{E}_R \subset \mathbf{E}_S$, claim (ii) follows for $\mathbf{A}_S$. Thanks to 6.11 and 7.2(iii) we have $\widetilde{\mathbf{A}}_{S_\infty} = \mathbf{A}_S \otimes_{\mathbf{A}_R} \widetilde{\mathbf{A}}_{R_\infty}$. Thus, claim (i) is clear. Certainly $\mathbf{A}_S \subset \mathbf{A}^{\mathcal{H}_S}$ ($\subset \widetilde{\mathbf{A}}^{\mathcal{H}_S} = \widetilde{\mathbf{A}}_{S_\infty}$ by 7.2(ii)). Since $\mathbf{A}_S$ is normal and p-adically complete and separated, it suffices to prove the equality modulo p. Since $\mathbf{A}/p\mathbf{A} \subset \mathbf{E}$ and $\mathbf{E}_S = \mathbf{E}^{\mathcal{H}_S}$ by 6.14, the conclusion follows. □

7.9. DEFINITION. – Let S be a normal R-subalgebra of R^{sep}, étale over $R[p^{-1}]$ such that S_∞ is an integral domain; see 2.6. Let $\mathrm{Rep}(\mathcal{G}_S)$ be the abelian tensor category of finitely generated $\mathbf{Z}_p$-modules endowed with a continuous action of $\mathcal{G}_S$.

Let $(\varphi, \Gamma_S) - \mathrm{Mod}_{\mathbf{A}_S}$ (resp. $(\varphi, \Gamma_S) - \mathrm{Mod}_{\mathbf{A}_S}^{\mathrm{et}}$) be the abelian tensor category of finitely generated $\mathbf{A}_S$-modules D endowed with

(i) a semi-linear action of Γ_S;

(ii) a semi-linear homomorphism φ commuting with Γ_S (resp. so that $\varphi \otimes 1 : D \otimes_{\mathbf{A}_S}^{\varphi} \mathbf{A}_S \to D$ is an isomorphism as $\mathbf{A}_S$-modules).

For any object M in $\mathrm{Rep}(\mathcal{G}_S)$, define

$$\mathfrak{D}(M) := (\mathbf{A} \otimes_{\mathbf{Z}_p} M)^{\mathcal{H}_S}.$$

It is an $\mathbf{A}_S$-module. It is endowed with a semi-linear action of $\Gamma_S = \mathcal{G}_S/\mathcal{H}_S$. The homomorphism φ on $\mathbf{A}$ defines a semi-linear action of φ on $\mathfrak{D}(M)$ commuting with the action of Γ_S. For any object D in $(\varphi,\Gamma_S)-\mathrm{Mod}_{\mathbf{A}_S}$ define

$$\mathcal{V}(D):=(\mathbf{A}\otimes_{\mathbf{A}_S}D)_{\varphi=\mathrm{Id}}.$$

It is a $\mathbf{Z}_p$-module. The action of $\mathcal{G}_S$ on $\mathbf{A}$ and on D (via $\mathcal{G}_S\to\Gamma_S$) induces an action of $\mathcal{G}_S$ on $\mathcal{V}(D)$.

7.10. LEMMA. – *Let D be an étale (φ,Γ_S)-module annihilated by p. Then, D is a locally free $\mathbf{E}_S$-module.*

Proof. – Since D is finitely generated and $\mathbf{A}_S$ is noetherian, that claim is equivalent to say that D is a flat $\mathbf{E}_S$-module. Let $\mathfrak{m}\subset\mathbf{E}_S$ be a maximal ideal and let $\widehat{\mathbf{E}_S}$ be the $\mathfrak{m}$-adic completion of $\mathbf{E}_S$. It suffices to prove that the $\mathfrak{m}$-adic completion $\widehat{D}$ of D is free as $\widehat{\mathbf{E}_S}$-module. Let d be the dimension of $D/\mathfrak{m}D$ as $\mathbf{E}_S/\mathfrak{m}$-vector space. The choice of lifts into D of a basis of $D/\mathfrak{m}D$ defines a map $f:\widehat{\mathbf{E}_S}^d\to\widehat{D}$ which is surjective by Nakayama's lemma. Since D is étale we have $\varphi\otimes 1:(D/\mathfrak{m}^nD)\otimes_{\mathbf{E}_S}(\mathbf{E}_S/\mathfrak{m}^{pn})\xrightarrow{\sim}D/\mathfrak{m}^{pn}D$ for very n. Using this and arguing by induction on n, we deduce that $(\mathbf{E}_S/\mathfrak{m}^{pn}\mathbf{E}_S)^d$ and $D/\mathfrak{m}^{pn}D$ have the same length as $\mathbf{E}_S$-modules for every $n\in\mathbf{N}$. Thus, f is an isomorphism. □

7.11. THEOREM (cf. [10, Thm. A.3.4.3]). – *Suppose that the equivalent conditions of* 5.1 *hold for every finite extension of R_∞ which is normal as a ring and is étale over $R_\infty[p^{-1}]$. Then, the functors $\mathfrak{D}$ and $\mathcal{V}$ are inverse one of the other and define an equivalence of abelian tensor categories between the category* $\mathrm{Rep}(\mathcal{G}_S)$ *and the category* $(\varphi,\Gamma_S)-\mathrm{Mod}^{\mathrm{et}}_{\mathbf{A}_S}$.

Proof (see [12, Prop. 4.1.1]). – By devissage it suffices to prove that p-torsion representations and p-torsion étale (φ,Γ_S)-modules are equivalent via the functors above. The fact that the functor $\mathfrak{D}$ is well defined and fully faithful and the fact that $\mathcal{V}\circ\mathfrak{D}=\mathrm{id}$ follow from étale descent due to 6.6 and 7.8; see [12, Prop. 4.1.1] for details. We prove by induction on n that, for every étale (φ,Γ_S)-module D annihilated by p^n, the group $\mathcal{V}(D)$ is finite and $\mathfrak{D}(\mathcal{V}(D))=D$. The case $n=1$ follows from [12, Prop. 4.1.1] and 7.10. Assume the claim holds for $n-1$. It suffices to prove that $\mathcal{V}(D)\to\mathcal{V}(D/p^{n-1}D)$ is surjective for every (φ,Γ_S)-module D annihilated by p^n. Indeed, if we denote by D' the kernel of $D\to D/p^{n-1}D$, it then follows that $0\to\mathcal{V}(D')\to\mathcal{V}(D)\to\mathcal{V}(D/p^{n-1}D)\to 0$ is exact. In particular, by the inductive hypothesis applied to D' and to $D/p^{n-1}D$, we get that $\mathcal{V}(D)$ is finite and that $\mathfrak{D}(\mathcal{V}(D))=D$. Suppose we are given an element $h\in\mathcal{V}(D/p^{n-1}D)$. Let $g\in\mathbf{A}_T\otimes_{\mathbf{A}_S}D$ be an element lifting h. We look for $s\in\mathbf{A}_T\otimes_{\mathbf{A}_S}D'$ such that $\varphi(g+s)=g+s$. Note that $t=\varphi(g)-g$ lies in $\mathbf{A}_T\otimes_{\mathbf{A}_S}D'$. To conclude we then have to solve the equation $\varphi(s)-s=t$ in $((\bigcup_T\mathbf{A}_T)\otimes_{\mathbf{A}_S}D')$. Since D' is annihilated by p, the latter is isomorphic to $(\bigcup_T\mathbf{E}_T)\otimes_{\mathbf{F}_p}\mathcal{V}(D')$ and Frobenius is $\varphi\otimes 1$. Since $\mathrm{H}^1(\bigcup_T\mathbf{E}_T,\mathbf{Z}/p\mathbf{Z})=0$, the equation $\varphi(s)-s=t$ admits solutions as claimed. □

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Appendix A. Criteria for normality

A.1. Notations–Assumptions

Let $\{R_n\}_{n\in\mathbf{N}}$ be a set of domains, totally ordered with respect to inclusion i.e., $R_0 \hookrightarrow R_1 \hookrightarrow \cdots \hookrightarrow R_n \hookrightarrow \cdots$. Let $L_n := \mathrm{Frac}(R_n)$ be the field of fractions of R_n for every $n \in \mathbf{N}$. Let

$$R_\infty := \varinjlim R_n \quad \text{and} \quad L_\infty := \varinjlim L_n = \mathrm{Frac}(R_\infty).$$

Let $(\mathfrak{p}) \subset R_0$ be a principal prime ideal. For each $n \in \mathbf{N}$ assume that

(i) the ring R_n is noetherian and normal and $\mathfrak{p}$-adically complete and separated;

(ii) there exists a unique prime ideal $\mathfrak{P}_n$ of R_n associated to $\mathfrak{p}$ and $\mathfrak{P}_n$ is a principal ideal;

(iii) R_{n+1} is finite as R_n-module.

Define $\mathfrak{P}_\infty := \varinjlim \mathfrak{P}_i$ and let $\mathbf{v}: L_\infty^* \to \mathbf{Q}$ be the unique valuation on L_∞ such that $\mathbf{v}(\mathfrak{p}) = 1$ and the induced valuation ring on L_∞ is the local ring $R_{\infty,\mathfrak{P}_\infty}$. For every $\delta \in \mathbf{v}(R_\infty \backslash \{0\})$ choose an element $\mathfrak{p}^\delta \in \mathfrak{P}_\infty$ such that $\mathbf{v}(\mathfrak{p}^\delta) = \delta$. We further require that if $\delta \in \mathbf{v}(R_n \backslash \{0\})$, then $p^\delta \in R_n$. Let

$$\widehat{R_\infty} := \lim_{\infty \leftarrow n} R_\infty/\mathfrak{p}^n R_\infty \quad \text{and} \quad \widehat{R_{\infty,\mathfrak{P}_\infty}} := \lim_{\infty \leftarrow n} R_{\infty,\mathfrak{P}_\infty}/\mathfrak{p}^n R_{\infty,\mathfrak{P}_\infty}$$

be the completion of R_∞ (resp. of $R_{\infty,\mathfrak{P}_\infty}$) with respect to the ideal generated by $\mathfrak{p}$. Assume furthermore that

(iv) there exist $N \in \mathbf{N}$ and an element $\mathfrak{p}^\varepsilon \in R_N$ of valuation ε such that $R_N/\mathfrak{p}^\varepsilon R_N$ is a ring of positive characteristic p and for every $n \geqslant N$ we have $R_{n+1}^p + \mathfrak{p}^\varepsilon R_{n+1} = R_n + \mathfrak{p}^\varepsilon R_{n+1}$, as subrings of R_{n+1}.

A.2. LEMMA. – *The map $\widehat{R_\infty} \to \widehat{R_{\infty,\mathfrak{P}_\infty}}$ is injective.*

Proof. – Let $n \in \mathbf{N}$. Consider the map $\pi: R_\infty/\mathfrak{p}^n R_\infty \to R_{\infty,\mathfrak{P}_\infty}/\mathfrak{p}^n R_{\infty,\mathfrak{P}_\infty}$. Let x be an element of R_∞ such that $\pi(x) = 0$. We conclude from (i) and (ii) of A.1 that $\mathfrak{p}^n R_\infty \cap R_i = \mathfrak{p}^n R_i$ for every $i \in \mathbf{N}$. Hence, there exists i such that $x \in R_i$ and there is $y \in R_i \backslash \mathfrak{P}_i$ so that $xy \in \mathfrak{p}^n R_i$. Thus, $\mathbf{v}(x\mathfrak{p}^{-n}) + \mathbf{v}(y) = \mathbf{v}(x\mathfrak{p}^{-n}) \geqslant 0$. It follows from (i) and (ii) of A.1 that $x\mathfrak{p}^{-n} \in R_i$ i.e., $x \in \mathfrak{p}^n R_i$. Hence, π is injective. □

A.3. LEMMA. – *The rings $\widehat{R_\infty}$ and $\widehat{R_{\infty,\mathfrak{P}_\infty}}$ are domains. There exists a unique valuation*

$$\hat{\mathbf{v}}: \mathrm{Frac}(\widehat{R_{\infty,\mathfrak{P}_\infty}})^* \to \mathbf{Q}$$

extending the valuation on L^*. Its valuation ring is $\widehat{R_{\infty,\mathfrak{P}_\infty}}$ and

$$\left\{ x \in \widehat{R_\infty}\left[\frac{1}{\mathfrak{p}}\right] \mid \hat{\mathbf{v}}(x) \geqslant 0 \right\} = \widehat{R_\infty}.$$

Proof. – Let m_n be a Cauchy sequence of elements of $R_{\infty,\mathfrak{P}_\infty}$ converging $\mathfrak{p}$-adically to an element m of $\widehat{R_{\infty,\mathfrak{P}_\infty}}$. If $\mathbf{v}(m_n) \to \infty$, then $m_n \to 0$ and $m = 0$. Hence, if $m \neq 0$, the sequence $\{\mathbf{v}(m_n)\}_n$ is bounded above by some constant α and

$$\hat{\mathbf{v}}(m) := \lim_{h\to\infty} \mathbf{v}(m_h)$$

is a well defined natural number. There exists $N\in\mathbf{N}$ such that $m_N\neq 0$ and $\mathbf{v}(m_n-m_N)\geqslant 2\alpha$ for $n\geqslant N$. Hence, $\mathbf{v}(m_n)=\min(\mathbf{v}(m_N),2\alpha)=\mathbf{v}(m_N)$ for $n\geqslant N$. Thus, $\hat{\mathbf{v}}(m)=\mathbf{v}(m_N)$. Using this it follows that $\hat{\mathbf{v}}$ defines a valuation on $\widehat{R_{\infty,\mathfrak{p}_\infty}}$. In particular, $\widehat{R_{\infty,\mathfrak{p}_\infty}}$ is a domain. By construction $\hat{\mathbf{v}}$ extends the valuation on $R_{\infty,\mathfrak{p}_\infty}$ and $\widehat{R_{\infty,\mathfrak{p}_\infty}}$ is its valuation ring. Its uniqueness is clear. By A.2 also $\widehat{R_\infty}$ is a domain. The last assertion is left to the reader. □

A.4. LEMMA. – *There exists a unique map*

$$\mathbf{w}: R_\infty/\mathfrak{P}_\infty R_\infty \to \widehat{R_\infty} \quad (\textit{resp. } \mathbf{w}: \mathrm{Frac}(R_\infty/\mathfrak{P}_\infty R_\infty)\to \widehat{R_{\infty,\mathfrak{p}_\infty}})$$

such that $\mathbf{w}(a)\equiv a \bmod \mathfrak{P}_\infty$ *and* $\mathbf{w}(a^p)=\mathbf{w}(a)^p$ *for every* a. *Furthermore,* $\mathbf{w}(ab)=\mathbf{w}(a)\cdot\mathbf{w}(b)$ *for every* a *and* b *and the composition*

$$R_\infty/\mathfrak{P}_\infty R_\infty \xrightarrow{\mathbf{w}} \widehat{R_\infty}\to R_\infty/\mathfrak{p}^\varepsilon R_\infty$$

$$(\textit{resp. } \mathrm{Frac}(R_\infty/\mathfrak{P}_\infty R_\infty)\xrightarrow{\mathbf{w}}\widehat{R_{\infty,\mathfrak{p}_\infty}}\to R_{\infty,\mathfrak{p}_\infty}/\mathfrak{p}^\varepsilon R_{\infty,\mathfrak{p}_\infty})$$

is a ring homomorphism with the property that for every $n\geqslant N$ *the image of* $R_n/\mathfrak{P}_n R_n$ (*resp.* $R_{n,\mathfrak{p}_n}/\mathfrak{P}_n R_{n,\mathfrak{p}_n}$) *is contained in* $R_n/\mathfrak{p}^\varepsilon R_n$ (*resp.* $R_{n,\mathfrak{p}_n}/\mathfrak{p}^\varepsilon R_{n,\mathfrak{p}_n}$).

Proof. – Note that $\widehat{R_\infty}$ (resp. $\widehat{R_{\infty,\mathfrak{p}_\infty}}$) is complete and separated with respect to the ideals $\{(\mathfrak{p}^N\mathfrak{P}_\infty)\}_N$ (resp. the ideals $\{(\mathfrak{p}^N\mathfrak{P}_\infty R_{\infty,\mathfrak{p}_\infty})\}_N$). It follows from (iv) of A.1 that the ring $\widehat{R_\infty}/\mathfrak{P}_\infty\widehat{R_\infty}$ (resp. $\widehat{R_{\infty,\mathfrak{p}_\infty}}/\mathfrak{P}_\infty\widehat{R_{\infty,\mathfrak{p}_\infty}}$) is a perfect domain of positive characteristic p i.e., the map $x\mapsto x^p$ defines an isomorphism of $R_\infty/\mathfrak{P}_\infty R_\infty$ (resp. $\widehat{R_{\infty,\mathfrak{p}_\infty}}/\mathfrak{P}_\infty\widehat{R_{\infty,\mathfrak{p}_\infty}}$). The existence of $\mathbf{w}$, with all the required properties apart from the claim on the image of $R_n/\mathfrak{P}_n R_n$ (resp. $R_{n,\mathfrak{p}_n}/\mathfrak{P}_n R_{n,\mathfrak{p}_n}$), follows from [15, II.5, Prop. 8]. We prove the last claim for R_∞, the proof for $R_{\infty,\mathfrak{p}_\infty}$ being entirely similar. We recall from loc. cit. the construction of $\mathbf{w}$. Let $\lambda\in R_n/\mathfrak{P}_n R_n$ with $n\geqslant N$. Denote by L_m the elements of $\widehat{R_\infty}$ reducing to $\lambda^{p^{-m}}\in R_\infty/\mathfrak{P}_\infty R_\infty$ and by $U_m:=\{x^{p^m}\mid x\in L_m\}$. Then, chosen $u_m\in U_m$ for every m, the sequence $\{u_m\}_{m\in\mathbf{N}}$ is a Cauchy sequence and its limit is $\mathbf{w}(\lambda)$. In our case we can choose $x_m\in L_m\cap R_{m+n}$ since the latter is not empty by A.1(iv). Then, the class of $u_m=x_m^{p^m}$ in $R_{m+n}/\mathfrak{p}^\varepsilon R_{m+n}$ lies in $R_n/\mathfrak{p}^\varepsilon R_n$ by A.1(iv). The conclusion follows. □

A.5. LEMMA. – *Every non-zero element* a *of* $\widehat{R_\infty}$ (*resp.* $\widehat{R_{\infty,\mathfrak{p}_\infty}}$) *can be written uniquely as*

$$a=\sum_n \mathbf{w}(a_n)\mathfrak{p}^{\delta_n}$$

with $a_n\in R_\infty/\mathfrak{P}_\infty R_\infty$ (*resp.* $a_n\in\mathrm{Frac}(R_\infty/\mathfrak{P}_\infty R_\infty)$) *so that the sequence* $\{\delta_n\mid a_n\neq 0\}$ *is strictly increasing and it is either finite or it converges to infinity.*

Proof. – For $a\in\widehat{R_\infty}$ write $\bar{a}$ for $a \bmod \mathfrak{P}_\infty\widehat{R_\infty}$. Let $m\geqslant N$ be an integer such that the image of a in $\widehat{R_\infty}/p^\varepsilon\widehat{R_\infty}$ lies in $R_m+p^\varepsilon\widehat{R_\infty}$. It follows from A.4 that we have

$$R_m/\mathfrak{p}^\varepsilon R_m\cong R_m/\mathfrak{P}_m R_m\oplus\left(\mathfrak{P}_m R_m/\mathfrak{p}^\varepsilon R_m\right)$$

i.e., $a=\mathbf{w}(a_0)+p^\delta b_1$ with $a_0:=\bar{a}$ and p^δ and $b_1\in R_m$. Applying the decomposition above to b_1 and proceeding by induction, we may write a as a sum $\sum_{n,\delta_n<\varepsilon}\mathbf{w}(a_n)\mathfrak{p}^{\delta_n}$, which is finite since the valuation $\mathbf{v}$ restricted to R_m is discrete by A.1(i). This implies that every element $a\in\widehat{R_\infty}$

can be written as a finite sum $a=\sum_{\delta<\varepsilon}\mathbf{w}(a_\delta)\mathfrak{p}^\delta+\mathfrak{p}^\varepsilon b$ for suitable elements $a_\delta\in R_\infty/\mathfrak{P}_\infty R_\infty$ and $b\in\widehat{R_\infty}$. Proceeding inductively define an increasing sequence of elements $\delta_n\in\mathbf{Q}_{>0}$ which is either finite or converges to ∞ and a sequence of elements $a_n\in R_\infty/\mathfrak{P}_\infty R_\infty$ such that $\sum_n\mathbf{w}(a_n)\mathfrak{p}^{\delta_n}$ converges $\mathfrak{p}$-adically to a. We remark that if $0\neq c\in R_\infty/\mathfrak{P}_\infty R_\infty$ then $\mathbf{v}(\mathbf{w}(c))=0$ since $\mathbf{w}(c)$ does not lie in $\mathfrak{P}_\infty R_\infty$. In particular, given $0\neq a\in\widehat{R_\infty}$ and a decomposition $a=\sum_n\mathbf{w}(a_n)\mathfrak{p}^{\delta_n}$ as in the lemma, $\mathbf{v}(a)=\min\{\delta_n\mid a_n\neq 0\}$. Using this the proof of the uniqueness follows. The details are left to the reader. Taking $R_{\infty,\mathfrak{P}_\infty}$ instead of R_∞ we deduce the existence and uniqueness of the $\mathfrak{p}$-adic expansion in $\widehat{R_{\infty,\mathfrak{P}_\infty}}$. □

A.6. PROPOSITION. – *If $R_n/\mathfrak{P}_nR_n$ is normal for every $n\in\mathbf{N}$, then $R_\infty/\mathfrak{P}_\infty R_\infty$ and $\widehat{R_\infty}$ are normal.*

Proof (cf. [13, Ch. 7, Thm. 34]). – Since $R_\infty/\mathfrak{P}_\infty R_\infty$ is the direct limit of the normal domains $\{R_n/\mathfrak{P}_nR_n\}_n$, it is normal. We are left to prove that $\widehat{R_\infty}$ is normal. Let x and y be non-zero elements of $\widehat{R_\infty}$. Let $a:=xy^{-1}$ be integral over $\widehat{R_\infty}$. In particular, the algebra $\widehat{R_\infty}[a]\subset\mathrm{Frac}(\widehat{R_\infty})$ is finite as $\widehat{R_\infty}$-module. The inclusion $\widehat{R_\infty}\subset\widehat{R_\infty}[a]$ is an isomorphism after $\bigotimes_{\widehat{R_\infty}}\mathrm{Frac}(\widehat{R_\infty})$. Hence, the cokernel is killed by some non-zero element $d\in\widehat{R_\infty}$. Hence, $dx^m\in y^m\widehat{R_\infty}$ for every $m\in\mathbf{N}$ i.e., a is almost integral in the sense of [13, Ch. 7, p. 115]. The $\mathfrak{p}$-adic valuation $\hat{\mathbf{v}}$ of a, see A.3, is non-negative. Write $a=\sum_\delta\mathbf{w}(a_\delta)\mathfrak{p}^\delta$ as an element of $\widehat{R_{\infty,\mathfrak{P}_\infty}}$ using A.5. We claim that $a_\delta\in R_\infty/\mathfrak{P}_\infty R_\infty$ for every δ. This implies that $a\in\widehat{R_\infty}$. We proceed by induction on the numberable set $\{\delta\mid a_\delta\neq 0\}$. Let $\delta(a)$ be the minimum of such set. Let $\delta(d)$ (resp. $\delta(x)$, $\delta(y)$) be the minimum of $\{\delta\mid d_\delta\neq 0\}$ (resp. $\{\delta\mid x_\delta\neq 0\}$, $\{\delta\mid y_\delta\neq 0\}$). By the uniqueness in A.5, we get that $d_{\delta(d)}x_{\delta(x)}^m\in y_{\delta(y)}^m R_\infty/\mathfrak{P}_\infty R_\infty$ for every $m\in\mathbf{N}$. Since $R_\infty/\mathfrak{P}_\infty R_\infty$ is the direct limit of the noetherian normal domains $\{R_n/\mathfrak{P}_nR_n\}_n$, this implies that $a_{\delta(a)}=\frac{x_{\delta(x)}}{y_{\delta(y)}}$ lies in $R_\infty/\mathfrak{P}_\infty R_\infty$ [13, Ch. 7, pp. 115–116]. Assume that the first n non-zero coefficients $a_{\delta_1},\dots,a_{\delta_n}$ belong to $R_\infty/\mathfrak{P}_\infty R_\infty$. Let $b=\sum_{i=1}^n\mathbf{w}(a_{\delta_i})\mathfrak{p}^{\delta_i}$. If $a=b$, we are done. Otherwise, note that $a-b$ is still integral over $\widehat{R_\infty}$. We conclude as in the base step of the induction that the first non-zero coefficient $(a-b)_\delta$ in the $\mathfrak{p}$-adic expansion of $a-b$ as an element of $\widehat{R_{\infty,\mathfrak{P}_\infty}}$ lies in $R_\infty/\mathfrak{P}_\infty R_\infty$. But such coefficient is the first non-zero coefficient $a_{\delta_{n+1}}$ in the $\mathfrak{p}$-adic expansion of a different from $a_{\delta_1},\dots,a_{\delta_n}$. Hence, the conclusion. □

A.7. LEMMA. – *Assume that R_∞ is free as R_i-module for some $i\in\mathbf{N}$. Then, the map $R_i\subset\widehat{R_\infty}$ is faithfully flat and for any finitely generated R_i-module M the map*

$$M\otimes_{R_i}\widehat{R_\infty}\to\lim_{\infty\leftarrow n}M\otimes_{R_i}\left(R_\infty/\mathfrak{p}^nR_\infty\right)$$

is an isomorphism of $\widehat{R_\infty}$-modules. In particular, if R_∞ is free as R_i-module for every i, then $R_\infty\to\widehat{R_\infty}$ is flat.

Proof. – For every $n\in\mathbf{N}$ we denote $M_n:=M/\mathfrak{p}^nM$. Since R_i is $\mathfrak{p}$-adically complete and separated and noetherian, if M is finitely generated, then $M\cong\lim_n M_n$ (here and below $\lim_n$ denotes the inverse limit taken over $n\in\mathbf{N}$). Let $0\to A\to B\to C\to 0$ be a sequence of finitely generated R_i-modules. Since R_∞ is a free R_i-module, one checks that the sequence above is exact if and only if

$$0\to\lim_n(A_n\otimes_{R_i}R_\infty)\to\lim_n(B_n\otimes_{R_i}R_\infty)\to\lim_n(C_n\otimes_{R_i}R_\infty)\to 0$$

is exact. Thus, the functor $M \mapsto \lim_n(M_n \otimes_{R_i} R_\infty)$ from the category of finitely generated R_i-modules to the category of $\widehat{R_\infty}$-modules is exact. By [2, Prop. 10.13] for any finitely generated R_i-module M the map $M \otimes_{R_i} \widehat{R_\infty} \to \lim_n(M_n \otimes_{R_i} R_\infty)$ is surjective. Reasoning as in the proof of [2, Prop. 10.13], we conclude that such map is an isomorphism. In particular, $R_i \subset \widehat{R_\infty}$ is faithfully flat.

Let $J \subset R_\infty$ be an ideal. Let $J_i := R_i \cap J$ for every $i \in I$. Since $R_i \subset \widehat{R_\infty}$ is faithfully flat, the natural map $J_i \otimes_{R_i} \widehat{R_\infty} \to \widehat{R_\infty}$ is injective. Consider the maps

$$\varinjlim (J_i \otimes_{R_i} \widehat{R_\infty}) \to J \otimes_{R_\infty} \widehat{R_\infty} \to \widehat{R_\infty}.$$

If R_∞ is a free R_i-module for every $i \in \mathbf{N}$, the composition of the above maps is injective. The LHS map is surjective. Hence, the RHS map is injective as well and $R_\infty \to \widehat{R_\infty}$ is flat as claimed. □

A.8. LEMMA. – *Assume that $\widehat{R_\infty}$ is normal. Let $R_\infty[\frac{1}{\mathfrak{p}}] \subset S$ be a finite and étale extension of subrings of* $\mathrm{Frac}(\widehat{R_\infty})$*. Then, $S \cap \widehat{R_\infty}$ is a normal integral extension of R_∞.*

Proof. – Define S_∞ to be the integral closure of R_∞ in S. Since $\widehat{R_\infty}$ is normal, it is a subring of $\widehat{R_\infty}$. Furthermore, $S_\infty[\frac{1}{\mathfrak{p}}] = S$. It suffices to prove that $S_\infty = S \cap \widehat{R_\infty}$. The inclusion $\subset$ is trivial; it remains to prove the inclusion $\supset$. Define $\mathfrak{P}_{S_\infty} := (\mathfrak{P}_\infty \widehat{R_\infty}) \cap S_\infty$.

Let $d \in S_\infty$. Then, there exists $i \in I$ such that d is integral over R_i and $\mathrm{Frac}(R_i)[d]$ (as subring of the total field of fractions $\mathrm{Frac}(S_\infty)$) is linearly disjoint from $\mathrm{Frac}(R_\infty)$ over $\mathrm{Frac}(R_i)$. Let S_d be the normalization of $R_i[d]$. Since $R_i \subset S_d$ is generically separable by the choice of i and by hypothesis, it follows from [5, Prop. 13.14] that S_d is finite as R_i-module. Thus, since R_i is noetherian and $\mathfrak{p}$-adically complete and separated by A.1(i), S_d is $\mathfrak{p}$-adically complete and separated. Since S_d is contained in $\widehat{R_\infty}$, it is a domain by A.3 and, in particular, it contains no non-trivial idempotents. Hensel's lemma implies that $S_d/\mathfrak{p}S_d$ contains no non-trivial idempotents. Let $s \in S_\infty$. If $\mathfrak{p}s \in S_d$, then $s \in \mathrm{Frac}(R_i)[d]$ and it is integral over R_∞ and, thus, over R_i by A.1(iii). Hence, $s \in S_d$. This proves that $\mathfrak{p}S_\infty \cap S_d = \mathfrak{p}S_d$. Since $S_\infty = \bigcup_d S_d$, we conclude that $S_\infty/\mathfrak{p}S_\infty$ is equal to the direct limit $\lim_d S_d/\mathfrak{p}S_d$. In particular, it contains no non-trivial idempotents. By construction we have homomorphisms $R_\infty \to S_\infty \to \widehat{R_\infty}$ whose composite is the natural inclusion. Then, $S_\infty/\mathfrak{p}S_\infty$ decomposes as the direct sum of R_∞-algebras $R_\infty/\mathfrak{p}R_\infty \oplus A_\infty$ with multiplication $(a,b)\cdot(\alpha,\beta) = (a\alpha, a\beta + b\alpha + b\beta)$. The image of 1 is the only possible non-zero idempotent i.e., $1 = (1,0)$.

For every $d \in S_\infty$ the ring $S_d \otimes_{R_i} R_{i,\mathfrak{P}_i}$ is normal, finite and torsion free as $R_{i,\mathfrak{P}_i}$-module. Thus, $S_d \otimes_{R_i} \widehat{R_{i,\mathfrak{P}_i}}$ is the product of complete discrete valuation rings $\prod_j S_{d,j}$. Note that the number of those is bounded above by the generic degree of $R_\infty \subset S_\infty$. Since $S_\infty = \bigcup_d S_d$, it has only finitely many prime ideals $\{\mathfrak{P}_{S_\infty,j}\}_j$ containing $\mathfrak{p}$ and the ring $S_\infty \otimes_{R_\infty} (\widehat{R_{\infty,\mathfrak{P}_\infty}}/\mathfrak{p}\widehat{R_{\infty,\mathfrak{P}_\infty}})$ is $\bigcup_d(\prod_j S_{d,j}/\mathfrak{p}S_{d,j})$ and, hence, it coincides with the product $\prod_j(S_{\infty,j}/\mathfrak{p}S_{\infty,j})$ of the quotients modulo $\mathfrak{p}$ of the localizations of S_∞ at the ideals $\mathfrak{P}_{S_\infty,j}$. Note that $1 = (1,\ldots,1)$ in this decomposition. The reduction modulo $\mathfrak{p}$ is also $\prod_j(S_{\infty,j}/\mathfrak{p}S_{\infty,j}) = \widehat{R_{\infty,\mathfrak{P}_\infty}}/\mathfrak{p}\widehat{R_{\infty,\mathfrak{P}_\infty}} \oplus (A_\infty \otimes_{R_\infty} \widehat{R_{\infty,\mathfrak{P}_\infty}})$. Since $1 \mapsto (1,0)$ by the above discussion, we conclude that S_∞ has only one prime over $\mathfrak{P}_\infty$ which must then coincide with $\mathfrak{P}_{S_\infty}$. Furthermore, $S_{\infty,\mathfrak{P}_{S_\infty}}$ is the valuation ring of $\mathrm{Frac}(S_\infty)$ associated to the restriction to S_∞ of the valuation $\hat{\mathbf{v}}$ on $\widehat{R_\infty}$.

Let $s \in S_\infty[\frac{1}{\mathfrak{p}}] \cap \widehat{R_\infty} = S \cap \widehat{R_\infty}$. Then, $\hat{\mathbf{v}}(s) \geqslant 0$. Hence, $s \in S_{\infty,\mathfrak{P}_{S_\infty}}$. Hence, s lies in the intersection of all valuation rings containing S_∞. Hence, it lies in S_∞ by [2, Cor. 5.22] as claimed. □

A.9. PROPOSITION. – *Let S_∞ be an integral R_∞-algebra and a normal ring. Assume that*
(1) *R_∞ is a free R_i-module for every $i \in \mathbf{N}$;*
(2) *$\widehat{R_\infty}$ is normal;*
(3) *$S_\infty[\frac{1}{\mathfrak{p}}]$ is a finite étale extensions of $R_\infty[\frac{1}{\mathfrak{p}}]$.*
Then, $S_\infty \otimes_{R_\infty} \widehat{R_\infty}$ is normal and $\mathfrak{p}$-torsion free.

Proof. – The morphism $S_\infty \to S_\infty$ defined by $b \mapsto \mathfrak{p} \cdot b$ is injective. It follows from A.7 that multiplication by $\mathfrak{p}$ is injective on $S_\infty \otimes_{R_\infty} \widehat{R_\infty}$. Hence, the morphism $S_\infty \otimes_{R_\infty} \widehat{R_\infty} \to (S_\infty \otimes_{R_\infty} \widehat{R_\infty})[\frac{1}{\mathfrak{p}}]$ is injective. It follows from (2) and (3) that $(S_\infty \otimes_{R_\infty} \widehat{R_\infty})[\frac{1}{\mathfrak{p}}]$ is normal.

Let c be an element in $S_\infty \otimes_{R_\infty} \operatorname{Frac}(\widehat{R_\infty})$, which is a product of fields by (3), integral over $S_\infty \otimes_{R_\infty} \widehat{R_\infty}$. Then, $c \in (S_\infty \otimes_{R_\infty} \widehat{R_\infty})[\frac{1}{\mathfrak{p}}]$ i.e., $\mathfrak{p}^n c \in S_\infty \otimes_{R_\infty} \widehat{R_\infty}$ for some $n \in \mathbf{N}$. Fix $N \in \mathbf{N}$ with $N \geqslant n$. There exists $b \in S_\infty$ such that $\mathfrak{p}^n c - b \in \mathfrak{p}^N S_\infty \otimes_{R_\infty} \widehat{R_\infty}$. Let $d := \mathfrak{p}^{-n} b$. We conclude from the integrality of c that d is integral over $S_\infty \otimes_{R_\infty} \widehat{R_\infty}$. It follows from the integrality of S_∞ over R_∞ that d is integral over $\widehat{R_\infty}$.

Let g be a monic polynomial of minimal degree having coefficients in $\widehat{R_\infty}$ and such that $g(d) = 0$. Let h be the minimal polynomial of d over $\operatorname{Frac}(\widehat{R_\infty})$. Then, g factors as $g = h \cdot h'$ for a suitable monic polynomial h' with coefficients in $\operatorname{Frac}(\widehat{R_\infty})$. In particular, the coefficients of h are integral over $\widehat{R_\infty}$ and, since $\widehat{R_\infty}$ is normal, they lie in $\widehat{R_\infty}$. Hence, $g = h$. Let S_g be the integral closure in $\widehat{R_\infty}[\frac{1}{\mathfrak{p}}]$ of the subring generated by $R_\infty[\frac{1}{\mathfrak{p}}]$ and the coefficients of g. Reasoning as above and using the normality of $R_\infty[\mathfrak{p}^{-1}]$ we deduce that the minimal polynomial q of d over L_∞ has coefficients in $R_\infty[\mathfrak{p}^{-1}]$. Then, g divides q. Since $d \in S_\infty[\mathfrak{p}^{-1}]$, we conclude from (3) that there exists a ring S_q, finite and étale as $R_\infty[\mathfrak{p}^{-1}]$-algebra, splitting q, and hence g, completely. Then, S_g is a subring of S_q, finite and étale as $R_\infty[\mathfrak{p}^{-1}]$-algebra. It follows from A.8 that $S_g \cap \widehat{R_\infty}$ (as a subring of $\widehat{R_\infty}[\mathfrak{p}^{-1}]$) is an integral extension of R_∞. Since the coefficients of g lie in $S_g \cap \widehat{R_\infty}$, we conclude that they are integral over R_∞. Thus, d is integral over R_∞ and, hence, $d \in S_\infty$. Therefore, $c \in S_\infty \otimes_{R_\infty} \widehat{R_\infty}$ as wanted. $\square$

Appendix B. Some deformation theory

We start with a $\mathbf{E}_R^+$-algebra $\mathbf{B}$, which is normal and finite as $\mathbf{E}_R^+$-module and such that $\mathbf{E}_R^+[\overline{\pi}_K^{-1}] \subset \mathbf{B}[\overline{\pi}_K^{-1}]$ is étale. We construct R_n-algebras B_n, for $n \gg 0$, with isomorphisms $B_n/p^\varepsilon B_n \xrightarrow{\sim} \mathbf{B}(n)/\overline{\pi}_0^\varepsilon \mathbf{B}(n)$. This is achieved lifting $\mathbf{B}(n)/\overline{\pi}_0^\varepsilon \mathbf{B}(n)$ as a R_n-module first (see B.4–B.6) and then proving that it inherits the structure of algebra (see B.7–B.8). We further prove that this construction is functorial in $\mathbf{B}$ (see B.10).

B.1. Notation

We use the notation of Section 6.7. Let $\overline{\pi}_0^\ell \in \mathbf{E}_R^+$ be the element killing the ramification of $\mathbf{E}_R^+ \to \mathbf{B}$ i.e., such that there exists an element of $\mathbf{B} \otimes_{\mathbf{E}_R^+} \mathbf{B}$ with image $\overline{\pi}_0^\ell \overline{\mathfrak{e}}_0$. Abusing the notation we still denote this element by $\overline{\pi}_0^\ell \overline{\mathfrak{e}}_0$. Let $\overline{b}_1, \ldots, \overline{b}_h$ be generators of $\mathbf{B}$ as $\mathbf{E}_R^+$-module. Write $\overline{\pi}_0^\ell \overline{\mathfrak{e}}_0 =: \overline{a}_1 \otimes \overline{b}_1 + \cdots + \overline{a}_h \otimes \overline{b}_h$ as an element of $\mathbf{B} \otimes_{\mathbf{E}_R^+} \mathbf{B}$.

Fix $n \in \mathbf{N}$. Then, $\overline{b}_1^{\frac{1}{p^n}}, \ldots, \overline{b}_h^{\frac{1}{p^n}}$ are generators of $\mathbf{B}(n)$ as $\mathbf{E}_R^+(n)$-module and $\overline{\pi}_0^{\frac{\ell}{p^n}} \overline{\mathfrak{e}}_n = \overline{a}_1^{\frac{1}{p^n}} \otimes \overline{b}_1^{\frac{1}{p^n}} + \cdots + \overline{a}_h^{\frac{1}{p^n}} \otimes \overline{b}_h^{\frac{1}{p^n}}$ in $\mathbf{B}(n)_{\mathbf{E}_R^+(n)} \mathbf{B}(n)$.

Denote by Tr_n the trace of the finite and étale extension $\mathbf{E}_R^+(n)[\overline{\pi}_K^{-1}] \subset \mathbf{B}(n)[\overline{\pi}_K^{-1}]$. Define the $\mathbf{E}_R^+(n)$-linear homomorphisms

$$\mathbf{B}(n) \xrightarrow{\overline{g}_n} \mathbf{E}_R^+(n)^h \xrightarrow{\overline{f}_n} \mathbf{B}(n),$$

where $\overline{g}_n(x) := (\mathrm{Tr}_n(x\overline{a}_1^{\frac{1}{p^n}}), \ldots, \mathrm{Tr}_n(x\overline{a}_h^{\frac{1}{p^n}}))$ and $\overline{f}_n(y_1,\ldots,y_h) = y_1\overline{b}_1^{\frac{1}{p^n}} + \cdots + y_h\overline{b}_h^{\frac{1}{p^n}}$; see 3.1. Note that $\overline{f}_n$ is surjective and that $\overline{f}_n \circ \overline{g}_n$ is multiplication by $\overline{\pi}_0^{\frac{\ell}{p^n}}$. Let

$$\overline{\alpha}_n := \overline{g}_n \circ \overline{f}_n.$$

It satisfies $\overline{\alpha}_n^2 = \overline{\pi}_0^{\frac{\ell}{p^n}} \overline{\alpha}_n$.

B.2. DEFINITION. – Fix n such that $0 < \frac{\ell}{p^n} < \varepsilon < 1$ and $R_n/p^\varepsilon R_n = \mathbf{E}_R^+(n)/\overline{\pi}_0^\varepsilon \mathbf{E}_R^+(n)$ (see 4.5(1)). A *good lift* of $\mathbf{B}(n)$ modulo $\overline{\pi}_0^\varepsilon$ is an R_n-algebra B_n and an isomorphism $\overline{\beta}_n : B_n/p^\varepsilon B_n \xrightarrow{\sim} \mathbf{B}(n)/\overline{\pi}_0^\varepsilon \mathbf{B}(n)$ as algebras over $R_n/p^\varepsilon R_n = \mathbf{E}_R^+(n)/\overline{\pi}_0^\varepsilon \mathbf{E}_R^+(n)$ such that B_n has no non-trivial p-torsion and there exist R_n-linear homomorphisms

$$g_n : B_n \to R_n^h,$$
$$f_n : R_n^h \to B_n$$

with f_n surjective and whose composite $f_n \circ g_n$ is multiplication by $p^{\frac{\ell}{p^n}}$.

B.3. PROPOSITION. – *Fix a positive integer n and $1 - \frac{22\ell}{p^n} > \varepsilon > \frac{22\ell}{p^n}$ such that $R_n/p^\varepsilon R_n = \mathbf{E}_R^+(n)/\overline{\pi}_0^\varepsilon \mathbf{E}_R^+(n)$. Then, a good lift of $\mathbf{B}(n)$ modulo $\overline{\pi}_0^\varepsilon$, in the sense of* B.2*, exists.*

Proof. – It is proven in the next lemmas. See especially B.7 and B.8. □

Regard $\overline{\alpha}_n$ as a $h \times h$-matrix with entries in $R_n/p^\varepsilon R_n$ identifying $\mathbf{E}_R^+(n)/\overline{\pi}_0^\varepsilon \mathbf{E}_R^+(n)$ with $R_n/p^\varepsilon R_n$ via the isomorphism $(y_0, y_1, \ldots) \mapsto y_0$; see 4.4(2). Let $Q(X)$ be the polynomial $X^2 - p^{\frac{\ell}{p^n}} X$ in the variable X. We have the following fundamental lemma, which is an instance of Hensel's lemma:

B.4. LEMMA. – *For every n such that $\varepsilon > \frac{2\ell}{p^n}$ there exists a matrix α_n in $\mathrm{M}_{h\times h}(R_n)$ satisfying $Q(\alpha_n) = 0$ and lifting $\overline{\alpha}_n \in \mathrm{M}_{h\times h}(R_n/p^\varepsilon R_n)$.*

Proof. – In what follows we write $\tau_n := p^{\frac{\ell}{p^n}}$. Let $\beta_0 \in \mathrm{M}_{h\times h}(R_n)$ be any lift of $\overline{\alpha}_n \in \mathrm{M}_{h\times h}(R_n/p^\varepsilon R_n)$. Then, $Q'(\beta_0) = 2\beta_0 - \tau_n$ and $Q'(\beta_0) \equiv 2\overline{\alpha}_n - \tau_n$ modulo $p^\varepsilon \mathrm{M}_{h\times h}(R_n)$. In particular, $Q'(\beta_0)^2 \equiv \tau_n^2$ modulo $p^\varepsilon \mathrm{M}_{h\times h}(R_n)$. Since R_n is noetherian and $\mathrm{M}_{h\times h}(R_n)$ is finite as R_n-module, $\mathrm{M}_{h\times h}(R_n)$ is p-adically complete and separated. Thus, $Q'(\beta_0)^2 = \tau_n^2 v$ where v is an invertible element of $\mathrm{M}_{h\times h}(R_n)$. Hence, $Q'(\beta_0)$ is not a zero divisor in $\mathrm{M}_{h\times h}(R_n)$ and, since $p^\varepsilon = Q'(\beta_0)^2 \frac{p^\varepsilon}{\tau_n^2} v^{-1}$, we have $p^\varepsilon \mathrm{M}_{h\times h}(R_n) \subset Q'(\beta_0)\frac{p^\varepsilon}{\tau_n^2} \mathrm{M}_{h\times h}(R_n)$.

Let A be the R_n-subalgebra of $\mathrm{M}_{h\times h}(R_n)$ of the elements commuting with β_0. Then, A is p-adically complete and separated as well. Put $\beta_1 = \beta_0$. For $m \geqslant 2$ we construct β_m in A such that $Q(\beta_m) \equiv 0$ modulo $Q'(\beta_0)^2 (\frac{p^\varepsilon}{\tau_n^2})^m \mathrm{M}_{h\times h}(R_n)$ and $\beta_m \equiv \beta_{m-1}$ mod $Q'(\beta_0)(\frac{p^\varepsilon}{\tau_n^2})^{m-1} \times \mathrm{M}_{h\times h}(R_n)$. Since $Q'(\beta_0)$ is not a zero divisor of $\mathrm{M}_{h\times h}(R_n)$, the sequence $\{\beta_m\}_m$ converges to an element $\beta \in A$ satisfying the requirements of the lemma.

Suppose that β_m has been constructed. We may write $Q(\beta_m) = -Q'(\beta_0)^2 (\frac{p^\varepsilon}{\tau_n^2})^m \delta$. Since $Q'(\beta_0)$ and $\frac{p^\varepsilon}{\tau_n^2}$ are not zero divisors in $\mathrm{M}_{h\times h}(R_n)$ and β_m lies in A, we have $\delta \in A$. Put $\beta_{m+1} = \beta_m + Q'(\beta_0)(\frac{p^\varepsilon}{\tau_n^2})^m \delta$. Since δ and β_m lie in A, we have

$Q(\beta_{m+1}) = Q(\beta_m) + (\beta_m\delta + \delta\beta_m)Q'(\beta_0)(\frac{p^\varepsilon}{\tau_n^2})^m - \tau_n\delta Q'(\beta_0)(\frac{p^\varepsilon}{\tau_n^2})^m$, up to terms congruent to zero modulo $Q'(\beta_0)^2(\frac{p^\varepsilon}{\tau_n^2})^{m+1}\mathrm{M}_{h\times h}(R_n)$. We need to check that $\beta_m\delta + \delta\beta_m - \tau_n\delta = Q'(\beta_0)\delta$ modulo $Q'(\beta_0)\frac{p^\varepsilon}{\tau_n}\mathrm{M}_{h\times h}(R_n)$. Remark that $\beta_m = \beta_0$ modulo $Q'(\beta_0)\frac{p^\varepsilon}{\tau_n}\mathrm{M}_{h\times h}(R_n)$. Thus, the equality we have to check becomes $(2\beta_0 - \tau_n)\delta = Q'(\beta_0)\delta$, which is trivially true. □

B.5. DEFINITION. – For every n with $1 - \frac{\ell}{p^n} > \varepsilon > \frac{\ell}{p^n}$ let $\alpha_n \in \mathrm{M}_{h\times h}(R_n)$ be a lift of $\overline{\alpha}_n$ modulo $p^{\varepsilon+\frac{\ell}{p^n}}$ such that $\alpha_n^2 - p^{\frac{\ell}{p^n}}\alpha_n = 0$ as in B.4. Let

$$B_n := \big(R_n^h/\big(\alpha_n - p^{\frac{\ell}{p^n}}\big)\big) \bmod p\text{-torsion},$$

where p-torsion is the R_n-submodule of $R_n^h/(\alpha_n - p^{\frac{\ell}{p^n}})$ of elements killed by some power of p. Denote by

$$f_n : R_n^h \to B_n$$

the natural projection. Since R_n is p-torsion free, there is a unique R_n-linear map

$$g_n : B_n \to R_n^h$$

through which α_n factors. Note that $f_n \circ g_n$ is multiplication by $p^{\frac{\ell}{p^n}}$.

B.6. The map $\overline{\beta}_n$

Consider the following diagram

$$\begin{array}{ccccc}
\mathbf{B}(n)/\bar\pi_0^{\varepsilon+\frac{\ell}{p^n}}\mathbf{B}(n) & \xrightarrow{\overline{g}_n} & (\mathbf{E}_R^+(n)/\bar\pi_0^{\varepsilon+\frac{\ell}{p^n}}\mathbf{E}_R(n))^h & \xrightarrow{\overline{f}_n} & \mathbf{B}(n)/\bar\pi_0^{\varepsilon+\frac{\ell}{p^n}}\mathbf{B}(n) \\
 & & \wr\downarrow & & \\
B_n/p^{\varepsilon+\frac{\ell}{p^n}}B_n & \xrightarrow{g_n} & (R_n/p^{\varepsilon+\frac{\ell}{p^n}}R_n)^h & \xrightarrow{f_n} & B_n/p^{\varepsilon+\frac{\ell}{p^n}}B_n
\end{array}$$

where the vertical isomorphism is induced by $\mathbf{E}_R^+(n)/\bar\pi_0^{\varepsilon+\frac{\ell}{p^n}}\mathbf{E}_R^+(n) \cong R_n/p^{\varepsilon+\frac{\ell}{p^n}}R_n$. Consider the homomorphisms of R_n-modules:

$$\overline{f}_n \circ g_n : B_n/p^{\varepsilon+\frac{\ell}{p^n}}B_n \to \mathbf{B}(n)/\bar\pi_0^{\varepsilon+\frac{\ell}{p^n}}\mathbf{B}(n)$$

and

$$f_n \circ \overline{g}_n : \mathbf{B}(n)/\bar\pi_0^{\varepsilon+\frac{\ell}{p^n}}\mathbf{B}(n) \to B_n/p^{\varepsilon+\frac{\ell}{p^n}}B_n.$$

The image of g_n modulo $p^{\varepsilon+\frac{\ell}{p^n}}$ coincides with the image of $\overline{\alpha}_n$ i.e., with the image of $\overline{g}_n$ modulo $\bar\pi_0^{\varepsilon+\frac{\ell}{p^n}}$. Hence, the image of $\overline{f}_n \circ g_n$ coincides with the image of $\overline{f}_n \circ \overline{g}_n$ which is multiplication by $\bar\pi_0^{\frac{\ell}{p^n}}$. Since $\mathbf{B}(n)$ has no $\bar\pi_K$-torsion, the map $\mathbf{B}(n)/\bar\pi_0^{\varepsilon}\mathbf{B}(n) \to \bar\pi_0^{\frac{\ell}{p^n}}(\mathbf{B}(n)/\bar\pi_0^{\varepsilon+\frac{\ell}{p^n}}\mathbf{B}(n))$ given by multiplication by $\bar\pi_0^{\frac{\ell}{p^n}}$ is an isomorphism. Define the homomorphism of R_n-modules

$$\overline{\beta}_n : B_n/p^{\varepsilon}B_n \to \mathbf{B}(n)/\bar\pi_0^{\varepsilon}\mathbf{B}(n)$$

by requiring that $\bar{\pi}_0^{\frac{\ell}{p^n}}\overline{\beta}_n := \overline{f}_n \circ g_n$. Analogously, since B_n has no non-trivial p-torsion by construction, there is a unique homomorphism of R_n-modules

$$\overline{\gamma}_n : \mathbf{B}(n)/\bar{\pi}_0^{\varepsilon}\mathbf{B}(n) \to B_n/p^{\varepsilon}B_n$$

such that $p^{\frac{\ell}{p^n}}\overline{\gamma}_n = f_n \circ \overline{g}_n$. Furthermore, $\overline{\gamma}_n \circ \overline{\beta}_n = \mathrm{Id}$ and $\overline{\beta}_n \circ \overline{\gamma}_n = \mathrm{Id}$. Hence, $\overline{\beta}_n$ is an isomorphism.

B.7. LEMMA. – *Let $n \in \mathbf{N}$ be such that $\varepsilon > \frac{22\ell}{p^n}$. Then, there exists an associative and commutative R_n-bilinear map $\mu_n : B_n \times B_n \to B_n$ lifting the multiplication $\overline{\mu}_n$ defined modulo p^{ε} via the isomorphism $\overline{\beta}_n : B_n/p^{\varepsilon}B_n \cong \mathbf{B}(n)/\bar{\pi}_0^{\varepsilon}\mathbf{B}(n)$.*

Proof. – We construct an increasing sequence $\{\delta_r\}_{r\in\mathbf{N}}$ of rational numbers with $\delta_0 = \varepsilon$ and $\delta_r \to \infty$ and, for every $r \in \mathbf{N}$, an associative and commutative R_n-bilinear map $\mu_n^r : B_n \times B_n \to B_n/p^{\delta_r}B_n$ such that $\mu_n^{r+1} \equiv \mu_n^r$ modulo $p^{\delta_r - \frac{14\ell}{p^n}}$. Since B_n is finite as R_n-module, it is p-adically complete and separated. Hence, $\mu_n := \lim_r \mu_n^r$ satisfies the requirement of the proposition.

Write $\tau_n := p^{\frac{\ell}{p^n}}$. We proceed by induction on r. Put $\delta_r = \varepsilon$ and $\mu_n^1 = \overline{\mu}_n$. Suppose that μ_n^r has been constructed. Consider a R_n-bilinear map

$$\xi : R_n^h \times R_n^h \to R_n^h$$

such that $f_n \circ \xi(a,b) \equiv \mu_n^r \circ (f_n \times f_n)(a,b)$ modulo p^{δ_r}. Define the map

$$\delta_n : (\tau_n B_n) \times (\tau_n B_n) \to B_n/p^{\delta_r+\varepsilon}B_n$$

as $\delta_n(\tau_n a, \tau_n b) := f_n \circ \xi \circ (g_n(a) \times g_n(b))$. Then, $\delta_n(\tau_n a, \tau_n b) \equiv \tau_n^2 \mu_n^r(a,b)$ modulo p^{δ_r}. Consider the map

$$\begin{aligned} b_0 \otimes b_1 \otimes b_2 \otimes b_3 \otimes b_4 \xrightarrow{\gamma_n} \; & \tau_n b_0 \cdot \big(\delta_n\big(\tau_n b_1, \tau_n \delta_n(\tau_n b_2, \tau_n b_3)\big) \\ & -\delta_n\big(\tau_n \delta_n(\tau_n b_1, \tau_n b_2), \tau_n b_3\big)\big) \cdot \tau_n b_4 \end{aligned}$$

for $b_i \in B_n$. Since μ_n^r is associative, the term in parenthesis lies in $p^{\delta_r}(B_n/p^{\delta_r+p^{\varepsilon}}B_n)$. Since B_n has no non-trivial p-torsion, this is isomorphic to $B_n/p^{\varepsilon}B_n$. The latter may be identified with $\mathbf{B}(n)/\bar{\pi}_0^{\varepsilon}\mathbf{B}(n)$ via $\overline{\beta}_n$. In particular, γ_n is well defined and we may view γ_n as a $\mathbf{E}_R^+(n)$-linear map

$$\gamma_n : \mathbf{B}(n) \otimes_{\mathbf{E}_R^+(n)} \mathbf{B}(n) \otimes_{\mathbf{E}_R^+(n)} \mathbf{B}(n) \otimes_{\mathbf{E}_R^+(n)} \mathbf{B}(n) \otimes_{\mathbf{E}_R^+(n)} \mathbf{B}(n) \to \mathbf{B}(n)/\bar{\pi}_0^{\varepsilon}\mathbf{B}(n).$$

Using that for every x and every $y \in B_n$ we have

$$\delta_n\big(\tau_n x, \tau_n p^{\delta_r} y\big) = \mu_n^1(\tau_n x, \tau_n y) \cdot p^{\delta_r} = \tau_n^2 x \cdot p^{\delta_r} y,$$

one checks that γ_n is a 3-cocycle for the Hochschild cohomology of $\mathbf{B}(n)$ considered with its natural structure of $\mathbf{B}(n) \otimes_{\mathbf{E}_R^+(n)} \mathbf{B}(n)$-bimodule i.e., that

$$\begin{aligned} & \gamma_n(c_0 c_1 \otimes c_2 \otimes \cdots \otimes c_5) - \gamma_n(c_0 \otimes c_1 c_2 \otimes \cdots \otimes c_5) + \cdots \\ & \quad + (-1)^4 \gamma_n(c_0 \otimes \cdots c_3 \otimes c_4 c_5) = 0 \end{aligned}$$

for every $c_i \in \mathbf{B}(n)$. In particular, γ_n defines a class in $\mathrm{H}^3(\mathbf{B}(n)/\mathbf{E}_R^+(n), \mathbf{B}(n)/\bar{\pi}_0^{\varepsilon}\mathbf{B}(n))$. By [6, Rmk. (v) p. 259] the latter Hochschild cohomology group is killed by τ_n. More precisely, consider the $\mathbf{E}_R^+(n)$-linear map

$$h_n : \mathbf{B}(n)^{\otimes^4_{\mathbf{E}_R^+(n)}} \to \mathbf{B}(n)^{\otimes^5_{\mathbf{E}_R^+(n)}} \xrightarrow{-\gamma_n} \mathbf{B}(n)/\bar{\pi}_0^{\varepsilon}\mathbf{B}(n)$$

where the map on the left is given by

$$c_0 \otimes \cdots \otimes c_3 \mapsto (\tau_n \bar{\mathfrak{e}}_n) \cdot c_0 \otimes \cdots \otimes c_3 = \sum_{i=1}^{h} \bar{a}_i^{\frac{1}{p^n}} \otimes \bar{b}_i^{\frac{1}{p^n}} c_0 \otimes \cdots \otimes c_3.$$

Then, $\tau_n\gamma_n$ is the coboundary $-dh_n$ i.e.,

$$\begin{aligned}&\tau_n^3\delta_n\big(\tau_n a, \tau_n\delta_n(\tau_n b, \tau_n c)\big) - \tau_n^3\delta_n\big(\tau_n\delta_n(\tau_n a, \tau_n b), \tau_n c\big)\\ &\quad = \big(-a \cdot h_n(b,c) + h_n(ab,c) - h_n(a,bc) + h_n(a,b)\cdot c\big)p^{\delta_r}\end{aligned}$$

for every a, b and $c \in B_n$; in the formula we write $h_n(x,y)$ for $h_n(1\otimes x \otimes y \otimes 1)$. Consider the map

$$m_n^{r+1} : \big(\tau_n^7 B_n\big) \times \big(\tau_n^7 B_n\big) \to \tau_n^7\big(B_n/p^{\delta_r+\varepsilon}B_n\big),$$

given by

$$m_n^{r+1}\big(\tau_n^7 a, \tau_n^7 b\big) := \tau_n^{12}\delta_n(\tau_n a, \tau_n b) + \tau_n^7 h_n(a,b)p^{\delta_r}.$$

One checks that it is associative, R_n-bilinear and reduces to $\tau_n^{14}\mu_n^r$ modulo $p^{\delta_r+\frac{14\ell}{p^n}}$. Furthermore, its image modulo p^{ε} is contained in $\tau_n^{14}(B_n/p^{\varepsilon}B_n)$ and $\varepsilon > \frac{14\ell}{p^n}$. Since B_n has no non-trivial p-torsion, $\mu_n^{r+1} := \tau_n^{-14}m_n^{r+1}$ is a well defined R_n-bilinear map with values in $B_n/p^{\delta_r+\varepsilon-\frac{14\ell}{p^n}}$, it reduces to μ_n^r modulo $p^{\delta_r-\frac{14\ell}{p^n}}$ and it is associative modulo $B_n/p^{\delta_r+\varepsilon-\frac{21\ell}{p^n}}$.

The commutation with an element of B_n defines a biderivation with values in $p^{\delta_r}B_n$ i.e., a class of $\mathrm{H}^1(\mathbf{B}(n)/\mathbf{E}_R^+(n), \mathbf{B}(n)/\bar{\pi}_0^{\varepsilon-\frac{21\ell}{p^n}}\mathbf{B}_n)$. Since the latter group is annihilated by τ_n and 1-coboundaries are trivial in our case, we conclude that $\tau_n\mu_n^{r+1}$ is commutative. Hence, μ_n^{r+1} is commutative modulo $B_n/p^{\delta_{r+1}}B_n$ with $\delta_{r+1} := \delta_r + \varepsilon - \frac{22\ell}{p^n}$. □

B.8. LEMMA. – *The notation is as in* B.7. *There is a unique element* $\mathbf{1}$ *in* B_n *such that* $\mu_n(x, \mathbf{1}) = x$ *for every* $x \in B_n$. *In particular, the* R_n*-linear map* $R_n \to B_n$ *defined by* $1 \mapsto \mathbf{1}$ *defines the structure of* R_n*-algebra on* B_n.

Proof. – Let $u \in B_n$ be a lift of 1 in $B_n/p^{\varepsilon}B_n \cong \mathbf{B}(n)/\bar{\pi}_0^{\varepsilon}\mathbf{B}(n)$. Then, $\mu_n(\_, u)$ defines an isomorphism on B_n modulo p^{ε} and, hence, on B_n by Nakayama's lemma. Thus, there exists $\mathbf{1} \in B_n$ such that $\mu_n(\mathbf{1}, u) = u$. Using the associativity of μ_n we get that $\mu_n(\mu_n(z, \mathbf{1}), u) = \mu_n(z, \mu_n(\mathbf{1}, u)) = \mu_n(z, u)$. Hence, $\mu_n(\mu_n(z, \mathbf{1}) - z, u) = 0$ for every $z \in B_n$ i.e., $\mu_n(z, \mathbf{1}) = z$. If there are two such elements $\mathbf{1}$ and $\mathbf{1}'$, using the commutativity of μ_n, we get that $\mathbf{1}' = \mu_n(\mathbf{1}', \mathbf{1}) = \mu_n(\mathbf{1}, \mathbf{1}') = \mathbf{1}$. □

B.9. Functoriality

Let $j : \mathbf{B} \to \mathbf{B}'$ be a homomorphism of $\mathbf{E}_R^+$-algebras which are normal and étale over $\mathbf{E}_R^+[\bar{\pi}_K^{-1}]$. Possibly enlarging ℓ, we may and will assume that $\bar{\pi}_0^{\frac{\ell}{p^n}}$ kills the ramification of $\mathbf{E}_R^+(n) \subset \mathbf{B}'(n)$ as well i.e., that $\bar{\pi}_0^{\frac{\ell}{p^n}}\bar{\mathfrak{e}}'_n \in \mathbf{B}'(n) \otimes_{E_R^+(n)} \mathbf{B}'(n)$. Let B_n and B'_m for $m \geqslant n$ be good lifts of $\mathbf{B}(n)$ and $\mathbf{B}'(m)$ modulo $\bar{\pi}_0^{\varepsilon}$.

B.10. PROPOSITION. – *If $\varepsilon > \frac{18\ell}{p^n}$ there is a unique homomorphism $u_{m,n}\colon B_n \to B'_m$ of R_n-algebras such that $u_{m,n}$ modulo p^ε is the map*

$$B_n/p^\varepsilon B_n \cong \mathbf{B}(n)/\overline{\pi}_0^\varepsilon \mathbf{B}(n) \xrightarrow{j} \mathbf{B}'(n)/\overline{\pi}_0^\varepsilon \mathbf{B}'(n) \cong B'_m/p^\varepsilon B'_m.$$

Proof. – As in B.7, we construct $u_{m,n}$ modulo p^{δ_r} with $\delta_0 = \varepsilon < \delta_1 < \cdots < \delta_r < \cdots$ a sequence of rational number going to infinity. We also show that the induced homomorphism $u_{m,n}\colon B_n \to B'_n$ is unique.

Denote by μ_n and μ'_m the multiplications on B_n and B'_m respectively. Write $\tau_n := \frac{\ell}{p^n}$. Suppose that $u_{m,n}$ has been constructed modulo p^{δ_r}. Denote such map by $u^r_{m,n}$. Let $\xi\colon R_n^h \to B'_m$ be a R_n-linear homomorphism lifting $u^r_{m,n} \circ f_n$. Let

$$\rho := \xi \circ g_n\colon B_n \to B'_m/p^{\delta_r+\varepsilon} B'_n.$$

Then, by [7, §2a], the map

$$b_0 \otimes b_1 \otimes b_2 \otimes b_3 \mapsto b_0 \cdot \big(\tau_n^2 \rho(b_1 b_2) - \tau_n \rho(b_1)\rho(b_2)\big) \cdot b_3$$

defines a class in the Hochschild cohomology group $\mathrm{H}^2(\mathbf{B}(n)/\mathbf{E}_R^+(n), \mathbf{B}'(m)/\overline{\pi}_0^\varepsilon \mathbf{B}'(m))$. Here, we identify $p^{\delta_r}(B'_m/p^{\delta_r+\varepsilon}B'_m) \cong \mathbf{B}'(m)/\overline{\pi}_0^\varepsilon \mathbf{B}'(m)$ and we give it the structure of $\mathbf{B}(n) \otimes_{\mathbf{E}_R^+(n)} \mathbf{B}(n)$-module via $j \otimes j$. As in B.7 one proves that τ_n annihilates the cohomology group $\mathrm{H}^2(\mathbf{B}(n)/\mathbf{E}_R^+(n), \mathbf{B}'(m)/\overline{\pi}_0^\varepsilon \mathbf{B}'(m))$. We then conclude that $\tau_n^4 u^r_{m,n}$ admits a R_n-linear lift $\gamma\colon B_n \to B'_m/p^{\delta_r+\varepsilon}B'_m$ satisfying $\mu'_m(\gamma(x),\gamma(y)) = \tau_n^4 \gamma(\mu_n(x,y))$ for every x and y in B_n.

Given two lifts u and u' the map $(b_0,b_1,b_2) \mapsto \tau_n^4 b_0 \cdot (u'-u)(b_1) \cdot b_2$ defines a class in $\mathrm{H}^1(\mathbf{B}(n)/\mathbf{E}_R^+(n), \mathbf{B}'(m)/\overline{\pi}_0^\varepsilon \mathbf{B}'(m))$. Indeed, for every x and y in $\mathbf{B}(n)$ we have

$$\begin{aligned}\tau_n^4(u'-u)(xy) &= u'(x)u'(y) - u(x)u(y)\\ &= u'(x)\big(u'(y)-u(y)\big) + \big(u'(x)-u(x)\big)u(y)\\ &= \tau_n^4 x \cdot (u'-u)(y) + (u'-u)(x) \cdot \tau_n^4 y.\end{aligned}$$

Such class is zero if and only if $\tau_n^4 u$ and $\tau_n^4 u'$ differ by an inner derivation i.e., they are equal. Since τ_n annihilates $\mathrm{H}^1(\mathbf{B}(n)/\mathbf{E}_R^+(n), \mathbf{B}'(m)/\overline{\pi}_0^\varepsilon \mathbf{B}'(m)))$, we conclude that $\tau_n^9 u^r_{m,n}$ admits a unique lift γ as above. Since $\varepsilon > \frac{18\ell}{p^n}$, the map $u^{r+1}_{m,n} := \frac{\gamma}{\tau_n^9}\colon B_n \to B'_m/p^{\delta_r+\varepsilon-\frac{18\ell}{p^n}}B'_m$ is well defined, it is a ring homomorphism, it lifts $u^r_{m,n}$ modulo $p^{\delta_r-\frac{9\ell}{p^n}}$ and it is R_n-linear. Furthermore, since $u^{r+1}_{m,n}(1)$ is an idempotent congruent to 1 modulo p^ε, we deduce from Hensel's lemma that $u^{r+1}_{m,n}(1) = 1$. □

Appendix C. Construction of $\mathbf{A}_R$

This section is devoted to the proof of 7.6. We follow closely [10, §A.3.2.2]. Define $\pi := \mathbf{w}(\epsilon) - 1$ where $\mathbf{w}(\epsilon) \in \widetilde{\mathbf{A}}^+_{R_\infty}$ is the Teichmüller lift of ϵ.

C.1. LEMMA. – *Let $\mathbf{A}^+$ be a subring of $\widetilde{\mathbf{A}}_{R_\infty}$ containing π and p-adically complete. For every $n \in \mathbf{N}$ assume that $\mathbf{A}^+/p^n\mathbf{A}^+$ injects in $\widetilde{\mathbf{A}}_{R_\infty}/p^n\widetilde{\mathbf{A}}_{R_\infty}$ and it is π-adically complete. Suppose also that the π-adic topology on $\mathbf{A}^+/p^n\mathbf{A}^+$ is finer than the topology induced by the weak topology on $\widetilde{\mathbf{A}}_{R_\infty}/p^n\widetilde{\mathbf{A}}_{R_\infty}$ and that for $n=1$ the two topologies coincide. Then,*

(a) *the π-adic topology on $\mathbf{A}^+/p^n\mathbf{A}^+$ coincides with the induced weak topology for every n. In particular, $\mathbf{A}^+/p^n\mathbf{A}^+$ is closed in $\widetilde{\mathbf{A}}_{R_\infty}/p^n\widetilde{\mathbf{A}}_{R_\infty}$;*

(b) *$\mathbf{A}^+$ is the inverse limit $\lim_n \mathbf{A}_n^+$, as a topological ring, and it is closed in $\widetilde{\mathbf{A}}_{R_\infty}$;*

(c) *the inverse limit $\lim_n \mathbf{A}_n^+[\pi^{-1}]$ is the topological closure of $\mathbf{A}^+[\pi^{-1}]$ in $\widetilde{\mathbf{A}}_{R_\infty}$ for the weak topology.*

Proof. – For every n denote $\mathbf{A}_n^+ := \mathbf{A}^+/p^n\mathbf{A}^+$ and $\widetilde{\mathbf{A}}_n^+ := \widetilde{\mathbf{A}}_{R_\infty}^+/p^n\widetilde{\mathbf{A}}_{R_\infty}^+$. The weak topology on $\widetilde{\mathbf{A}}_{R_\infty}/p^n\widetilde{\mathbf{A}}_{R_\infty}$ has $\{\mathbf{w}(\varepsilon-1)^h\widetilde{\mathbf{A}}_n^+\}_h$ as a fundamental system of neighborhoods. Since $\pi^{p^n}-\mathbf{w}(\varepsilon-1)^{p^n}\in p^n\widetilde{\mathbf{A}}_{R_\infty}$, then $\{\pi^h\widetilde{\mathbf{A}}_n^+\}_h$ is a fundamental system of neighborhoods as well.

(a) By assumption for every $r\in\mathbf{N}$ there exists an integer $h(n,r)\geqslant r$ such that $\pi^{h(n,r)}\mathbf{A}_n^+\subset(\pi^r\widetilde{\mathbf{A}}_n^+)\cap\mathbf{A}_n^+$. We have to prove that there exists $t(n,r)\geqslant r$ such that $(\pi^{t(n,r)}\widetilde{\mathbf{A}}_n^+)\cap\mathbf{A}_n^+\subset\pi^r\mathbf{A}_n^+$. We proceed by induction on n. For $n=1$ this is an assumption. Define $t(n+1,r)$ as the maximum between $t(1,r)$ and $t(n,h(n+1,t(1,r)))$. Clearly $t(n+1,r)\geqslant r$. Let $x\in(\pi^{t(n+1,r)}\widetilde{\mathbf{A}}_{n+1}^+)\cap\mathbf{A}_{n+1}^+$. Then, the image of x in $\widetilde{\mathbf{A}}_n^+$ lies in $\pi^{h(n+1,t(1,r))}\mathbf{A}_n^+$ by inductive hypothesis. Then, there exists $y\in\mathbf{A}_{n+1}^+$ such that $x-\pi^{h(n+1,t(1,r))}y$ lies in $p^n\mathbf{A}_{n+1}^+\cap\pi^{t(1,r)}\widetilde{\mathbf{A}}_{n+1}^+$. Since $p^n\widetilde{\mathbf{A}}_{n+1}^+\cong\widetilde{\mathbf{A}}_1^+$, the latter is identified with $\mathbf{A}_1^+\cap\pi^{t(1,r)}\widetilde{\mathbf{A}}_1^+$ which is contained in $\pi^r\mathbf{A}_1^+$. Hence, x lies in $\pi^r\mathbf{A}_{n+1}^+$ as wanted.

(b) The ring $\widetilde{\mathbf{A}}_{R_\infty}$ coincides with the inverse limit $\lim_n\widetilde{\mathbf{A}}_{R_\infty}/p^n\widetilde{\mathbf{A}}_{R_\infty}$, as a topological ring. Since $\mathbf{A}^+$ is p-adically complete, this implies the claim.

(c) $\mathbf{A}_n^+[\pi^{-1}]$ has $\{\pi^h\mathbf{A}_n^+\}_{h\in\mathbf{Z}}$ as a fundamental system of neighborhoods for the topology induced from $\widetilde{\mathbf{A}}_{R_\infty}/p^n\widetilde{\mathbf{A}}_{R_\infty}$ by (a). In particular, it is closed for the weak topology. Thus, $\lim_n\mathbf{A}_n^+[\pi^{-1}]$ is closed for the weak topology as well. On the other hand, the topological closure of $\mathbf{A}^+[\pi^{-1}]$ in $\widetilde{\mathbf{A}}_{R_\infty}$ for the weak topology is p-adically complete since p-adic convergence in $\widetilde{\mathbf{A}}_{R_\infty}$ implies convergence for the weak topology. The conclusion follows. □

C.2. LEMMA. – *The notation is as in* C.1. *Let $\{z_i\}_{i\in I}$ be a subset of $\widetilde{\mathbf{A}}_{R_\infty}$ of elements whose classes modulo p lie in $\widetilde{\mathbf{E}}_{R_\infty}^+$. Assume that* (a) *either I is a finite set or that* (b) *$z_i^{-1}\in\mathbf{A}^+$ for every $i\in I$. Denote by $\mathbf{A}^+\{y_i\}_{i\in I}$ the ring of power series in the variables y_i convergent for the weak topology on $\mathbf{A}^+$. Then,*

(i) *for every n we have $\mathbf{A}^+\{y_i\}_{i\in I}/p^n\mathbf{A}^+\{y_i\}_{i\in I}\cong(\mathbf{A}^+/p^n\mathbf{A}^+)\{y_i\}_{i\in I}$;*

(ii) *the $\mathbf{A}^+$-linear map $\mathbf{A}^+[y_i]_{i\in I}\to\widetilde{\mathbf{A}}_{R_\infty}$, sending $y_i\mapsto z_i$ for every i, extends to a unique map $\mathbf{A}^+\{y_i\}_{i\in I}\to\widetilde{\mathbf{A}}_{R_\infty}$;*

(iii) *the image of $\mathbf{A}^+\{y_i\}_{i\in I}\to\widetilde{\mathbf{A}}_{R_\infty}/p^n\widetilde{\mathbf{A}}_{R_\infty}$ coincides with the ring of π-adically convergent power series $(\mathbf{A}^+/p^n\mathbf{A}^+)\{z_i\}_{i\in I}$;*

(iv) *the π-adic topology on $(\mathbf{A}^+/p^n\mathbf{A}^+)\{z_i\}_{i\in I}$ is finer than the topology induced by the weak topology on $\widetilde{\mathbf{A}}_{R_\infty}/p^n\widetilde{\mathbf{A}}_{R_\infty}$.*

Proof. – (i) The topology on $\mathbf{A}^+$ induced from the weak topology on $\widetilde{\mathbf{A}}_{R_\infty}$ coincides with the inverse limit topology $\lim_n\mathbf{A}_n^+$. By assumption, the topology on $\mathbf{A}_n^+$ induced from the weak topology on $\widetilde{\mathbf{A}}_{R_\infty}/p^n\widetilde{\mathbf{A}}_{R_\infty}$ is the π-adic topology. In particular, the π-adic convergence on $\mathbf{A}^+$ implies convergence for the weak topology. Since $\mathbf{A}^+$ is closed, it is π-adically complete. The claim follows.

(ii)–(iv) Note that $\pi^{p^n}-\mathbf{w}(\varepsilon-1)^{p^n}$ lies in $p^n\widetilde{\mathbf{A}}_{R_\infty}$. In particular, the weak topology on $\widetilde{\mathbf{A}}_{R_\infty}/p^n\widetilde{\mathbf{A}}_{R_\infty}$ has $\{\pi^h\widetilde{\mathbf{A}}_{R_\infty}^+\}_h$ as fundamental system of neighborhoods and the ring $\widetilde{\mathbf{A}}_{R_\infty}^+/p^n\widetilde{\mathbf{A}}_{R_\infty}^+$ is π-adically complete. By assumption $\mathbf{w}(z_i)$ is in $\widetilde{\mathbf{A}}_{R_\infty}^+$ and $z_i^{p^n}=\mathbf{w}(z_i)^{p^n}$ modulo $p^n\widetilde{\mathbf{A}}_{R_\infty}$ for every i. By the assumption on the topology of $\mathbf{A}_n^+$ there exists $h\in\mathbf{N}$ such that $\pi^h \mathbf{A}_n^+ \subset \widetilde{\mathbf{A}}_{R_\infty}^+/p^n\widetilde{\mathbf{A}}_{R_\infty}^+$. In particular, the power series in the elements $\{z_i\}_{i\in I}$ with coefficients in $\mathbf{A}_n^+$ and convergent for the π-adic topology, which coincide in case (b) with the power series in the elements $\{z_i^{p^n}\}_{i\in I}$, converge in $\widetilde{\mathbf{A}}_{R_\infty}/p^n\widetilde{\mathbf{A}}_{R_\infty}$ as well. Furthermore, the π-adic topology is finer than the weak topology induced from $\widetilde{\mathbf{A}}_{R_\infty}/p^n\widetilde{\mathbf{A}}_{R_\infty}$. □

The first step in the construction of $\mathbf{A}_R$ is the case of $R^0_{\mathbf{W}(k)} = \mathbf{W}(k)\{T_1^{\pm 1}, \ldots, T_d^{\pm 1}\}$.

C.3. Construction of $\mathbf{A}_{R^0_{\mathbf{W}(k)}}$

Note that $\mathbf{E}^+_{\mathbf{W}(k)} \cong k[\![\epsilon - 1]\!]$ by 4.6 and $\mathbf{E}^+_{R_{\mathbf{W}(k)^0}} = \mathbf{E}^+_{\mathbf{W}(k)}\{x_1^{\pm 1}, \ldots, x_d^{\pm 1}\}$ by 4.7. The weak topology on $\widetilde{\mathbf{A}}^+_{R^0_{\mathbf{W}(k)_\infty}}/p^n\widetilde{\mathbf{A}}^+_{R^0_{\mathbf{W}(k)_\infty}}$ is the $\mathbf{w}(\varepsilon - 1)$-adic topology. It is the π-adic topology as well because $\pi - \mathbf{w}(\varepsilon - 1)$ lies in $p\widetilde{\mathbf{A}}^+_{R^0_{\mathbf{W}(k)_\infty}}$. For every n define $\mathbf{A}_n^+$ as the π-adic completion of the ring

$$\mathbf{W}_n(k)\big[\mathbf{w}(\epsilon), \mathbf{w}(x_1)^{\pm 1}, \ldots, \mathbf{w}(x_d)^{\pm 1}\big]$$

generated over $\mathbf{W}_n(k)$ by the Teichmüller lifts $\mathbf{w}(\epsilon)$ and $\mathbf{w}(x_j)^{\pm 1}$ for $j = 1, \ldots, d$. It is a subring $\widetilde{\mathbf{A}}^+_{R^0_{\mathbf{W}(k)_\infty}}/p^n\widetilde{\mathbf{A}}^+_{R^0_{\mathbf{W}(k)_\infty}}$, since the latter is π-adically complete, and the π-adic topology on $\mathbf{A}_n^+$ is finer than the topology induced from $\widetilde{\mathbf{A}}^+_{R^0_{\mathbf{W}(k)_\infty}}/p^n\widetilde{\mathbf{A}}^+_{R^0_{\mathbf{W}(k)_\infty}}$. It follows from 4.15 that for $n = 1$ the two topologies coincide. Define $\mathbf{A}^+_{R^0_{\mathbf{W}(k)}}$ as the inverse limit $\lim_n \mathbf{A}_n^+$. Since $\mathbf{A}_m^+/p^n\mathbf{A}_m^+ \cong \mathbf{A}_n^+$ for every $m \geqslant n$, we have $\mathbf{A}^+_{R^0_{\mathbf{W}(k)}}/p^n\mathbf{A}^+_{R^0_{\mathbf{W}(k)}} \cong \mathbf{A}_n^+$. Thus, $\mathbf{A}^+_{R^0_{\mathbf{W}(k)}}$ satisfies the assumptions of C.1 and it is closed in $\widetilde{\mathbf{A}}^+_{R^0_{\mathbf{W}(k)_\infty}}$.

Each $\mathbf{A}_n^+$ is stable under the actions of φ and Γ_R on $\widetilde{\mathbf{A}}_{R^0_{\mathbf{W}(k)_\infty}}/p^n\widetilde{\mathbf{A}}_{R^0_{\mathbf{W}(k)_\infty}}$. Hence, φ and Γ_R act continuously on $\mathbf{A}^+_{R^0_{\mathbf{W}(k)}}$ by restriction from $\widetilde{\mathbf{A}}_{R^0_{\mathbf{W}(k)_\infty}}$. Eventually, define $\mathbf{A}_{R^0_{\mathbf{W}(k)}}$ as the closure for the weak topology of $\mathbf{A}_{R^0_{\mathbf{W}(k)}}[\pi^{-1}]$. It has the properties claimed in 7.6 thanks to C.1.

C.4. Construction of $\mathbf{A}_{R^0}$

Since $\mathbf{E}^+_{\mathbf{W}(k)} \subset \mathbf{E}^+_V$ is a finite extension of dvrs generically separable and with perfect residue fields, it is monogenic i.e., $\mathbf{E}^+_V \cong \mathbf{E}^+_{\mathbf{W}(k)}[t]/(\overline{f})$ with $\overline{f}(t)$ monic of degree m. Let $f \in \mathbf{A}^+_{R^0_{\mathbf{W}(k)}}[T]$ be a lift of $\overline{f}$ as a monic polynomial of degree m. Define $\mathbf{A}^+_{R^0} := \mathbf{A}^+_{R^0_{\mathbf{W}(k)}}[T]/(f(T))$ and $\mathbf{A}_{R^0} := \mathbf{A}^+_{R^0} \otimes_{\mathbf{A}^+_{R^0_{\mathbf{W}(k)}}} \mathbf{A}_{R^0_{\mathbf{W}(k)}}$. Note that $\mathbf{A}_{R^0}$ is finite and free as $\mathbf{A}_{R^0_{\mathbf{W}(k)}}$-module and $\mathbf{A}_{R^0}/p\mathbf{A}_{R^0} = \mathbf{E}_{R^0_{\mathbf{W}(k)}}[t]/(\overline{f}) = \mathbf{E}_{R^0}$. The latter is an étale $\mathbf{E}_{R^0_{\mathbf{W}(k)}}$-algebra of degree m. In particular, $\mathbf{A}_{R^0}$ is the unique étale $\mathbf{A}_{R^0_{\mathbf{W}(k)}}$-algebra lifting the $\mathbf{E}_{R^0_{\mathbf{W}(k)}}$-algebra $\mathbf{E}_{R^0}$. By étaleness there is a unique homomorphism of $\mathbf{A}_{R^0_{\mathbf{W}(k)}}$-algebras $\mathbf{A}_{R^0} \to \widetilde{\mathbf{A}}_{R^0_\infty}$ lifting the inclusion $\mathbf{E}_{R^0} \subset \widetilde{\mathbf{E}}_{R^0_\infty}$. It is injective since it is injective modulo p and $\mathbf{A}_{R^0_{\mathbf{W}(k)}}$, and hence $\mathbf{A}_{R^0}$, is p-adically separated. Furthermore, by uniqueness of étale lifts $\mathbf{A}_{R^0} \subset \widetilde{\mathbf{A}}_{R^0_\infty}$ is stable under the actions of Γ_R and φ.

Since $\mathbf{A}^+_{R^0}$ coincides with the image $\mathbf{A}^+_{R^0_{\mathbf{W}(k)}}\{T\} \to \widetilde{\mathbf{A}}_{R^0_\infty}$ and since $\mathbf{A}^+_{R^0}/p\mathbf{A}^+_{R^0} = \mathbf{E}^+_{R^0}$, we conclude from C.2 that $\mathbf{A}^+_{R^0}$ satisfies the assumptions of C.1. In particular, $\mathbf{A}^+_{R^0}$ and $\mathbf{A}_{R^0}$ are closed in $\widetilde{\mathbf{A}}_{R^0_\infty}$.

We now construct $\mathbf{A}_R$. We proceed as follows. Let $R^0 \subset R^1 \subset R$ be extensions obtained iterating the operations given in 7.6. Note that R^1 fulfills the hypotheses of 2.2. Suppose that we have constructed $\mathbf{A}^+_{R^1} \subset \widetilde{\mathbf{A}}_{R^1_\infty}$ satisfying the assumptions of C.1 and such that $\mathbf{A}^+_{R^1}/p\mathbf{A}^+_{R^1} \cong \mathbf{E}^+_{R^1}$ and the closure $\mathbf{A}_{R^1}$ of $\mathbf{A}^+_{R^1}[\pi^{-1}]$ in $\widetilde{\mathbf{A}}_{R^1_\infty}$ for the weak topology is stable under φ and Γ_R. Let $R^1 \subset R^2$ be obtained applying (ét), (loc) or (comp) of 7.6. We then construct $\mathbf{A}^+_{R^2} \subset \widetilde{\mathbf{A}}_{R^2_\infty}$ satisfying the assumptions of C.1 and such that $\mathbf{A}^+_{R^2}/p\mathbf{A}^+_{R^2} \cong \mathbf{E}^+_{R^2}$ and the closure $\mathbf{A}_{R^2}$ of $\mathbf{A}^+_{R^2}[\pi^{-1}]$ in $\widetilde{\mathbf{A}}_{R^2_\infty}$ for the weak topology is stable under φ and Γ_R. In particular, $\mathbf{A}_{R^2}$ has the properties given in 7.4.

C.5. Case (ét)

Let N be as in 4.5. Since $R^1_N/p^\varepsilon R^1_N \subset R^2_N/p^\varepsilon R^2_N$ is étale by assumption, $R^2_N/p^\varepsilon R^2_N \cong R^1_N/p^\varepsilon R^1_N[z_1,\ldots,z_h]/J$ with $J=(g_1,\ldots,g_h)$ and $J/J^2 \cong R^2_N/p^\varepsilon R^2_N\,dz_1 \oplus \cdots \oplus R^2_N/p^\varepsilon R^2_N\,dz_h$ by the Jacobian criterion of étaleness; see [11, 0.22.6.1]. Since $\mathbf{E}^+_{R^2}$ is π-adically complete, there is a unique prime over π and it is principal and π is not a zero divisor in $\mathbf{E}^+_{R^2}$, we get from 4.5(1) that $\mathbf{E}^+_{R^2}$ is the quotient of the ring of power series $\mathbf{E}^+_{R^1}\{\overline{y}_1,\ldots,\overline{y}_h\}$, convergent for the π-adic topology, modulo an ideal $\overline{I}=(\overline{f}_1,\ldots,\overline{f}_h)$ such that $\overline{I}/\overline{I}^2 \cong \mathbf{E}^+_{R_2}\,d\overline{y}_1 \oplus \cdots \oplus \mathbf{E}^+_{R_2}\,d\overline{y}_h$ as $\mathbf{E}^+_{R^2}$-modules.

Let $\mathbf{A}^+_{R^1}\{y_1,\ldots,y_h\}$ be the ring of power series convergent with respect to the weak topology on $\mathbf{A}^+_{R^1}$. We have $\mathbf{A}^+_{R^1}\{y_1,\ldots,y_h\}/p\mathbf{A}^+_{R^1}\{y_1,\ldots,y_h\} \xrightarrow{\sim} \mathbf{E}^+_{R^1}\{\overline{y}_1,\ldots,\overline{y}_h\}$ by C.2. Let $f_1,\ldots,f_h$ be lifts of $\overline{f}_1,\ldots,\overline{f}_h$ to $\mathbf{A}^+_{R^1}\{y_1,\ldots,y_h\}$ and define $\mathbf{A}^+_{R^2} := \mathbf{A}^+_{R^1}\{y_1,\ldots,y_h\}/I$ with $I=(f_1,\ldots,f_h)$. Modulo p it is isomorphic to $\mathbf{E}^+_{R^2}$ and, in particular, it is noetherian. Furthermore, $I/I^2 \cong \mathbf{A}^+_{R^2}\,dy_1 \oplus \cdots \oplus \mathbf{A}^+_{R^2}\,dy_h$ as $\mathbf{A}^+_{R^2}$-modules since this holds modulo p. Thus, by C.2(ii) there is a unique homomorphism of $\mathbf{A}^+_{R^1}$-algebras $\mathbf{A}^+_{R^2} \to \widetilde{\mathbf{A}}_{R^2_\infty}$ lifting the natural inclusion $\mathbf{E}^+_{R^2} \subset \widetilde{\mathbf{E}}_{R^2_\infty}$. It is injective since it is injective modulo p and $\mathbf{A}^+_{R^2}$ is p-adically separated. By C.2 it satisfies the assumptions of C.1. In particular, the closure $\mathbf{A}_{R^2}$ of $\mathbf{A}^+_{R^2}[\pi^{-1}]$ in $\widetilde{\mathbf{A}}_{R^2_\infty}$ for the weak topology coincides with $\mathbf{A}_{R^1}\{y_1,\ldots,y_h\}/I$ so that $\mathbf{A}_{R^2}$ is formally étale as $\mathbf{A}_{R^1}$-algebra. Since $\mathbf{A}_{R^1}$ is stable under Γ_R and φ, then $\mathbf{A}_{R^2}$ is stable under the actions of Γ_R and φ on $\widetilde{\mathbf{A}}_{R^2_\infty}$.

C.6. Case (loc)

In this case R^2 is the p-adic completion of the localization of R^1 with respect to a multiplicative system S. Since $R^2_n = R^2 \otimes_{R^1} R^1_n$ and R^1_n is finite and free as R^1-module, R^2_n is the p-adic completion of $R^1_n[S^{-1}]$. It follows from 4.5(1) that $\mathbf{E}^+_{R^2}$ is the π-adic completion of the localization of $\mathbf{E}^+_{R^1}$ with respect to a multiplicative system $\overline{U}$ stable under φ and Γ_R. Let $U \subset \mathbf{A}^+_{R^1}$ be the set of elements reducing to $\overline{U}$ modulo p. Using C.2 define $\mathbf{A}^+_{R^2}$ as the subring of $\widetilde{\mathbf{A}}_{R^2_\infty}$ given by $\mathbf{A}^+_{R^1}\{u^{-1}\}_{u\in U}$. By loc. cit. it satisfies the hypotheses of C.1 and $\mathbf{A}^+_{R^2}/p\mathbf{A}^+_{R^2} \cong \mathbf{E}^+_{R^2}$. Let $\mathbf{A}_{R^2}$ be the closure of $\mathbf{A}^+_{R^2}[\pi^{-1}]$ in $\widetilde{\mathbf{A}}_{R^2_\infty}$ for the weak topology. By C.1 it is also the closure of the localization of $\mathbf{A}_{R^1}$ with respect to the subset of elements reducing to $\overline{U}$ modulo p. In particular, since $\overline{U}$ and $\mathbf{A}_{R^1}$ are stable under φ and Γ_R, also $\mathbf{A}_{R^2}$ is stable under the actions of φ and Γ_R.

We are left to discuss Case (comp) i.e., R^2 is the completion with respect to an ideal J containing p. Since $R^2_n = R^2 \otimes_{R^1} R^1_n$ and R^1_n is finite and free as R^1-module, R^2_n is the completion of R^1_n with respect to JR^1_n. With the notation of 4.5(1) define $\overline{I}$ to be kernel of $\mathbf{E}^+_{R^1} \to R^1_N/(J+p^\varepsilon R^1_N)$. Since it contains $\bar{\pi}_0^{p^n\varepsilon}$, the completion of $\mathbf{E}^+_{R^1}$ with respect to the ideal $\overline{I}$ is $\bar{\pi}_0^{\varepsilon}$-adically complete and, by 4.5(1), it is isomorphic to $\mathbf{E}^+_{R^2}$ modulo any power of $\bar{\pi}_0^{\varepsilon}$. In particular, it coincides with $\mathbf{E}^+_{R^2}$. Furthermore, $\overline{I}$ is stable under the actions of φ and Γ_R. We start with the following:

C.7. LEMMA. – *For every $s \in \mathbf{N}$ the ring $\mathbf{A}^+_{R^1}/p^s\mathbf{A}^+_{R^1}$ is I-adically separated.*

Proof. – Since $\mathbf{A}^+_{R^1}/p^s\mathbf{A}^+_{R^1}$ is noetherian, by [2, Thm. 10.17] the intersection $\bigcap_n I^n$ consists of elements killed by some element of $1+I$. It suffices to show that $1+I$ does not contain zero divisors. This is proven by induction on s using that $p^n\mathbf{A}^+_{R^1}/p^{n+1}\mathbf{A}^+_{R^1} \cong \mathbf{E}^+_{R^1}$ for every integer $n \geqslant 0$ and the latter is a domain and, in particular, it does not have any non-trivial zero divisors. □

C.8. Case (comp)

Let $\mathbf{A}^+_{R^2}$ be the (p,I)-adic completion of $\mathbf{A}^+_{R^1}$. Then, $\mathbf{A}^+_{R^2}/p\mathbf{A}^+_{R^2} \cong \mathbf{E}^+_{R^2}$. Since $\mathbf{A}^+_{R^1}$ is noetherian, the extension $\mathbf{A}^+_{R^1} \to \mathbf{A}^+_{R^2}$ is flat. Since p is not a zero divisor on $\widetilde{\mathbf{A}}_{R^1_\infty}$, it is not a zero divisor on $\mathbf{A}^+_{R^2} \otimes_{\mathbf{A}^+_{R^1}} \widetilde{\mathbf{A}}_{R^1_\infty}$ either. Consider the p-adic completion $\widetilde{\mathbf{A}}_2$ of $\mathbf{A}^+_{R^2} \otimes_{\mathbf{A}^+_{R^1}} \widetilde{\mathbf{A}}_{R^1_\infty}$. Modulo p it coincides with $\mathbf{E}^+_{R^2} \otimes_{\mathbf{E}^+_{R^1}} \widetilde{\mathbf{E}}_{R^1_\infty}$ and p is not a zero divisor on $\widetilde{\mathbf{A}}_2$. Frobenius φ on $\mathbf{E}^+_{R^2}$ is injective by 4.5(2). Since $\mathbf{E}^+_{R^1} \subset \widetilde{\mathbf{E}}_{R^1_\infty} = \widetilde{\mathbf{E}}^+_{R^1_\infty}[\bar{\pi}_0^{-\varepsilon}]$ is flat by 4.15(2), also $\varphi\otimes 1$ is injective on $\mathbf{E}^+_{R^2} \otimes_{\mathbf{E}^+_{R^1}} \widetilde{\mathbf{E}}_{R^1_\infty}$. By 4.15(2) Frobenius is an automorphism on $\widetilde{\mathbf{E}}_{R^1_\infty}$. Thus, Frobenius on $\mathbf{E}^+_{R^2} \otimes_{\mathbf{E}^+_{R^1}} \widetilde{\mathbf{E}}_{R^1_\infty}$, which coincides with $\varphi\otimes\varphi$, is injective. Note that $\mathbf{E}^{+,\mathrm{perf}}_{R^2} = \mathbf{E}^+_{R^2} \otimes_{\mathbf{E}^+_{R^1}} \mathbf{E}^{+,\mathrm{perf}}_{R^1}$ by 4.7. Hence, $\varphi\otimes\varphi$ is surjective on $\mathbf{E}^+_{R^2} \otimes_{\mathbf{E}^+_{R^1}} \widetilde{\mathbf{E}}_{R^1_\infty}$ i.e., the latter is a perfect ring. In particular, $\widetilde{\mathbf{A}}_2$ coincides with the ring of Witt vectors $\mathbf{W}(\mathbf{E}^+_{R^2} \otimes_{\mathbf{E}^+_{R^1}} \widetilde{\mathbf{E}}_{R^1_\infty})$ by [15, II.5, Thm. 5]. We then get a homomorphism of $\mathbf{A}^+_{R^1}$-algebras

$$\rho\colon \mathbf{A}^+_{R^2} \to \mathbf{A}^+_{R^2} \otimes_{\mathbf{A}^+_{R^1}} \widetilde{\mathbf{A}}_{R^1_\infty} \to \widetilde{\mathbf{A}}_2 \cong \mathbf{W}\big(\mathbf{E}^+_{R^2} \otimes_{\mathbf{E}^+_{R^1}} \widetilde{\mathbf{E}}_{R^1_\infty}\big) \to \mathbf{W}(\widetilde{\mathbf{E}}_{R^2_\infty}) = \widetilde{\mathbf{A}}_{R^2_\infty}.$$

Since ρ is injective modulo p and $\mathbf{A}^+_{R^2}$ is p-adically separated, ρ is injective.

We prove that $\mathbf{A}^+_{R^2}$ satisfies the assumptions of C.1. By construction $\mathbf{A}^+_{R^2}$ is p-adically complete, $\mathbf{A}^+_{R^2}/p^n\mathbf{A}^+_{R^2}$ injects in $\widetilde{\mathbf{A}}_{R^2_\infty}/p^n\widetilde{\mathbf{A}}_{R^2_\infty}$ and it is π-adically complete. We are left to show that the π-adic topology on $\mathbf{A}^+_{R^2}/p^n\mathbf{A}^+_{R^2}$ is finer than the weak topology induced from $\widetilde{\mathbf{A}}_{R^2_\infty}/p^n\widetilde{\mathbf{A}}_{R^2_\infty}$. It follows from 4.7 and from 4.15, applied to the extension $\mathbf{W}(k) \subset V$, that there exists $t \in \mathbf{N}$ such that for every m

$$\big(\mathbf{E}^+_{R^2}\big)^{(p^m)} = \mathbf{E}^+_{R^2}\big[x_1^{\frac{1}{p^m}}, \ldots, x_d^{\frac{1}{p^m}}\big] \otimes_{\mathbf{E}^+_{\mathbf{W}(k)}} \big(\mathbf{E}^+_V\big)^{(p^m)} \subset \frac{1}{\pi^t}\mathbf{E}^+_{R^2}\big[\varepsilon^{\frac{1}{p^m}}, x_1^{\frac{1}{p^n}}, \ldots, x_d^{\frac{1}{p^m}}\big].$$

For every $z \in \mathbf{E}^+_{R^2}$ the Teichmüller lift $\mathbf{w}(z)^{\frac{1}{p^m}}$ of $z^{\frac{1}{p^m}}$ in $\widetilde{\mathbf{A}}_2/p^n\widetilde{\mathbf{A}}_2$ coincides with the p^n-th power of any lift of $z^{\frac{1}{p^{m+n}}}$. Hence, in $\widetilde{\mathbf{A}}_{R^2_\infty}/p^n\widetilde{\mathbf{A}}_{R^2_\infty}$ we have

$$\pi^{tp^n}\mathbf{W}_n\big(\mathbf{E}^{+,\mathrm{perf}}_{R^2}\big) \subset \big(\mathbf{A}^+_{R^2}/p^n\mathbf{A}^+_{R^2}\big)\big[\mathbf{w}(\varepsilon)^{\frac{1}{p^\infty}}, \mathbf{w}(x_1)^{\frac{1}{p^\infty}}, \ldots, \mathbf{w}(x_d)^{\frac{1}{p^\infty}}\big].$$

Passing to π-adic completions we then get that

$$\pi^{tp^n}\widetilde{\mathbf{A}}^+_{R^2_\infty}/p^n\widetilde{\mathbf{A}}^+_{R^2_\infty} \subset \big(\mathbf{A}^+_{R^2}/p^n\mathbf{A}^+_{R^2}\big)\big\{\mathbf{w}(\varepsilon)^{\frac{1}{p^\infty}}, \mathbf{w}(x_1)^{\frac{1}{p^\infty}}, \ldots, \mathbf{w}(x_d)^{\frac{1}{p^\infty}}\big\}.$$

Since $\mathbf{E}^+_{R^2}\{\varepsilon^{\frac{1}{p^\infty}}, x_1^{\frac{1}{p^\infty}}, \ldots, x_d^{\frac{1}{p^\infty}}\} \subset \widetilde{\mathbf{E}}^+_{R^2_\infty}$, one proves by induction on $m \leqslant n$ that

$$\pi^{mtp^n}\left(\mathbf{A}^+_{R^2}/p^m\mathbf{A}^+_{R^2}\right)\left\{\mathbf{w}(\varepsilon)^{\frac{1}{p^\infty}}, \mathbf{w}(x_1)^{\frac{1}{p^\infty}}, \ldots, \mathbf{w}(x_d)^{\frac{1}{p^\infty}}\right\} \subset \pi^{tp^n}\widetilde{\mathbf{A}}^+_{R^2_\infty}/p^m\widetilde{\mathbf{A}}^+_{R^2_\infty}.$$

Hence, $\pi^{(n-1)tp^n}(\mathbf{A}^+_{R^2}/p^n\mathbf{A}^+_{R^2}) \subset \widetilde{\mathbf{A}}^+_{R^2_\infty}/p^n\widetilde{\mathbf{A}}^+_{R^2_\infty}$. The claim follows.

Let $\mathbf{A}_{R^2}$ be the closure of $\mathbf{A}^+_{R^2}[\pi^{-1}]$ in $\widetilde{\mathbf{A}}_{R^2_\infty}$ for the weak topology. We prove that $\mathbf{A}_{R^2}$ is stable under φ and Γ_R. By C.1 and the assumption on $\mathbf{A}^+_{R^1}$, we have $\mathbf{A}_{R^1}/p^n\mathbf{A}_{R^1} = (\mathbf{A}^+_{R^1}/p^n\mathbf{A}^+_{R^1})[\pi^{-1}]$ and the weak topology induced from $\widetilde{\mathbf{A}}_{R^1_\infty}/p^n\widetilde{\mathbf{A}}_{R^1_\infty}$ coincides with the one induced from the π-adic topology on $\mathbf{A}^+_{R^1}/p^n\mathbf{A}^+_{R^1}$. Since φ and $\gamma \in \Gamma_R$ act continuously, the image of $\mathbf{A}^+_{R^1}/p^n\mathbf{A}^+_{R^1}$ under φ (resp. γ) is contained in $\frac{1}{\pi^h}(\mathbf{A}^+_{R^1}/p^n\mathbf{A}^+_{R^1})$ for some h. If $x \in I$, since $\overline{I}$ is stable under φ (resp. Γ_R), there exists $y \in I$ such that $\varphi(x) \equiv y$ (resp. $\gamma(x) \equiv y$) modulo p. Then, y^{p^n} coincides with $\varphi(x)^{p^n}$ (resp. $\gamma(x)^{p^n}$) in $\widetilde{\mathbf{A}}_{R^1_\infty}/p^n\widetilde{\mathbf{A}}_{R^1_\infty}$. We conclude that the system of neighborhoods $\left\{I^s\mathbf{A}^+_{R^1}\right\}_s$ is preserved by φ (resp. γ). It follows from C.7 that Cauchy sequences relative to the given system of neighborhoods have at most one limit in $\widetilde{\mathbf{A}}_{R^2_\infty}/p^n\widetilde{\mathbf{A}}_{R^2_\infty}$. In particular, $\mathbf{A}^+_{R^1}/p^n\mathbf{A}^+_{R^1} \to \widetilde{\mathbf{A}}_{R^2_\infty}/p^n\widetilde{\mathbf{A}}_{R^2_\infty}$ extends in at most one way to $(\mathbf{A}^+_{R^2}/p^n\mathbf{A}^+_{R^2})[\pi^{-1}]$. Since ρ defines such an extension and φ (resp. γ) preserves the ring $\mathbf{A}_{R^1}$ and is continuous for the topology defined by the system $\left\{I^s\mathbf{A}^+_{R^1}\right\}_s$, then $(\mathbf{A}^+_{R^2}/p^n\mathbf{A}^+_{R^2})[\pi^{-1}]$ is stable under φ (resp. γ). We conclude from C.1 that $\mathbf{A}_{R^2}$ is stable under φ and Γ_R.

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