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Hypersonic similarity for the two dimensional steady potential flow with large data

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Abstract

In this paper, we establish the first rigorous mathematical result on the validation of the hypersonic similarity globally, which is also called the Mach-number independence principle, for the two dimensional steady potential flow. The hypersonic similarity is equivalent to the Van Dyke's similarity theory, that is, if the hypersonic similarity parameter K is fixed, the shock solution structures (after scaling) are consistent, when the Mach number of the flow is sufficiently large. One of the difficulty is that after scaling, the solutions are usually of large data since the perturbation of the hypersonic flow is usually not small related to the sonic speed. In order to make it, we first develop a modified Glimm scheme to construct the approximate solutions with large data and find fine structure of the elementary wave curves to obtain the global existence of entropy solutions with large data, for fixed K and sufficiently large Mach number of the incoming flow M_∞. Finally, we further show that for a fixed hypersonic similarity parameter K, if the Mach number $M_\infty \to \infty$, the solutions obtained above approach to the solution of the corresponding initial-boundary value problem of the hypersonic small-disturbance equations. Therefore, the Van Dyke's similarity theory is verified rigorously for the first time.

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Keywords: Hypersonic similarity; Steady potential flow; Hypersonic flow; Glimm scheme; BV solutions with large data

1. Introduction and main result

The flow is called hypersonic when the Mach number of the flow is bigger than five. Since 1940s, there are many studies on the hypersonic flow (see [20] for example) due to many applications in areodynamics and engineering. The main difficulty on the study of the hypersonic flow is that the density is relatively very small compared to the speed,

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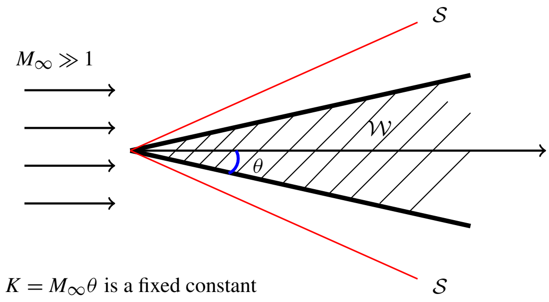

Fig. 1. Hypersonic flow past over a slender wedge.

so like the fluids behaviour near the vacuum, all the characteristics are close to each other and the shock layer is thin. On the other hand, there is one important feature of the hypersonic flow, which is called the hypersonic similarity. This property is of great significance on both the theoretical and experimental research of the thin shock layer for the hypersonic flow (see [3] for more details).

Let θ be the wedge angle and let M_∞ be the Mach number of the incoming flow (see Fig. 1). Define the similarity parameter (see (127.3) in Landau-Lifschitz [14, Page 482] for more details),

$$K = M_\infty\theta. \tag{1.1}$$

Physically, the hypersonic similarity means that for a fixed similarity parameter K, the flow structures are similar under scaling if the Mach number M_∞ is sufficiently large. Actually, after scaling, the flows with the same similarity parameter K are governed approximately by the same equation, which is called the hypersonic small-disturbance equations and was first developed by Tsien [20] for the two-dimensional steady irrotational flow and the three-dimensional axially symmetric steady flow. Recently, Qu-Yuan-Zhao [19] studied a different problem, the hypersonic limit, in which there is no hypersonic similarity structures since the wedge angle θ is fixed such that the similarity parameter K changes for all M_∞ and tends to the infinity for the hypersonic limit $M_\infty \to \infty$.

The hypersonic small-disturbance equations and the hypersonic similarity are derived as follows. Suppose the hypersonic flow is governed by

$$\begin{cases} \partial_x(\rho u) + \partial_y(\rho v) = 0, \\ \partial_x v - \partial_y u = 0, \end{cases} \tag{1.2}$$

where the density ρ and the velocity (u, v) satisfy the following Bernoulli's law:

$$\frac{1}{2}(u^2 + v^2) + \frac{\rho^{\gamma-1}}{\gamma - 1} = B_\infty := \frac{1}{2}U_\infty^2 + \frac{\rho_\infty^{\gamma-1}}{\gamma - 1}. \tag{1.3}$$

For the problem of the hypersonic flow onto a solid slender-body with boundary $y = \pm\tau b_0 x$, without loss of the generality, let us only consider the lower half space domain, *i.e.*, in the region that $x \ge 0$ and $y \le \tau b_0 x$ with a fixed constant $b_0 < 0$ in Fig. 1. The incoming flows are given by

$$(\rho, u, v)\big|_{x=0,\, y\le 0} = \big(\rho_0, u_0, v_0\big)(y). \tag{1.4}$$

Along the boundary, the flow satisfies the impermeable slip boundary condition, *i.e.*,

$$(u, v)\cdot(\tau b_0, -1) = 0. \tag{1.5}$$

Let U_∞ be a sufficiently large number. Let

$$a_\infty := \tau M_\infty = \tau U_\infty \rho_\infty^{\frac{1-\gamma}{2}}.$$

Obviously, if K is fixed, then a_∞ is fixed too. So a_∞ is also called the hypersonic similarity parameter (see Chapter 4 in [3]). As done in [3,11], we define the following scaling:

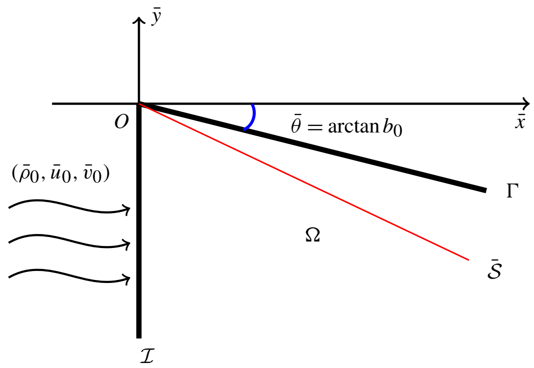

Fig. 2. Hypersonic similarity law.

$$x = \bar{x}, \quad y = \tau \bar{y}, \quad u = U_\infty(1 + \tau^2 \bar{u}), \quad v = U_\infty \tau \bar{v}, \quad \rho = \rho_\infty \bar{\rho}, \tag{1.6}$$

and substitute (1.6) into equations (1.2) and (1.3) to obtain

$$\begin{cases} \partial_{\bar{x}}\big(\bar{\rho}(1+\tau^2\bar{u})\big) + \partial_{\bar{y}}(\bar{\rho}\bar{v}) = 0, \\ \partial_{\bar{x}}\bar{v} - \partial_{\bar{y}}\bar{u} = 0, \\ \bar{u} + \frac{1}{2}(\bar{v}^2 + \tau^2\bar{u}^2) + \frac{\bar{\rho}^{\gamma-1}-1}{(\gamma-1)a_\infty^2} = 0. \end{cases} \tag{1.7}$$

The solid boundary is now given by $\bar{y} = b_0\bar{x}$. Then, the corresponding fluid domain and its boundary are given by (see Fig. 2)

$$\Omega = \{(\bar{x}, \bar{y}) : \bar{x} > 0, \ \bar{y} < b_0\bar{x}\}, \quad \Gamma = \{(\bar{x}, \bar{y}) : \bar{x} > 0, \ \bar{y} = b_0\bar{x}\}.$$

The unit normal of Γ is $\mathbf{n} = \mathbf{n}(\bar{x}, b_0\bar{x}) = \frac{(b_0,-1)}{\sqrt{1+b_0^2}}$. Initial condition (1.4) becomes

$$(\bar{\rho}, \bar{u}, \bar{v})\big|_{\mathcal{I}} = \big(\bar{\rho}_0, \bar{u}_0, \bar{v}_0\big)(\bar{y}), \qquad \mathcal{I} = \{\bar{x} = 0, \ \bar{y} \le 0\}. \tag{1.8}$$

Along Γ, condition (1.5) now becomes

$$\big((1+\tau^2\bar{u}), \bar{v}\big) \cdot \mathbf{n}\big|_\Gamma = 0. \tag{1.9}$$

Physically, the hypersonic similarity is, for a fixed similarity parameter a_∞, the structure of solutions of (1.7)-(1.9) is persistent if M_∞ is large (or τ is small). Mathematically, the structure of solutions of (1.7)-(1.9) should be investigated by the simpler equation via neglecting the terms involving τ^2, that is the hypersonic small-disturbance equations

$$\begin{cases} \partial_{\bar{x}}\bar{\rho} + \partial_{\bar{y}}(\bar{\rho}\bar{v}) = 0, \\ \partial_{\bar{x}}\bar{v} - \partial_{\bar{y}}\bar{u} = 0, \\ \bar{u} + \frac{1}{2}\bar{v}^2 + \frac{\bar{\rho}^{\gamma-1}-1}{(\gamma-1)a_\infty^2} = 0, \end{cases} \tag{1.10}$$

with initial data (1.8) and boundary condition that

$$\bar{v}\big|_\Gamma = b_0. \tag{1.11}$$

It is also called the Van Dyke's similarity theory. So if the Van Dyke's similarity theory can be justified rigorously, then the study of the two-dimensional steady hypersonic flow can be much simplified by studying of the hypersonic small-disturbance equations (1.10), because we do not face the difficulty that the characteristics are so close. On the other hand, since for the hypersonic flow, the perturbation of the velocity $(\bar{u}, \bar{v})$ is usually not small related to the sonic speed, so the solutions of (1.7) and (1.10) are usually with large data in the physical applications.

In this paper, we are going to show the Van Dyke's similarity theory rigorously. First, since the flow concerned moves along the wedge from left to right, *i.e.*, $1+\tau^2\bar{u}>0$, then from the third equation of (1.7), we have

$$\bar{u}(\bar{\rho},\bar{v};\tau^2)=\frac{1}{\tau^2}\Big(\sqrt{1-t\tau^2}-1\Big), \tag{1.12}$$

where

$$t=\frac{2\big(\bar{\rho}^{\gamma-1}-1\big)}{(\gamma-1)a_\infty^2}+\bar{v}^2. \tag{1.13}$$

Then, substituting (1.12) into the first two equations of (1.7), we get

$$\begin{cases}\partial_{\bar{x}}\big(\bar{\rho}(1+\tau^2\bar{u})\big)+\partial_{\bar{y}}(\bar{\rho}\bar{v})=0, & \text{in} \quad \Omega,\\ \partial_{\bar{x}}\bar{v}-\partial_{\bar{y}}\bar{u}=0, & \text{in} \quad \Omega.\end{cases} \tag{1.14}$$

Similarly, substituting the third equation in (1.10) into the second equation in (1.10), we have

$$\begin{cases}\partial_{\bar{x}}\bar{\rho}+\partial_{\bar{y}}(\bar{\rho}\bar{v})=0, & \text{in} \quad \Omega,\\ \partial_{\bar{x}}\bar{v}+\partial_{\bar{y}}\big(\frac{1}{2}\bar{v}^2+\frac{\bar{\rho}^{\gamma-1}-1}{(\gamma-1)a_\infty^2}\big)=0, & \text{in} \quad \Omega,\end{cases} \tag{1.15}$$

where $(\bar{\rho},\bar{v})$ satisfies the initial condition (1.18) and the boundary condition (1.11).

To unify equations (1.14) and (1.15), we rewrite $(\bar{\rho},\bar{v})$ as $(\bar{\rho}^{(\tau)},\bar{v}^{(\tau)})$, where (1.15) corresponds to the case that $\tau=0$. Let $U^{(\tau)}=(\bar{\rho}^{(\tau)},\bar{v}^{(\tau)})$ and

$$W(U^{(\tau)},\tau^2)=\Big(\bar{\rho}^{(\tau)}\big(1+\tau^2\bar{u}^{(\tau)}\big),\bar{v}^{(\tau)}\Big),\quad F(U^{(\tau)},\tau^2)=\Big(\bar{\rho}^{(\tau)}\bar{v}^{(\tau)},-\bar{u}^{(\tau)}\Big). \tag{1.16}$$

Then, equations (1.14) and (1.15) can be rewritten as

$$\partial_{\bar{x}}W(U^{(\tau)},\tau^2)+\partial_{\bar{y}}F(U^{(\tau)},\tau^2)=0, \tag{1.17}$$

with the initial condition

$$U^{(\tau)}\big|_{\mathcal{I}}=U_0(y), \tag{1.18}$$

and the boundary condition

$$\Big((1+\tau^2\bar{u}^{(\tau)}),\bar{v}^{(\tau)}\Big)\cdot(-b_0,1)\Big|_\Gamma=0. \tag{1.19}$$

Now, we will introduce the definition of the entropy solutions of problem (1.17)–(1.19).

Definition 1.1 *(Entropy solutions).* A weak solution $U^{(\tau)}\in\big(BV_{loc}(\Omega)\cap L^1_{loc}(\Omega)\big)^2$ of the initial-boundary value problem (1.17)–(1.19) in $\Omega\subset\mathbb{R}^2_+$ is called an entropy solution, if for any convex entropy pair $(\mathcal{E},\mathcal{Q})$, that is, $\nabla\mathcal{Q}(W^{(\tau)},\tau)=\nabla\mathcal{E}(W^{(\tau)},\tau^2)\nabla F(U(W^{(\tau)}),\tau^2)$ and $\nabla^2\mathcal{E}(W^{(\tau)},\tau^2)\geq 0$, the entropy inequality holds: For any $\phi\in C_0^\infty(\mathbb{R}^2)$ with $\phi\geq 0$,

$$\begin{aligned}&\iint_\Omega\Big(\mathcal{E}(W^{(\tau)},\tau^2)\partial_{\bar{x}}\phi+\mathcal{Q}(W^{(\tau)},\tau^2)\partial_{\bar{y}}\phi\Big)dxdy+\int_{-\infty}^0\mathcal{E}(W_0^{(\tau)},\tau^2)\phi(0,y)dy\\&\qquad+\int_\Gamma(\mathcal{E}(W^{(\tau)},\tau^2),\mathcal{Q}(W^{(\tau)},\tau^2))\cdot\mathbf{n}ds\geq 0,\end{aligned} \tag{1.20}$$

where $W_0^{(\tau)}=W(U_0,\tau^2)$ and $\mathbf{n}$ is the unit inner normal on boundary Γ.

The main result in this paper is stated as follows.

Theorem 1.1 *(Main theorem). Suppose that ρ_* and ρ^* are two constant states with $0 < \rho_* < \rho^* < \infty$ and $\bar{\rho}_0$ is the given initial density satisfying that $\bar{\rho}_0 \in [\rho_*, \rho^*]$. There exist constants $C > 0$, $\gamma_0 \in (1, 2)$ and $\varepsilon_0 > 0$ such that for any $\gamma \in [1, \gamma_0]$ and $\tau \in (0, \varepsilon_0)$, if*

$$(\gamma - 1 + \tau^2)\Big(T.V.\big\{(\bar{\rho}_0, \bar{v}_0) :\ (-\infty, 0]\big\} + \|b_0\|_{L^\infty}\Big) \le C, \tag{1.21}$$

then initial-boundary value problem (1.17)–(1.19) admits a global entropy solutions $(\bar{\rho}^{(\tau)}, \bar{v}^{(\tau)})$ with bounded total variations defined for all $\bar{x} > 0$, i.e.*,*

$$\sup_{\bar{x}>0} T.V.\big\{(\bar{\rho}^{(\tau)}, \bar{v}^{(\tau)})(\bar{x}, \cdot); (-\infty, b_0\bar{x}]\big\} + \sup_{\bar{x}>0} \|(\bar{\rho}^{(\tau)}, \bar{v}^{(\tau)})(\bar{x}, \cdot)\|_{L^\infty((-\infty, b_0\bar{x}])} \le \tilde{C}, \tag{1.22}$$

where the constant $\tilde{C} > 0$ is independent of $\gamma - 1$ and τ. Moreover, as $\tau \to 0$,

$$(\bar{\rho}^{(\tau)}, \bar{v}^{(\tau)}) \to (\bar{\rho}^{(0)}, \bar{v}^{(0)}) = (\bar{\rho}, \bar{v}), \qquad in \quad L^1(\Omega \cap B_{\bar{R}}(O)), \tag{1.23}$$

for any $\bar{R} > 0$, where $B_{\bar{R}}(O) = \big\{(\bar{x}, \bar{y}) : \bar{x}^2 + \bar{y}^2 \le \bar{R}\big\}$ and $(\bar{\rho}, \bar{v})$ is the entropy solution of the initial-boundary value problem (1.17)–(1.19) with $\tau = 0$, which satisfies that

$$\sup_{\bar{x}>0} T.V.\big\{(\bar{\rho}, \bar{v})(\bar{x}, \cdot); (-\infty, b_0\bar{x}]\big\} + \sup_{\bar{x}>0} \|(\bar{\rho}, \bar{v})(\bar{x}, \cdot)\|_{L^\infty((-\infty, b_0\bar{x}])} < \infty. \tag{1.24}$$

Remark 1.1. When $\tau = 0$, the convex entropy pair $(\mathcal{E}(W^{(\tau)}, \tau^2), \mathcal{Q}(W^{(\tau)}, \tau^2))$ can be taken of the form

$$\mathcal{E}(W^{(0)}, 0) = \frac{\rho v^2}{2} + \frac{\rho^{\gamma-1} - 1}{a_\infty \gamma(\gamma - 1)}, \qquad \mathcal{Q}(W^{(0)}, 0) = v\mathcal{E}(W^{(0)}, 0). \tag{1.25}$$

So the entropy solution $(\bar{\rho}, \bar{v})$ of problem (1.17)–(1.19) with $\tau = 0$ satisfies the entropy inequality

$$\partial_{\bar{x}}\mathcal{E}(W^{(0)}, 0) + \partial_{\bar{y}}\mathcal{Q}(W^{(0)}, 0) \le 0, \tag{1.26}$$

in the distribution sense.

Remark 1.2. Once the solution $(\bar{\rho}^{(\tau)}, \bar{v}^{(\tau)})$ of problem (1.17)–(1.19) is obtained, it is easy to obtain the solutions $(\bar{\rho}^{(\tau)}, \bar{u}^{(\tau)}, \bar{v}^{(\tau)})$ of problem (1.7)–(1.9) by solving $\bar{u}^{(\tau)}$ directly from equation (1.12). Therefore, in this paper, we are devoted to showing Theorem 1.1.

In this paper, we will give the first rigorous mathematical proof on the Van Dyke's similarity theory. More precisely, we will prove that solution $U^{(\tau)}$ of the initial boundary value problem (1.17)–(1.19) with large data has a limit U as $\tau \to 0$, where U is a solution of the initial boundary value problem (1.15), (1.18) and (1.11), *i.e.*, problem (1.17)–(1.19) with $\tau = 0$. To achieve this, we first establish the global existence of entropy solutions of the initial boundary value problem (1.17)–(1.19) for fixed τ with large data.

The main difficulty is that we can't apply the results in [17,18,22] directly, because equations (1.17) is different from the ones that considered in [17,18,22]. Moreover, the boundary condition (1.19) is Neumann type which is also different from the one studied in [18], which is the Dirichlet boundary type. As far as we know, there is no result on the steady supersonic Euler flow with large data. In order to deal with it, we first need to study fine structures of the elementary wave curves carefully and then derive the local wave interaction estimates. Fortunately, we find the fine structures to allow us to establish the wave interaction estimates as well as the estimates of the elementary waves reflection on the boundary. Based on them, we can choose weights $\mathscr{K}_b$ and C_* (see (4.17) below) to construct a modified Glimm's type functional and then shows it monotonicity decreasing. Now, we can follow the standard arguments to show the global existence of entropy solutions of the initial-boundary value problem (1.17)–(1.19) with uniformly bound in the BV norm independent of τ provided that (1.21) holds. Finally, by the uniformly bounds, we can further extact a subsequence to show that its limit as $\tau \to 0$ is actually a entropy solution of problem (1.17)–(1.19) with $\tau = 0$. It justifies the Van Dyke's similarity theory rigorously.

There are many literatures on the global existence of the entropy solutions of small data in the BV space for the one dimensional hyperbolic conservation laws since J. Glimm's original paper [12] in 1960s. There are also many literatures on the BV solutions of the two dimensional steady supersonic Euler flow with small data (see [6–8,10, 13,23–25]). However, there are few results on the global existence of weak solutions with large data due to the nonlinearity of the system. As far as we know, only systems with special structures can be dealt with. One of the most important example is the one dimensional isothermal gas dynamic system. The global existence of the entropy solutions of this system with large data has been proved by Nishida [16] in 1968. Then Nishida-Smoller extended the existence result to the isentropic case with the assumption that γ is sufficiently close to 1 in [17,18]. Later on, the existence result was extended to the non-isentropic case by Liu in [15]. Recently, Askura-Corli [1,2] proved these results by using the wave-front tracking method and see also [4,5,9] for the related results.

The rest of this paper is organized as follows. In Section 2, we study some basic structure for system (1.17) near $\tau = 0$, including the Riemann invariants, the fine properties of the elementary wave curves, as well as the solutions for the Riemann problem including the boundary. As a byproducts, we also give some basic structure for system (1.15), (*i.e.* $\tau = 0$) involving the Riemann invariants, the fine properties of the elementary wave curves, as well as the solutions for the Riemann problem including the boundary. Section 3 is devoted to the analysis of the local wave interaction estimates of various type. In Section 4, we construct the approximate solutions by the modified Glimm scheme, introduce the modified Glimm-type functional by choosing some weights, and then show that it is a decreasing functional, which leads to the global existence of the entropy solutions to the initial-boundary value problem (1.17)–(1.19) with large data by a standard procedure. Finally, we show that as $\tau \to 0$, it approaches to the solutions of the initial-boundary value problem (1.15), (1.18) and (1.11). In the appendix, we prove Lemma 2.9, Lemma 3.1, and Lemma 3.5.

Finally, we remark that in what follows, for the notational simplicity, we will denote $U^{(\tau)} = (\bar{\rho}^{(\tau)}, \bar{v}^{(\tau)})$ *and* $(\bar{x}, \bar{y})$ *as* $U = (\rho, v)$ *and* (x, y)*, respectively.*

2. Riemann problem of the initial-boundary value problem (1.17)-(1.19)

In this section, we will study the basic structure of system (1.17) and then consider the corresponding Riemann solutions.

2.1. Riemann invariants and the shock curves of equations (1.17)

In this subsection, we study some basic structures of the Riemann solutions of system (1.17) of large data. By direct computation, the eigenvalues of system (1.17) are

$$\begin{aligned} \lambda_-(U,\tau^2) &= \frac{v\sqrt{1-t\tau^2} - a_\infty^{-1}\rho^{\frac{\gamma-1}{2}}\sqrt{1-(\gamma-1)^{-1}(\gamma+1)a_\infty^{-2}\rho^{\gamma-1}\tau^2}}{1-(t+a_\infty^{-2}\rho^{\gamma-1})\tau^2}, \\ \lambda_+(U,\tau^2) &= \frac{v\sqrt{1-t\tau^2} + a_\infty^{-1}\rho^{\frac{\gamma-1}{2}}\sqrt{1-(\gamma-1)^{-1}(\gamma+1)a_\infty^{-2}\rho^{\gamma-1}\tau^2}}{1-(t+a_\infty^{-2}\rho^{\gamma-1})\tau^2}, \end{aligned} \tag{2.1}$$

and the corresponding right eigenvectors are

$$\begin{aligned} r_-(U,\tau^2) &= \Big(-a_\infty^2\rho^{-\frac{\gamma-1}{2}}\big(\lambda_-(U,\tau^2)+\partial_v u(\rho,v,\tau^2)\big),\ a_\infty^2\rho^{-\frac{\gamma-1}{2}}\partial_\rho u(\rho,v,\tau^2)\Big), \\ r_+(U,\tau^2) &= \Big(-a_\infty^2\rho^{-\frac{\gamma-1}{2}}\big(\lambda_+(U,\tau^2)+\partial_v u(\rho,v,\tau^2)\big),\ a_\infty^2\rho^{-\frac{\gamma-1}{2}}\partial_\rho u(\rho,v,\tau^2)\Big). \end{aligned} \tag{2.2}$$

For $u(\rho, v, \tau^2)$, we have the following lemma.

Lemma 2.1. *If* $\gamma > 1$*, then we have*

$$\partial_\rho u(\rho,v,\tau^2) = -\frac{\rho^{\gamma-2}}{a_\infty^2\sqrt{1-t\tau^2}}, \quad \partial_v u(\rho,v,\tau^2) = -\frac{v}{\sqrt{1-t\tau^2}}, \tag{2.3}$$

and

$$\partial^2_{\rho\rho}u(\rho,v,\tau^2) = -\frac{(\gamma-2)\rho^{\gamma-3}\Big(1-\big(t-(\gamma-2)^{-1}a_\infty^{-2}\rho^{\gamma-1}\big)\tau^2\Big)}{a_\infty^2(1-\tau^2 t)^{3/2}},$$
$$\partial^2_{\rho v}u(\rho,v,\tau^2) = -\frac{\rho^{\gamma-2}v\tau^2}{a_\infty^2(1-\tau^2 t)^{3/2}},\quad \partial^2_{vv}u(\rho,v,\tau^2) = -\frac{1-2(\gamma-1)^{-1}a_\infty^{-2}(\rho^{\gamma-1}-1)\tau^2}{(1-\tau^2 t)^{3/2}}, \tag{2.4}$$

where t is defined by (1.13).

Proof. First, by (1.13), we have

$$\frac{\partial t}{\partial\rho} = \frac{2\rho^{\gamma-2}}{a_\infty^2},\quad \frac{\partial t}{\partial v} = 2v.$$

From u, we also get that

$$1+\tau^2 u(\rho,v,\tau^2) = \sqrt{1-t\tau^2}.$$

So it follows that

$$\tau^2\partial_\rho u(\rho,v,\tau^2) = -\frac{1}{2}(1-t\tau^2)^{-1/2}\tau^2\frac{\partial t}{\partial\rho},\quad \tau^2\partial_v u(\rho,v,\tau^2) = -\frac{1}{2}(1-t\tau^2)^{-1/2}\tau^2\frac{\partial t}{\partial v},$$

which gives $\partial_\rho u(\rho,v,\tau^2)$ and $\partial_v u(\rho,v,\tau^2)$, respectively. With (2.3), we can further take derivatives with respect to ρ, v to derive (2.4). This completes the proof of the lemma. □

Remark 2.1. By Lemma 2.1 and (2.1), we have that

$$\lambda_\pm(U,\tau^2)+\partial_v u(\rho,v,\tau^2) = \frac{a_\infty^{-1}\rho^{\frac{\gamma-1}{2}}\left(a_\infty^{-1}\rho^{\frac{\gamma-1}{2}}v\tau^2 \pm \sqrt{\Big(1-(\gamma-1)^{-1}(\gamma+1)a_\infty^{-2}\rho^{\gamma-1}\tau^2\Big)(1-t\tau^2)}\right)}{\Big(1-\big(t+a_\infty^{-2}\rho^{\gamma-1}\big)\tau^2\Big)\sqrt{1-t\tau^2}}. \tag{2.5}$$

Lemma 2.2. *For the eigenvalues λ_+ and λ_-, we have*

$$\lambda_\pm(U,0) = v\pm\frac{\rho^{\frac{\gamma-1}{2}}}{a_\infty},\quad \lambda_\pm(U,0)+\partial_v u(\rho,v,0) = \pm\frac{\rho^{\frac{\gamma-1}{2}}}{a_\infty}, \tag{2.6}$$

and

$$r_\pm(U,0) = (a_\infty, \pm\rho^{\frac{\gamma-3}{2}}). \tag{2.7}$$

Moreover,

$$\partial_\rho\lambda_\pm(U,0) = \pm\frac{(\gamma-1)\rho^{\frac{\gamma-3}{2}}}{2a_\infty},\qquad \partial_v\lambda_\pm(U,0) = 1. \tag{2.8}$$

Proof. Firstly, by the definition of t, (2.6) and (2.7) follow directly from (2.1) and (2.2). For $\partial_\rho\lambda_\pm(U,0)$, note that the characteristic equation of system (1.17) is

$$\big(1+\tau^2(u+\rho\partial_\rho u)\big)\lambda^2 + \big((1+\tau^2 u)\partial_v u - v\big)\lambda + \rho\partial_\rho u - v\partial_v u = 0. \tag{2.9}$$

Taking derivative on (2.9) with respect to ρ to obtain that

$$\big[2\big(1+(u+\rho\partial_\rho u)\tau^2\big)\lambda + (1+\tau^2 u)\partial_v u - v\big]\partial_\rho\lambda + \tau^2\lambda^2\partial_\rho\big(u+\rho\partial_\rho u\big)$$
$$+\big[\tau^2\partial_\rho u\partial_v u + (1+\tau^2 u)\partial^2_{\rho v}u\big]\lambda + \rho\partial^2_{\rho\rho}u - v\partial^2_{\rho v}u + \partial_\rho u = 0.$$

So take $\tau=0$, we have

$$\partial_\rho \lambda(U,0) = -\frac{\rho\partial_{\rho\rho}u(\rho,v,0)+\partial_\rho u(\rho,v,0)}{2\lambda(U,0)+\partial_v u(\rho,v,0)-v},$$

which gives the expression of $\partial_\rho\lambda_\pm(U,0)$ with the help of Lemma 2.1 and (2.6).

In the same way, we can also take derivatives on (2.9) with respect to v to have

$$\partial_v \lambda(U,0) = \frac{\big(1-\partial^2_{vv}u(\rho,v,0)\big)\lambda(U,0)+v\partial^2_{vv}u(\rho,v,0)+\partial_v u(\rho,v,0)}{2\lambda(U,0)+\partial_v u(\rho,v,0)-v},$$

which implies the expression of $\partial_v\lambda_\pm(U,0)$ by employing Lemma 2.1 and (2.6) again. □

Let

$$\omega(U,\tau^2) = \big(\omega_-,\omega_+\big)(U,\tau^2) \quad (\text{or } \omega=(\omega_-,\omega_+)) \tag{2.10}$$

be the Riemann invariants satisfying

$$\nabla_U\omega_\pm(U,\tau^2)\cdot r_\pm(U,\tau^2)=0.$$

Without loss of the generality, we can assume $\omega_\pm(U,\tau^2)$ is defined by solving the following two equations

$$\partial_\rho\omega_\pm(U,\tau^2) := -a_\infty^2\rho^{-\frac{\gamma-1}{2}}\partial_\rho u(\rho,v,\tau^2) = \frac{\rho^{\frac{\gamma-3}{2}}}{\sqrt{1-t\tau^2}}, \tag{2.11}$$

and

$$\begin{aligned}\partial_v\omega_\pm(U,\tau^2) &:= -a_\infty^2\rho^{-\frac{\gamma-1}{2}}\big(\lambda_\pm(U,\tau^2)+\partial_v u(\rho,v,\tau^2)\big)\\ &= -\frac{\rho^{\frac{\gamma-1}{2}}v\tau^2 \pm a_\infty\sqrt{\Big(1-(\gamma-1)^{-1}(\gamma+1)a_\infty^{-2}\rho^{\gamma-1}\tau^2\Big)(1-t\tau^2)}}{\Big(1-\big(t+a_\infty^{-2}\rho^{\gamma-1}\big)\tau^2\Big)\sqrt{1-t\tau^2}}.\end{aligned} \tag{2.12}$$

Remark 2.2. For $\tau=0$, $\omega_\pm(U,0)$ can be expressed explicitly as

$$r := \omega_-(U,0) = a_\infty v + \frac{2(\rho^{\frac{\gamma-1}{2}}-1)}{\gamma-1}, \quad s := \omega_+(U,0) = -a_\infty v + \frac{2(\rho^{\frac{\gamma-1}{2}}-1)}{\gamma-1}. \tag{2.13}$$

Lemma 2.3. *For $\rho>0$, there exists a constant $\epsilon_1>0$ sufficiently small such that for any $\tau\in(0,\epsilon_1)$, $U=(\rho,v)$ can be represented as a function of ω. Moreover, the map $U=(\rho,v)\mapsto\omega=(\omega_-(U,\tau^2),\omega_+(U,\tau^2))$ is bijective for any fixed parameter $\rho>0$ and sufficiently small parameter τ^2. Moreover*

$$\nabla_{\omega_-}U\big|_{\tau=0} = \Big(\frac12\rho^{-\frac{\gamma-3}{2}}, \frac{1}{2a_\infty}\Big), \quad \nabla_{\omega_+}U\big|_{\tau=0} = \Big(\frac12\rho^{-\frac{\gamma-3}{2}}, -\frac{1}{2a_\infty}\Big). \tag{2.14}$$

Proof. By Lemma 2.1 and Remark 2.1,

$$\begin{aligned}\det\Big(\nabla_U\omega_-(U,\tau^2),\ \nabla_U\omega_+(U,\tau^2)\Big)\Big|_{\tau=0} &= \frac{a_\infty^2}{\rho^{\frac{\gamma-1}{2}}}\begin{vmatrix}-\partial_\rho u(\rho,v,0) & -\lambda_-(U,0)-\partial_v u(\rho,v,0)\\ -\partial_\rho u(\rho,v,0) & -\lambda_+(U,0)-\partial_v u(\rho,v,0)\end{vmatrix}\\ &= \frac{2\rho^{\gamma-2}}{a_\infty}>0.\end{aligned}$$

So it follows from the implicit function theorem that there exists a constant $\epsilon_1>0$ sufficiently small such that for any $\tau\in(0,\epsilon_1)$, U can be solved as a function of ω.

Next, we are going to prove (2.14). Taking derivatives as follows and let $\tau=0$

$$\begin{cases}\partial_\rho\omega_-(U,0)\frac{\partial\rho}{\partial\omega_-}\big|_{\tau=0}+\partial_v\omega_-(U,0)\frac{\partial v}{\partial\omega_-}\big|_{\tau=0}=1,\\ \partial_\rho\omega_+(U,0)\frac{\partial\rho}{\partial\omega_-}\big|_{\tau=0}+\partial_v\omega_+(U,0)\frac{\partial v}{\partial\omega_-}\big|_{\tau=0}=0,\end{cases}$$

which gives the expression of $\nabla_U \omega_-\big|_{\tau=0}$ in (2.14) by (2.11) and (2.12). In the same way, one can also get the expression of $\nabla_U \omega_+\big|_{\tau=0}$. We omit the argument for the shortness. □

Now, we are going to study the elementary wave curves to system (2.1) globally. Based on Lemma 2.3, we will use ω_-, ω_+ as the variables in the phase plane for the convenience.

The elementary wave curves consist of the rarefaction wave curve and the shock wave curve. First, for the rarefaction wave curve, one of the Riemann invariants corresponding to $\lambda_+(U,\tau^2)$ or $\lambda_-(U,\tau^2)$ is a constant. We denoted the rarefaction wave by $\mathcal{R}_1$ (or $\mathcal{R}_2$) corresponding to $\lambda_+(U,\tau^2)$ (or $\lambda_-(U,\tau^2)$). So, in the phase plane, the rarefaction wave curves $\mathcal{R}_1$ and $\mathcal{R}_2$ which pass through $\omega_0=(\omega_{-,0},\omega_{+,0})=(\omega_-,\omega_+)(U_0,\tau^2)$ are

$$\mathcal{R}_1:\ \omega_+=\omega_{+,0},\ \ \omega_->\omega_{-,0} \qquad \mathcal{R}_2:\ \omega_-=\omega_{-,0},\ \ \omega_+<\omega_{+,0}. \tag{2.15}$$

Next, let us consider the shock wave curves for system (4.2). The shock solutions are the Riemann solutions satisfying the following Rankine-Hugoniot conditions on the shock with shock speed $\sigma(\tau^2)$:

$$\sigma(\tau^2)[W(U,\tau^2)]=[F(U,\tau^2)], \tag{2.16}$$

where the bracket $[\cdot]$ stands for the difference of the value of the quality across the discontinuity. In addition, across the shock, the following Lax geometry entropy conditions hold:

$$\lambda_-(U,\tau^2)<\sigma_-(\tau^2)<\lambda_-(U_0,\tau^2),\ \ or\ \ \lambda_+(U,\tau^2)<\sigma_+(\tau^2)<\lambda_+(U_0,\tau^2), \tag{2.17}$$

where $\sigma_-(\tau^2)$ and $\sigma_+(\tau^2)$ are the shock speeds corresponding to $\lambda_-(U,\tau^2)$ and $\lambda_+(U,\tau^2)$, respectively. Actually, entropy condition (2.17) implies that

$$\rho>\rho_0,\ \ v<v_0,\ \ or\ \ \rho<\rho_0,\ \ v<v_0. \tag{2.18}$$

Eliminating $\sigma(\tau^2)$ from the R-H condition (2.16) yields

$$(\rho v-\rho_0 v_0)(v-v_0)=\Big(\rho-\rho_0+\tau^2\big(\rho u(\rho,v,\tau^2)-\rho_0 u(\rho_0,v_0,\tau^2)\big)\Big)\big(u(\rho_0,v_0,\tau^2)-u(\rho,v,\tau^2)\big). \tag{2.19}$$

Let $\alpha=\rho/\rho_0$ with $\rho_0>0$ and define

$$\begin{aligned}\mathscr{F}(\alpha,v,U_0;\tau^2)=&(\alpha v-v_0)(v-v_0)-\Big(\alpha-1+\tau^2\big(\alpha u(\rho_0\alpha,v,\tau^2)-u(\rho_0,v_0,\tau^2)\big)\Big)\\&\times\big(u(\rho_0,v_0,\tau^2)-u(\rho_0\alpha,v,\tau^2)\big).\end{aligned} \tag{2.20}$$

Then equation (2.19) is equivalent to equation $\mathscr{F}(\alpha,v,U_0;\tau^2)=0$. First, we will study some properties for $\mathscr{F}$ when $\tau=0$.

Lemma 2.4. *For $\mathscr{F}$ defined by (2.20) and for $\gamma>1$, equation $\mathscr{F}(\alpha,v,U_0;0)=0$ admits a unique solution v satisfying that*

$$v=v_0-\sqrt{\frac{2\rho_0^{\gamma-1}(\alpha-1)(\alpha^{\gamma-1}-1)}{(\gamma-1)a_\infty^2(\alpha+1)}}. \tag{2.21}$$

Moreover, we have

$$\begin{aligned}&\frac{\partial\mathscr{F}}{\partial\alpha}\Big|_{\tau=0}=-\frac{\rho_0^{\gamma-1}\Big(2(\alpha^{\gamma-1}-1)+(\gamma-1)\alpha^{\gamma-2}(\alpha^2-1)\Big)}{(\gamma-1)a_\infty^2(\alpha+1)},\\&\frac{\partial\mathscr{F}}{\partial v}\Big|_{\tau=0}=-\sqrt{\frac{2\rho_0^{\gamma-1}(\alpha^2-1)(\alpha^{\gamma-1}-1)}{(\gamma-1)a_\infty^2}},\\&\frac{\partial^2\mathscr{F}}{\partial\alpha^2}\Big|_{\tau=0}=-a_\infty^{-2}\rho_0^{\gamma-1}\alpha^{\gamma-3}(\gamma\alpha+2-\gamma),\\&\frac{\partial^2\mathscr{F}}{\partial\alpha\partial v}\Big|_{\tau=0}=v-v_0,\qquad \frac{\partial^2\mathscr{F}}{\partial v^2}\Big|_{\tau=0}=\alpha-1.\end{aligned} \tag{2.22}$$

Proof. (2.21) can be obtained by the direct computation together with the entropy condition (2.18). For (3.7), first for $\frac{\partial \mathscr{F}}{\partial \alpha}$, by the direct computation

$$\begin{aligned}\frac{\partial \mathscr{F}}{\partial \alpha} &= v(v-v_0) + u(\rho, v, \tau^2) - u(\rho_0, v_0, \tau^2) + \rho_0(\alpha-1)\partial_\rho u(\rho, v, \tau^2)\\ &\quad + \Big(u(\rho, v, \tau^2) + \rho_0\alpha\partial_\rho u(\rho, v, \tau^2)\Big)\Big(u(\rho, v, \tau^2) - u(\rho_0, v_0, \tau^2)\Big)\tau^2\\ &\quad + \rho_0\Big(\alpha u(\rho, v, \tau^2) - u(\rho_0, v_0, \tau^2)\Big)\partial_\rho u(\rho, v, \tau^2)\tau^2.\end{aligned}$$

So it follows from Lemma 2.1 that

$$\begin{aligned}\frac{\partial \mathscr{F}}{\partial \alpha}\Big|_{\tau=0} &= v(v-v_0) + u(\rho, v, \tau^2) - u(\rho_0, v_0, \tau^2) + \rho_0(\alpha-1)\partial_\rho u(\rho, v, \tau^2)\\ &= v(v-v_0) - \frac{1}{2}\left(v^2 + \frac{2(\rho^{\gamma-1}-1)}{(\gamma-1)a_\infty^2} - v_0^2 - \frac{2(\rho_0^{\gamma-1}-1)}{(\gamma-1)a_\infty^2}\right) - a_\infty^{-2}\rho_0^{\gamma-1}(\alpha-1)\alpha^{\gamma-2}.\end{aligned}$$

Thus the expression of $\frac{\partial \mathscr{F}}{\partial \alpha}\Big|_{\tau=0}$ in (2.22) follows with the help of (2.21).

Next, taking derivative on $\mathscr{F}$ with respect to v

$$\begin{aligned}\frac{\partial \mathscr{F}}{\partial v} &= 2\alpha v - (\alpha+1)v_0 + (\alpha-1)\partial_v u(\rho, v, \tau^2)\\ &\quad + \Big(2\alpha u(\rho, v, \tau^2) - (\alpha+1)u(\rho_0, v_0, \tau^2)\Big)\partial_v u(\rho, v, \tau^2)\tau^2.\end{aligned}$$

So

$$\frac{\partial \mathscr{F}}{\partial v}\Big|_{\tau=0} = 2\alpha v - (\alpha+1)v_0 + (\alpha-1)\partial_v u(\rho, v, 0) = (\alpha+1)(v-v_0).$$

Hence, the expression of $\frac{\partial \mathscr{F}}{\partial v}\Big|_{\tau=0}$ in (2.22) follows by (2.21) again.

In the same way as done for deriving the expression of $\frac{\partial \mathscr{F}}{\partial \rho}\Big|_{\tau=0}$ and $\frac{\partial \mathscr{F}}{\partial v}\Big|_{\tau=0}$, we can further take derivatives on $\frac{\partial \mathscr{F}}{\partial \rho}$ and $\frac{\partial \mathscr{F}}{\partial v}$ with respect to α and v and let $\tau = 0$, then (2.22) follows from Lemma 2.1. □

Remark 2.3. When $\tau = 0$, it follows from Remark 2.2, entropy condition (2.18), and the straightforward calculation that

$$S_1: \begin{cases} s_0 - s = \sqrt{\frac{2}{\gamma-1}}\rho_0^{\frac{\gamma-1}{2}}\left\{-\sqrt{\frac{(1-\alpha)(1-\alpha^{\gamma-1})}{\alpha+1}} + \sqrt{\frac{2}{\gamma-1}}(1-\alpha^{\frac{\gamma-1}{2}})\right\}\\ r_0 - r = \sqrt{\frac{2}{\gamma-1}}\rho_0^{\frac{\gamma-1}{2}}\left\{\sqrt{\frac{(1-\alpha)(1-\alpha^{\gamma-1})}{\alpha+1}} + \sqrt{\frac{2}{\gamma-1}}(1-\alpha^{\frac{\gamma-1}{2}})\right\}\end{cases} \quad 0 < \alpha \le 1, \tag{2.23}$$

and

$$S_2: \begin{cases} s_0 - s = \sqrt{\frac{2}{\gamma-1}}\rho_0^{\frac{\gamma-1}{2}}\left\{-\sqrt{\frac{(\alpha-1)(\alpha^{\gamma-1}-1)}{\alpha+1}} - \sqrt{\frac{2}{\gamma-1}}(\alpha^{\frac{\gamma-1}{2}}-1)\right\}\\ r_0 - r = \sqrt{\frac{2}{\gamma-1}}\rho_0^{\frac{\gamma-1}{2}}\left\{\sqrt{\frac{(\alpha-1)(\alpha^{\gamma-1}-1)}{\alpha+1}} - \sqrt{\frac{2}{\gamma-1}}(\alpha^{\frac{\gamma-1}{2}}-1)\right\}\end{cases} \quad \alpha \ge 1. \tag{2.24}$$

Remark 2.4. When $\gamma = 1$ and $\tau = 0$, S_1 and S_2 are of the following forms:

$$S_1: \begin{cases} s_0 - s = -\sqrt{-\frac{2(1-\alpha)}{\alpha+1}\ln\alpha} - \ln\alpha,\\ r_0 - r = \sqrt{-\frac{2(1-\alpha)}{\alpha+1}\ln\alpha} - \ln\alpha,\end{cases} \quad 0 < \alpha \le 1, \tag{2.25}$$

and

$$S_2: \begin{cases} s_0 - s = -\sqrt{\frac{2(\alpha-1)}{\alpha+1}\ln\alpha} - \ln\alpha, \\ r_0 - r = \sqrt{\frac{2(\alpha-1)}{\alpha+1}\ln\alpha} - \ln\alpha, \end{cases} \quad \alpha \geq 1. \tag{2.26}$$

Eliminating α, one has

$$r_0 - r - (s_0 - s) = 2\sqrt{\frac{1 - e^{-\frac{1}{2}(r_0 - r + s_0 - s)}}{1 + e^{-\frac{1}{2}(r_0 - r + s_0 - s)}}\Big(r_0 - r + s_0 - s\Big)}, \tag{2.27}$$

where $r_0 - r + s_0 - s \geq 0$ for the S_1 wave, and $r_0 - r + s_0 - s \leq 0$ for the S_2 wave.

Now, we will give the existence and properties of the shock wave curves near $\tau = 0$.

Lemma 2.5. *There exists a small constant $0 < \epsilon_2 < \epsilon_1$ such that for any $\tau \in (0, \epsilon_2)$, v can be solved as a function of α, U_0, τ^2 from equation $\mathscr{F}(\alpha, v, U_0; \tau^2) = 0$,* i.e., *$v = \varphi(\alpha, U_0, \tau^2)$. Moreover,*

$$\frac{\partial\varphi}{\partial\alpha}\Big|_{\tau=0} = -\sqrt{\frac{\rho_0^{\gamma-1}}{2(\gamma-1)a_\infty^2}}\frac{2(\alpha^{\gamma-1} - 1) + (\gamma-1)\alpha^{\gamma-2}(\alpha^2 - 1)}{\sqrt{(\alpha-1)(\alpha^{\gamma-1}-1)(\alpha+1)^3}}, \tag{2.28}$$

and

$$\begin{aligned} \frac{\partial^2\varphi}{\partial\alpha^2}\Big|_{\tau=0} = & \frac{1}{4}\sqrt{\frac{2\rho_0^{\gamma-1}}{(\gamma-1)a_\infty^2}}\Big(\sqrt{(\alpha^2-1)(\alpha^{\gamma-1}-1)}(\alpha^2-1)(\alpha^{\gamma-1}-1)(\alpha+1)\Big)^{-1} \\ & \times\Big(2(\gamma-1)\alpha^{\gamma-3}\big((2-\gamma)\alpha^2 - 2\alpha + \gamma - 2\big)(\alpha^2-1)(\alpha^{\gamma-1}-1) \\ & \quad + 4(2\alpha-1)(\alpha^{\gamma-1}-1)^2 + (\gamma-1)^2\alpha^{2(\gamma-2)}(\alpha^2-1)^2\Big). \end{aligned} \tag{2.29}$$

Proof. When $\alpha = 1$, it is easy to see that $\rho = \rho_0$ and $v = v_0$. Now, we only consider the case that $\alpha \neq 1$. Let

$$G(\alpha, v, U_0; \tau^2) = \frac{\mathscr{F}(\alpha, v, U_0; \tau^2)}{\alpha - 1}.$$

By $(2.22)_2$,

$$\frac{\partial G}{\partial v}\Big|_{\tau=0} = \frac{(\alpha+1)(v - v_0)}{\alpha - 1} = -\sqrt{\frac{2\rho_0^{\gamma-1}}{(\gamma-1)a_\infty^2}}\frac{\sqrt{(\alpha^2-1)(\alpha^{\gamma-1}-1)}}{\alpha-1}.$$

Then, we know $\frac{\partial G}{\partial v}\Big|_{\tau=0} > 0$ for $0 < \alpha < 1$, $\frac{\partial G}{\partial v}\Big|_{\tau=0} < 0$ for $\alpha > 1$, and

$$\lim_{\alpha\to\pm1}\frac{\partial G}{\partial v}\Big|_{\tau=0} = \mp\frac{\sqrt{2\rho_0^{\gamma-1}}}{a_\infty} \neq 0.$$

Therefore, by Lemma 2.4 and the implicit function theorem, there exists a small constant $0 < \epsilon_2 < \epsilon_1$ such that for any $\tau \in (0, \epsilon_2)$, equation $G(\alpha, v, U_0; \tau^2) = 0$ admits a unique solution $v = \varphi(\alpha, U_0, \tau^2)$. It implies that $\mathscr{F}(\alpha, \varphi(\alpha, U_0, \tau^2), U_0; \tau^2) = 0$.

Next, let us compute $\frac{\partial\varphi}{\partial\alpha}\Big|_{\tau=0}$. Notice that $\mathscr{F}(\alpha, \varphi(\alpha, U_0, \tau^2), U_0; \tau^2) = 0$. Taking derivative on it with respect to α yields that

$$\frac{\partial\mathscr{F}(\alpha, v; \tau^2)}{\partial\alpha} + \frac{\partial\mathscr{F}(\alpha, v; \tau^2)}{\partial v}\frac{\partial\varphi}{\partial\alpha} = 0. \tag{2.30}$$

Let $\tau = 0$, then we can obtain (2.28), by Lemma 2.3.

Finally, taking derivatives with respect to α again on (2.30) yields that

$$\frac{\partial^2\varphi}{\partial\alpha^2}=-\frac{\partial^2_{\alpha\alpha}\mathscr{F}(\alpha,v;\tau^2)+2\partial^2_{\alpha v}\mathscr{F}(\alpha,v;\tau^2)\partial_\alpha\varphi+\partial^2_{vv}\mathscr{F}(\alpha,v;\tau^2)(\partial_\alpha\varphi)^2}{\partial_v\mathscr{F}(\alpha,v;\tau^2)}. \tag{2.31}$$

So, by Lemma 2.3 and (2.22), we have (2.29). This completes the proof. □

Next, we are going to study the shock wave curves in the Riemann invariants coordinates. First, we have the following properties for $\omega_\pm$.

Lemma 2.6. *For $\gamma\in[1,2]$, there exists a small constant $0<\epsilon_3<\epsilon_2$ such that for any $\tau\in(0,\epsilon_3)$, along the shock wave curve $v=\varphi(\alpha,U_0;\tau^2)$,*

$$\frac{\partial(\omega_{-,0}-\omega_-)}{\partial\alpha}<0,\qquad for\quad 0<\alpha<1, \tag{2.32}$$

and

$$\frac{\partial(\omega_{+,0}-\omega_+)}{\partial\alpha}<0,\qquad for\quad \alpha>1, \tag{2.33}$$

where ω_- and ω_+ are defined by (2.11) and (2.12), and $\omega_{\pm,0}=\omega_\pm(U_0,\tau^2)$.

Proof. We only prove (2.32) here since we can treat ω_+ in the same way. By the definition of ω_-, along the shock wave curve,

$$\begin{aligned}\frac{\partial(\omega_{-,0}-\omega_-)}{\partial\alpha}&=-\Big(\rho_0\frac{\partial\omega_-}{\partial\rho}+\frac{\partial\omega_-}{\partial v}\frac{\partial\varphi}{\partial\alpha}\Big)\\&=a_\infty^2\rho^{-\frac{\gamma-1}{2}}\Big(\rho_0\partial_\rho u(\rho,v,\tau^2)+\big(\lambda_-(U,\tau^2)+\partial_v u(\rho,v,\tau^2)\big)\frac{\partial\varphi}{\partial\alpha}\Big).\end{aligned}$$

So, by Lemma 2.1, Lemma 2.2 and Lemma 2.5, we obtain that

$$\begin{aligned}\frac{\partial(\omega_{-,0}-\omega_-)}{\partial\alpha}\Big|_{\tau=0}=&-\frac{\rho_0^{\frac{\gamma-1}{2}}}{2\sqrt{2(\gamma-1)(1-\alpha)(1-\alpha^{\gamma-1})(1+\alpha)^3}}\\&\times\Big(2(1-\alpha^{\gamma-1})+(\gamma-1)(1-\alpha^2)\alpha^{\gamma-2}\\&\qquad+\sqrt{2(\gamma-1)(1-\alpha)(1-\alpha^{\gamma-1})(1+\alpha)^3\alpha^{\gamma-3}}\Big)\\&<0,\end{aligned}$$

for $0<\alpha<1$. It completes the proof of the lemma. □

Denote

$$\beta_-=\omega_{-,0}-\omega_-,\quad \beta_+=\omega_{+,0}-\omega_+. \tag{2.34}$$

By Lemma 2.6 and the implicit function theorem, α can be regarded as a function of β_- or β_+, *i.e.*, $\alpha=\alpha_1(\beta_-,U_0;\tau^2)$ and $\alpha=\alpha_2(\beta_+,U_0;\tau^2)$. So along the shock wave curves,

$$\beta_+=\Phi_1(\beta_-,U_0;\tau^2):=\omega_{+,0}-\omega_+(\alpha_1(\beta_-,U_0;\tau^2),\tau^2), \tag{2.35}$$

which is called the $\mathcal{S}_1$ shock curve, or

$$\beta_-=\Phi_2(\beta_+,U_0;\tau^2):=\omega_{-,0}-\omega_-(\alpha_2(\beta_+,U_0;\tau^2),\tau^2), \tag{2.36}$$

which is called the $\mathcal{S}_2$ shock curve.

For the $\mathcal{S}_1$ shock wave curve, we have the following lemma.

Lemma 2.7. *For $\gamma \in [1,2]$ and $0 < \alpha < 1$, there exists a constant $\epsilon_4 > 0$ sufficiently small such that for $\tau \in (0, \epsilon_4)$, the shock curve $\mathcal{S}_1$ starting at $(\omega_{-,0}, \omega_{+,0})$ is*

$$\beta_+ = \Phi_1(\beta_-, U_0; \tau^2) = \int_0^{\beta_-} \Psi_1(\alpha, U_0; \tau^2)\Big|_{\alpha=\alpha_1(\beta, U_0;\tau^2)} d\beta, \tag{2.37}$$

where $\beta_- = \omega_{-,0} - \omega_- > 0$. Moreover,

$$0 < \frac{\partial \Phi_1(\beta_-, U_0; 0)}{\partial \beta_-} < 1, \qquad \frac{\partial^2 \Phi_1(\beta_-, U_0; 0)}{\partial \beta_-^2} > 0. \tag{2.38}$$

Finally, if $\alpha > \varepsilon_0 > 0$, then

$$C_1 \varepsilon_0 < \frac{\partial \Phi_1(\beta_-, U_0; \tau^2)}{\partial \beta_-} < 1, \tag{2.39}$$

where $C_1 > 0$ is a constant depending only on the data and ε_0, and independent of τ.

Proof. By (2.35) and Lemma 2.6, we can define

$$\Psi_1(\alpha, U_0; \tau^2) := \frac{\partial \Phi_1(\beta_-, U_0; \tau^2)}{\partial \beta_-}.$$

So (2.37) follows. Moreover,

$$\frac{\partial \Phi_1(\beta_-, U_0; \tau^2)}{\partial \beta_-} = \frac{\frac{\partial(\omega_{+,0}-\omega_+)}{\partial \alpha}}{\frac{\partial(\omega_{-,0}-\omega_-)}{\partial \alpha}} = \frac{\rho_0 \partial_\rho u(\rho, v, \tau^2) + \big(\lambda_+(U, \tau^2) + \partial_v u(\rho, v, \tau^2)\big)\partial_\alpha \varphi}{\rho_0 \partial_\rho u(\rho, v, \tau^2) + \big(\lambda_-(U, \tau^2) + \partial_v u(\rho, v, \tau^2)\big)\partial_\alpha \varphi}.$$

With the help of (2.5) and (2.28), we have

$$\begin{aligned}\Psi_1\big|_{\tau=0} &= \frac{\rho_0 \partial_\rho u(\rho, v, \tau^2)\big|_{\tau=0} + \big(\lambda_+(U, \tau^2) + \partial_v u(\rho, v, \tau^2)\big)\partial_\alpha \varphi\big|_{\tau=0}}{\rho_0 \partial_\rho u(\rho, v, \tau^2)\big|_{\tau=0} + \big(\lambda_-(U, \tau^2) + \partial_v u(\rho, v, \tau^2)\big)\partial_\alpha \varphi\big|_{\tau=0}} \\ &= -\frac{2(1-\alpha^{\gamma-1}) + (\gamma-1)(1-\alpha^2)\alpha^{\gamma-2} - \sqrt{2(\gamma-1)(1-\alpha)(1-\alpha^{\gamma-1})(1+\alpha)^3\alpha^{\gamma-3}}}{2(1-\alpha^{\gamma-1}) + (\gamma-1)(1-\alpha^2)\alpha^{\gamma-2} + \sqrt{2(\gamma-1)(1-\alpha)(1-\alpha^{\gamma-1})(1+\alpha)^3\alpha^{\gamma-3}}}.\end{aligned}$$

By Lemma 2.6, we know that $\beta_- = \omega_{-,0} - \omega_-$ is monotonically decreasing with respect to α when $0 < \alpha \le 1$. Notice that $\beta_- = 0$ when $\alpha = 1$. Therefore, for $0 < \alpha < 1$, $\beta_- = \omega_{-,0} - \omega_- > 0$.

Next, let us consider $\frac{\partial^2 \Phi_1(\beta_-, U_0; \tau^2)}{\partial \beta_-^2}$. Note that

$$\frac{\partial \Psi_1(\alpha, U_0; \tau^2)}{\partial \alpha} = \Big(\rho_0 \partial_\rho u(\rho, v, \tau^2) + \big(\lambda_-(U, \tau^2) + \partial_v u(\rho, v, \tau^2)\big)\partial_\alpha \varphi\Big)^{-2} \mathcal{J}(U, \tau^2),$$

where

$$\begin{aligned}\mathcal{J}(U, \tau^2) = {} & \rho_0^2\Big(\big(\lambda_- - \lambda_+\big)\partial^2_{\rho\rho} u(\rho, v, \tau^2) + \big(\partial_\rho \lambda_+ - \partial_\rho \lambda_-\big)\partial_\rho u(\rho, v, \tau^2)\Big)\partial_\alpha \varphi \\ & + \rho_0\Big(2\big(\lambda_- - \lambda_+\big)\partial^2_{\rho v} u(\rho, v, \tau^2) + \big(\lambda_- + \partial_v u(\rho, v, \tau^2)\big)\partial_\rho \lambda_+ \\ & - \big(\lambda_+ + \partial_v u(\rho, v, \tau^2)\big)\partial_\rho \lambda_- + \big(\partial_v \lambda_+ - \partial_v \lambda_-\big)\partial_\rho u(\rho, v, \tau^2)\Big)(\partial_\alpha \varphi)^2 \\ & + \Big(\big(\partial_v \lambda_+ + \partial^2_{vv} u(\rho, v, \tau^2)\big)\big(\lambda_- + \partial_v u(\rho, v, \tau^2)\big) \\ & - \big(\partial_v \lambda_- + \partial^2_{vv} u(\rho, v, \tau^2)\big)\big(\lambda_+ + \partial_v u(\rho, v, \tau^2)\big)\Big)(\partial_\alpha \varphi)^3 + \rho_0\big(\lambda_+ - \lambda_-\big)\partial_\rho u(\rho, v, \tau^2)\partial^2_{\alpha\alpha} \varphi.\end{aligned}$$

When $\tau = 0$, by Lemma 2.1 and Lemma 2.5, we get

$$\mathcal{J}(U,0) = a_\infty^{-3}\rho_0^{\frac{3(\gamma-1)}{2}}\alpha^{\frac{3\gamma-7}{2}}\Big((\gamma-3)\partial_\alpha\varphi\big|_{\tau=0} - 2\alpha\partial^2_{\alpha\alpha}\varphi\big|_{\tau=0}\Big)$$
$$= \frac{2\big[(\gamma+1)\alpha^2-2\alpha+3-\gamma\big]\Big(\frac{1-\alpha^{\gamma-1}}{\gamma-1}\Big)^2 - 4\alpha^{\gamma-1}(1-\alpha^2)\frac{1-\alpha^{\gamma-1}}{\gamma-1}+\alpha^{\gamma-2}(\alpha^2-1)^2}{\sqrt{2(\gamma-1)(1-\alpha)(1-\alpha^{\gamma-1})(1+\alpha)^3}(1-\alpha^2)(1-\alpha^{\gamma-1})}$$
$$\times\Big(-a_\infty^4(\gamma-1)^2\rho_0^{2(\gamma-1)}\alpha^{\frac{3\gamma-7}{2}}\Big).$$

On the other hand, we also have

$$\Big(\rho_0\partial_\rho u(\rho,v,\tau^2) + \big(\lambda_-(U,\tau^2)+\partial_v u(\rho,v,\tau^2)\big)\partial_\alpha\varphi\Big)^{-2}$$
$$= \frac{2(\gamma-1)a_\infty^4\rho_0^{-2(\gamma-1)}\alpha^{-(\gamma-1)}(1-\alpha)(1-\alpha^{\gamma-1})(1+\alpha)^3}{\Big(2(1-\alpha^{\gamma-1})+(\gamma-1)\alpha^{\gamma-2}(1-\alpha^2)+\sqrt{2(\gamma-1)(1-\alpha)(1-\alpha^{\gamma-1})(1+\alpha)^3\alpha^{\gamma-3}}\Big)^2}.$$

With the above two equalities, we have

$$\frac{\partial\Psi_1}{\partial\alpha}\Big|_{\tau=0} = -\frac{(\gamma-1)^2\alpha^{\frac{\gamma-5}{2}}\sqrt{2(\gamma-1)(1-\alpha)(1-\alpha^{\gamma-1})}(1+\alpha)}{(1-\alpha)(1-\alpha^{\gamma-1})}$$
$$\times\frac{2\big[(\gamma+1)\alpha^2-2\alpha+3-\gamma\big]\Big(\frac{1-\alpha^{\gamma-1}}{\gamma-1}\Big)^2 - 4\alpha^{\gamma-1}(1-\alpha^2)\frac{1-\alpha^{\gamma-1}}{\gamma-1}+\alpha^{\gamma-2}(\alpha^2-1)^2}{\Big(2(1-\alpha^{\gamma-1})+(\gamma-1)\alpha^{\gamma-2}(1-\alpha^2)+\sqrt{2(\gamma-1)(1-\alpha)(1-\alpha^{\gamma-1})(1+\alpha)^3\alpha^{\gamma-3}}\Big)^2}.$$

Let

$$J(\alpha,\gamma) := 2\big[(\gamma+1)\alpha^2-2\alpha+3-\gamma\big]\Big(\frac{1-\alpha^{\gamma-1}}{\gamma-1}\Big)^2 - 4(1-\alpha^2)\alpha^{\gamma-1}\Big(\frac{1-\alpha^{\gamma-1}}{\gamma-1}\Big)+\big(1-\alpha^2\big)^2\alpha^{\gamma-2}.$$

Note that

$$\Delta = \Big(-4(1-\alpha^2)\alpha^{\gamma-1}\Big)^2 - 8\big[(\gamma+1)\alpha^2-2\alpha+3-\gamma\big]\big(1-\alpha^2\big)^2\alpha^{\gamma-2}$$
$$= 8\big(1-\alpha^2\big)^2\alpha^{\gamma-2}\big[2\alpha^\gamma-(\gamma+1)\alpha^2+2\alpha-3+\gamma\big]$$
$$= 8\big(1-\alpha^2\big)^2\alpha^{\gamma-2}\Delta_0(\alpha,\gamma),$$

where $\Delta_0(\alpha,\gamma) = 2\alpha^\gamma-(\gamma+1)\alpha^2+2\alpha-3+\gamma$. Obviously, $\Delta_0(1,\gamma)=0$, and for $0<\alpha<1$ and $1\le\gamma\le 2$, we have $\partial_\alpha\Delta_0(\alpha,\gamma) = 2\gamma\alpha(\alpha^{\gamma-2}-1)+2(1-\alpha)>0$. So $\Delta<0$ when $0<\alpha<1$ and $1\le\gamma\le 2$. Therefore, $J(\alpha,\gamma)>0$ when $0<\alpha<1$ and $1\le\gamma\le 2$. Thus

$$\frac{\partial\Psi_1}{\partial\alpha}\Big|_{\tau=0} < 0,$$

when $0<\alpha<1$ and $1\le\gamma\le 2$. So for $0<\alpha<1$ and $1\le\gamma\le 2$,

$$\frac{\partial^2\Phi_1(\beta_-,U_0;0)}{\partial\beta_-^2} = \Big(\frac{\partial(\omega_{-,0}-\omega_-)}{\partial\alpha}\Big)^{-1}\Big|_{\tau=0}\frac{\partial\Psi_1(\alpha,U_0;\tau^2)}{\partial\alpha}\Big|_{\tau=0} > 0.$$

Moreover, because $\Psi_1(1,U_0;0)=0$, and $\big|\Psi_1\big|_{\tau=0}\big|<1$ for $0<\alpha<1$, we have

$$0 < \frac{\partial\Phi_1(\beta_-,U_0;0)}{\partial\beta_-} < 1. \tag{2.40}$$

Finally, for $\alpha>\varepsilon_0>0$, we can choose $\epsilon_4>0$ sufficiently small and a constant $C_1>0$ independent of τ such that when $\tau\in(0,\epsilon_4)$, $C_1\varepsilon_0<\Psi_1(\alpha,U_0;\tau^2)<1$. □

Based on the proof, actually, when $\tau = 0$, we have the following lemma.

Lemma 2.8. *If $\gamma \in [1,2]$, then the shock curve S_1 starting at (r_0, s_0) can be written as*

$$s_0 - s = g_1(r_0 - r,\ \rho_0) = \int_0^{r_0 - r} h_1(\alpha)\Big|_{\alpha=\alpha_1\left(\frac{\beta}{\rho_0^{(\gamma-1)/2}}\right)} d\beta, \tag{2.41}$$

where $0 \le \frac{\partial g_1(\beta,\rho_0)}{\partial \beta} < 1$, $\frac{\partial^2 g_1(\beta,\rho_0)}{\partial^2 \beta} \ge 0$, *and* $\beta = r_0 - r \ge 0$.

Next, let us consider the properties of the shock wave curve $\mathcal{S}_2$.

Lemma 2.9. *If $\gamma \in [1,2]$ and $\alpha > 1$, there exists a constant $\epsilon_5 > 0$ sufficiently small such that for $\tau \in (0, \epsilon_5)$, the shock wave curve $\mathcal{S}_2$ starting at $(\omega_{-,0}, \omega_{+,0})$ can be expressed as*

$$\beta_- = \Phi_2(\beta_+, U_0; \tau^2) = \int_0^{\beta_+} \Psi_2(\alpha, U_0; \tau^2)\Big|_{\alpha=\alpha_2(\beta, U_0;\tau^2)} d\beta, \tag{2.42}$$

where $\beta_+ = \omega_{+,0} - \omega_+ < 0$ and

$$0 < \frac{\partial \Phi_2(\beta_+, U_0; 0)}{\partial \beta_+} < 1, \qquad \frac{\partial^2 \Phi_2(\beta_+, U_0; 0)}{\partial \beta_+^2} < 0. \tag{2.43}$$

Moreover, if $\alpha < \varepsilon_0^{-1}$, it holds that

$$0 < \frac{\partial \Phi_2(\beta_+, U_0; \tau^2)}{\partial \beta_+} < 1 - C_2 \varepsilon_0, \tag{2.44}$$

where constant $C_2 > 0$, depending on the data and ε_0, is independent of τ.

Because the proof is similar to the one of Lemma 2.7, we postpone the proof to be stated in *Appendix A* for the shortness. Based on the proof, we actually have the following lemma for $\tau = 0$.

Lemma 2.10. *If $\gamma \in [1,2]$, then the shock curve S_2 starting at (r_0, s_0) can be rewritten as*

$$r_0 - r = g_2(s_0 - s,\ \rho_0) \equiv \int_0^{s_0 - s} h_2(\alpha)\Big|_{\alpha=\alpha_2\left(\frac{\beta}{\rho_0^{(\gamma-1)/2}}\right)} d\beta, \tag{2.45}$$

where $0 < \frac{\partial g_2(\beta,\rho_0)}{\partial \beta} < 1$, $\frac{\partial^2 g_2(\beta,\rho_0)}{\partial^2 \beta} < 0$, *and* $\beta = s_0 - s \le 0$.

2.2. Riemann solutions of equations (1.17)

Based on Lemma 2.7 and Lemma 2.8, for any constant state $\omega_L = (\omega_{-,L}, \omega_{+,L})$, let

$$z_1 = \omega_{-,L} - \omega_- \qquad \text{and} \qquad z_2 = \omega_{+,L} - \omega_+.$$

Define

$$\begin{aligned} &\mathscr{H}_1^{(1)}(z_1, \omega_L; \tau^2) = -z_1 + \omega_{-,L}, \\ &\mathscr{H}_1^{(2)}(z_1, \omega_L; \tau^2) = \begin{cases} -\Phi_1(z_1, U_L; \tau^2) + \omega_{+,L}, & z_1 > 0, \\ \omega_{+,L}, & z_1 < 0, \end{cases} \end{aligned} \tag{2.46}$$

and

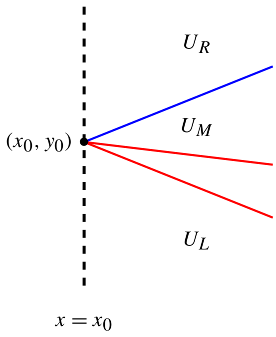

Fig. 3. Riemann problem without boundary.

$$\mathscr{H}_2^{(1)}(z_2, \omega_L; \tau^2) = \begin{cases} -\Phi_2(z_2, U_L; \tau^2) + \omega_{-,L}, & z_2 < 0, \\ \omega_{-,L}, & z_2 > 0, \end{cases} \tag{2.47}$$
$$\mathscr{H}_2^{(2)}(z_2, \omega_L; \tau^2) = -z_2 + \omega_{+,L},$$

where functions Φ_1 and Φ_2 are given in Lemma 2.7 and Lemma 2.8, respectively. Let

$$\begin{aligned} &\mathscr{H}_1(z_1, \omega_L; \tau^2) = \big(\mathscr{H}_1^{(1)}, \mathscr{H}_1^{(2)}\big)(z_1, \omega_L; \tau^2), \\ &\mathscr{H}_2(z_2, \omega_L, \tau^2) = \big(\mathscr{H}_2^{(1)}, \mathscr{H}_2^{(2)}\big)(z_2, \omega_L; \tau^2), \end{aligned} \tag{2.48}$$

and finally denote

$$\mathscr{H}(z, \omega_L; \tau^2) =: \mathscr{H}_1(z_1, \mathscr{H}_2(z_2, \omega_L; \tau^2); \tau^2), \quad z = (z_1, z_2). \tag{2.49}$$

Then, we can parameterize the 1-waves by z_1 and parameterize the 2-waves by z_2. For the case that $\tau = 0$, we set $z := \mathbf{z} = (z_1, z_2)$ and $\omega := \omega^0 = (r, s)$.

Now, let us consider the Riemann problem of (1.17) with large initial data at $x = x_0$

$$U(x, y)\big|_{x=x_0} = \begin{cases} U_L, & y < y_0, \\ U_R, & y > y_0, \end{cases} \tag{2.50}$$

where $U_L = (\rho_L, v_L)$ and $U_R = (\rho_R, v_R)$ are two given constant states satisfying $\rho_L > 0$ and $\rho_R > 0$ (see Fig. 3). We have the following proposition that gives the solvability and the invariant region of the Riemann problem of (1.17) and (2.50).

Proposition 2.1. *Suppose that $\omega_{-,L} + \omega_{+,R} > -\frac{4-\varepsilon_0}{\gamma-1}$ for some constant $0 < \varepsilon_0 < 4$, then there exists a sufficiently small constant $\epsilon_6 > 0$ such that for any $\tau \in [0, \epsilon_6)$, Riemann problem (1.17) and (2.50) admits a unique piecewise smooth solution $U(x, y)$ without the vacuum state. Moreover solution $U(x, y)$ satisfies*

$$\omega_-(U(x, y), \tau^2) + \omega_+(U(x, y), \tau^2) \geq \omega_{-,L} + \omega_{+,R}, \tag{2.51}$$

where $\omega_{\pm,L} = \omega_\pm(U_L, \tau^2)$ and $\omega_{\pm,R} = \omega_\pm(U_R, \tau^2)$.

Proof. The existence of the solutions of Riemann problem (1.17) and (2.50) is equivalent to the existence of solutions z of the following equation,

$$\omega_R = \mathscr{H}(z, \omega_L; \tau^2). \tag{2.52}$$

From (2.49), we know that

$$\det\big(\nabla_z \mathscr{H}\big)(z, \omega_L; \tau^2) = \det\Big(\nabla_{\mathscr{H}_2}\mathscr{H}_1 \cdot \nabla_{z_2}\mathscr{H}_2, \ \ \nabla_{z_1}\mathscr{H}_1\Big)(z, \omega_L; \tau^2).$$

Based on the sign of z_1 and z_2, we divide the proof into three cases for checking the sign of the determinant above to show the existence of solution z of equation (2.52).

Case (i). $z_1 > 0$ and $z_2 < 0$. By the definition of $\mathscr{H}_1^{(1)}$ and $\mathscr{H}_1^{(2)}$, we know that

$$\mathscr{H}_2^{(1)}(z_2, U_L; \tau^2) = -\Phi_2(z_2, U_L; \tau^2) + \omega_{-,L}, \quad \mathscr{H}_2^{(2)}(z_2, U_L; \tau^2) = -z_2 + \omega_{+,L},$$

and

$$\mathscr{H}_1^{(1)}(z_1, \mathscr{H}_2(z_2, \omega_L; \tau^2); \tau^2) = -z_1 + \mathscr{H}_2^{(1)}(z_2, \omega_L; \tau^2),$$
$$\mathscr{H}_1^{(2)}(z_1, \mathscr{H}_2(z_2, \omega_L; \tau^2); \tau^2) = -\Phi_1\Big(z_1, U\big(\mathscr{H}_2(z_2, \omega_L; \tau^2)\big); \tau^2\Big) + \mathscr{H}_2^{(2)}(z_2, \omega_L; \tau^2).$$

So

$$\begin{aligned}
&\nabla_{\mathscr{H}_2}\mathscr{H}_1 \cdot \nabla_{z_2}\mathscr{H}_2 \\
&= \begin{pmatrix} 1 & 0 \\ -\nabla_U\Phi_1 \cdot \partial_{\mathscr{H}_2^{(1)}}U & -\nabla_U\Phi_1 \cdot \partial_{\mathscr{H}_2^{(2)}}U + 1 \end{pmatrix} \cdot \Big(-\partial_{z_2}\Phi_2(z_2, U_L; \tau^2),\ -1\Big)^\top \\
&= \Big(-\partial_{z_2}\Phi_2(z_2, U_L; \tau^2),\ \nabla_U\Phi_1 \cdot \partial_{\mathscr{H}_2^{(1)}}U + \nabla_U\Phi_1 \cdot \partial_{\mathscr{H}_2^{(2)}}U \cdot \partial_{z_2}\Phi_2(z_2, U_L; \tau^2) - 1\Big)^\top,
\end{aligned}$$

and

$$\nabla_{z_1}\mathscr{H}_1 = \Big(-\partial_{z_1}\Phi_1\big(z_1, U(\mathscr{H}_2(z_2, \omega_L; \tau^2)); \tau^2\big),\ -1\Big)^\top.$$

Note that it follows from Lemma 2.3 that,

$$\begin{aligned}
\nabla_U\Phi_1 \cdot \partial_{\mathscr{H}_2^{(1)}}U\Big|_{\tau=0} &= \nabla_U\Phi_1 \cdot \partial_{\mathscr{H}_2^{(2)}}U\Big|_{\tau=0} \\
&= \frac{1}{2}\rho^{\frac{3-\gamma}{2}}\partial_\rho\Phi_1\Big(z_1, U\big(\mathscr{H}_2(z_2, \omega(U_L, 0); 0)\big); 0\Big) \\
&= \frac{\gamma-1}{4}\partial_{\rho^{\frac{\gamma-1}{2}}}\Phi_2\Big(z_1, U\big(\mathscr{H}_1(z_2, \omega(U_L, 0); 0)\big); 0\Big) \\
&= -\frac{1}{2}\partial_{z_1}\Phi_1\Big(z_1, U\big(\mathscr{H}_2(z_2, \omega(U_L, 0); 0)\big); 0\Big).
\end{aligned}$$

Then, by Lemma 2.7 and Lemma 2.8,

$$\begin{aligned}
\det\big(\nabla_z\mathscr{H}\big)(z, \omega_L; \tau^2)\Big|_{\tau=0} &= \det\Big(\nabla_{\mathscr{H}_2}\mathscr{H}_1 \cdot \nabla_{z_2}\mathscr{H}_2,\ \ \nabla_{z_1}\mathscr{H}_1\Big)(z, \omega_L; \tau^2)\big|_{\tau=0} \\
&= -1 + \frac{1}{2}\partial_{z_1}\Phi_1\Big(z_1, U\big(\mathscr{H}_2(z_2, \omega(U_L, 0); 0)\big); 0\Big) \cdot \Big(\partial_{z_2}\Phi_2\big(z_2, U_L; 0\big) - 1\Big) \\
&< -1.
\end{aligned}$$

Thus, for τ sufficiently small, we can get the existence of solution z of equation (2.52) by applying the implicit function theorem. Moreover, by the signs of z_1 and z_2,

$$\omega_+(U(x, y), \tau^2) = \omega_{+,R} - \Phi_1(z_1, U\big(\mathscr{H}_2(z_2, \omega_L; \tau^2)\big); \tau^2) > \omega_{+,R}, \quad \text{for} \quad z_1 > 0,$$

and

$$\omega_-(U(x, y), \tau^2) = \Phi_1\Big(z_2, U_L; \tau^2\Big) + \omega_{-,L} > \omega_{-,L}, \quad \text{for} \quad z_2 < 0,$$

which leads to the estimate (2.51).

Case (ii). $z_1 < 0$ and $z_2 < 0$ (or $z_1 > 0$ and $z_2 > 0$). Without loss of the generality, we only consider the case that $z_1 < 0$ and $z_2 < 0$, since the other case can be treated in the same way. For the case that $z_1 < 0$ and $z_2 < 0$, notice that

$$\mathscr{H}_2^{(1)}(z_2, U_L; \tau^2) = -\Phi_2(z_2, U_L; \tau^2) + \omega_{-,L}, \quad \mathscr{H}_2^{(2)}(z_2, U_L; \tau^2) = -z_2 + \omega_{+,L},$$

and

$$\mathscr{H}_1^{(1)}(z_1, \mathscr{H}_2(z_2, \omega_L; \tau^2); \tau^2) = -z_1 + \mathscr{H}_2^{(1)}(z_2, \omega_L; \tau^2),$$
$$\mathscr{H}_1^{(2)}(z_1, \mathscr{H}_2(z_2, \omega_L; \tau^2); \tau^2) = -z_2 + \omega_{+,L}.$$

So

$$\det\big(\nabla_z \mathscr{H}\big)(z, \omega_L; \tau^2)\Big|_{\tau=0} = \Phi_2(z_2, U_L; \tau^2) - 1 < -C\varepsilon_0,$$

and

$$\omega_{-,R} = -z_1 - \Phi_2(z_2, U_L; \tau^2) + \omega_{-,L}, \quad \omega_{+,R} = -z_2 + \omega_{+,L}.$$

Hence we can obtain the existence of solution z of equation (2.52) directly, and

$$\omega_-(U(x,y), \tau^2) > \omega_{-,L} \quad \text{and} \quad \omega_+(U(x,y), \tau^2) = \omega_{+,R},$$

which leads to the estimate (2.51).

Case (iii). $z_1 < 0$ and $z_2 > 0$. In this case, notice that

$$\mathscr{H}_2^{(1)}(z_2, \omega_L; \tau^2) = \omega_{-,L}, \quad \mathscr{H}_2^{(2)}(z_2, \omega_L; \tau^2) = -z_2 + \omega_{+,L},$$

and

$$\mathscr{H}_1^{(1)}(z_1, \mathscr{H}_2(z_2, \omega_L; \tau^2); \tau^2) = -z_1 + \mathscr{H}_2^{(1)}(z_2, \omega_L; \tau^2),$$
$$\mathscr{H}_1^{(2)}(z_1, \mathscr{H}_2(z_2, \omega_L; \tau^2); \tau^2) = \mathscr{H}_2^{(2)}(z_2, \omega_L; \tau^2).$$

Then

$$z_1 = \omega_{-,L} - \omega_{-,R} \qquad \text{and} \qquad z_2 = \omega_{+,L} - \omega_{+,R}.$$

So we obtain the existence of solution z directly, and in this case it is easy to see

$$\omega_-(U, \tau^2) + \omega_+(U, \tau^2) = \omega_{-,L} + \omega_{+,R}.$$

Moreover, notice that

$$\begin{aligned}\rho^{\frac{\gamma-1}{2}}\Big|_{\tau=0} &= \frac{\gamma-1}{4}\Big(\omega_-(U,0) + \omega_+(U,0)\Big) + 1 \\ &\geq \frac{\gamma-1}{4}\Big(\omega_+(U_R,0) + \omega_-(U_L,0)\Big) + 1 \\ &> \hat{C} > 0.\end{aligned}$$

Based on this fact and combining the arguments for *Cases* (i)-(iii) together, we can choose $\epsilon_6 > 0$ sufficiently small such that for $\tau \in (0, \epsilon_6)$, equation (2.52) and then Riemann problem (1.17) and (2.50) admits a unique solution z without the vacuum states. Moreover, estimate (2.51) follows. It completes the proof of the Proposition. □

Next, let us study the Riemann problem involving boundary. Define

$$\Omega_0 = \{(x,y) : \ x_0 \leq x < x_1, \ y \leq b_0(x - x_0) + y_0\},$$

$$\Gamma_0 = \{(x,y) : \ x_0 \leq x < x_1, y = b_0(x - x_0) + y_0\}.$$

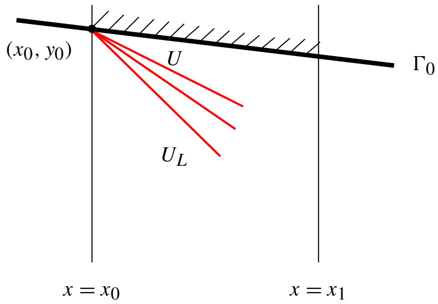

Fig. 4. Riemann problem with boundary.

Let us consider the following Riemann problem (see Fig. 4):

$$\begin{cases} \partial_x W(U,\tau^2) + \partial_y F(U,\tau^2) = 0, & in \ \ \Omega_0, \\ U(x,y) = U_L, & on \ \ \Omega_0 \cap \{x = x_0\}, \\ v(x,y) = \big(1+\tau^2 u(\rho,v,\tau^2)\big) b_0, & on \ \ \Gamma_0, \end{cases} \tag{2.53}$$

where $b_0 < 0$ and $U_L = (\rho_L, v_L)$ is a given constant state satisfying $\rho_L > 0$.

We have the following lemma on the solvability of Riemann problem (2.53).

Proposition 2.2. *Assume that $\omega_{-,L} - a_\infty b_0 > -\frac{2-\varepsilon_0}{\gamma-1}$ for some $0 < \varepsilon_0 < 2$, then there exists a small constant $\epsilon_7 > 0$ such that for any $\tau \in [0, \epsilon_7)$, Riemann problem (2.53) admits a unique piecewise smooth solution $U(x,y)$ consists of a single 2-shock or a 2-rarefaction wave without the vacuum states. Here, $\omega_{-,L} = \omega_-(U_L, \tau^2)$.*

Proof. It is easy to see that the existence of solutions of Riemann problem (2.53) is equivalent to the existence of solutions z_2 of the following system

$$\begin{cases} \omega = \mathscr{H}_2(z_2, \omega_L; \tau^2), \\ v = \mathcal{V}(\omega, \tau^2) = \big(1+\tau^2 u(\rho,v,\tau^2)\big) b_0. \end{cases} \tag{2.54}$$

Let

$$\mathscr{G}(z_2, \omega_L, b_0; \tau^2) = \mathcal{V}(\mathscr{H}_2(z_2,\omega_L;\tau^2), \tau^2) - \big(1+\tau^2 u(\rho,v,\tau^2)\big) b_0,$$

and consider equation $\mathscr{G}(z_2, \omega_L, b_0; \tau^2) = 0$ for τ sufficiently small.

Note that

$$\mathscr{G}\big(z_2, \omega(U_L,0), b_0; 0\big) = \frac{1}{2a_\infty}\Big(\mathscr{H}_2^{(1)}(z_2, \omega(U_L,0); 0) - \mathscr{H}_2^{(2)}(z_2, \omega(U_L,0); 0)\Big) - b_0.$$

If $b_0 < v_L$, *i.e.*, $\omega_-(U_L,0) - \omega_+(U_L,0) > 2a_\infty b_0$, then

$$\mathscr{G}\big(z_2, \omega(U_L,0), b_0; 0\big) = \frac{1}{2a_\infty}\Big(z_2 - \Phi_2(z_2, U_L; 0) + \omega_-(U_L,0) - \omega_+(U_L,0)\Big) - b_0.$$

By Lemma 2.8, we get that

$$\frac{\partial \mathscr{G}\big(z_2, \omega(U_L,0), b_0; 0\big)}{\partial z_2} = \frac{1}{2a_\infty}\Big(1 - \partial_{z_2}\Phi_2(z_2, U_L; 0)\Big) > \frac{C_2 \varepsilon_0}{2a_\infty} > 0.$$

On the other hand, we notice that $\mathscr{G} \in C^2$ with respect to z_2, then

$$\mathscr{G}(0, \omega(U_L,0), b_0; 0) = \frac{1}{2a_\infty}\big(\omega_-(U_L,0) - \omega_+(U_L,0)\big) - b_0 > 0,$$

and

$$\lim_{z_2\to-\infty}\mathscr{G}(z_2,\omega(U_L,0),b_0;0)=\frac{1}{2a_\infty}\lim_{z_2\to-\infty}\left(1-\frac{\Phi_2(z_2,U_L;0)}{z_2}\right)z_2+\frac{\omega_-(U_L,0)-\omega_+(U_L,0)}{2a_\infty}-b_0$$
$$=-\infty.$$

So, by the intermediate value theorem and the implicit function theorem, there exists a small constant $\epsilon_7'>0$ such that when $\tau\in[0,\epsilon_7')$, equation (2.54) admits a unique solution $z_2<0$ which consists of a shock wave belonging to the second family. There is no vacuum state, which can be verified by the observation that $\omega_-(U,0)>\omega_-(U_L,0)$, which leads to

$$\begin{aligned}\rho^{\frac{\gamma-1}{2}}\Big|_{\tau=0}&=\frac{\gamma-1}{4}\Big(\omega_+(U,0)+\omega_-(U,0)\Big)+1\\&=\frac{\gamma-1}{2}\Big(\omega_-(U,0)-a_\infty b_0\Big)+1\\&>\frac{\gamma-1}{2}\Big(\omega_-(U_L,0)-a_\infty b_0\Big)+1\\&>\tilde{C}.\end{aligned}$$

Second, if $b_0>v_L$, *i.e.*, $\omega_-(U_L,0)-\omega_+(U_L,0)<2a_\infty b_0$, then states U_L and U are connected by a 2-rarefaction wave R_2. So, by (2.47) and (2.54), we know that

$$\omega_-(U,0)=\omega_-(U_L,0),\quad \omega_+(U,0)=\omega_-(U_L,0)-2a_\infty b_0.$$

This also gives that

$$\rho^{\frac{\gamma-1}{2}}\Big|_{\tau=0}=\frac{\gamma-1}{2}\Big(\omega_-(U_L,0)-a_\infty b_0\Big)+1>\tilde{C},$$

which means that the vacuum states dose not appear. Moreover,

$$\mathscr{G}(z_2,\omega_L,b_0;\tau^2)=\mathcal{V}((\omega_{-,L},-z_2+\omega_{+,L}),\tau^2)-\big(1+\tau^2u(\rho,v,\tau^2)\big)b_0,$$

so

$$\frac{\partial\mathscr{G}(z_2,\omega_L,b_0;\tau^2)}{\partial z_2}\Big|_{\tau=0}=\frac{1}{2a_\infty}>0.$$

Hence by the implicit function theorem, there exists a small constant $\epsilon_7''>0$ such that for $\tau\in[0,\epsilon_7'')$, equation (2.54) admits a unique solution $z_2>0$ such that U_L and U are connected by a 2-rarefaction wave $\mathcal{R}_2$ without the vacuum state.

Finally, take $\epsilon_7=\min\{\epsilon_7',\epsilon_7''\}$, then when $\tau\in[0,\epsilon_7)$, we can get the existence of solutions of Riemann problem (2.53) without the vacuum states. □

3. Local interaction estimates

In order to control the total variation of the approximate solutions which will be constructed in the next section, we need to study the local interaction estimates of the elementary waves of large data. Firstly, let us consider the estimates on the difference of the Riemann invariance of the same family along the corresponding shock wave curve. Let us consider them for the case that $\tau=0$ first. By Remark 2.2, as shown in Fig. 5, let

$$\begin{aligned}&r_0:=a_\infty v_0+\frac{2(\rho_0^{\frac{\gamma-1}{2}}-1)}{\gamma-1}=a_\infty v_1+\frac{2(\rho_1^{\frac{\gamma-1}{2}}-1)}{\gamma-1},\quad s_0:=-a_\infty v_0+\frac{2(\rho_0^{\frac{\gamma-1}{2}}-1)}{\gamma-1},\\&r:=a_\infty v+\frac{2(\rho^{\frac{\gamma-1}{2}}-1)}{\gamma-1}=a_\infty v_2+\frac{2(\rho_2^{\frac{\gamma-1}{2}}-1)}{\gamma-1},\quad s:=-a_\infty v+\frac{2(\rho^{\frac{\gamma-1}{2}}-1)}{\gamma-1},\end{aligned}\tag{3.1}$$

and

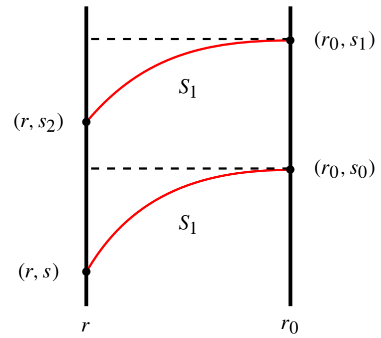

Fig. 5. Lemma 3.1.

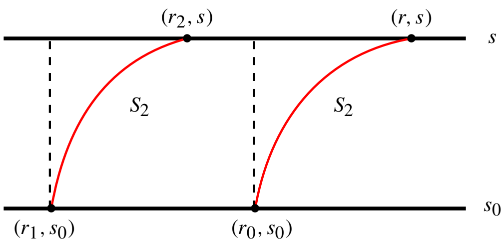

Fig. 6. Lemma 3.2.

$$s_1 := -a_\infty v_1 + \frac{2(\rho_1^{\frac{\gamma-1}{2}} - 1)}{\gamma - 1}, \quad s_2 := -a_\infty v_2 + \frac{2(\rho_2^{\frac{\gamma-1}{2}} - 1)}{\gamma - 1}. \tag{3.2}$$

Then we have the following lemmas.

Lemma 3.1. *Suppose $\tau = 0$ and $s_1 > s_0$. For two S_1 shock wave curves starting at points (r_0, s_1) and (r_0, s_0) and ending at points (r, s_2) and (r, s) respectively, if $0 < \rho_* < \rho_i < \rho^* < \infty$ for $i = 0$ and 1, then there exists a constant $C_3 > 0$ depending only on ρ_* and ρ^*, such that*

$$0 \le (s_0 - s) - (s_1 - s_2) \le C_3(\gamma - 1)(s_1 - s_0)(r_0 - r). \tag{3.3}$$

The proof of this lemma will be given in the appendix since it is similar to the one in [17]. Similarly, we also have the estimate on the difference of r on S_2 (see Fig. 6).

Lemma 3.2. *Assume $\tau = 0$ and $r_0 > r_1$. For two S_2 shock wave curves starting at points (r_1, s_0) and (r_0, s_0), and ending at points (r_2, s) and (r, s), respectively, if $0 < \rho_* < \rho_i < \rho^* < \infty$ for $i = 0$ and 1, then there exists a constant $C_3' > 0$ depending only on ρ_* and ρ^*, such that*

$$0 \le (r - r_0) - (r_2 - r_1) \le C_3'(\gamma - 1)(r_0 - r_1)(s - s_0). \tag{3.4}$$

Now, let us consider the case $\tau \ne 0$ in the following lemmas.

Lemma 3.3. *(see Fig. 7) Assume $\omega_{+,1} > \omega_{+,0}$. For two $\mathcal{S}_1$ shock wave curves starting at points $(\omega_{-,0}, \omega_{+,1})$ and $(\omega_{-,0}, \omega_{+,0})$ corresponding to (ρ_1, v_1) and (ρ_0, v_0) respectively, and ending at points $(\omega_-, \omega_{+,2})$ and (ω_-, ω_+) corresponding to (ρ_2, v_2) and (ρ, v), respectively. If $0 < \rho_* < \rho_i < \rho^* < \infty$ for $i = 0$ and 1, then there exists a constant $C_4 > 0$ depending only on ρ_* and ρ^*, such that*

$$\omega_{+,0} - \omega_+ - (\omega_{+,1} - \omega_{+,2}) \le C_4(\gamma - 1 + \tau^2)(\omega_{+,1} - \omega_{+,0})(\omega_{-,0} - \omega_-). \tag{3.5}$$

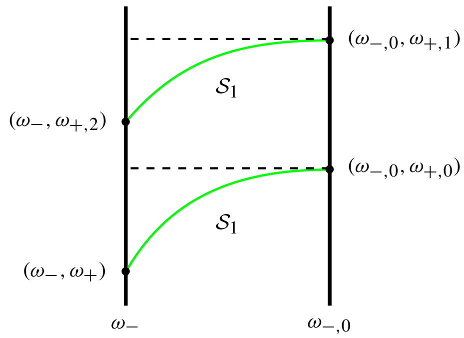

Fig. 7. Lemma 3.3.

Proof. Let $\Delta\omega_- = \omega_{-,0} - \omega_-$, $\Delta\omega_+ = \omega_{+,1} - \omega_{+,0}$, and let $\omega_{+,2} = \omega^*(\Delta\omega_-, \Delta\omega_+, \tau^2)$. For the case that $\tau = 0$, by Lemma 3.1,

$$\omega^*(0,0,0) - \omega^*(\Delta\omega_-, 0, 0) - \omega^*(0, \Delta\omega_+, 0) + \omega^*(\Delta\omega_-, \Delta\omega_+, 0) = \mathcal{O}(1)(\gamma - 1)\Delta\omega_-\Delta\omega_+.$$

Therefore, we have

$$\begin{aligned}
&\omega_{+,0} - \omega_+ - (\omega_{+,1} - \omega_{+,2}) \\
&= \omega^*(0,0,\tau^2) - \omega^*(\Delta\omega_-, 0, \tau^2) - \Big(\omega^*(0, \Delta\omega_+, \tau^2) - \omega^*(\Delta\omega_-, \Delta\omega_+, \tau^2)\Big) \\
&= \omega^*(0,0,\tau^2) - \omega^*(0,0,0) - \Big(\omega^*(\Delta\omega_-, 0, \tau^2) - \omega^*(\Delta\omega_-, 0, 0)\Big) \\
&\quad - \Big(\omega^*(0, \Delta\omega_+, \tau^2) - \omega^*(0, \Delta\omega_-, 0)\Big) + \omega^*(\Delta\omega_-, \Delta\omega_+, \tau^2) - \omega^*(\Delta\omega_-, \Delta\omega_+, 0) \\
&\quad + \omega^*(0,0,0) - \omega^*(\Delta\omega_-, 0, 0) - \omega^*(0, \Delta\omega_+, 0) + \omega^*(\Delta\omega_-, \Delta\omega_+, 0) \\
&= \tau^2 \int_0^1 \mathbf{e}(\Delta\omega_-, \Delta\omega_+, \chi\tau^2) d\chi + \mathcal{O}(1)(\gamma - 1)\Delta\omega_-\Delta\omega_+,
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{e}(\Delta\omega_-, \Delta\omega_+, \chi\tau^2) &= \partial_{\tau^2}\omega^*(0,0,\chi\tau^2) - \partial_{\tau^2}\omega^*(\Delta\omega_-, 0, \chi\tau^2) \\
&\quad - \partial_{\tau^2}\omega^*(0, \Delta\omega_+, \chi\tau^2) + \partial_{\tau^2}\omega^*(\Delta\omega_-, \Delta\omega_+, \chi\tau^2) \\
&= \mathcal{O}(1)\Delta\omega_-\Delta\omega_+.
\end{aligned}$$

Combining the above two estimates together, we have (3.5). □

Similarly, we also have the estimate on the difference of ω_- on $\mathcal{S}_2$ shock wave curves.

Lemma 3.4. *(see Fig. 8) Assume $\omega_{-,0} > \omega_{-,1}$. For $\mathcal{S}_2$ shock wave curves starting at points $(\omega_{-,1}, \omega_{+,0})$ and $(\omega_{-,0}, \omega_{+,0})$ corresponding to (ρ_1, v_1) and (ρ_0, v_0) respectively, and ending at points $(\omega_{-,2}, \omega_+)$ and (ω_-, ω_+) corresponding to (ρ, v) and (ρ_2, v_2), respectively. If $0 < \rho_* < \rho_i < \rho^* < \infty$ for $i = 0$ and 1, then there exists a constant $C_4' > 0$ depending only on ρ_* and ρ^*, such that*

$$\omega_- - \omega_{-,0} - (\omega_{-,2} - \omega_{-,1}) \le C_4'(\gamma - 1 + \tau^2)(\omega_{-,0} - \omega_{-,1})(\omega_+ - \omega_{+,0}). \tag{3.6}$$

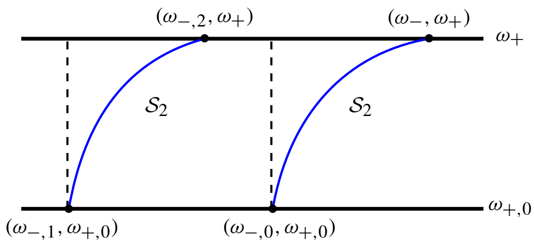

Fig. 8. Lemma 3.4.

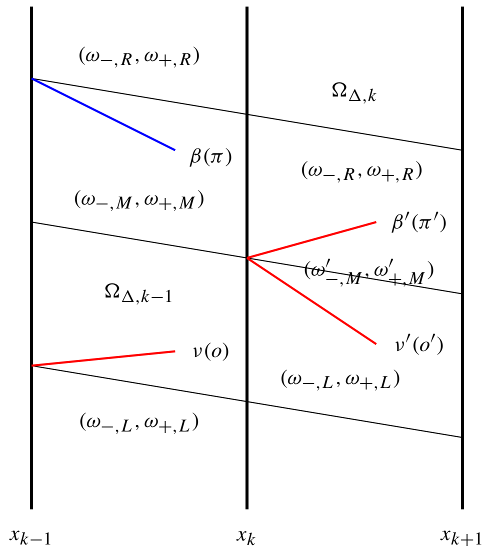

Fig. 9. Local interaction estimates away from the boundary.

Now we are ready to introduce the local interaction estimates case by case (see Fig. 9). Let ν and ν' be the wave strength of shock wave $\mathcal{S}_1$ before and after the interaction. Let β and β' be the wave strength of shock wave $\mathcal{S}_2$ before and after the interaction. And let o, π and o', π' be the wave strength of rarefaction wave $\mathcal{R}_1$ and $\mathcal{R}_2$ before and after the interaction respectively.

Lemma 3.5. *Let $\gamma \in [1, 2]$, and let $0 < \hat{\rho} < \check{\rho} < \infty$. Then, for $\rho \in [\hat{\rho}, \check{\rho}]$, there exist positive constants $C_0 > 0$, $C_5 > 0$ and $\delta \in (0, 1)$ independent of γ, β, ν and ρ, such that the following interaction estimates hold:*

(1) *For the case that $\mathcal{S}_2 + \mathcal{S}_1 \to \mathcal{S}'_1 + \mathcal{S}'_2$,* i.e., *for the wave strength interaction that $\beta + \nu \to \nu' + \beta'$, one of the following estimates holds:*

$$
\begin{aligned}
&(a)\ |\nu'| + |\beta'| \le |\beta| + |\nu| + C_5(\gamma - 1 + \tau^2)|\beta||\nu|,\\
&(b)\ |\nu'| = |\nu| - \zeta, \quad |\beta'| \le |\beta| + C_5(\gamma - 1 + \tau^2)|\beta||\nu| + \eta, \qquad (3.7)\\
&(c)\ |\beta'| = |\beta| - \zeta, \quad |\nu'| \le |\nu| + C_5(\gamma - 1 + \tau^2)|\beta||\nu| + \eta,
\end{aligned}
$$

where $0 \le \eta \le \delta\zeta$ and $\zeta > 0$ is a constant;

(2) *For the case that $\mathcal{S}_2 + \mathcal{R}_1 \to \mathcal{R}'_1 + \mathcal{S}'_2$,* i.e., *for the wave interaction that $\beta + o \to o' + \beta'$, we have $|\beta'| = |\beta|$;*

(3) *For the case that $\mathcal{S}_2 + \mathcal{S}_2 \to \mathcal{R}'_1 + \mathcal{S}'_2$,* i.e., *for the wave interaction that $\beta_1 + \beta_2 \to o' + \beta'$, we have $|\beta'| = |\beta_1| + |\beta_2|$;*

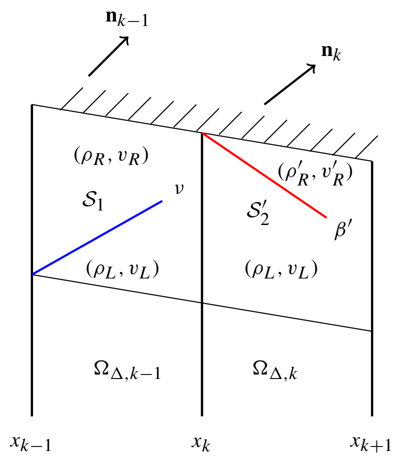

Fig. 10. $\mathcal{S}_1$ wave hits the boundary and $\mathcal{S}'_2$ wave reflects.

(4) *For the case that* $\mathcal{S}_2 + \mathcal{R}_2 \to \mathcal{S}'_1 + \mathcal{S}'_2$ *(or* $\mathcal{S}_2 + \mathcal{R}_1 \to \mathcal{R}'_1 + \mathcal{S}'_2$*),* i.e.*, for the wave interaction that* $\beta + \pi \to \nu' + \beta'$ *(or* $\beta + o \to o' + \beta'$*), there exist* 1*-shock wave* ν_0 *and* 2*-shock wave* β_0 *such that the wave interaction* $\beta_0 + \nu_0 \to \nu' + \beta'$ *is the same as the one in (1) and the following estimate hold:*

$$|\nu_0| + |\beta_0| \le |\beta| - C_0|\nu_0|;$$

(5) *For the case that* $\mathcal{R}_2 + \mathcal{S}_2 \to \mathcal{S}'_1 + \mathcal{S}'_2$ *(or* $\mathcal{R}_2 + \mathcal{S}_1 \to \mathcal{S}'_1 + \mathcal{R}'_2$*),* i.e.*, for the wave interaction that* $0 + \beta \to \nu' + \beta'$ *(or* $\pi + \nu \to \nu' + 0'$*), we have* $|\nu'| + |\beta'| \le |\beta| - C_0|\nu'|$*;*

(6) *For the case that* $\mathcal{R}_2 + \mathcal{R}_1 \to \mathcal{R}'_1 + \mathcal{R}'_2$, i.e.*, for the wave interaction that* $\pi + o \to o' + \pi'$*, we have* $|o| + |\pi| = |o' + |\pi'|$*;*

(7) *For the case that* $\mathcal{S}_1 + \mathcal{R}_1 \to \mathcal{S}'_1 + \mathcal{S}'_2$, i.e.*, for the wave interaction that* $\nu + o \to \nu' + \beta'$*, we have* $|\nu'| + |\beta'| \le |\nu| - C_0|\beta'|$*;*

(8) *For the case that* $\mathcal{S}_1 + \mathcal{S}_1 \to \mathcal{S}'_1 + \mathcal{R}'_2$, i.e.*, for the wave interaction that* $\nu_1 + \nu_2 \to \nu' + \pi'$*, we have* $|\nu'| = |\nu_1| + |\nu_2|$.

Proof. We postpone the proof into the appendix because it is similar to the one in [17]. □

Next, let us consider the interaction estimates near the boundary. First, we study the case that $\mathcal{S}_1$ wave hit the boundary and then reflects (see Fig. 10).

Lemma 3.6. *Let* $\gamma \in [1, 2]$*,* $0 < \hat{\rho} < \check{\rho} < \infty$ *and* $b_0 < 0$*. Suppose that the constant states* $U_L, U_R \in \mathcal{O}(U_\infty)$ *with* $\rho_L, \rho_R \in [\hat{\rho}, \check{\rho}]$*, satisfies that*

$$v_R = (1 + \tau^2 u_R) b_0, \qquad \omega_{+,R} = -\Phi_1(\nu, U_L, \tau^2) + \omega_{+,L}. \tag{3.8}$$

Then, for constant state $U'_R \in \mathcal{O}(U_\infty)$ *with* $\rho'_R \in [\hat{\rho}, \check{\rho}]$ *which satisfies that*

$$v'_R = (1 + \tau^2 u'_R) b_0, \qquad \omega'_{-,R} = -\Phi_2(\beta', U_L, \tau^2) + \omega_{-,L}, \tag{3.9}$$

we have

$$\beta' = K_b \nu, \tag{3.10}$$

where

$$K_b = -1 + \mathcal{O}(1)(\gamma - 1 + \tau^2), \tag{3.11}$$

with the bound $\mathcal{O}(1)$ *depending only on the system and* U_L.

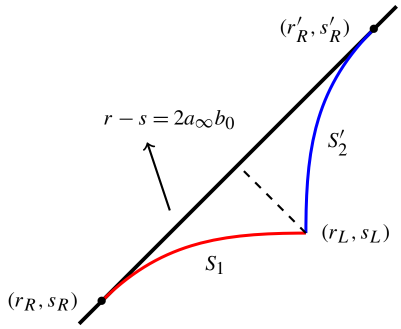

Fig. 11. S_1 wave hits on the boundary and S'_2 wave reflects.

Proof. Denote

$$\mathscr{L}_0(\beta', \nu, \gamma - 1, \tau^2) := (1 + \tau^2 u_R)v'_R - (1 + \tau^2 u'_R)v_R. \tag{3.12}$$

When $\gamma = 1$ and $\tau = 0$, (3.12) is reduced to

$$\mathscr{L}_0(\beta', \nu, \gamma - 1, \tau^2)\Big|_{\gamma=1,\tau=0} = \frac{1}{2a_\infty}\Big(\beta' + \nu + g(-\beta') - g(\nu)\Big),$$

where $g(\nu) := \Phi_1(\nu, U_L; \tau^2)\Big|_{\gamma=1,\tau=0}$ with $0 < g' < 1$, $g'' > 0$, and $\Phi_2(\beta', U_L; \tau^2)\Big|_{\gamma=1,\tau=0} = -g(-\beta')$. In this case, equation $\mathscr{L}_0(\beta', \nu, \gamma - 1, \tau^2)\Big|_{\gamma=1,\tau=0} = 0$ admits a unique solution $\beta' = -\nu$. Note that

$$\frac{\partial \mathscr{L}_0(\beta', \nu, 0, 0)}{\partial \beta'}\Big|_{\gamma=1,\tau=0,\beta'=-\nu} = \frac{1}{2a_\infty}\Big(1 - g'(-\nu)\Big) > C > 0,$$

where constant C depends only on $\hat{\rho}$ and $\check{\rho}$. So it follows from the implicit function theorem that β' can be solved as a C^2 function of ν, $\gamma - 1$, τ^2, b_0 and U_L. Moreover,

$$\beta' = \beta'(\nu, \gamma - 1, \tau^2) = \nu \int_0^1 \partial_\nu \beta'(\chi\nu, \gamma - 1, \tau^2) d\chi,$$

where we have used the fact that $\beta'(0, \gamma - 1, \tau^2) = 0$ (Fig. 11).

Since $\beta'(\nu, 0, 0) = -\nu$, then $\partial_\nu \beta'(\nu, 0, 0) = -1$, which gives that

$$\beta' = \int_0^1 \Big(\partial_\nu \beta'(\chi\nu, \gamma - 1, \tau^2) - \partial_\nu \beta'(\chi\nu, 0, 0)\Big) d\chi\, \nu - \nu = \big(-1 + \mathcal{O}(1)(\gamma - 1 + \tau^2)\big)\nu.$$

So by taking $K_b = -1 + \mathcal{O}(1)(\gamma - 1 + \tau^2)$, we have equality (3.10) (Fig. 12). □

Now, let us consider the local interaction estimates near the boundary.

Lemma 3.7. *Let $\gamma \in [1, 2]$, and let $0 < \hat{\rho} < \check{\rho} < \infty$. Suppose constant states $U_L, U_M, U_R \in \mathcal{O}(U_\infty)$ with $\rho_L, \rho_M, \rho_R \in [\hat{\rho}, \check{\rho}]$, satisfy that*

$$v_R = (1 + \tau^2 u_R)b_0, \quad \omega_R = \mathscr{H}_2(z_2, \omega_M, \tau^2), \quad \omega_M = \mathscr{H}_1(z_1, \omega_L, \tau^2). \tag{3.13}$$

Then, there exist constants $C_{b0} > 0$, $C_{b1} > 0$ and $C_6 > 0$ independent of γ, τ, z_1, z_2 such that for any constant state $U'_R \in \mathcal{O}(U_\infty)$ with $\rho'_R \in [\hat{\rho}, \check{\rho}]$ which satisfies that

$$v'_R = (1 + \tau^2 u'_R)b_0, \qquad \omega'_R = \mathscr{H}_2(z'_2, \omega_L, \tau^2), \tag{3.14}$$

the following interaction estimates hold:

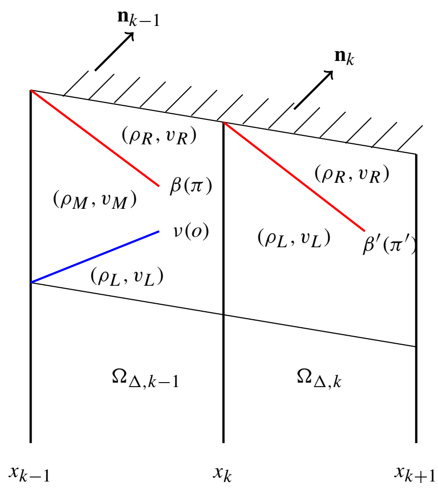

Fig. 12. Local interaction estimates near the boundary.

(1) *For the case that* $\mathcal{S}_1 + \mathcal{S}_2 \to \mathcal{S}_2'$, i.e., *for the wave strength interaction that* $\nu + \beta \to \beta'$, *it holds that*

$$|\beta'| \le K_{b0}|\nu| + |\beta| + C_6(\gamma - 1 + \tau^2)|\beta||\nu|, \tag{3.15}$$

with

$$K_{b0}\Big|_{\gamma=1,\tau=0} = 1 + C_{b0}. \tag{3.16}$$

(2) *For the case that* $\mathcal{R}_1 + \mathcal{S}_2 \to \mathcal{S}_2'$ *(or* $\mathcal{R}_1 + \mathcal{S}_2 \to \mathcal{R}_2'$*)*, i.e., *for the wave interaction that* $o + \beta \to \beta'$, *it holds that*

$$|\beta'| \le |\beta| + C_6(\gamma - 1 + \tau^2)|\beta||o| - C_{b1}|o|. \tag{3.17}$$

(3) *For the case that* $\mathcal{S}_1 + \mathcal{R}_2 \to \mathcal{S}_2'$ *(or* $\mathcal{S}_1 + \mathcal{R}_2 \to \mathcal{R}_2'$*)*, i.e., *for the wave interaction that* $\nu + \pi \to \beta'$ *(or* $\beta + o \to \beta'$*), it holds that*

$$|\beta'| \le K_{b1}|\nu| + C_6(\gamma - 1 + \tau^2)|\nu|^2, \tag{3.18}$$

where

$$K_{b1}\Big|_{\gamma=1,\tau=0} = 1. \tag{3.19}$$

(4) *For the case that* $\mathcal{R}_1 + \mathcal{R}_2 \to \mathcal{R}_2'$, i.e., *for the wave interaction that* $o + \pi \to \pi'$, *it holds that* $|o| + |\pi| = |\pi'|$.

Proof. For the notational simplicity, for $\gamma = 1$ and $\tau = 0$, let

$$\Phi_1(\alpha, U; \tau^2)\big|_{\gamma=1,\tau=0} =: g(\alpha), \tag{3.20}$$

for some $\alpha > 0$ and $U \in \mathcal{O}(U_\infty)$. Then function g satisfies the properties that

$$0 < g'(\alpha) < 1, \quad g''(\alpha) > 0, \tag{3.21}$$

for $\alpha > 0$. As shown in Remark 2.4,

$$\Phi_2(\beta, U; \tau^2)\big|_{\gamma=1,\tau=0} = -g(-\beta), \tag{3.22}$$

for some $\beta < 0$ and $U \in \mathcal{O}(U_\infty)$.

For the first case $\mathcal{S}_1 + \mathcal{S}_2 \to \mathcal{S}_2'$, from (3.13)-(3.14), we have

$$v_R = (1 + \tau^2 u_R)b_0, \quad \omega_{-,M} - \omega_{-,R} = \Phi_2(\beta, U_M, \tau^2), \quad \beta = \omega_{+,M} - \omega_{+,R}, \tag{3.23}$$

$$\omega_{+,L} - \omega_{+,M} = \Phi_1(\nu, U_L, \tau^2), \quad \nu = \omega_{-,L} - \omega_{-,M}, \tag{3.24}$$

and

$$v'_R = (1 + \tau^2 u'_R) b_0, \quad \omega_{-,L} - \omega'_{-,R} = \Phi_2(\beta', U_L, \tau^2), \quad \beta' = \omega_{+,L} - \omega'_{+,R}. \tag{3.25}$$

Then,

$$(1 + \tau^2 u_R) v'_R = (1 + \tau^2 u'_R) v_R, \tag{3.26}$$

where $(u_R, v_R) = (u, v)(\beta, \nu, \gamma - 1, \tau^2, U_L)$ and $(u'_R, v'_R) = (u', v')(\beta', \gamma - 1, \tau^2, U_L)$. Let

$$\mathscr{L}_1(\beta', \beta, \nu, \gamma - 1, \tau^2, b_0, U_L) := (1 + \tau^2 u_R) v'_R - (1 + \tau^2 u'_R) v_R. \tag{3.27}$$

When $\gamma = 1$ and $\tau = 0$, equation (3.27) is

$$\mathscr{L}_1\big|_{\gamma=1,\tau=0} = \frac{1}{2a_\infty}\Big(\beta' + g(-\beta') - \beta - g(-\beta) + \nu - g(\nu)\Big), \tag{3.28}$$

for $\beta < 0$ and $\nu > 0$.

Notice that $\frac{\partial(\mathscr{L}_1|_{\gamma=1,\tau=0})}{\partial\beta'} = \frac{1-g'(-\beta')}{2a_\infty} > 0$, $\lim_{\beta'\to-\infty} \mathscr{L}_1\big|_{\gamma=1,\tau=0} = -\infty$, and

$$\mathscr{L}_1\big|_{\gamma=1,\tau=0,\beta'=\beta-\nu} = \frac{1}{2a_\infty}\Big(g(\nu-\beta) - g(-\beta) - g(\nu)\Big) = -\beta\nu \int_0^1\int_0^1 g''(\xi\nu - \tilde{\xi}\beta) d\xi d\tilde{\xi} > 0.$$

So equation $\mathscr{L}_1(\beta', \beta, \nu, 0, 0, b_0, U_L) = 0$ admits a unique root β'_0. By Lemma 2.3,

$$\frac{\partial \mathscr{L}_1(\beta', \beta, \nu, \gamma - 1, \tau^2, b_0, U_L)}{\partial \beta'}\Big|_{\gamma=1,\tau=0,\beta'=\beta'_0} = \frac{1}{2a_\infty}\Big(1 - g'(-\beta'_0)\Big) > C > 0,$$

for some $C > 0$ depends only on the $\hat{\rho}$ and $\check{\rho}$.

Therefore, it follows from the implicit function theorem that β' can be solved as a C^2 function of $\beta, \nu, \gamma - 1, \tau^2, b_0$ and U_L, that is

$$\begin{aligned}
\beta' &= \beta'(\beta, \nu, \gamma - 1, \tau^2) \\
&= \beta'(0, \nu, \gamma - 1, \tau^2) + \beta'(\beta, 0, \gamma - 1, \tau^2) + \mathcal{O}(\beta, \nu, \gamma - 1, \tau^2)\beta\nu \\
&= K_b\nu + \beta + \mathcal{O}(\beta, \nu, \gamma - 1, \tau^2)\beta\nu,
\end{aligned}$$

where coefficient K_b is given by (3.10) in Lemma 3.6. Moreover,

$$\beta'_0 := \beta'(\beta, \nu, 0, 0) = \beta'(0, \nu, 0, 0) + \beta'(\beta, 0, 0, 0) + \mathcal{O}(\beta, \nu, 0, 0)\beta\nu = -\nu + \beta + \mathcal{O}(\beta, \nu, 0, 0)\beta\nu.$$

Subtracting the two identities above implies that

$$\beta' = \beta'_0 + \mathcal{O}(1)(\gamma - 1 + \tau^2)\nu + \mathcal{O}(1)(\gamma - 1 + \tau^2)\beta\nu. \tag{3.29}$$

So the remaining task is to estimate β'_0 more carefully (see Fig. 13) for the case that $\gamma = 1$ and $\tau = 0$. By (3.23)-(3.26), we have the relation that

$$\beta'_0 + \nu - \beta = g(\nu) + g(-\beta) - g(-\beta'_0), \tag{3.30}$$

where $\nu = \big(\omega_{-,L} - \omega_{-,M}\big)\big|_{\gamma=1,\tau=0} > 0$, $\beta = \big(\omega_{+,M} - \omega_{+,R}\big)\big|_{\gamma=1,\tau=0} < 0$ and $\beta'_0 = \big(\omega_{+,L} - \omega'_{+,R}\big)\big|_{\gamma=1,\tau=0} < 0$. Direct computation shows that

$$\begin{aligned}
g(\nu) - g(-\beta) - g(-\beta'_0) &= g(\nu - \beta) - g(-\beta'_0) + g(\nu) + g(-\beta) - g(\nu - \beta) \\
&\geq g'(\xi_1)(\nu - \beta + \beta'_0) + g(-\beta) - g(\nu - \beta) \\
&\geq g'(\xi_1)(\nu - \beta + \beta'_0) + g'(\xi_2)(-\nu),
\end{aligned}$$

where $\xi_1 \in (-\beta'_0, \nu - \beta)$ and $\xi_2 \in (\nu - \beta, -\beta')$. This together with (3.30) yields that

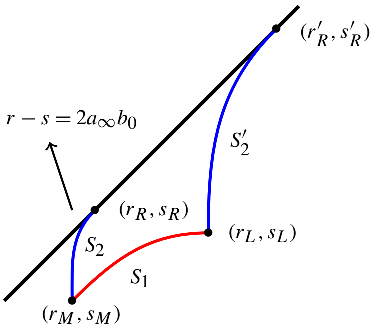

Fig. 13. S_1 and S_2 waves interaction and reflection on the boundary.

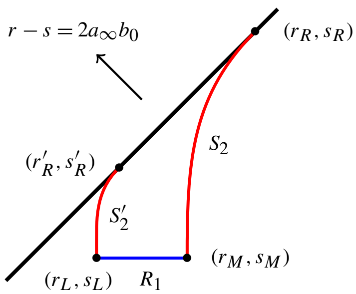

Fig. 14. R_1 and S_2 waves interaction and reflection on the boundary.

$$-\beta'_0 - \nu + \beta \le \frac{g'(\xi_2)}{1 - g'(\xi_1)}\nu.$$

Let $C_{b0} = \sup_{\xi_1 \in (-\beta'_0, \nu-\beta), \xi_2 \in (\nu-\beta, -\beta')} \frac{g'(\xi_2)}{1-g'(\xi_1)}$, then we have

$$|\beta'_0| \le (1 + C_{b0})|\nu| + \beta.$$

So it follows from (3.29) that

$$|\beta'| \le \Big(1 + C_{b0} + \mathcal{O}(1)(\gamma - 1 + \tau^2)\Big)|\nu| + \beta + \mathcal{O}(1)(\gamma - 1 + \tau^2)|\nu||\beta|.$$

This completes the proof for the first case.

Next, for the second case $\mathcal{R}_1 + \mathcal{S}_2 \to \mathcal{S}'_2$, note that β and β' satisfy (3.23) and (3.25),

$$\omega_{+,M} = \omega_{+,L}, \quad o = \omega_{-,L} - \omega_{-,M} < 0, \qquad \text{on } \mathcal{R}_1, \tag{3.31}$$

and equality (3.26) holds on the boundaries Γ_k and Γ_{k+1} with $(u_R, v_R) = (u, v)(\beta, o, \gamma - 1, \tau^2, U_L)$ and $(u'_R, v'_R) = (u', v')(\beta', \gamma - 1, \tau^2, U_L)$. Let

$$\mathscr{L}_2(\beta', \beta, o, \gamma - 1, \tau^2, b_0, U_L) := (1 + \tau^2 u_R)v'_R - (1 + \tau^2 u'_R)v_R.$$

As done for the first case, similarly, it follows from the implicit function theorem that β' can be solved as a C^2 function of $\beta, \nu, \gamma - 1, \tau^2, b_0, U_L$ with the estimate that

$$|\beta'| \le |\beta'(\beta, o, 0, 0)| + \mathcal{O}(1)(\gamma - 1 + \tau^2)|\beta||o|. \tag{3.32}$$

Now, we will estimate $\beta'(\beta, 0, 0, 0)$ (see Fig. 14). Let $\beta'_1 = \beta'(\beta, 0, 0, 0)$. Then

$$\beta'_1 - \beta + g(-\beta'_1) - g(-\beta) = |o|.$$

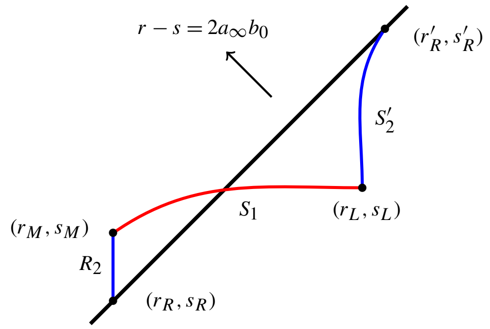

Fig. 15. S_1 and R_2 waves interaction and reflection on the boundary.

By the mean value theorem, we further have

$$\beta'_1-\beta=\frac{|o|}{1-g'(\xi_3)},\quad \xi_3\in(-\beta,-\beta'_1),$$

which implies that

$$|\beta'_1|\le|\beta|-C_{b1}|o|,$$

where $C_{b1}=\inf_{\xi_3\in(-\beta,-\beta'_1)}\frac{1}{1-g'(\xi_3)}$. This together with (3.32) yields estimate (3.17).

For the third case that $\mathcal{S}_1+\mathcal{R}_2\to\mathcal{S}'_2$, we know that (3.24) and (3.25) hold on $\mathcal{S}_1$ and $\mathcal{S}'_2$,

$$\omega_{-,M}=\omega_{-,R},\quad \pi=\omega_{+,M}-\omega_{+,R}>0,\qquad \text{on } \mathcal{R}_2,$$

and equality (3.26) holds on boundaries Γ_k and Γ_{k+1} with $(u_R,v_R)=(u,v)(\pi,\nu,\gamma-1,\tau^2,U_L)$ and $(u'_R,v'_R)=(u',v')(\beta',\gamma-1,\tau^2,U_L)$. Let

$$\mathscr{L}_3(\beta',\beta,o,\gamma-1,\tau^2,b_0,U_L):=(1+\tau^2u_R)v'_R-(1+\tau^2u'_R)v_R.$$

Then similarly as done for the first case, by the implicit function theorem, β' can be solved as a C^2 function of $\pi,\nu,\gamma-1,\tau^2,b_0,U_L$, with the following estimate

$$|\beta'|\le|\beta'(\pi,\nu,0,0)|+\mathcal{O}(1)(\gamma-1+\tau^2)|\nu|+\mathcal{O}(1)(\gamma-1+\tau^2)|\nu|^2. \tag{3.33}$$

For the term $\beta'_2=\beta'(\pi,\nu,0,0)$ (see Fig. 15), we have that

$$\beta'_2+\nu=g(\nu)-g(-\beta'_2)+\pi\ge g'(\xi_4)(\beta'_2+\nu),\quad \xi_4\in(-\beta'_2,\nu)$$

which implies that $|\beta'_2|\le|\nu|$. Thus, it with (3.33) yields estimate (3.18).

Finally, for the fourth case that $\mathcal{R}_1+\mathcal{R}_2\to\mathcal{R}'_2$, estimate is obvious since across the rarefaction waves the strength of the waves is unchanged. □

4. Global entropy solutions with large data

In this section, we first construct the approximate solution for the initial-boundary value problem (1.17)–(1.19) by employing the modified Glimm scheme in an approximate domain Ω_Δ which will be defined below, and then show the existence of global entropy solutions with large data.

4.1. Modified Glimm scheme for the problem (1.17)–(1.19)

Since $T.V.(U_0)<\infty$, limits $\lim_{y\to\pm\infty}U_0(y)$ exist, which are denoted by $U_\pm$. Let

$$\mathcal{O}(U_\pm)=\big\{U:|U-U_-|+|U-U_+|<4T.V.(U_0)\big\}. \tag{4.1}$$

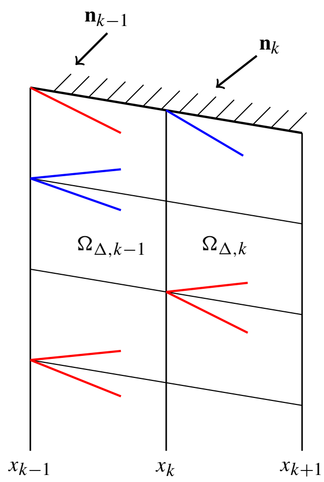

Fig. 16. The modified Glimm scheme.

Let Δx be the mesh length in the x-direction. Choose a set of points $\{A_k\}_{k=0}$ with $A_k = (x_k, b_k) = (k\Delta x, b_0 k\Delta x)$ on the straight boundary $y = b_0 x$ in order. As shown in Fig. 16, define

$$\begin{aligned}
&b_\Delta(x) = b_k + (x - x_k)b_0, \quad \forall x \in [k\Delta x,\ (k+1)\Delta x),\ k \geq 0,\\
&\Omega_{\Delta,k} = \{(x, y) :\ k\Delta x \leq x < (k+1)\Delta x,\ y < b_\Delta(x)\},\\
&\Gamma_{\Delta,k} = \{(x, y) :\ k\Delta x \leq x < (k+1)\Delta x,\ y = b_\Delta(x)\},\\
&\Omega_\Delta = \bigcup_{k\geq 0} \Omega_{\Delta,k}, \quad \Gamma_\Delta = \bigcup_{k\geq 0} \Gamma_{\Delta,k}.
\end{aligned} \tag{4.2}$$

Let $\mathbf{n}_k$ be the outer unit normal vector to $\Gamma_{\Delta,k}$ as

$$\mathbf{n}_k = \frac{(b_{k+1} - b_k, -x_{k+1} + x_k)}{\sqrt{(b_{k+1} - b_k)^2 + (x_{k+1} - x_k)^2}} = \frac{(b_0, -1)}{\sqrt{1 + b_0^2}}. \tag{4.3}$$

We choose the mesh length in the y-direction as Δy such that the following Courant-Friedrichs-Lewy condition holds:

$$\frac{\Delta y}{\Delta x} < \sup_{U\in\mathcal{O}(U_\pm),\tau\in(0,\epsilon_*)} \left\{\max_{l=\pm} |\lambda_l(U, \tau^2)|\right\} - b_0, \tag{4.4}$$

where $\epsilon_* = \min\{\epsilon_6, \epsilon_7\}$, and ϵ_6 and ϵ_7 are given by Proposition 2.1 and Proposition 2.2, respectively.

For any non-negative integer k and negative integer n, *i.e.*, for $k \geq 0$ and $n \leq -1$, define

$$y_{k,n} = b_k + (2n + 1 + \sigma_k)\Delta y, \tag{4.5}$$

where σ_k is randomly chosen in $(-1, 1)$. Then, let

$$P_{k,n} = (x_k, y_{k,n}), \tag{4.6}$$

be the mesh points and define the approximate solutions $U_{\Delta,\sigma}(x, y)$ in Ω_Δ for any $\sigma = (\sigma_0, \sigma_1, \cdots)$ via the Glimm Scheme inductively as follows.

Step 1. For $k = 0$, we approximate the initial data by piecewise constant functions.

$$U_{\Delta,\sigma}(x = 0, y) = \begin{cases} U_0(y_{0,n}), & b_k + 2(n+1)\Delta y \leq y \leq b_k + 2n\Delta y,\\ U_0(y_{0,n+1}), & b_k + 2(n+2)\Delta y \leq y \leq b_k + 2(n+1)\Delta y, \end{cases} \tag{4.7}$$

where $U_0(y_{0,n})$ and $U_0(y_{0,n+1})$ are constant states.

Step 2. Assume the approximate solution $U_{\Delta,\sigma}(x, y)$ has been defined in $\Omega_\Delta \cap \{0 < x < x_k\}$ for $k > 0$. Then, for any $n \le -1$ and $y \in (b_k + 2n\Delta y, b_k + 2(n+1)\Delta y)$, define $U^0_{k,n}$ by

$$U^0_{k,n} = U_{\Delta,\sigma}(x_k-, y_{k,n}). \tag{4.8}$$

Now, we first solve the Riemann problem in the diamond $T_{k,0}$ whose vertices are (x_k, b_k), $(x_k, b_k - \Delta y)$, (x_{k+1}, b_k) and $(x_{k+1}, b_k - \Delta y)$ with initial data $U_{\Delta,\sigma} = U^0_{k,0}$, that is

$$\begin{cases} \partial_x W(U_{k,0}, \tau^2) + \partial_y F(U_{k,0}, \tau^2) = 0, & \text{in } T_{k,0}, \\ U_{k,0}|_{x=x_k} = U^0_{k,0}, & \text{on } \{b_k - \Delta y < y < b_k\}, \\ \big((1 + \tau^2 u(\rho_{k,0}, v_{k,0}, \tau^2)), v_{k,0}\big) \cdot \mathbf{n}_k = 0, & \text{on } \Gamma_k. \end{cases} \tag{4.9}$$

Then, we can obtain the Riemann solution $U_{k,0}$ in $T_{k,0}$ by Proposition 2.2. Define

$$U_{\Delta,\sigma} = U_{k,0}, \quad \text{in } T_{k,0}. \tag{4.10}$$

Next, we solve the Riemann problem in each diamond $T_{k,n}$ for $n \le -2$ whose vertices are $(x_k, b_k + 2n\Delta y)$, $(x_k, b_k + 2(n+1)\Delta y)$, $(x_{k+1}, b_k + 2n\Delta y)$ and $(x_{k+1}, b_k + 2(n+1)\Delta y)$

$$\begin{cases} \partial_x W(U_{k,n}, \tau^2) + \partial_y F(U_{k,n}, \tau^2) = 0, & \text{in } T_{k,n}, \\ U_{k,n}|_{x=x_k} = \begin{cases} U^0_{k,n}, & b_k + 2n\Delta y < y < b_k + 2(n+1)\Delta y, \\ U^0_{k,n-1}, & b_k + 2(n-1)\Delta y < y < b_k + 2n\Delta y. \end{cases} \end{cases} \tag{4.11}$$

By Proposition 2.1, Riemann problem (4.11) admits a Riemann solution $U_{k,n}$ in $T_{k,n}$. Define

$$U_{\Delta,\sigma} = U_{k,n}, \quad \text{in } \ T_{k,n}. \tag{4.12}$$

Therefore, we can construct the approximate solution $U_{\Delta,\sigma}(x, y)$ globally provided that we can obtain the uniform bound of the approximate solutions, which will be the main goal in the next subsection.

4.2. Glimm-type functional and the global existence of entropy solutions

In this subsection, we will introduce the weighted Glimm-type functional and apply the functional to show the convergence of the approximation solutions and then obtain the global existence of entropy solutions of problem (1.17)-(1.19) of large data. To obtain it, as done in [12], we introduce mesh curves J which is space-like, and consists of the line segments jointing the random points $P_{k,n}$ one by one in the order of n. Obviously, region Ω_Δ is the union of the diamonds whose boundaries are the line segments of the mesh curves with four adjacent random points as their vertices. Moreover, J divides the region Ω_Δ into two subregions denoted by J^- and J^+, where J^- denotes the subregion containing the y-axis and $J^+ = \Omega_\Delta \backslash J^-$. Now we can define the order of the mesh curves.

Definition 4.1. Assume that I and J are two mesh curves, we call $J > I$ if and only if every mesh point of the curve J is either on I or contained in I^+. Moreover, if $J > I$ and every mesh points of J except one lie on I, then we call J is an immediate successor to I.

For the approximate solution $U_{\Delta,\sigma}(x, y)$, let $S_j(J)$, where $j = 1$ or 2, be the set of j-shock waves which go across the mesh curve J. Let $S(J) := S_1(J) \cap S_2(J)$. Define the Glimm-type functional

$$F(J) = L(J) + 4C_*(\gamma - 1 + \tau^2)Q(J), \tag{4.13}$$

where

$$L(J) = \mathscr{K}_b L_1(J) + L_2(J), \tag{4.14}$$

$$L_1(J) = \sum \big\{|\alpha| : \alpha \in S_1(J)\big\}, \quad L_2(J) = \sum \big\{|\beta| : \alpha \in S_2(J)\big\}, \tag{4.15}$$

and

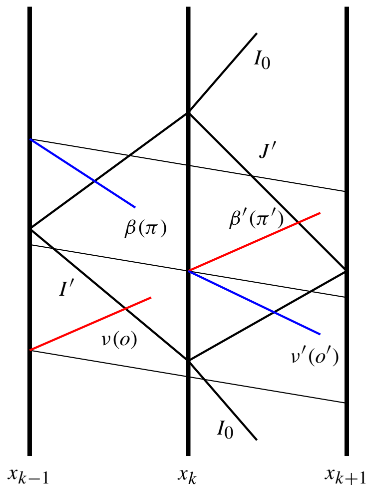

Fig. 17. Λ lies in Ω_Δ.

$$Q(J) = \sum \{|\alpha||\beta| : \alpha \in S_1(J),\ \beta \in S_2(J) \text{ and } \alpha, \beta \text{ are approaching}\}. \tag{4.16}$$

For the definition of the Glimm's type functional, we remark that different from the small data case, our definition in (4.15) and (4.16) for the large data case does not included the rarefaction waves. It is because the strength of the rarefaction waves can be controlled by the strength of the shock waves.

Constants $\mathscr{K}_b$ and C_* satisfy that

$$\max\{K_{b0},\ K_{b1},\ 1\} < \mathscr{K}_b < \min\{\frac{1}{\delta},\ 1 + C_0,\ 4\}, \qquad C_* > \max\{C_5,\ C_6, \mathscr{K}_b\}, \tag{4.17}$$

where constants δ, C_0, C_5 and K_{b0}, K_{b1}, C_6 are given in Lemma 3.5 and Lemma 3.7, respectively.

Then, we have the following lemma for functional $F(J)$, which ensures the uniform bound of the approximate solutions.

Lemma 4.1. *Suppose that I and J are any two space-like mesh curves satisfying $J > I$. There exists a constant $C_7 > 0$ depending only on C_0 and δ, such that if $C_*(\gamma - 1 + \tau^2)F(I) \le C_7$, then it holds that*

$$F(J) < F(I). \tag{4.18}$$

Proof. Without loss of the generality, we only consider the case that J is an immediate successor to I, since the other cases can be treated easily by the induction method. Let Λ be the diamond between I and J, *i.e.*, $\Lambda = I' \cup J'$, where $I = I_0 \cup I'$ and $J = I_0 \cup J'$. The proof is devided into two cases depending the location of Λ.

Case 1. Λ lies in the interior of Ω_Δ (see Fig. 17). Let us start with case (1) as listed in Lemma 3.5. For the subcase (a), we have

$$L(J) - L(I) \le C_5(\mathscr{K}_b + 1)(\gamma - 1 + \tau^2)|\beta||\nu|.$$

For $Q(J)$, we have that

$$\begin{aligned} Q(J) - Q(I) &= Q(J', I_0) + Q(I_0) - Q(I', I_0) - Q(I_0) - Q(I') \\ &\le \sum_{\mu \in S(I)} |\mu|\big(|\beta'| + |\nu'| - |\beta| - |\nu|\big) - |\beta||\nu| \\ &\le \Big(C_5(\gamma - 1 + \tau^2)F(I) - 1\Big)|\beta||\nu|. \end{aligned}$$

Then

$$
\begin{aligned}
F(J)-F(I) &= C_5(\mathscr{K}_b+1)(\gamma-1+\tau^2)|\beta||\nu| + 4C_*(\gamma-1+\tau^2)\big(C_5(\gamma-1+\tau^2)F(I)-1\big)|\beta||\nu| \\
&\le 2C_*(\gamma-1+\tau^2)\Big(2C_*(\gamma-1+\tau^2)F(I)-1\Big)|\beta||\nu|.
\end{aligned}
$$

Therefore, if we choose $(\gamma-1+\tau^2)F(I) < \frac{1}{2C_*}$, then we have $F(J) < F(I)$.

Next, let us consider subcase (b) of case (1) as listed in Lemma 3.5. By Lemma 3.5, we have

$$
\begin{aligned}
L(J)-L(I) &\le -\mathscr{K}_b\zeta + C_5(\gamma-1+\tau^2)|\beta||\nu| + \eta \\
&\le -(\mathscr{K}_b-\delta)\zeta + C_5(\gamma-1+\tau^2)|\beta||\nu|,
\end{aligned}
$$

and

$$
\begin{aligned}
Q(J)-Q(I) &= Q(J',I_0)+Q(I_0)-Q(I',I_0)-Q(I_0)-Q(I') \\
&\le \sum_{\mu\in S(I)} |\mu|\big(|\beta'|-|\beta|\big) + \sum_{\mu'\in S(I)} |\mu'|\big(|\nu'|-|\nu|\big) - |\beta||\nu| \\
&\le \sum_{\mu\in S(I)} |\mu|\big(\eta + C_5(\gamma-1+\tau^2)|\beta||\nu|\big) - \sum_{\mu'\in S(I)} |\mu'|\zeta - |\beta||\nu| \\
&\le \Big(\delta\zeta + C_5(\gamma-1+\tau^2)|\beta||\nu|\Big)F(I) - |\beta||\nu|.
\end{aligned}
$$

Then

$$
\begin{aligned}
&F(J)-F(I) \\
&\le -(\mathscr{K}_b-\delta)\zeta + C_5(\gamma-1+\tau^2)|\beta||\nu| + 4C_*(\gamma-1+\tau^2)\Big(\big(\delta\zeta + C_3(\gamma-1+\tau^2)|\beta||\nu|\big)F(I) - |\beta||\nu|\Big) \\
&\le 4\delta\zeta\Big(C_*(\gamma-1+\tau^2)F(I) - \frac{\mathscr{K}_b-\delta}{4\delta}\Big) + 4C_*(\gamma-1+\tau^2)\Big(C_*(\gamma-1+\tau^2)F(I) - \frac{3}{4}\Big)|\beta||\nu|.
\end{aligned}
$$

Therefore, if $(\gamma-1+\tau^2)F(I) < \min\big\{\frac{3}{4C_*}, \frac{\mathscr{K}_b-\delta}{4\delta C_*}\big\}$, then $F(J) < F(I)$.

Finally, let us consider subcase (c) of case (1) at listed in Lemma 3.5. Note that

$$
L(J)-L(I) \le -(1-\mathscr{K}_b\delta)\zeta + \mathscr{K}_b C_5(\gamma-1+\tau^2)|\beta||\nu|,
$$

and

$$
\begin{aligned}
Q(J)-Q(I) &= Q(J',I_0)+Q(I_0)-Q(I',I_0)-Q(I_0)-Q(I') \\
&\le \sum_{\mu\in S(I)} |\mu|\big(|\nu'|-|\nu|\big) + \sum_{\mu'\in S(I)} |\mu'|\big(|\beta'|-|\beta|\big) - |\beta||\nu| \\
&\le \sum_{\mu\in S(I)} |\mu|\big(\eta + C_5(\gamma-1+\tau^2)|\beta||\nu|\big) - \sum_{\mu'\in S(I)} |\mu'|\zeta - |\beta||\nu| \\
&\le \Big(\delta\zeta + C_5(\gamma-1+\tau^2)|\beta||\nu|\Big)F(I) - |\beta||\nu|.
\end{aligned}
$$

So, we deduce that

$$
\begin{aligned}
&F(J)-F(I) \\
&\le -(1-\mathscr{K}_b\delta)\zeta + \mathscr{K}_b C_5(\gamma-1+\tau^2)|\beta||\nu| + 4C_*(\gamma-1+\tau^2)\Big(\big(\delta\zeta + C_5(\gamma-1+\tau^2)|\beta||\nu|\big)F(I) - |\beta||\nu|\Big) \\
&\le 4\delta\zeta\Big(C_*(\gamma-1+\tau^2)F(I) - \frac{1-\mathscr{K}_b\delta}{4\delta}\Big) + 4C_*(\gamma-1+\tau^2)\Big(C_*(\gamma-1+\tau^2)F(I) - \frac{4-\mathscr{K}_b}{4}\Big)|\beta||\nu|.
\end{aligned}
$$

If we choose $(\gamma-1+\tau^2)F(I) < \min\big\{\frac{1-\mathscr{K}_b\delta}{4\delta C_*}, \frac{4-\mathscr{K}_b}{4C_*}\big\}$, then $F(J) < F(I)$.

For case (2) as listed in Lemma 3.5, we have

$$L(J)-L(I)=0, \quad Q(J)-Q(I)=Q(J',I_0)+Q(I_0)-Q(I',I_0)-Q(I_0)=0.$$

Therefore $F(J)=F(I)$.

Next, let us consider case (3) as listed in Lemma 3.5. By Lemma 3.5, we have

$$L(J)-L(I)=0,$$

and

$$\begin{aligned} Q(J)-Q(I) &= Q(J',I_0)+Q(I_0)-Q(I',I_0)-Q(I_0)-Q(I') \\ &\leq \sum_{\mu\in S(I)} |\mu|\big(|\beta'|-|\beta_1|-|\beta_2|\big)-|\beta_1||\beta_2| \\ &= -|\beta_1||\beta_2|<0. \end{aligned}$$

So $F(J)<F(I)$.

Now, for case (4) as listed in Lemma 3.5, with the notations introduced in Lemma 3.1, we introduce a new mesh curve $\tilde{J}$ between the mesh curves I and J such that we have the local wave interaction $\beta+0\to\beta_0+\nu_0$ from I to $\tilde{J}$, and the local wave interaction $\beta_0+\nu_0\to\nu'+\beta'$ from $\tilde{J}$ to J. Then by Lemma 3.5, we have

$$F(J)<F(\tilde{J}),$$

provided that $(\gamma-1+\tau^2)F(\tilde{J})<\frac{1}{2C_*}$. Next, we also have that

$$\begin{aligned} L(\tilde{J})-L(I) &\leq (\mathscr{K}_b-1-C_0)|\nu_0|, \\ Q(\tilde{J})-Q(I) &\leq \sum_{\mu\in S(I)} |\mu|\big(|\beta_0|+|\nu_0|-|\beta|\big)+|\beta_0||\nu_0| \\ &\leq -C_0|\nu_0|F(I)+|\beta_0||\nu_0|. \end{aligned}$$

So

$$\begin{aligned} F(\tilde{J})-F(I) &\leq (\mathscr{K}_b-1-C_0)|\nu_0|+4C_*(\gamma-1+\tau^2)\Big(-C_0|\nu_0|F(I)+|\beta_0||\nu_0|\Big) \\ &\leq |\nu_0|\Big(4C_*(\gamma-1+\tau^2)|\beta_0|+(\mathscr{K}_b-C_0)-4C_*(\gamma-1+\tau^2)C_0F(I)\Big) \\ &\leq 4|\nu_0|\Big(C_*(\gamma-1+\tau^2)F(I)-\frac{C_0+1-\mathscr{K}_b}{4}\Big). \end{aligned}$$

Then, if we choose $(\gamma-1+\tau^2)F(I)<\frac{C_0+1-\mathscr{K}_b}{4C_*}$, then $F(\tilde{J})<F(I)$. Therefore,

$$F(J)<F(\tilde{J})<F(I).$$

For case (5) as listed in Lemma 3.5, we have that

$$L(J)-L(I)\leq \mathscr{K}_b|\nu'|+|\beta'|-|\beta|\leq(\mathscr{K}_b-1-C_0)|\nu'|<0,$$

and

$$Q(J)-Q(I)\leq \sum_{\mu\in S(I)}|\mu|\big(|\nu'|+|\beta'|-|\beta|\big)\leq -C_0F(I)|\nu'|.$$

It follows that $F(J)<F(I)$.

For case (6) as listed in Lemma 3.5, obviously, we have $F(J)=F(I)$.

For case (7) as listed in Lemma 3.5, we have

$$L(J)-L(I)\leq \mathscr{K}_b(|\nu'|-|\nu|)+|\beta'|\leq(1-\mathscr{K}_b-C_0\mathscr{K}_b)|\beta'|<0,$$

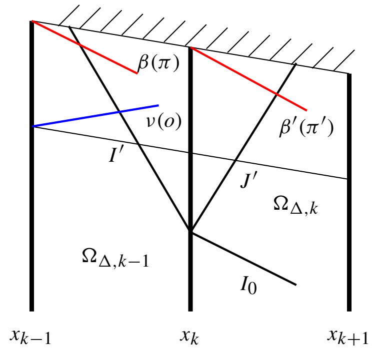

Fig. 18. Λ covers part of the approximate boundary Γ_Δ.

and

$$Q(J)-Q(I)\le\sum_{\mu\in S(I)}|\mu|\big(|\nu'|+|\beta'|-|\nu|\big)\le -C_0F(I)|\beta'|.$$

It implies that

$$F(J)-F(I)\le\Big(1-\mathscr{K}_b-C_0\mathscr{K}_b-4C_*C_0(\gamma-1+\tau^2)F(I)\Big)|\beta'|<0.$$

Therefore, $F(J)<F(I)$.

Finally, for case (8) as listed in Lemma 3.5, it can be treated similarly as the argument above for case (3) at listed in Lemma 3.5 to obtain (4.18).

Case 2. Λ covers part of the approximate boundary Γ_Δ (see Fig. 18). For case (1) as listed in Lemma 3.7, we have $L_2(J)-L_2(I)\le K_b|\nu|+C_6(\gamma-1+\tau^2)|\nu||\beta|$ and $L_1(J)-L_1(I)\le -|\nu|$. So

$$L(J)-L(I)\le -(\mathscr{K}_b-K_b)|\nu|+C_6(\gamma-1+\tau^2)|\nu||\beta|.$$

For $Q(J)$, we have that

$$\begin{aligned}Q(J)-Q(I)&=Q(J',I_0)+Q(I_0)-Q(I',I_0)-Q(I_0)-Q(I')\\&\le\sum_{\mu\in S(I)}|\mu|\big(|\beta'|-|\beta|-|\nu|\big)-|\beta||\nu|\\&\le\big(K_b-1\big)F(I)|\nu|+\Big(C_6(\gamma-1+\tau^2)F(I)-1\Big)|\beta||\nu|.\end{aligned}$$

Then, it follows from the estimates of $L(J)$ and $Q(J)$ that

$$\begin{aligned}F(J)-F(I)&\le\Big(4C_*(K_b-1)(\gamma-1+\tau^2)F(I)-(\mathscr{K}_b-K_b)\Big)|\nu|\\&\quad+C_6(\gamma-1+\tau^2)|\nu||\beta|+4C_*(\gamma-1+\tau^2)\Big(C_6(\gamma-1+\tau^2)F(I)-1\Big)|\beta||\nu|\\&\le\Big(2C_*K_b(\gamma-1+\tau^2)F(I)-(\mathscr{K}_b-K_b)\Big)|\nu|\\&\quad+4C_*(\gamma-1+\tau^2)\Big(C_*(\gamma-1+\tau^2)F(I)-\frac{3}{4}\Big)|\beta||\nu|.\end{aligned}$$

Therefore, if we choose $(\gamma-1+\tau^2)F(I)<\min\{\frac{3}{4C_*},\frac{\mathscr{K}_b-K_b}{2K_bC_*}\}$, then $F(J)<F(I)$.

Next, let's consider case (2) as listed in Lemma 3.7. Note that

$$L(J)-L(I)\le C_6(\gamma-1+\tau^2)|\beta||o|-C_{b1}|o|$$

and

$$\begin{aligned}Q(J)-Q(I) &= Q(J',I_0)+Q(I_0)-Q(I',I_0)-Q(I_0)-Q(I')\\ &\le \sum_{\mu\in S(I)}|\mu|\big(|\beta'|-|\beta|\big)\\ &\le \Big(C_6(\gamma-1+\tau^2)|\beta||o|-C_{b1}|o|\Big)F(I)\\ &\le \Big(C_6(\gamma-1+\tau^2)F(I)-C_{b1}\Big)F(I)|o|.\end{aligned}$$

So if $C_*(\gamma-1+\tau^2)F(I)\le C_{b1}$, then

$$\begin{aligned}F(J)-F(I) &\le C_6(\gamma-1+\tau^2)|\beta||o|-C_{b1}|o|\\ &\quad +4C_*(\gamma-1+\tau^2)\Big(C_6(\gamma-1+\tau^2)F(I)-C_{b1}\Big)F(I)|o|\\ &\le \Big(C_*(\gamma-1+\tau^2)F(I)-C_{b1}\Big)|o|\\ &\quad +4C_*(\gamma-1+\tau^2)\Big(C_*(\gamma-1+\tau^2)F(I)-C_{b1}\Big)F(I)|o|\\ &\le 0.\end{aligned}$$

Finally, let us consider case (3) as listed in Lemma 3.7. By direct computations,

$$L(J)-L(I)\le -\big(\mathscr{K}_b-K_{b1}\big)|\nu|+C_6(\gamma-1+\tau^2)|\nu|^2$$

and

$$\begin{aligned}Q(J)-Q(I) &= Q(J',I_0)+Q(I_0)-Q(I',I_0)-Q(I_0)-Q(I')\\ &\le \sum_{\mu\in S(I)}|\mu|\big(|\beta'|-|\nu|\big)\\ &\le \Big((K_{b1}-1)|\nu|+C_6(\gamma-1+\tau^2)|\nu|^2\Big)F(I).\end{aligned}$$

So

$$\begin{aligned}F(J)-F(I) &\le -\big(\mathscr{K}_b-K_{b1}\big)|\nu|+C_6(\gamma-1+\tau^2)|\nu|^2\\ &\quad +4C_*(\gamma-1+\tau^2)F(I)\Big((K_{b1}-1)|\nu|+C_6(\gamma-1+\tau^2)|\nu|^2\Big)\\ &\le \Big(-\big(\mathscr{K}_b-K_{b1}\big)+C_6(\gamma-1+\tau^2)F(I)+4C_*(K_{b1}-1)(\gamma-1+\tau^2)F(I)\\ &\quad +4C_*C_6(\gamma-1+\tau^2)^2F^2(I)\Big)|\nu|\\ &\le \Big(-\big(\mathscr{K}_b-K_{b1}\big)+4K_{b1}C_*(\gamma-1+\tau^2)F(I)+\big(2C_*(\gamma-1+\tau^2)F(I)\big)^2\Big)|\nu|.\end{aligned}$$

So, if we choose $(\gamma-1+\tau^2)F(I)\le\min\{\frac{1}{C_*},\frac{\mathscr{K}_b-K_{b1}}{4K_{b1}C_*}\}$, then $F(J)-F(I)\le 0$.
Based on all the arguments above, let

$$\begin{aligned}C_7=\min\Bigg\{&\frac{1}{2},\ \min\Big\{\frac{3}{4},\frac{\mathscr{K}_b-\delta}{4\delta}\Big\},\ \min\Big\{\frac{1-\mathscr{K}_b\delta}{4\delta},\frac{4-\mathscr{K}_b}{4}\Big\},\ \frac{1+C_0-\mathscr{K}_b}{4},\\ &\min\Big\{\frac{3}{4},\frac{\mathscr{K}_b-K_b}{2K_b}\Big\},\ C_{b1},\ \min\Big\{1,\frac{\mathscr{K}_b-K_{b1}}{4K_b}\Big\}\Bigg\}.\end{aligned}\tag{4.19}$$

So if $(\gamma-1+\tau^2)F(I)<\frac{C_7}{C_*}$, we can get estimate (4.18). □

Let O stand for the initial mesh curve, *i.e.*, for any mesh curve J, we have $O \le J$. Then, by Lemma 4.1, we know that if $C_*(\gamma - 1 + \tau^2)F(O) < C_7$, then

$$F(J) < F(O).$$

Next choose $\gamma_0 \in (1,2)$ and $\epsilon_* > 0$ such that $C_*(\gamma_0 - 1 + \epsilon_*^2)L(O) < 1$ and $C_*(\gamma_0 - 1 + \epsilon_*^2)F(O) < C_7$. Then for any $\gamma \in [1, \gamma_0]$ and $\tau \in (0, \epsilon_*)$, we have

$$F(J) < F(O) = L(O) + 4C_*(\gamma - 1 + \tau^2)Q(O) \le L(O) + 4C_*(\gamma - 1 + \tau^2)L^2(O) < 5L(O).$$

Notice that $L(O) \le C_{**}\big(T.V.\{U_0(\cdot); (-\infty, 0]\} + \|b_0\|_{L^\infty}\big)$ for some constant $C_{**} > 0$ depending only $\mathscr{K}_b$ and C_*. So by the standard argument, (see [12,21]), we have the following proposition.

Proposition 4.1. *Suppose that $\rho_0 \in [\rho_*, \rho^*]$ for some constant states ρ_* and ρ^* with $0 < \rho_* < \rho^* < \infty$. Then there exist constants $C_8 > 0$, $\gamma_0 \in (1,2)$ and $\epsilon_* > 0$ such that for any $\gamma \in [1, \gamma_0]$, $\tau \in (0, \epsilon_*)$ and $\sigma \in \prod_{k=0}^{\infty}(-1,1)$ if*

$$(\gamma - 1 + \tau^2)\Big(T.V.\big\{U_0(\cdot); (-\infty, 0]\big\} + \|b_0\|_{L^\infty}\Big) \le C_8, \tag{4.20}$$

then, a sequence of global approximate solutions $U_{\Delta,\sigma}(x,y)$ for all $(x,y) \in \Omega_\Delta$ is constructed via the Glimm scheme as given in §4.1. Moreover, there exist positive constants $C_9 > 0$ and $C_{10} > 0$ which is independent of Δ and σ such that

$$\sup_{x>0} T.V.\big\{U_{\Delta,\sigma}(x,\cdot); (-\infty, b_0 x]\big\} + \sup_{x>0} \|U_{\Delta,\sigma}(x,\cdot)\|_{L^\infty((-\infty, b_0 x])} \le C_9, \tag{4.21}$$

and

$$\int_{-\infty}^{0} \big|U_{\Delta,\sigma}(x_1, y + b_0 x_1) - U_{\Delta,\sigma}(x_2, y + b_0 x_2)\big| dy \le C_{10}\Big(\Delta x + |x_1 - x_2|\Big), \tag{4.22}$$

for any $x_1, x_2 > 0$.

Proposition 4.1 implies the compactness of the approximate solutions $\{U_{\Delta,\sigma}(x,y)\}$ in L^1_{loc} (see Theorem 2.4 of Chapter 2 in [4]). Then, by the standard arguments as done in [12,21,10,24,25], we can obtain the global existence of the entropy solutions of initial boundary value problem (1.17)–(1.19).

Theorem 4.1. *Assume that the range of the initial density ρ_0 lies in the interval $[\rho_*, \rho^*]$ for some constants ρ_* and ρ^* with $0 < \rho_* < \rho^* < \infty$. There exist constants $C_{11} > 0$, $C_{12} > 0$, $C_{13} > 0$ independent of γ, τ, and $\gamma_0 \in (1,2)$, $\epsilon_* > 0$ and a null set $\mathcal{N}$ such that for any $\gamma \in [1, \gamma_0]$, $\tau \in (0, \epsilon_*)$ and $\sigma \in \Big(\prod_{k=0}^{\infty}(-1,1)\backslash\mathcal{N}\Big)$ if*

$$(\gamma - 1 + \tau^2)\Big(T.V.\big\{U_0(\cdot); (-\infty, 0]\big\} + \|b_0\|_{L^\infty}\Big) \le C_{11}, \tag{4.23}$$

then, there exist a subsequence $\{\Delta_i\}_{i=0}^{\infty}$ and a function $U_\sigma(x,y)$ with bounded total variation such that $U_{\Delta_i,\sigma} \to U_\sigma(x,y)$ in $L^1_{loc}((-\infty, b_0 x])$ as $\Delta_i \to 0$ for every $x > 0$. The function $U_\sigma(x,y)$ is a global entropy solution of the initial boundary value problem (1.17)–(1.19) with the properties that

$$\sup_{x>0} T.V.\big\{U_\sigma(x,\cdot); (-\infty, b_0 x]\big\} + \sup_{x>0} \|U_\sigma(x,\cdot)\|_{L^\infty((-\infty, b_0 x])} \le C_{12}, \tag{4.24}$$

and

$$\int_{-\infty}^{0} \big|U_\sigma(x_1, y + b_0 x_1) - U_\sigma(x_2, y + b_0 x_2)\big| dy \le C_{13}|x_1 - x_2|, \quad \forall x_1, x_2 > 0. \tag{4.25}$$

Remark 4.1. As the notations introduced in the last sentence in the introduction, *i.e.*, in Section 1, solution $U_\sigma(x,y)$ to the initial boundary value problem (1.17)–(1.19) which are obtained in Theorem 4.1 actually depends on τ. So in order to pass the limit $\tau \to 0$ to prove Theorem 1.1, we will use the notations $U_\sigma^{(\tau)}(\bar{x}, \bar{y})$ and $(\bar{x}, \bar{y})$ again as done in the introduction except the last sentence.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. First, the global existence of the entropy solutions $U_\sigma^{(\tau)}$ to the initial boundary value problem (1.17)–(1.19) follows from the Theorem 4.1. Since solution $U_\sigma^{(\tau)}$ satisfies estimates (4.24) and (4.25) which is independent of τ, we can further apply the Helly's compactness theorem to obtain a subsequence $\{\tau_i\}_{i=1}^\infty$ such that $U_\sigma^{(\tau_i)}$ converges to $U_\sigma^{(0)}$ *a.e.* in Ω as $\tau_i \to 0$. Hence, $U_\sigma^{(\tau_i)} \to U_\sigma^{(0)}$ in $L^1(\Omega \cap B_{\bar{R}}(O))$ as $\tau_i \to 0$ for any $\bar{R} > 0$, where $B_{\bar{R}}(O) = \{(\bar{x}, \bar{y}) : \bar{x}^2 + \bar{y}^2 \leq \bar{R}^2\}$. Then, by the definition of entropy (1.20), we can show that $U_\sigma^{(0)}$ is an entropy solution to the initial-boundary value problem (1.15), (1.18) and (1.11) with $(\mathcal{E}(W^{(0)}, 0), \mathcal{Q}(W^{(0)}, 0))$, defined by (1.25), being its convex entropy pair with entropy inequality (1.26) in the distribution sense. This completes the proof of Theorem 1.1. □

Declaration of competing interest

There is no competing interest.

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Appendix A. Proof of Lemma 2.9

In this section, we are going to prove Lemma 2.9.

Proof. By Lemma 2.6 and the implicit function theorem, we define

$$\Psi_2(\alpha, U_0; \tau^2) := \frac{\partial \Phi_2(\beta_+, U_0; \tau^2)}{\partial \beta_+}.$$

Then (2.42) follows. Furthermore, by the straightforward calculation,

$$\frac{\partial \Phi_2(\beta_+, U_0; \tau^2)}{\partial \beta_+} = \frac{\frac{\partial(\omega_{-,0} - \omega_-)}{\partial \alpha}}{\frac{\partial(\omega_{+,0} - \omega_+)}{\partial \alpha}} = \frac{\rho_0 \partial_\rho u(\rho, v, \tau^2) + \big(\lambda_-(U, \tau^2) + \partial_v u(\rho, v, \tau^2)\big)\partial_\alpha \varphi}{\rho_0 \partial_\rho u(\rho, v, \tau^2) + \big(\lambda_+(U, \tau^2) + \partial_v u(\rho, v, \tau^2)\big)\partial_\alpha \varphi}.$$

When $\tau = 0$, it follows from Lemma 2.1, Remark 2.1 and Lemma 2.5 that

$$\Psi_2\big|_{\tau=0} = -\frac{2(\alpha^{\gamma-1} - 1) + (\gamma - 1)(\alpha^2 - 1)\alpha^{\gamma-2} - \sqrt{2(\gamma-1)(\alpha-1)(\alpha^{\gamma-1}-1)(\alpha+1)^3\alpha^{\gamma-3}}}{2(\alpha^{\gamma-1} - 1) + (\gamma - 1)(\alpha^2 - 1)\alpha^{\gamma-2} + \sqrt{2(\gamma-1)(\alpha-1)(\alpha^{\gamma-1}-1)(\alpha+1)^3\alpha^{\gamma-3}}}.$$

By Lemma 2.5, we know that $\beta_+ = \omega_{+,0} - \omega_+$ is monotonically decreasing with respect to α when $\alpha > 1$. Note that $\beta_+ = 0$ when $\alpha = 1$, so $\beta_+ = \omega_{+,0} - \omega_+ > 0$ when $\alpha > 1$.

Next, let us consider $\frac{\partial^2 \Phi_2(\beta_+, U_0; \tau^2)}{\partial \beta_+^2}$. Note that

$$\frac{\partial \Psi_2(\alpha, U_0; \tau^2)}{\partial \alpha} = \Big(\rho_0 \partial_\rho u(\rho, v, \tau^2) + \big(\lambda_+(U, \tau^2) + \partial_v u(\rho, v, \tau^2)\big)\partial_\alpha \varphi\Big)^{-2} \tilde{\mathcal{J}}(U, \tau^2),$$

where

$$\tilde{\mathcal{J}}(U,\tau^2)=\rho_0^2\Big(\big(\lambda_+-\lambda_-\big)\partial^2_{\rho\rho}u(\rho,v,\tau^2)+\big(\partial_\rho\lambda_--\partial_\rho\lambda_+\big)\partial_\rho u(\rho,v,\tau^2)\Big)\partial_\alpha\varphi$$
$$+\rho_0\Big(2\big(\lambda_+-\lambda_-\big)\partial^2_{\rho v}u(\rho,v,\tau^2)+\big(\lambda_++\partial_v u(\rho,v,\tau^2)\big)\partial_\rho\lambda_-$$
$$-\big(\lambda_-+\partial_v u(\rho,v,\tau^2)\big)\partial_\rho\lambda_++\big(\partial_v\lambda_--\partial_v\lambda_+\big)\partial_\rho u(\rho,v,\tau^2)\Big)(\partial_\alpha\varphi)^2$$
$$+\Big(\big(\partial_v\lambda_-+\partial^2_{vv}u(\rho,v,\tau^2)\big)\big(\lambda_++\partial_v u(\rho,v,\tau^2)\big)$$
$$-\big(\partial_v\lambda_++\partial^2_{vv}u(\rho,v,\tau^2)\big)\big(\lambda_-+\partial_v u(\rho,v,\tau^2)\big)\Big)(\partial_\alpha\varphi)^3+\rho_0\big(\lambda_--\lambda_+\big)\partial_\rho u(\rho,v,\tau^2)\partial^2_{\alpha\alpha}\varphi.$$

So, for $\tau=0$, by Lemma 2.1 and Lemma 2.5, we have

$$\frac{\partial\Psi_2}{\partial\alpha}\Big|_{\tau=0}=\frac{(\gamma-1)^2\alpha^{\frac{\gamma-5}{2}}\sqrt{2(\gamma-1)(\alpha-1)(\alpha^{\gamma-1}-1)}(\alpha+1)}{(\alpha-1)(\alpha^{\gamma-1}-1)}$$
$$\times\frac{2\big[(\gamma+1)\alpha^2-2\alpha+3-\gamma\big]\Big(\frac{\alpha^{\gamma-1}-1}{\gamma-1}\Big)^2-4\alpha^{\gamma-1}(1-\alpha^2)\frac{1-\alpha^{\gamma-1}}{\gamma-1}+\alpha^{\gamma-2}(\alpha^2-1)^2}{\Big(2(\alpha^{\gamma-1}-1)+(\gamma-1)\alpha^{\gamma-2}(\alpha^2-1)+\sqrt{2(\gamma-1)(\alpha-1)(\alpha^{\gamma-1}-1)(\alpha+1)^3\alpha^{\gamma-3}}\Big)^2}.$$

Define

$$\tilde{J}(\alpha,\gamma):=2\big[(\gamma+1)\alpha^2-2\alpha+3-\gamma\big]\Big(\frac{\alpha^{\gamma-1}-1}{\gamma-1}\Big)^2-4(\alpha^2-1)\alpha^{\gamma-1}\Big(\frac{\alpha^{\gamma-1}-1}{\gamma-1}\Big)+\big(\alpha^2-1\big)^2\alpha^{\gamma-2}.$$

Similar as the argument in the proof of Lemma 2.7, we can show that $\tilde{J}(\alpha,\gamma)>0$ when $\alpha>1$ and $1\le\gamma\le2$. Thus, we have $\frac{\partial\Psi_2}{\partial\alpha}\Big|_{\tau=0}>0$ when $\alpha>1$ and $1\le\gamma\le2$. So

$$\frac{\partial^2\Phi_2(\beta_+,U_0;0)}{\partial\beta_+^2}=\Big(\frac{\partial(\omega_{+,0}-\omega_+)}{\partial\alpha}\Big)^{-1}\Big|_{\tau=0}\frac{\partial\Psi_2(\alpha,U_0;\tau^2)}{\partial\alpha}\Big|_{\tau=0}<0,$$

for $\alpha>1$ and $1\le\gamma\le2$. Moreover, by the facts that $\Psi_2(1,U_0;0)=0$ and that $|\Psi_2\big|_{\tau=0}|<1$ for $\alpha>1$, we have $0<\Psi_2\big|_{\tau=0}<1$. For given ε_0, we can choose $\epsilon_5>0$ sufficiently small and a positive constant C_2 independent of τ such that for $\tau\in(0,\epsilon_5)$ and $\alpha<\varepsilon_0^{-1}$

$$0<\Psi_2(\alpha,U_0;\tau^2)<1-C_2\varepsilon_0.$$

This completes the proof of the lemma. □

Appendix B. Proof of lemmas of the local interaction estimates

First, let us give the proof of Lemma 3.1.

Proof of Lemma 3.1. Let $\Delta r=r_0-r$ and $\Delta s=s_1-s_0$. Notice that

$$\rho_0-\rho_1=\frac{\gamma-1}{4}(s_0-s_1)\le0.$$

Hence by Lemma 2.2, for $\xi\in\Big(\frac{\beta}{\rho_1^{\frac{\gamma-1}{2}}},\frac{\beta}{\rho_0^{\frac{\gamma-1}{2}}}\Big)$, we have

$$s_0-s-(s_1-s_2)=\int_0^{\Delta r}\frac{\partial h_1}{\partial\alpha}\Big|_{\alpha=\alpha_1(\xi)}\frac{\partial\alpha}{\partial\xi}\Big(\frac{\beta}{\rho_0^{\frac{\gamma-1}{2}}}-\frac{\beta}{\rho_1^{\frac{\gamma-1}{2}}}\Big)d\beta\ge0.$$

So in order to show (3.3), we only need to show

$$(s_0-s)-(s_1-s_2)\le C_3(\gamma-1)(s_1-s_0)(r_0-r).\tag{B.1}$$

Let $s_2 = s^*(\Delta r, \Delta s; \gamma - 1)$. Then, by Lemma 2.9 and Lemma 2.10, we know that s^* is a C^2-function of Δr, Δs and $\gamma - 1$.

For $\gamma = 1$ and $\alpha = \frac{\rho_1}{\rho_0}$, we have

$$\Delta s = -\sqrt{-\frac{2(1-\alpha)}{1+\alpha}\ln\alpha - \ln\alpha}, \quad \Delta r = \sqrt{-\frac{2(1-\alpha)}{1+\alpha}\ln\alpha - \ln\alpha}.$$

Notice that

$$\frac{\partial \Delta r}{\partial \alpha} = -\frac{-2\alpha\ln\alpha + 1 - \alpha^2 - \sqrt{-2(1-\alpha)(1+\alpha)^3\ln\alpha}}{\sqrt{-2(1-\alpha)(1+\alpha)^3\ln\alpha}} < 0.$$

Then, by the implicit function theorem, α is a function of Δr as $\alpha = \alpha(\Delta r)$, which is independent on ρ_0 and ρ_1. Hence Δs is a function of Δr which is independent on ρ_0 and ρ_1. Based on this observation, we thus deduce that for $\gamma = 1$

$$s^*(0,0;0) - s^*(\Delta r, 0; 0) - \big(s^*(0, \Delta s; 0) - s^*(\Delta r, \Delta s; 0)\big) = 0.$$

So

$$\begin{aligned}
&s_0 - s - \big(s_1 - s_2\big) \\
&= s^*(0,0;\gamma-1) - s^*(\Delta r, 0; \gamma-1) - \big(s^*(0,\Delta s;\gamma-1) - s^*(\Delta r, \Delta s; \gamma-1)\big) \\
&= s^*(0,0;\gamma-1) - s^*(0,0;0) - \big(s^*(\Delta r, 0;\gamma-1) - s^*(\Delta r, 0; 0)\big) \\
&\quad - \big(s^*(0,\Delta s;\gamma-1) - s^*(0,\Delta s;0)\big) + \big(s^*(\Delta r,\Delta s;\gamma-1) - s^*(\Delta r, \Delta s; 0)\big) \\
&= (\gamma-1)\int_0^1 e(\Delta r, \Delta s; \chi(\gamma-1))d\chi,
\end{aligned} \tag{B.2}$$

where

$$\begin{aligned}
e(\Delta r, \Delta s; \chi(\gamma-1)) &= \partial_{\gamma-1}s^*(0,0;\chi(\gamma-1)) - \partial_{\gamma-1}s^*(\Delta r, 0; \chi(\gamma-1)) \\
&\quad - \partial_{\gamma-1}s^*(0,\Delta s;\chi(\gamma-1)) + \partial_{\gamma-1}s^*(\Delta r, \Delta s; \chi(\gamma-1)) \\
&= O(1)\Delta r \Delta s.
\end{aligned}$$

Substituting the estimate for $e(\Delta r, \Delta s; \chi(\gamma-1))$ into (B.2), we proved (3.3). It completes the proof. □

Now, we will give the proof of Lemma 3.5.

Proof. We will show this Lemma case by case.

First, let us study the first case. In this case, an $\mathcal{S}_2$ shock wave from the left with wave strength β interacts with an $\mathcal{S}_1$ shock wave from the right with wave strength ν. Both of them enter into Λ. Denote by ν' and β' the wave strength of the resulting shock waves $\mathcal{S}_1'$ and $\mathcal{S}_2'$ issuing out from Λ after the wave interaction.

Let us consider the estimate in the (ω_-, ω_+) plane. Let $(\omega_{-,L}, \omega_{+,L})$, $(\omega_{-,M}, \omega_{+,M})$, $(\omega_{-,R}, \omega_{+,R})$ be the left, middle and right states before the wave interaction, *i.e.*, $(\omega_{-,L}, \omega_{+,L})$ and $(\omega_{-,M}, \omega_{+,M})$ are connected by $\mathcal{S}_1$ shock, and $(\omega_{-,M}, \omega_{+,M})$ and $(\omega_{-,R}, \omega_{+,R})$ are connected by $\mathcal{S}_2$ shock. Let $(\omega'_{-,M}, \omega'_{+,M})$ be middle state after the wave interaction which is uniquely determined by the shock curves $\mathcal{S}_1'$ and $\mathcal{S}_2'$ which issue from $(\omega_{-,L}, \omega_{+,L})$ and $(\omega_{-,R}, \omega_{+,R})$ respectively (See Fig. 19).

In order to derive the wave interaction estimate, as shown in Fig. 20, we consider the wave curves $\hat{\mathcal{S}}_1$ and $\hat{\mathcal{S}}_2$ instead of the wave curves $\mathcal{S}_1'$ and $\mathcal{S}_2'$, such that the wave curves $\hat{\mathcal{S}}_1$ and $\hat{\mathcal{S}}_2$, issuing from $(\omega_{-,L}, \omega_{+,L})$ and $(\omega_{-,R}, \omega_{+,R})$ respectively, intersect at point $(\hat{\omega}_{-,M}, \hat{\omega}_{+,M})$. By Lemma 2.2 and Lemma 2.3, we know that the wave curves $\hat{\mathcal{S}}_1$ and $\hat{\mathcal{S}}_2$, the straight lines $\omega_- = \hat{\omega}_{-,M}$ and $\omega_+ = \hat{\omega}_{+,M}$, and the wave curves $\mathcal{S}_1'$ and $\mathcal{S}_2'$ together form the boundaries of subregions I, II and III. Moreover, $(\omega'_{-,M}, \omega'_{+,M})$ must lie in one of them.

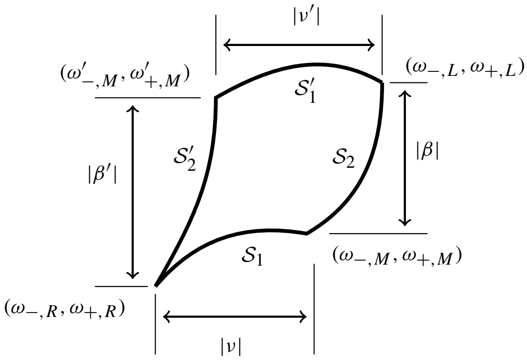

Fig. 19. Interactions between $\mathcal{S}_2$ and $\mathcal{S}_1$ waves.

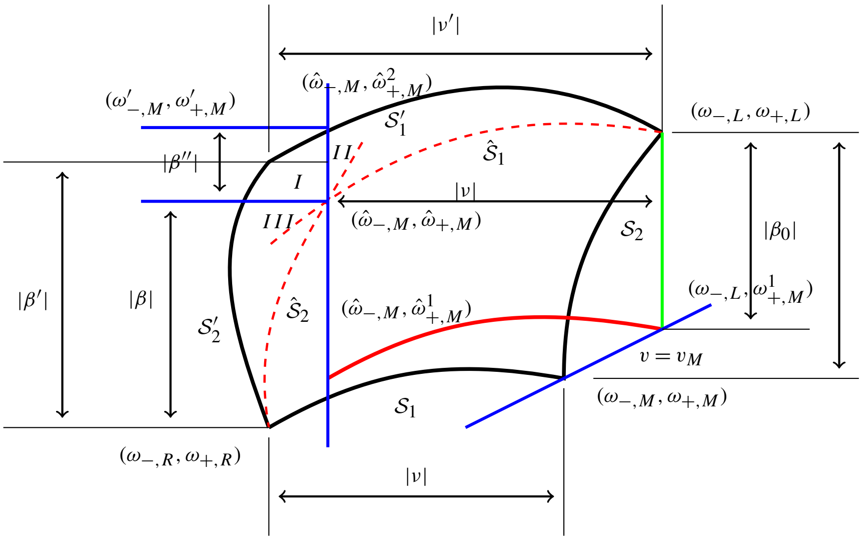

Fig. 20. Case that $(\omega'_{-,M},\omega'_{+,M})$ lies in region I.

We first consider the case that $(\omega'_{-,M},\omega'_{+,M})$ lies in the region I, *i.e.*, $\omega'_{-,M}<\hat{\omega}_{-,M}$ and $\omega'_{+,M}>\hat{\omega}_{+,M}$ (See Fig. 20). In this case, we know that

$$|\beta'|-|\beta|\le|\beta''|,\quad \omega_{+,L}-\omega^1_{+,L}=\hat{\omega}_{+,M}-\hat{\omega}^1_{+,M}=|\beta_0|. \tag{B.3}$$

Notice that

$$\begin{aligned}\omega^1_{+,M}-\hat{\omega}^1_{+,M}-(\omega_{+,L}-\hat{\omega}^2_{+,M})&=\hat{\omega}^2_{+,M}-\hat{\omega}^1_{+,M}-(\omega_{+,L}-\omega^1_{+,M})\\&=\hat{\omega}^2_{+,M}-\hat{\omega}^1_{+,M}-(\hat{\omega}_{+,M}-\hat{\omega}^1_{+,M})\\&=|\beta''|.\end{aligned} \tag{B.4}$$

By Lemma 2.4, there exists a constant $C_5>0$ such that

$$\begin{aligned}\omega^1_{+,M}-\hat{\omega}^1_{+,M}-(\omega_{+,L}-\hat{\omega}^2_{+,M})&\le C_5(\gamma-1+\tau^2)|\nu||\beta_0|\\&\le C_5(\gamma-1+\tau^2)|\nu||\beta|.\end{aligned} \tag{B.5}$$

Then combing (B.3)-(B.5) together, we have that

$$|\beta'|-|\beta|\le C_5(\gamma-1+\tau^2)|\nu||\beta|.$$

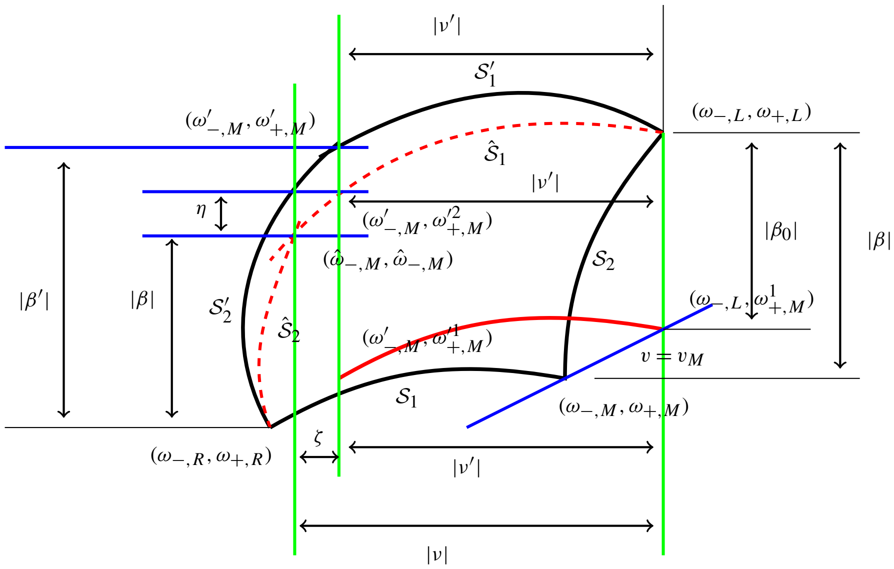

Fig. 21. Case that $(\omega'_{-,M}, \omega'_{+,M}) \in II$.

By the same way and by Lemma 2.5, one can also show that

$$|\nu'| - |\nu| \le C_5(\gamma - 1 + \tau^2)|\nu||\beta|.$$

Therefore, we show estimate (a) for the first case in Lemma 3.5.

Next, let us consider the case that $(\omega'_{-,M}, \omega'_{+,M}) \in II$. As shown in Fig. 21, we can see that

$$|\nu'| = |\nu| - \zeta > 0,$$

and

$$|\beta'| - |\beta| - \eta = \omega'_{+,M} - \omega'^{2}_{+,M},$$

where $\zeta > 0$ is a constant.

For the estimate of $\omega'_{+,M} - \omega'^{2}_{+,M}$, by Lemma 2.4, there exists a constant $C_5 > 0$ such that

$$\begin{aligned}\omega'_{+,M} - \omega'^{2}_{+,M} &= \omega'_{+,M} - \omega'^{1}_{+,M} - (\omega'^{2}_{+,M} - \omega'^{1}_{+,M})\\ &= \omega^{1}_{+,M} - \omega'^{1}_{+,M} - (\omega_{+,M} - \omega'_{+,M})\\ &\le C_5(\gamma - 1 + \tau^2)|\nu'||\beta_0|\\ &\le C_5(\gamma - 1 + \tau^2)|\nu||\beta|.\end{aligned}$$

For the estimate of η, by Lemma 2.2, we have

$$\eta = \Phi_1(|\nu|, U_L; \tau^2) - \Phi_1(|\nu| - \zeta, U_L; \tau^2) = \Phi'_1(\xi_5, U_L; \tau^2)\zeta, \quad \text{where } \xi_5 \in (|\nu| - \zeta, |\nu|),$$

which implies that $\eta \le \delta\zeta$ by taking

$$\delta = \sup_{\xi_5 \in (|\nu|-\zeta, |\nu|), \rho_L \in (\hat{\rho}, \check{\rho})} \Phi'_1(\xi_5, U_L; \tau^2) \in (0, 1).$$

Therefore, combing the estimates above together, we can get estimate (b) in Lemma 3.5 for the second case.

Finally, by a similar argument as the one for the second case, for the case that $(\omega'_{-,M}, \omega'_{+,M}) \in III$, we can obtain the estimate (c) in Lemma 3.5.

It completes the proof of case (1).

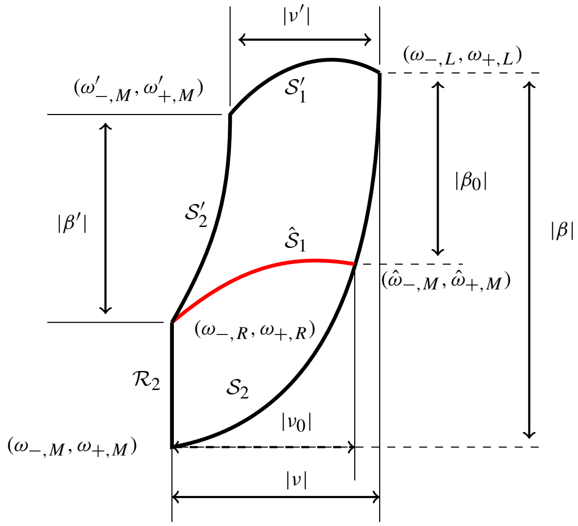

Fig. 22. Interactions between $\mathcal{S}_2$ and $\mathcal{R}_2$ waves.

Now, let us study case (2).

Similar to case (1), let $(\omega_{-,L}, \omega_{+,L})$, $(\omega_{-,M}, \omega_{+,M})$, $(\omega_{-,R}, \omega_{+,R})$ be the left, middle and right states before the wave interaction and let $(\omega'_{-,M}, \omega'_{+,M})$ be the middle state after the wave interaction which is uniquely determined by the rarefaction wave $\mathcal{R}'_1$ and the shock wave $\mathcal{S}'_2$. Notice that $\omega_{+,L} = \omega'_{+,M}$ and $\omega_{+,M} = \omega_{+,R}$, then by the monotonicity of function Φ_2, we have that

$$|\beta'| = |\omega_{+,R} - \omega'_{+,M}| = |\omega_{+,M} - \omega_{+,L}| = |\beta|.$$

The proof of the estimates for case (3) is similar to the one for case (2). In fact, by the monotonicity of function Φ_2, we have $\omega_{+,L} < \omega_{+,M} < \omega_{+,R}$ and $\omega'_{+,M} = \omega_{+,L}$. Then

$$|\beta'| = |\omega_{+,R} - \omega'_{+,M}| = |\omega_{+,R} - \omega_{+,L}| = |\omega_{+,R} - \omega_{+,M}| + |\omega_{+,M} - \omega_{+,L}| = |\beta_1| + |\beta_2|.$$

Next, let us consider case (4). As shown in Fig. 22, we can find a shock wave $\hat{\mathcal{S}}_1$ such that $\mathcal{S}_2 + \hat{\mathcal{S}}_1 \to \mathcal{S}'_1 + \mathcal{S}'_2$, and then one can follow the argument for the proof of case (1) exactly to have that

$$|\nu'| \le |\nu_0| + C_5(\gamma - 1 + \tau^2)|\nu_0||\beta_0|, \quad |\beta'| \le |\beta_0| + C_5(\gamma - 1 + \tau^2)|\nu_0||\beta_0|.$$

Now, we will consider the estimate between ν and ν_0, and the estimate between β and β_0. By Lemma 2.3, we can have that

$$\begin{aligned}|\nu_0| &= \omega_{-,L} - \omega_{-,M} - (\omega_{-,L} - \hat{\omega}_{-,M})\\ &= \Phi_2(|\beta|, U_L; \tau^2) - \Phi_2(|\beta_0|, U_L; \tau^2)\\ &= \Phi'_2(\xi_6, U_L; \tau^2)(\beta - \beta_0), \quad \xi_6 \in (|\beta_0|, |\beta|),\end{aligned}$$

which implies that

$$\begin{aligned}|\nu_0| + |\beta_0| &= |\beta| - \Big(\frac{1}{\Phi'_2(\xi_6, U_L; \tau^2)} - 1\Big)|\nu_0|\\ &\le |\beta| - \Big(\frac{1}{\Phi'_2(\xi_6, U_L; \tau^2)} - 1\Big)|\nu_0|.\end{aligned}$$

Notice that $\lim_{|\beta|\to+\infty}\Phi_2'(|\beta|,U_L;\tau^2)=1$, then we get that

$$\lim_{|\beta|\to+\infty}\Big(\frac{1}{\Phi_2'(\xi_6,U_L;\tau^2)}-1\Big)=0,$$

which implies that

$$C_0:=\inf_{\xi_6\in\{(\omega_-,\omega_+):\ 0<\hat{\rho}<\rho<\check{\rho}\},0<\hat{\rho}<\rho_L<\check{\rho}}\Big(\frac{1}{\Phi_2'(\xi_6,U_L;\tau^2)}-1\Big)>0.$$

Now we continue to study case (5), that is the wave interaction between $\mathcal{R}_2$ and $\mathcal{S}_2$. Let $(\omega_{-,L},\omega_{+,L})$, $(\omega_{-,M},\omega_{+,M})$, $(\omega_{-,R},\omega_{+,R})$, and $(\omega'_{-,M},\omega'_{+,M})$ be defined similarly as before. Then

$$\begin{aligned}|\nu'|&=\omega'_{-,M}-\omega_{-,R}-(\omega_{-,M}-\omega_{-,R})\\&=\Phi_2(-|\beta'|,U_R;\tau^2)-\Phi_2(-|\beta|,U_R;\tau^2)\\&=\Phi_2'(\xi_6,U_R;\tau^2)(|\beta|-|\beta'|),\quad \xi_6\in(-|\beta|,-|\beta'|).\end{aligned}$$

So

$$|\nu'|+|\beta'|=|\beta|-\Big(\frac{1}{\Phi_2'(\xi_6,U_L;\tau^2)}-1\Big)|\nu'|.$$

Based on the proof for case (4), we know that

$$C_0:=\inf_{\xi_6\in\{(\omega_-,\omega_+):0<\hat{\rho}<\rho<\check{\rho}\},0<\hat{\rho}<\rho_L<\check{\rho}}\Big(\frac{1}{\Phi_2'(\xi_6,U_L;\tau^2)}-1\Big)>0.$$

The estimate in case (6) is obviously.

Now, we will prove the estimate for case (7). Similarly, let $(\omega_{-,L},\omega_{+,L})$, $(\omega_{-,M},\omega_{+,M})$ and $(\omega_{-,R},\omega_{+,R})$ be the three states before the wave interaction, and let $(\omega'_{-,M},\omega'_{+,M})$ be the middle state after the wave interaction. Then

$$\begin{aligned}|\beta'|&=\omega'_{+,M}-\omega_{+,R}=\omega_{+,L}-\omega_{+,M}-(\omega_{+,L}-\omega'_{+,M})\\&=\Phi_1(|\nu|,U_L;\tau^2)-\Phi_1(|\nu'|,U_L;\tau^2)\\&=\Phi_1'(\xi_7,U_L;\tau^2)(|\nu|-|\nu'|),\quad \xi_7\in(|\nu'|,|\nu|).\end{aligned}$$

So

$$|\nu'|+|\beta'|=|\nu|-\Big(\frac{1}{\Phi_1'(\xi_7,U_L;\tau^2)}-1\Big)|\nu'|.$$

Again, we know that

$$C_0:=\inf_{\xi_7\in\{(\omega_-,\omega_+):0<\hat{\rho}<\rho<\check{\rho}\},0<\hat{\rho}<\rho_L<\check{\rho}}\Big(\frac{1}{\Phi_1'(\xi_7,U_L;\tau^2)}-1\Big)>0.$$

Finally, the proof of the estimate for case (8) is exactly the same as the one for case (3).

This completes the proof of this lemma. □

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