Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN 37 (2020) 1245–1270

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Statistical stability of mostly expanding diffeomorphisms

Martin Andersson [a,*], Carlos H. Vásquez [b,1]

[a] *Universidade Federal Fluminense, Departamento de Matemática Aplicada, Rua Professor Marcos Waldemar de Freitas Reis, s/n, 24210-201, Niterói, Brazil*

[b] *Instituto de Matemática, Pontificia Universidad Católica de Valparaíso, Blanco Viel 596, Cerro Barón, Valparaíso, Chile*

Received 13 December 2017; received in revised form 30 September 2019; accepted 16 April 2020
Available online 29 May 2020

Abstract

We study how physical measures vary with the underlying dynamics in the open class of C^r, $r > 1$, strong partially hyperbolic diffeomorphisms for which the central Lyapunov exponents of every Gibbs u-state is positive. If transitive, such a diffeomorphism has a unique physical measure that persists and varies continuously with the dynamics.

A main ingredient in the proof is a new Pliss-like Lemma which, under the right circumstances, yields frequency of hyperbolic times close to one. Another novelty is the introduction of a new characterization of Gibbs cu-states. Both of these may be of independent interest.

The non-transitive case is also treated: here the number of physical measures varies upper semi-continuously with the diffeomorphism, and physical measures vary continuously whenever possible.

Résumé

Nous étudions comment les mesures physiques varient avec la dynamique sous-jacente, dans la classe ouverte des difféomorphismes C^r, $r > 1$, fortement partiellement hyperboliques pour lesquelles les exposants de Lyapunov centraux de tout u-état de Gibbs sont positifs. Lorsque transitifs, de tels difféomorphismes possédent une unique mesure physique qui persiste et varie continûment avec la dynamique.

Un des ingrédients principaux de la preuve est un nouveau lemme de type Pliss qui, appliqué dans le contexte adéquat, implique que la fréquence des temps hyperboliques est proche de un. Une autre nouveauté est l'introduction d'une nouvelle caractérisation des cu-états de Gibbs. Chacun de ses deux aspects ayant leur propre intérêt.

Le cas non transitif est aussi traité : dans ce contexte, le nombre de mesures physiques est une fonction semi-continue supérieure du difféomorphisme, et les mesures physiques varient continûment sous des hypothèses naturelles.

MSC: primary 37D30, 37C40, 37D25

* Corresponding author.
E-mail addresses: nilsmartin@id.uff.br (M. Andersson), carlos.vasquez@pucv.cl (C.H. Vásquez).

1 C.V. was partially supported by FONDECYT 1171427, CONICYT, Chile.

https://doi.org/10.1016/j.anihpc.2020.04.007

Keywords: Partial hyperbolicity; Lyapunov exponents; SRB measures; Stable ergodicity; Statistical stability

1. Introduction

The present work deals with the question of continuity of physical measures in the setting of partially hyperbolic diffeomorphisms, i.e., with a uniformly contracting bundle E^s, a central bundle E^c, and a uniformly expanding bundle E^u. Here, by *physical measure* we mean a Borel probability μ for which the basin

$$B(\mu) = \{x \in M : \frac{1}{n}\sum_{k=0}^{n-1} \delta_{f^k(x)} \to \mu\}$$

has positive Lebesgue measure.

The particular context under consideration here is that in which the central direction is mostly expanding, by which we mean that every Gibbs u-state has positive central Lyapunov exponents. Here, by *Gibbs u-states* we mean invariant probabilities absolutely continuous with respect to Lebesgue measure along the partition in strong unstable manifolds.

Mostly expanding diffeomorphisms include some very important examples from smooth ergodic theory, among which we have Derived-from-Anosov diffeomorphisms [8] as well as the well-known examples of partially hyperbolic diffeomorphisms with pathological center foliation due to Shub and Wilkinson (see [26] and [30, Theorem D]). The above definition of mostly expanding partially hyperbolic diffeomorphisms was introduced in [8]. It is important to note that it differs from the homonymous notion considered in the influential work [3]. The reason for this change in terminology is that the concept of mostly expanding introduced in [8] (and considered here) is a closer analogue of the notion of *mostly contracting*, introduced in [12]. The original notion of mostly expanding used in [3] will be referred to by us as *non uniformly expanding along the center-unstable direction*, or simply the *NUE-condition*. As showed in [8,30], a partially hyperbolic diffeomorphism which is mostly expanding will necessarily satisfy the NUE-condition. That is,

$$\text{Mostly expanding} \implies \text{NUE-condition}$$

(but not the other way around). Another important feature of the mostly expanding condition is that it is open in the C^r topology for $r > 1$ — something that is not true for the more general NUE-condition (see [8] for proofs and counter-examples). The regularity $r > 1$ here may be any real number. If r is not an integer, i.e. $r = k + \alpha$ for some $k \in \mathbb{N}$ and $\alpha \in (0, 1)$, then by C^r diffeomorphism we mean a C^k diffeomorphism whose k-th derivative is α-Hölder, and the topology considered is the one induced by a metric which, in charts, can be written as the sum of the C^k metric and the α-Hölder metric on the k-th derivative.

The NUE-condition was introduced as a way to guarantee the existence and finitude of physical measures. Since the mostly expanding condition is open and implies the NUE-condition, it provides an ideal setting for the following much sought after situation:

Theorem 1.1 *([8]). Let $f : M \to M$ be a C^r, $r > 1$, partially hyperbolic diffeomorphism on a compact manifold. Suppose that every Gibbs u-state of f has positive central Lyapunov exponent (i.e. f is mostly expanding). Then there exists a C^r neighborhood $\mathcal{U}$ of f such that every $g \in \mathcal{U}$ is mostly expanding and has a finite number of physical measures whose basins together cover a full Lebesgue measure set in M.*

Theorem 1.1 begs the question of how the physical measures of a mostly expanding diffeomorphism react to small deterministic perturbations. Will the number of physical measures remain the same? If so, do they vary continuously with the dynamics? Such questions have been answered quite satisfactorily for the analogous notion of mostly contracting diffeomorphisms. A partially hyperbolic diffeomorphism is said to be *mostly contracting* if all its Gibbs u-states have negative central Lyapunov exponents. Just like the mostly expanding condition, it is C^r open for any $r > 1$. (This was noted in [15] and proved in detail in [7].) Moreover it was proved in [7] (see also [17]) that

(i) the number of physical measures of mostly contracting diffeomorphisms varies upper semi-continuously with the dynamics, and

(ii) physical measures vary continuously in the weak* topology under perturbations that don't change the number of physical measures.

In [29,17] Dolgopyat, Viana and Yang gave a detailed explanation of how bifurcations occur and an exhaustive set of examples. They have also proved a form of continuity of the basins of physical measures.

The aim of this work is to complete the picture by proving similar results for mostly expanding diffeomorphisms. We start by considering the case in which the physical measure is unique.

Theorem A. *Let $f : M \to M$ be a C^r, $r > 1$, transitive partially hyperbolic diffeomorphism of type $TM = E^s \oplus E^c \oplus E^u$ such that every Gibbs u-state has positive central Lyapunov exponents. Then there is a C^r neighborhood $\mathcal{U}$ of f such that every $g \in \mathcal{U}$ has a unique physical measure μ_g. Moreover μ_g varies continuously with g in the weak* topology.*

To our knowledge, there are no known examples of mostly expanding diffeomorphisms with more than one physical measure. Yet we can show that if such examples exist (and we believe they do), then their physical measures behave just like in the mostly contracting case:

Theorem B. *Let $f : M \to M$ be a C^r, $r > 1$, partially hyperbolic diffeomorphism of type $TM = E^s \oplus E^c \oplus E^u$ (not necessarily transitive) such that every Gibbs u-state has positive central Lyapunov exponents. Then the number of physical measures depends upper semi-continuously on g and physical measures vary continuously in the weak* topology on any subset $\mathcal{C} \subset \mathcal{U}$ on which the number of physical measures is constant.*

Theorem A is in fact a corollary of Theorem B, but it is by far the case of greatest interest and therefore deserves to be in the spotlight.

In spite of the strong analogy between the notions of mostly expanding and mostly contracting, there is at least one important difference between the two. In the former case, the basin of a physical measure is an open set, modulo a Lebesgue null set [8, Lemma 4.5]. That is very different from the case of mostly contracting, in which one may have the phenomenon of intermingled basins of attraction [20]. It also means that, for mostly expanding diffeomorphisms, transitivity is sufficient to guarantee uniqueness of the physical measure.

Some comments on terminology are pertinent. By statistical stability one usually means a situation where all physical measures persist and vary continuously with small perturbations on the dynamics. One can therefore say that a diffeomorphism satisfying the hypotheses of Theorem A is statistically stable. For the situation in Theorem B one can likewise talk about statistical stability within some suitable parameter space (in which the number of physical measures remain constant). On the other hand, it is possible to weaken the notion of statistical stability, saying that a diffeomorphism $f : M \to M$ is C^r-*weakly statistically stable* if, given any neighborhood U of the closed convex hull of the physical measures of f, there exists a C^r neighborhood $\mathcal{U}$ of f such that, given any $g \in \mathcal{U}$, every physical measure of g belongs to U. To appreciate the difference between statistical stability and weak statistical stability, it is instructive to look at the examples from [17, Section 3.3]. There, Dolgopyat, Viana and Yang produce examples of mostly contracting diffeomorphisms for which the number of physical measures decreases under arbitrarily small perturbations. The closed convex hull of the physical measures of the perturbed system lie in a neighborhood of a proper subsymplex of the closed convex hull of the physical measures of the unperturbed system. The unperturbed system is therefore weakly statistically stable but not statistically stable.

Theorem C. *Let $f : M \to M$ be a C^r, $r > 1$, partially hyperbolic diffeomorphism of type $TM = E^s \oplus E^c \oplus E^u$ such that every Gibbs u-state has positive central Lyapunov exponents. Then f is weakly statistically stable.*

In a recent paper [30], Yang proved that having positive central Lyapunov exponents with respect to every Gibbs u-state is a C^1 open property. It is therefore natural to ask whether the physical measures vary continuously with the dynamics in the C^1 topology. We do not know.

In another recent work [22] Cao, Mi and Yang prove existence and finiteness of physical measures for partially hyperbolic diffeomorphisms f with dominated splitting $TM = E^{cs} \oplus E^{cu} \oplus E^u$ (with uniform expansion in E^u) satisfying a mixture of *mostly contracting* and mostly expanding behavior. That is, every Gibbs u-state has positive

Lyapunov exponents in the E^{cu} bundle (i.e. f is mostly expanding along E^{cu}) and negative Lyapunov exponents in the E^{cs} bundle (i.e. f is mostly contracting along E^{cs}). All results in the present work can be extended to the setting in [22] using our results and following the ideas in the proof of Theorem C in [27].

There are several works on statistical stability related to our current setting, most notably [6,5,2] dealing with classes of non-invertible systems with non-uniformly expanding behavior. In [6] Alves proves the existence of measures absolutely continuous with respect to Lebesgue and then, [5,2] Alves and Viana give sufficient conditions to obtain an stronger version of statistical stability: continuous variation (in L^1 norm) of densities of absolutely continuous invariant measures. Those works rely on techniques rather different from ours, namely the Ruelle transfer operator and an induced piecewise expanding Markovian map. The latter approach, inspired by the works [31,32], consists of considering a disc $\Delta \subseteq M$ and an induced first return map $F_f : \Delta \to \Delta$. If F_f is piecewise expanding, then it has an invariant absolutely continuous measure μ_F. If, in addition, the return times $R_f : \Delta \to \mathbb{N}$ of f to Δ are integrable with respect to Lebesgue measure, then

$$\mu_f = \sum_{j=0}^{\infty} f_*^j(\mu_F|\{R_f > j\})$$

is an absolutely continuous f-invariant finite measure. In order to obtain statistical stability, Alves and Viana [5] impose a condition of uniformity (with respect to f) on the integral (with respect to Lebesgue) of the tail: given $\varepsilon > 0$, there is $N = N(\varepsilon) \in \mathbb{N}$ such that for every g close enough of f:

$$\sum_{j=N}^{\infty} \text{Leb}(\{R_g > j\}) < \epsilon.$$

A similar approach (and uniform condition on the tail) is used in [4]. It is not clear to what generality such uniform tail conditions hold. All works cited above apply to robust classes of maps where the uniformity of tail behavior is implicit in the calculations that shows that they are non-uniformly expanding in the first place. Yet there doesn't seem to be any good reason why it would be intrinsic to non-uniform expansion, or that one can find a universal mechanism that gives rise to it.

The present work is based on an approach used by the second author in [27]. Briefly speaking, the author considers a partially hyperbolic diffeomorphisms $f : M \to M$ with a Df-invariant splitting $TM = E^s \oplus E^c$ and satisfying the NUE-condition on the center direction (here $E^u = \{0\}$ is taken trivial). In this case, as showed in [3], there exists a finite number of physical measures. The author considers a sequence f_n converging to f in the C^r topology, $r > 1$. Since partial hyperbolicity is an open condition (and the bundles vary continuously with the diffeomorphism), each f_n also has a splitting of the tangent bundle $TM = E_n^s \oplus E_n^c$, $n \geq 1$. Assuming that f_n satisfies a uniform NUE-condition on the center direction, it is possible to choose a sequence μ_n of physical measures of each f_n and consider an accumulation point μ. Restricted to a "fixed bounded cylinder", the author proves that μ is the sum of measures $\nu + \eta$, with ν non-zero, such that ν is a combination of physical measures of f and η is like a "singular remainder". This is done using a construction from [3], in which one shows that μ_n can be approximated inside a "bounded cylinder" by a sum $\nu_n + \eta_n$ with the mass $|\nu_n|$ of μ_n bounded away from zero (uniformly with respect to n), and such that ν_n has a disintegration along center-unstable manifolds with uniform bounds on the densities of its conditional measures. Therefore ν_n accumulates (in the cylinder) on a measure ν with the same properties and then it is in the convex hull of the physical measures. By tacitly assuming uniformity of tail behavior of return maps to the cylinder, the author concludes that $\eta_n \to 0$ and so $\mu = \nu$.

Our main finding in this work is that, by replacing the NUE-condition by the slightly stronger mostly expanding condition, we are able to rid ourselves completely of any assumptions about tail behavior. In fact, our strategy is essentially the same as in [27], but with the important improvement that $|\nu_n|$ can be taken to be not only bounded away from zero, but arbitrarily close to one. The magic occurs because the following new version of the classical Pliss Lemma:

Lemma A ***(Pliss-Like Lemma)****. Let $L < \gamma < \Gamma$ and suppose that $a_1, \dots a_N$ are numbers such that $a_i \geq L$ for every $1 \leq i \leq N$. Let $\kappa > 0$ be a number such that*

$$\#\{i \in \{1, \dots, N\} : a_i < \Gamma\} \leq \kappa N$$

and write $\theta = 1 - \kappa \frac{\Gamma - L}{\Gamma - \gamma}$. *Then there exist* $1 < n_1 < n_2 < \ldots < n_m \le N$, *with* $m \ge \theta N$, *such that*

$$\sum_{j=n+1}^{n_i} a_j \ge \gamma(n_i - n) \tag{1.1}$$

for every $1 \le i \le m$ *and every* $0 \le n < n_i$.

Let us comment on the relation between Lemma A and the Classical Pliss Lemma from [24] (see also [21, Lemma 11.8] or [3, Lemma 3.1]). First of all, despite their similarities, both Lemmas are independent, i.e. none implies the other. Both Lemma A and the Classical Pliss Lemma estimate how often (frequency θ) certain partial averages of the finite sequence a_i are above a certain threshold. In the Classical Pliss Lemma, θ is a function of an upper bound of the a_i and its average. Unless these become close, θ cannot be taken close to 1. In contrast, Lemma A considers a lower bound of the a_i and a number Γ slightly smaller than the average so that a good fraction $(1-\kappa)$ of the numbers a_i are larger than Γ. If κ is small, then θ is close to one. That is the case whenever the sequence a_i has small variation (most of its members are close to its mean) and the lower bound is not too far from the mean. In particular, it applies to Birkhoff sums of observables of the form

$$\varphi(x) = \frac{1}{\ell} \int \log \|Df^\ell | E^c_x\| d\mu,$$

where f is a partially hyperbolic diffeomorphism, μ is an ergodic f-invariant measure, and ℓ is some large integer. Indeed, Pliss' Lemma has been intimately linked with the so called "hyperbolic times" introduced by Alves in [6]. Loosely speaking, hyperbolic times are numbers (times) associated to point whose orbit has non-uniformly hyperbolic behavior. These times corresponding to iterates along the orbit for which the non-uniform hyperbolicity behaves as uniform hyperbolicity for large chunks of the orbit. Pliss' Lemma guarantees that an orbit on which a diffeomorphism is, say, asymptotically expanding in some direction, will have hyperbolic times on a set of iterates that correspond to positive frequency. Many of the difficulties related to hyperbolic times arise from the fact that the frequency of hyperbolic times provided by the Pliss' Lemma is only positive, but not necessarily close to one. This is in fact the main difficulty in the current work, and we overcome it by replacing the Pliss' Lemma by Lemma A and applying it to iterates of the diffeomorphism under consideration.

Another important ingredient are Gibbs cu-states and their properties. They were introduced in [3] using the fact that, in the presence of positive Lyapunov exponents, there are Pesin invariant unstable manifolds. Thus Gibbs cu-states are a non-uniform version of Gibbs u-states: While the definition of Gibbs u-states involves the strong unstable foliation, we say that an invariant probability measure μ is a *Gibbs* cu*-state* if the conditional measures of μ along the corresponding local *unstable Pesin's manifolds* are almost everywhere absolutely continuous with respect to Lebesgue measure on these manifolds. Gibbs cu-states are natural candidates to be physical measures. In fact, in our setting, a measure is physical if and only if it is an ergodic Gibbs cu-state (see [8, Lemma 4.4]). In Section 2.1 we provide more details about the properties of Gibbs cu-states, their relation to physical measures, and a complete toolbox to be used in our proof.

Here is an outline of our arguments.

(i) Compactness of the set of Gibbs u-states (Proposition 2.3) provides us with uniform bounds, in a robust fashion, on the Lyapunov exponents of these (see (3.8)).
(ii) We use Lemma A to prove that an iterate of a mostly expanding diffeomorphism has hyperbolic times with frequency arbitrarily close to one (see Proposition 3.1). This is a considerable improvement on the positive but possibly small frequency of hyperbolic times used in most arguments with a similar flavor.
(iii) The abundance of hyperbolic times given by our Pliss-like Lemma is used to prove that, in our setting, limits of Gibbs cu-states are Gibbs cu-states (Theorem D). This convergence is tricky to prove rigorously. We overcome this difficulty by introducing a useful characterization of Gibbs cu-states which does not directly involve disintegration of the measure (Theorem 4.1).
(iv) Ergodic Gibbs cu-states are physical measures (Proposition 2.6).
(v) Finally, distinct ergodic Gibbs cu-states cannot get too close to each others; therefore they must either stay apart or collapse into one ergodic Gibbs cu-states. This gives upper semi-continuity (see Section 5).

As already mentioned above, Theorem A can be applied to a number of important examples. Certain Derived from Anosov diffeomorphisms, like the ones described in [8, Section 6], are mostly expanding with a unique physical measure, and therefore statistically stable. Also Theorem A can be applied to generic C^∞ perturbations f of the time one map of a hyperbolic geodesic flow on a surface M. In this case, it was proved in [16] that either f of its inverse f^{-1} is mostly expanding and so our results can be applied.

Another family of examples where our theorems apply are the examples provided by [26, Theorem 1]. In this case, it was proved in [30] that if f is a C^r, $r > 1$, accessible, volume preserving, partially hyperbolic diffeomorphism with one-dimensional center and the center exponent (with respect to the volume measure) is positive, then it is mostly expanding.

2. Some background

2.1. Dominated splitting and partial hyperbolicity

Let M be a closed Riemannian manifold. We denote by $\|\cdot\|$ the norm obtained from the Riemannian structure and by m the normalized volume measure on M induced by the Riemannian structure. We often refer to m as "the Lebesgue measure on M". Moreover, if D is a submanifold of M we denote by vol_D the volume measure on D induced by the Riemannian structure and by m_D its normalization, i.e. $m_D = \mathrm{vol}_D / |\mathrm{vol}_D|$.

A diffeomorphism $f : M \to M$ has a *dominated splitting* $F \prec G$ if there is a Df-invariant decomposition $TM = F \oplus G$ into complementary subbundles of TM of constant dimensions, and $N \geq 1$ such that

$$\|Df^N|F_x\| \cdot \|Df^{-N}|G_{f^N(x)}\| < 1$$

for every $x \in M$. Any such splitting is necessarily continuous.

A diffeomorphism $f : M \to M$ is *partially hyperbolic* if there exists a continuous Df-invariant splitting

$$TM = E^s \oplus E^c \oplus E^u$$

such that $E^s \prec (E^c \oplus E^u)$ and $(E^s \oplus E^c) \prec E^u$ are both dominated splittings and, moreover, there exists $N \geq 1$ such that $\|Df^N|E^s\| < 1$ and $\|Df^{-N}|E^u\| < 1$. We always assume that $\dim E^\sigma \geq 1$, $\sigma = s, c, u$ unless stated otherwise.

We denote by $\mathcal{PH}^r(M)$, $r \geq 1$, the set of C^r partially hyperbolic diffeomorphisms defined on M. The set $\mathcal{PH}^r(M)$ is open in the C^r topology and the bundles vary continuously with the diffeomorphism [18, Corollary 2.17]. For partially hyperbolic diffeomorphisms, it is a well-known fact that there are foliations $\mathcal{F}^\sigma$ tangent to the distributions E^σ for $\sigma = s, u$ [19]. The leaf of $\mathcal{F}^\sigma$ containing x will be called $W^\sigma(x)$, for $\sigma = s, u$.

2.2. Gibbs u-states, Gibbs cu-states, and physical measures

In the following $f : M \to M$ is a C^r, $r > 1$, partially hyperbolic diffeomorphism with Df-invariant splitting

$$TM = E^s \oplus E^c \oplus E^u.$$

Denote by $\mathbb{M}^1(M)$ the set of Borel probability measures defined on M provided with the weak* topology. We denote by $\mathbb{M}^1_f(M)$ the set of f-invariant probability measures. It is well known that $\mathbb{M}^1_f(M)$ is non empty, convex, closed (and so compact) subset of $\mathbb{M}^1(M)$.

Notation: Throughout this work, the "Riemannian topology" of M, the C^r topology on $\mathrm{Diff}^r(M)$ and the weak* topology on $\mathbb{M}^1(M)$ will be involved. To avoid confusion we use

- capital letters $U, V, \ldots$ to denote subsets of M,
- calligraphic capital letters $\mathcal{U}, \mathcal{V}, \ldots$ to denote subsets of $\mathrm{Diff}^r(M)$,
- board capital letters $\mathbb{U}, \mathbb{V}, \ldots$ to denote subsets of $\mathbb{M}^1(M)$, and
- boldface capital letters $\mathbf{U}, \mathbf{V}, \ldots$ to denote subsets of $\mathrm{Diff}^r(M) \times \mathbb{M}^1(M)$.

A measure $\mu \in \mathbb{M}^1_f(M)$ is a *Gibbs u-state* if the conditional measures of μ with respect to the partition into local strong-unstable manifolds are absolutely continuous with respect to Lebesgue measure along the corresponding local strong-unstable manifold.

We denote by $\mathbb{G}\text{ibbs}^{\mathrm{u}}(f)$ the subset of Gibbs u-states for f. If $\mathcal{U} \subseteq \mathcal{PH}^r(M)$, we denote by

$$\mathbb{G}\text{ibbs}^{\mathrm{u}}(\mathcal{U}) := \{(g, \mu) : g \in \mathcal{U} \text{ and } \mu \in \mathbb{G}\text{ibbs}^{\mathrm{u}}(g)\}$$

For future reference, we list some relevant properties of Gibbs u-states.

In the follows, if $g : M \to M$ is Borel measurable and ν is a Borel probability measure defined on M, $g_*\nu$ denotes the Borel probability measure $\nu \circ g^{-1}$.

Proposition 2.1. ***[Pesin, Sinai; [23]]*** *If f is a C^r partially hyperbolic diffeomorphism, $r > 1$, then there exists a Gibbs u-state. More precisely, if D is a disk of dimension $\dim(E^u)$ inside a strong unstable leaf, then every accumulation point of the sequence of probability measures*

$$\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} f_*^k m_D$$

is a Gibbs u-state with densities with respect to the volume measure along the strong unstable leaves satisfying

$$\frac{\rho(x)}{\rho(y)} = \prod_{n=0}^{\infty} \frac{\det(Df^{-1}|E^u_{f^{-n}(x)})}{\det(Df^{-1}|E^u_{f^{-n}(y)})}, \tag{2.1}$$

for any points x, y in the same unstable plaque. As a consequence, the density ρ along the strong unstable leaves are uniformly bounded away from zero and infinity.

Clearly, convex combinations of Gibbs u-states are Gibbs u-states. Recall that if μ is any f-invariant measure, the limit

$$\mu_x = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \delta_{f^k(x)}$$

exists and is ergodic μ-almost everywhere, and

$$\int \varphi\, d\mu = \int \left(\int \varphi\, d\mu_x \right) d\mu(x)$$

for every continuous function $\varphi : M \to \mathbb{R}$.

Proposition 2.2 *([11, Lemma 11.13]). Let $f : M \to M$ be a C^r partially hyperbolic diffeomorphism, $r > 1$. If μ is a Gibbs u-state, then μ_x is a Gibbs u-state for μ-almost every x. In other words, every Gibbs u-state μ is a convex combination of ergodic Gibbs u-states.*

To complete our knowledge about the structure of the set of Gibbs u-states we have:

Proposition 2.3 *([11, Remark 11.15], [13, Theorem 5]). Let $f : M \to M$ be a C^r partially hyperbolic diffeomorphism, $r > 1$. Then the set $\mathbb{G}\text{ibbs}^{\mathrm{u}}(f) \subseteq \mathbb{M}^1(M)$ is closed (and so compact) and convex. Moreover, the map $\mathcal{PH}^r(M) \ni f \mapsto \mathbb{G}\text{ibbs}^{\mathrm{u}}(f) \subset \mathbb{M}^1(M)$ is upper semicontinuous.*

The last statement in the proposition above is equivalent to say that, given any sufficiently small C^r neighborhood $\mathcal{U}$ of f, the set $\mathbb{G}\text{ibbs}^{\mathrm{u}}(\mathcal{U})$ is closed in $\mathcal{U} \times \mathbb{M}^1(M)$.

Given a partially hyperbolic diffeomorphism $f : M \to M$, the *minimum central Lyapunov exponents* is the measurable function

$$\begin{aligned} \lambda^c(f, \cdot) &: M \to \mathbb{R} \\ &x \mapsto \liminf_{n\to\infty} \frac{1}{n} \log \|(Df^n|E^c_x)^{-1}\|^{-1} \end{aligned}$$

We say that f has *positive central Lyapunov exponents* with respect to the invariant measure μ if $\lambda^c(f,x) > 0$ μ-almost everywhere. We find it convenient to write $E^{cu} = E^c \oplus E^u$. To say that f has positive central Lyapunov exponents with respect to μ is then the same thing to say that

$$\liminf_{n\to\infty} \frac{1}{n} \log \|(Df^n|E_x^{cu})^{-1}\|^{-1} > 0$$

μ-almost everywhere. Indeed, it follows from the definition of partial hyperbolicity that $\|Df^{-1}|E_x^u\| < \|Df^{-1}|E_x^c\|$ for every $x \in M$. Hence

$$\liminf_{n\to\infty} \frac{1}{n} \log \|(Df^n|E_x^{c})^{-1}\|^{-1} = \liminf_{n\to\infty} \frac{1}{n} \log \|(Df^n|E_x^{cu})^{-1}\|^{-1}.$$

Recall that a C^r partially hyperbolic diffeomorphism $f: M \to M$, $r > 1$, is *mostly expanding* if all the Gibbs u-states of the diffeomorphism have positive central Lyapunov exponents. We denote by $\mathcal{U}_{\mathcal{ME}} \subset \mathcal{PH}^r(M)$ the set of C^r mostly expanding partially hyperbolic diffeomorphism, with $r > 1$. It was showed in [8, Theorem B] that $\mathcal{U}_{\mathcal{ME}}$ is an open set in the C^r topology, $r > 1$.

For our purpose, it is useful to consider the *integrated minimum central Lyapunov exponent* defined by

$$\begin{aligned}\hat{\lambda}^c : \mathbb{G}\mathrm{ibbs}^{\mathrm{u}}(\mathcal{U}_{\mathcal{ME}}) &\to \mathbb{R}\\ (g,\mu) &\mapsto \int \lambda^c(g,x)\, d\mu(x)\end{aligned}$$

Proposition 2.4 *([8, Proposition 3.4]). The function $\hat{\lambda}^c$ is lower semicontinuous.*

Following [3], if f is mostly expanding, we say that an f-invariant measure μ is a *Gibbs cu-state* if μ has positive Lyapunov exponent and the conditional measures of μ along the corresponding local (Pesin) center-unstable manifolds are almost everywhere absolutely continuous with respect to Lebesgue measure on these manifolds. For each $f \in \mathcal{U}_{\mathcal{ME}}$, we denote by $\mathbb{G}\mathrm{ibbs}^{\mathrm{cu}}(f)$ the subset of Gibbs cu-states. If $\mathcal{U} \subseteq \mathcal{U}_{\mathcal{ME}}$, we denote by

$$\mathbb{G}\mathrm{ibbs}^{\mathrm{cu}}(\mathcal{U}) := \{(g,\mu) : g \in \mathcal{U} \text{ and } \mu \in \mathbb{G}\mathrm{ibbs}^{\mathrm{cu}}(g)\}.$$

Every Gibbs cu-state is in fact a Gibbs u-state with positive central Lyapunov exponents, although the converse is not true (see the example in [8, page 8]). We also have the following analogue of Proposition 2.2.

Proposition 2.5 *([27, Lemma 2.4]). Let $f: M \to M$ be a C^r partially hyperbolic diffeomorphism, $r > 1$. Then every Gibbs cu-state μ is a convex combination of ergodic Gibbs cu-states.*

It is rather straightforward to see that an ergodic Gibbs cu-state must be a physical measure. In fact it is a special case of the situation treated in [25] (and which has become an underlying paradigm for much of smooth ergodic theory), in which it was shown that if an ergodic measure has only non-zero Lyapunov exponents and is absolutely continuous along Pesin's unstable manifolds, then it is a physical measure. The next results say that, in the present context, the converse is also true.

Proposition 2.6 *([8, Lemma 4.4]). Let $f: M \to M$ be a C^r mostly expanding partially hyperbolic diffeomorphism, $r > 1$. Then the set of physical measures coincides with the set of ergodic Gibbs cu-states.*

A central result in this paper is the following Gibbs cu-states version of Proposition 2.3.

Theorem D. *Let $\mathcal{U}_{\mathcal{ME}}$ be the C^r open set of mostly expanding diffeomorphisms, $r > 1$. Then, the map $\mathcal{U}_{\mathcal{ME}} \ni f \mapsto \mathbb{G}\mathrm{ibbs}^{\mathrm{cu}}(f) \subset \mathbb{M}^1(M)$ is upper semicontinuous.*

Theorem D says that for every sequence f_n, $n \geq 1$, converging to $f \in \mathcal{U}_{\mathcal{ME}}$ in the C^r topology, and every sequence $\mu_n \in \mathbb{G}\mathrm{ibbs}^{\mathrm{cu}}(f_n)$, all accumulation points of μ_n (in the weak* topology) belong to the set $\mathbb{G}\mathrm{ibbs}^{\mathrm{cu}}(f)$. Notice that Theorem D is not a consequence of Proposition 2.3. When dealing with Gibbs u-states, one considers disintegration along unstable manifolds (which are defined at every point, are tangent to E^u and have uniform size). On the other hand, when one deals with Gibbs cu-states, one considers disintegration along Pesin unstable manifolds (which are defined almost everywhere, are tangent to $E^c \oplus E^u$, and do not have uniform size). Theorem D is new and Sections 3 and 4 are entirely dedicated to the proof of this result.

3. Uniform estimates of non-uniform hyperbolicity

The apparently paradoxical title of this section reflects much of the spirit of non-uniform hyperbolicity in the presence of dominated splittings and partial hyperbolicity. Unlike 'genuine' non-uniformly hyperbolic systems, in which the angle between stable and unstable bundles may be arbitrarily small, these often allow some explicit form of robustness. An important manifestation of such robustness properties is that the measure of sets on which certain degrees of hyperbolicity hold may be uniformly bounded away from zero or even uniformly close to one. Central to this theme is our Pliss-like Lemma A which we shall now prove.

3.1. Proof of Lemma A

Just as in Mañé's proof of Pliss' Lemma [21, Lemma 11.8], we define a function $S : \{0, \dots, N\} \to \mathbb{R}$ by taking $S(0) = 0$ and $S(n) = \sum_{j=1}^{n} a_j - n\gamma$ for $1 \le n \le N$. Defining $1 < n_1 < \cdots < n_k \le N$ as the maximal sequence such that $S(n_i) \ge S(n)$ holds for every $0 \le n < n_i$ and $i = 1, \dots, k$, one may easily check that the n_i satisfy (1.1). It remains to show that $k \ge \theta N$.

We set $F = \{i \in \{1, \dots, N\} : a_i < \Gamma\}$ and write $\{1, \dots, N\} \setminus \{n_1, \dots, n_k\}$ as the finite union $\bigcup_{\alpha\in\Lambda} I_\alpha$ of pairwise disjoint intervals in $\mathbb{N}$ with cardinality $|I_\alpha|$. Note that

$$\sum_{i\in I_\alpha} a_i < |I_\alpha|\gamma \tag{3.1}$$

for every $\alpha \in \Lambda$, for else the maximality of the sequence n_i would be violated. We can bound a_i from below by either L or Γ, depending on whether or not i belongs to F. Therefore

$$\sum_{i\in I_\alpha} a_i = \sum_{i\in I_\alpha\cap F} a_i + \sum_{i\in I_\alpha\cap F^c} a_i \ge |I_\alpha \cap F|L + |I_\alpha \cap F^c|\Gamma. \tag{3.2}$$

Combining (3.1) and (3.2) we obtain

$$|I_\alpha \cap F|L + |I_\alpha \cap F^c|\Gamma < |I_\alpha|\gamma. \tag{3.3}$$

Using the identity $|I_\alpha| = |I_\alpha \cap F| + |I_\alpha \cap F^c|$, rearranging terms, and summing over α, (3.3) becomes

$$(\Gamma - L)\sum_{\alpha\in\Lambda} |I_\alpha \cap F| > (\Gamma - \gamma)\sum_{\alpha\in\Lambda} |I_\alpha|. \tag{3.4}$$

Recall that $\{I_\alpha : \alpha \in \Lambda\}$ is the family of disjoint intervals in $\{1, \dots, N\} \setminus \{n_1, \dots, n_k\}$. In particular,

$$\sum_{\alpha\in\Lambda} |I_\alpha| = N - k. \tag{3.5}$$

Moreover, by hypotheses we have $|F| \le \kappa N$, and then

$$\sum_{\alpha\in\Lambda} |I_\alpha \cap F| \le |F| \le \kappa N. \tag{3.6}$$

Combining (3.4) with (3.5) and (3.6) gives

$$\begin{aligned}(\Gamma - L)\kappa N &> (\Gamma - \gamma)(N - k)\\ &> (\Gamma - \gamma)N - (\Gamma - \gamma)k\end{aligned} \tag{3.7}$$

Rearranging terms in (3.7) we obtain

$$(\Gamma - \gamma)k > \left[(\Gamma - \gamma) - (\Gamma - L)\kappa\right]N$$

and then

$$k > \left[1 - \frac{\Gamma - L}{\Gamma - \gamma}\kappa\right] N$$

shows that $k > N\theta$, where $\theta = \left[1 - \frac{\Gamma - L}{\Gamma - \gamma}\kappa\right]$.

3.2. *Abundance of hyperbolic times*

We recall (see [3]) that, given $0 < \sigma < 1$, an integer $n \geq 0$ is a σ*-hyperbolic time* for $x \in M$ if

$$\prod_{j=n-k+1}^{n} \|Df^{-1}|E^{cu}_{f^j(x)}\| \leq \sigma^k \quad \text{for all } 1 \leq k \leq n.$$

Recall from section 2.2 that if $f \in \mathcal{U}_{\mathcal{ME}}$ and $\mu \in \mathbb{G}\mathrm{ibbs}^{\mathrm{u}}(f)$ then

$$\lambda^c(f, x) = \liminf_{n\to\infty} \frac{1}{n} \log \|(Df^n|E^{cu}_x)^{-1}\|^{-1} > 0$$

μ-almost everywhere. The integrated minimum central Lyapunov exponent $\hat{\lambda}^c(f, \cdot)$ is therefore positive on $\mathbb{G}\mathrm{ibbs}^{\mathrm{u}}(f)$. Since $\mathbb{G}\mathrm{ibbs}^{\mathrm{u}}(f)$ is compact (Proposition 2.3) and $\hat{\lambda}^c$ is lower semi-continuous on $\mathbb{G}\mathrm{ibbs}^{\mathrm{u}}(f)$ (Proposition 2.4), there is a positive lower bound for $\hat{\lambda}^c(f, \mu)$ on $\mathbb{G}\mathrm{ibbs}^{\mathrm{u}}(f)$. We can therefore fix some $0 < \sigma < 1$ such that

$$0 < \log \sigma^{-1} < \inf_{\mu \in \mathbb{G}\mathrm{ibbs}^{\mathrm{u}}(f)} \hat{\lambda}^c(f, \mu) \tag{3.8}$$

and write

$$\tau^\ell_x(f, \sigma) = \{n \in \mathbb{N} : n \text{ is a } \sigma^\ell \text{ hyperbolic time for } x \text{ under } f^\ell\}.$$

When it is not necessary to emphasize the dependency of σ, by simplicity we write $\tau^\ell_x(f, \sigma) = \tau^\ell_x(f)$.

The next result says that the frequency of hyperbolic times can be taken arbitrarily close to one. The price to pay is that we may have to take a large iterate of f.

First we fix some notation. In what follows, we use bold capital letters **U**, **V**, ... to denote open sets of the (fibered) space $\mathbb{G}\mathrm{ibbs}^{\mathrm{u}}(\mathcal{U}_{\mathcal{ME}})$ in the topology induced by the product topology on $\mathrm{Diff}^r(M) \times \mathbb{M}^1(M)$.

Proposition 3.1. *Let $f : M \to M$, $r > 1$, be a C^r mostly expanding diffeomorphism, μ a Gibbs u-state of f, and let $0 < \sigma < 1$ be such that (3.8) holds. Then, given any $\epsilon > 0$, there exists a neighborhood* **U** *of (f, μ) in $\mathbb{G}\mathrm{ibbs}^{\mathrm{u}}(\mathcal{U}_{\mathcal{ME}})$ and some natural number $\ell_0 \geq 1$ such that for every $(g, \nu) \in$* **U***, and every $\ell \geq \ell_0$, there is a set $A \subset M$ with $\nu(A) > 1 - \epsilon$ such that*

$$\liminf_{N\to\infty} \frac{|\tau^\ell_x(g, \sigma) \cap \{1, \dots, N\}|}{N} \geq 1 - \epsilon$$

for every $x \in A$.

Before proving Proposition 3.1 we need an auxiliary result. There is a well known characterization of weak* convergence of probability measures on a compact metric space, saying that a sequence of measures μ_n converges to μ if and only if $\liminf_{n\to\infty} \mu_n(U) \geq \mu(U)$ whenever U is an open set. In other words, the function

$$\mathbb{M}^1(M) \ni \mu \mapsto \mu(U) \in \mathbb{R}$$

is lower semi-continuous whenever $U \subset M$ is open. Lemma 3.2 can be seen as a slight variation of that.

Let $C^0(M, \mathbb{R})$ be the space of continuous functions $M \to \mathbb{R}$ endowed with the C^0 topology.

Lemma 3.2. *For any $\varphi \in C^0(M, \mathbb{R})$ denote by $U_\varphi \subseteq M$ the (open) set on which φ is positive. Then the map*

$$C^0(M, \mathbb{R}) \times \mathbb{M}^1(M) \ni (\varphi, \mu) \mapsto \mu(U_\varphi) \in \mathbb{R}$$

is lower semi-continuous in the product topology on $C^0(M, \mathbb{R}) \times \mathbb{M}^1(M)$.

The proof is straightforward but included for the sake of completeness.

Proof. Fix some pair $(\varphi, \mu) \in C^0(M, \mathbb{R}) \times \mathbb{M}^1(M)$ and an arbitrary $\epsilon > 0$. We need to show that there are open neighborhood $\mathcal{U} \subseteq C^0(M, \mathbb{R})$ of φ and $\mathbb{U} \subseteq \mathbb{M}^1(M)$ of μ such that $\nu(U_\phi) > \mu(U_\varphi) - \epsilon$ for every $(\phi, \nu) \in \mathcal{U} \times \mathbb{U}$.

By regularity of μ there is some compact set $C \subset U_\varphi$ such that $\mu(C) > \mu(U_\varphi) - \epsilon$. Since φ is positive on C, it follows by compactness that we can find some number β that satisfies $0 < \beta < \inf_{x \in C} \varphi(x)$. Observe that $U_{\varphi-\beta} \supset C$.

Let $\mathcal{U}$ be the open ball of radius β around φ in $C^0(M, \mathbb{R})$. Thus if $\phi \in \mathcal{U}$ and $x \in U_{\varphi-\beta}$ we have $\phi(x) > \varphi(x) - \beta > 0$. Hence

$$C \subset U_{\varphi-\beta} \subset U_\phi$$

for every $\phi \in \mathcal{U}$.

Let $\rho : M \to [0, 1]$ be a continuous function satisfying $\rho|C = 1$ and $\rho|U^c_{\varphi-\beta} = 0$. In particular, for every $\phi \in \mathcal{U}$,

$$\mu(U_\phi) \geq \mu(U_{\varphi-\beta}) \geq \int \rho \, d\mu \geq \mu(C) > \mu(U_\varphi) - \epsilon$$

Now let $\mathbb{U}$ be the open neighborhood of μ in $\mathbb{M}^1(M)$ defined by

$$\mathbb{U} = \{\nu \in \mathbb{M}^1(M) : \int \rho \, d\nu > \mu(U_\varphi) - \epsilon\}.$$

Then, if $(\phi, \nu) \in \mathcal{U} \times \mathbb{U}$ we have

$$\nu(U_\phi) \geq \nu(U_{\varphi-\beta}) \geq \int \rho \, d\nu \geq \nu(C) > \nu(U_\varphi) - \epsilon. \quad \square$$

Proof of Proposition 3.1. Recall the choice of $0 < \sigma < 1$ in (3.8). We write $\gamma = \log \sigma^{-1}$ and fix some Γ with

$$\gamma < \Gamma < \inf_{\mu \in \mathbb{G}\mathrm{ibbs}^{\mathrm{u}}(f)} \hat{\lambda}^c(f, \mu). \tag{3.9}$$

We also fix some

$$L < \inf_{x \in M} \log \|Df^{-1}|E^{cu}_x\|^{-1}.$$

For every $\ell \geq 1$ and $g \in \mathcal{U}_{\mathcal{ME}}$, we define the family $\{U^\ell_g\}$ of open sets in M by

$$U^\ell_g = \{x \in M : \frac{1}{\ell} \log \|(Dg^\ell|E^{cu}_x)^{-1}\|^{-1} > \Gamma\}$$

For every $\mu \in \mathbb{G}\mathrm{ibbs}^{\mathrm{u}}(f)$, from (3.9) we have

$$\Gamma < \int \lambda^c(f, x) d\mu = \int \liminf \frac{1}{\ell} \log \|(Df^\ell|E^{cu})^{-1}\|^{-1} d\mu$$

Since $\lambda^c(f, \cdot)$ is f-invariant, if μ is ergodic, then for μ-almost every $x \in M$

$$\lambda^c(f, x) = \liminf \frac{1}{\ell} \log \|(Df^\ell|E^{cu}_x)^{-1}\|^{-1} > \Gamma. \tag{3.10}$$

If μ is not ergodic, then we write μ as convex combination of ergodic measures (see Proposition 2.3), all of which are Gibbs u-states satisfying (3.10). So any convergent subsequence of $\frac{1}{\ell} \log \|(Df^\ell|E^{cu}_x)^{-1}\|^{-1}$ must converges μ-almost everywhere to some limit larger than Γ. Hence, we must have

$$\lim_{\ell \to \infty} \mu(U^\ell_f) = 1.$$

Fix $\epsilon > 0$. Take

$$\kappa = \epsilon \frac{\Gamma - \gamma}{\Gamma - L} \tag{3.11}$$

and choose $\ell_0 \geq 1$ so that for every $\ell \geq \ell_0$ we have

$$\mu(U_f^\ell) > 1 - \epsilon\kappa.$$

It follows from Lemma 3.2 that the set

$$\mathbf{U} = \{(g,\nu) : \nu(U_g^\ell) > 1 - \epsilon\kappa \text{ and } \inf_{x\in M} \log \|Dg^{-1}|E_x^{cu}\|^{-1} > L\}$$

is open in $\mathbb{G}\mathrm{ibbs}^{\mathrm{u}}(\mathcal{U}_{\mathcal{ME}})$.

Pick any pair $(g,\nu) \in \mathbf{U}$. We shall prove that (g,ν) satisfies the conclusion of Proposition 3.1.

Consider the function

$$F(x) = \lim_{n\to\infty} \frac{1}{n} \#\{0 \le k \le n-1 : g^{\ell k}(x) \in U_g^\ell\}$$

of the frequency of visits to the set U_g^ℓ. By Birkhoff's Ergodic Theorem it is well defined ν-almost everywhere and satisfies

$$\int F \, d\nu = \nu(U_g^\ell) > 1 - \epsilon\kappa. \tag{3.12}$$

Let $A = \{x \in M : F(x) > 1-\kappa\}$. Chebyshev's inequality and (3.12) gives

$$\nu(M \setminus A) = \nu(\{x : 1 - F(x) \ge \kappa\}) \le \frac{1}{\kappa} \int 1 - F \, d\nu < \frac{\epsilon\kappa}{\kappa} = \epsilon.$$

In other words, $\nu(A) > 1-\epsilon$ and the proof will be complete once we have proved that for every $x \in A$,

$$\liminf_{N\to\infty} \frac{|\tau_x^\ell(g) \cap \{1,\dots,N\}|}{N} \ge 1 - \epsilon.$$

To this end, fix $x \in A$ and $N_0 \ge 1$ is such that

$$\frac{1}{N} \#\{0 \le k \le N-1 : g^{\ell k}(x) \in U_g^\ell\} > 1 - \kappa \tag{3.13}$$

for every $N \ge N_0$. Let

$$a_i = \frac{1}{\ell} \log \|(Dg^\ell | E^{cu}_{g^{\ell(i-1)}(x)})^{-1}\|^{-1}.$$

Then (3.13) implies that

$$\#\{i \in \{1,\dots,N\} : a_i < \Gamma\} \le \#\{i \in \{1,\dots,N\} : a_i \le \Gamma\} < \kappa N.$$

We can therefore conclude from Lemma A that, for $\epsilon > 0$, there exist $\kappa > 0$ defined by (3.11) and there exist $1 < n_1 < \dots < n_m \le N$ with $m > (1 - \kappa \frac{\Gamma - L}{\Gamma - \gamma})N = (1-\epsilon)N$ such that

$$\sum_{j=n+1}^{n_i} \frac{1}{\ell} \log \|(Dg^\ell | E^{cu}_{g^{\ell(j-1)}(x)})^{-1}\|^{-1} = \frac{1}{\ell} \log \prod_{j=n+1}^{n_i} \|Dg^{-\ell} | E^{cu}_{g^{\ell j}(x)}\|^{-1} \ge \gamma(n_i - n), \tag{3.14}$$

for every $1 \le i \le m$ and every $0 \le n < n_i$. Writing $k = n_i - n$ and remembering that $\gamma = \log \sigma^{-1}$, (3.14) may be more conveniently expressed by

$$\prod_{j=n_i-k+1}^{n_i} \|Dg^{-\ell} | E^{cu}_{g^{\ell j}(x)}\| \le \sigma^{\ell k}$$

for every $1 \le i \le m$ and every $1 \le k \le n_i$. That is, each n_i is a σ^ℓ hyperbolic time for x under g^ℓ. □

3.3. Pesin blocks of uniform measure

We now change our focus a bit. Fix $0 < \sigma < 1$. Instead of considering hyperbolic times of a given point x, we consider the set

$$\Lambda_\ell^n(f,\sigma) = \{x \in M \,:\, \prod_{j=0}^{k-1} \|Df^{-\ell}|E^{cu}_{f^{-\ell j}(x)}\| \le \sigma^{\ell k},\ \forall 1 \le k \le n\} \tag{3.15}$$

of points which are hyperbolic time iterates of some other point. We are particularly interested in the set

$$\Lambda_\ell(f,\sigma) = \bigcap_{n\ge 1} \Lambda_\ell^n(f,\sigma), \tag{3.16}$$

which we call a *Pesin like block* of f. When it is not necessary to emphasize the dependency of σ, we write $\Lambda_\ell^n(f) = \Lambda_\ell^n(f,\sigma)$ or $\Lambda_\ell(f) = \Lambda_\ell(f,\sigma)$ respectively in order to simplify notation.

Remark 3.1. The Pesin like blocks $\Lambda_\ell(f)$ are different from the Pesin blocks $\mathrm{Bl}(\ell, f^{-1})$ considered by Avila and Bochi in [9]. For example, for points in $\Lambda_\ell(f)$, the Lyapunov exponent in the E^{cu} bundle is bounded below by a fixed number $\log \sigma^{-1}$, whereas for points in $\mathrm{Bl}(\ell, f^{-1})$, they are bounded below by $1/\ell$. Our notion is therefore more restrictive, and suitable to a situation where Lyapunov exponents are almost everywhere bounded away from zero with respect to a relevant set of measures (which is not the case in [9]). A main ingredient in our work is that $\Lambda_\ell(f)$ has large μ-measure for large ℓ and $\mu \in \mathbb{G}\mathrm{ibbs}^{\mathrm{cu}}(f)$ in a way which is uniform in a neighborhood of f (see Lemma 3.4). It is for this reason that we have proved the Pliss-like Lemma (Lemma A). Avila and Bochi obtain similar results for the set $\mathrm{Bl}(\ell, f^{-1})$ using a very elegant application of the Maximal Ergodic Theorem. The current work could perhaps be made a few pages shorter by working with $\mathrm{Bl}(\ell, f^{-1})$ rather than $\Lambda_\ell(f)$ and making use of their results. However, we think that our estimates on the size of $\Lambda_\ell(f)$ is of independent interest, as well as being more intuitive for those who are used to arguments involving Pliss' Lemma.

Lemma 3.3. *Given $f : M \to M$ mostly expanding, $\mu \in \mathbb{G}\mathrm{ibbs}^{\mathrm{u}}(f)$, a number $0 < \sigma < 1$ satisfying (3.8) and $\epsilon > 0$, there exist a neighborhood $\mathbf{U}$ of (f,μ) in $\mathbb{G}\mathrm{ibbs}^{\mathrm{u}}(\mathcal{U}_{\mathcal{ME}})$ and an integer $\ell_0 \ge 1$ such that $\nu(\Lambda_\ell(g,\sigma)) > 1-\epsilon$ for every $(g,\nu) \in \mathbf{U}$ and every $\ell \ge \ell_0$.*

Proof. Fix $(f,\mu) \in \mathbb{G}\mathrm{ibbs}^{\mathrm{u}}(\mathcal{U}_{\mathcal{ME}})$ and $\epsilon > 0$ arbitrarily. Choose some $\epsilon' > 0$ small enough that

$$(1-\epsilon')^2 > 1-\epsilon. \tag{3.17}$$

Proposition 3.1 guarantees the existence of an open neighborhood $\mathbf{U}$ of (f,μ) in $\mathbb{G}\mathrm{ibbs}^{\mathrm{u}}(\mathcal{U}_{\mathcal{ME}})$ and a positive integer $\ell_0 \ge 1$ such that, given any $(g,\nu) \in \mathbf{U}$, there is some set $A \subset M$, with $\nu(A) > 1-\epsilon'$, such that

$$\liminf_{N\to\infty} \frac{|\tau_x^\ell(g) \cap \{1,\dots,N\}|}{N} > 1-\epsilon'$$

for every $x \in A$. We will prove that if (g,ν) belongs to $\mathbf{U}$, then $\nu(\Lambda_\ell^n(g)) > 1-\epsilon$. Let

$$A_n = \{x \in M : \inf_{k\ge n} \frac{|\tau_x^\ell(g) \cap \{1,\dots,k\}|}{k} > 1-\epsilon'\}.$$

Note that A_n is an increasing sequence of measurable sets such that $A \subset \cup A_n$ and, by our choice of $\mathbf{U}$ and ℓ, we have that $\nu(\bigcup_{n\in\mathbb{N}} A_n) > 1-\epsilon'$. Therefore, we can (and do) fix some integer $N \ge 1$ such that $\nu(A_N) > 1-\epsilon'$. Likewise, let

$$B_n = \{x \in M : \frac{|\tau_x^\ell(g) \cap \{1,\dots,n\}|}{n} > 1-\epsilon'\}.$$

The sequence B_n does not have to be increasing, but we do have $B_n \supset A_n$ for every $n \in \mathbb{N}$ so that, in particular,

$$\nu(B_N) > 1-\epsilon'. \tag{3.18}$$

We denote by χ_B the characteristic function of a Borelian set $B \subseteq M$. Observe that

$$\sum_{n=1}^{N} \chi_{\Lambda_\ell^n(g)} \circ g^{n\ell}(x) = |\tau_x^\ell \cap \{1, \dots, N\}|$$

for every $x \in M$. Consequently

$$\begin{aligned}
\nu(\Lambda_\ell^n(g)) &= \int \chi_{\Lambda_\ell^n(g)}\, d\nu \\
&= \int \frac{1}{N} \sum_{n=1}^{N} \chi_{\Lambda_\ell^n(g)} \circ g^{n\ell}\, d\nu \\
&\geq \int_{B_N} \frac{1}{N} \sum_{n=1}^{N} \chi_{\Lambda_\ell^n(g)} \circ g^{n\ell}\, d\nu \\
&\geq \int_{B_N} 1 - \epsilon'\, d\nu.
\end{aligned}$$

Recalling (3.18) and (3.17), we obtain

$$\nu(\Lambda_\ell^n(g)) > \int_{B_N} 1 - \epsilon'\, d\nu > (1 - \epsilon')^2 > 1 - \epsilon.$$

Recall that $\Lambda_\ell(g) = \bigcap_n \Lambda_\ell^n(g)$, and that the $\Lambda_\ell^n(g)$ form a nested decreasing sequence in n. The proof follows readily. □

Next Lemma is an improvement of Lemma 3.3. It says that, not only can the number ℓ_0 in Lemma 3.3 be taken uniform in a neighborhood of the pair (f, μ) in $\mathbb{G}\text{ibbs}^{\text{u}}(\mathcal{U}_{\mathcal{ME}})$; it can indeed be chosen so that it holds simultaneously for every Gibbs u-state of every diffeomorphism in a neighborhood of f.

Lemma 3.4. *Let $f : M \to M$ be a C^r mostly expanding diffeomorphism, $r > 1$. Given any $\epsilon > 0$, there exists ℓ_0 and a C^r neighborhood $\mathcal{U}$ of f such that $\nu(\Lambda_\ell(g)) > 1 - \epsilon$ for every $\ell \geq \ell_0$, $g \in \mathcal{U}$ and $\nu \in \mathbb{G}\text{ibbs}^{\text{u}}(g)$.*

Proof. Fix $\epsilon > 0$. Since $\mathbb{G}\text{ibbs}^{\text{u}}(f)$ is compact (see Proposition 2.3), it follows from Lemma 3.3 that there are open sets $\mathbf{U}_1, \dots, \mathbf{U}_n \subset \mathbb{G}\text{ibbs}^{\text{u}}(\mathcal{U}_{\mathcal{ME}})$, where $\mathbf{U}_i = \mathcal{U}_i \times \mathbb{U}_i$, and integers $\ell_1, \dots, \ell_n$ such that

(i) $\mathbb{G}\text{ibbs}^{\text{u}}(f) \subset \mathbb{U}_1 \cup \ldots \cup \mathbb{U}_n$, and
(ii) $\nu(\Lambda_\ell(g)) > 1 - \epsilon$ whenever $(g, \nu) \in \mathbf{U}_i$ for some $i = 1, \dots, n$ and every $\ell \geq \ell_i$.

Let $\mathcal{U} = \mathcal{U}_1 \cap \ldots \cap \mathcal{U}_n$ and $\mathbb{U} = \mathbb{U}_1 \cup \ldots \cup \mathbb{U}_n$. Then

$$\mathbb{G}\text{ibbs}^{\text{u}}(f) \subset \mathcal{U} \times \mathbb{U} \subset \bigcup_{i=1}^{n} \mathcal{U}_i \times \mathbb{U}_i.$$

It follows from Proposition 2.3 that, upon possibly reducing $\mathcal{U}$, we may (and do) suppose that $\mathbb{G}\text{ibbs}^{\text{u}}(g) \subset \mathcal{U} \times \mathbb{U}$ for every $g \in \mathcal{U}$. Let $\ell_0 = \prod_{i=1}^{n} \ell_i$. Given any $g \in \mathcal{U}$ and any $\nu \in \mathbb{G}\text{ibbs}^{\text{u}}(g)$ there exists some $i = 1, \dots, n$ such that $(g, \nu) \in \mathcal{U}_i \times \mathbb{U}_i$. Since $\ell_0 \geq \ell_i$, for every $i = 1, \dots, n$, from (ii) above we have $\nu(\Lambda_\ell(g)) > 1 - \epsilon$ for every $\ell \geq \ell_0$. □

3.4. *Unstable manifolds and uniform densities*

Central to our argument is that the size of local unstable manifolds can be controlled on the sets $\Lambda_\ell(f, \sigma)$, uniformly in a neighborhood of a given mostly expanding diffeomorphism. This was done from scratch in [9, Theorem 4,7] using graph transforms. In this work some further properties of unstable manifolds are needed which are not stated in [9, Theorem 4,7].

Theorem 3.5. *Let $f : M \to M$ be a C^r mostly expanding diffeomorphism, with $r > 1$. Then, given any $\ell \in \mathbb{N}$, there are a C^r neighborhood $\mathcal{U}$ of f and numbers $r = r(\ell) > 0$, $C = C(\ell) \geq 0$, $\delta = \delta(\ell) > 0$ for which the following holds:*

(i) Given any $g \in \mathcal{U}$ and $x \in \Lambda_\ell(g, \sigma)$, there is a C^1 embedded disk $W_r^{cu}(g, x)$ of dimension $\dim E^{cu}$ and radius $r > 0$, centered at x, such that

$$T_y W_r^{cu}(g, x) = E_y^{cu}(g)$$

for every $y \in W_r^{cu}(g, x)$.

(ii) $W_r^{cu}(g, x)$ depends continuously on both x and g in the C^1 topology;

(iii) $W_r^{cu}(g, x) \subset \Lambda_\ell(g, \sigma^{1/2})$;

(iv) if $y \in W_r^{cu}(g, x)$, then

$$\operatorname{dist}(f^{-n}(x), f^{-n}(y)) \leq C\sigma^{n/2} \operatorname{dist}(x, y)$$

for every $n \geq 0$;

(v) if $y, z \in \Lambda_\ell(g, \sigma) \cap B_\delta(x)$, then either

$$W_r^{cu}(g, y) \cap W_r^{cu}(g, z) = \emptyset$$

or

$$W_r^{cu}(g, y) \cap B_{2\delta}(x) = W_r^{cu}(g, z) \cap B_{2\delta}(x).$$

We give a complete proof of Theorem 3.5. Our approach is a combination of the methods of [3] and [1]. In particular, we make use of so-called locally invariant plaque families introduced in [19, Theorem 5.5]. Unfortunately, [19] only considers plaque families of a single diffeomorphism, but it is implicit in the construction that these plaques depend continuously, not only on the point in the manifold, but also as a function of the diffeomorphism in question. We cite a version of the plaque family theorem due to [14, Lemma 3.5] that takes this into account.

Theorem 3.6 *([19,14]). Suppose that a diffeomorphism $f : M \to M$ has a dominated splitting $F \prec G$. Then there exist a C^1 neighborhood $\mathcal{U}$ of f, a number $\rho > 0$, and a continuous family of embeddings*

$$\mathcal{U} \times M \ni (g, x) \mapsto \Phi_{g,x} \in \operatorname{Emb}^1(\mathbb{R}^{\dim(F)}, M)$$

such that for every $g \in \mathcal{U}$ and every $x \in M$ we have

- *$D\Phi_{g,x}(0) = F_x(g)$, and*
- *$g(\Phi_{g,x}(B_\rho(0))) \subset \Phi_{g,g(x)}(\mathbb{R}^{\dim F})$.*

We make some remarks and set some notation that will be used in the proof of Theorem 3.5. Fix a mostly expanding diffeomorphism $f : M \to M$. The inverse of f has a dominated splitting $E^{cu} \prec E^s$. Hence, according to Theorem 3.6 there are a C^1 neighborhood $\mathcal{U}$ of f, a number $\rho > 0$ and a family $\{\Phi_{g,x}\}_{g\in\mathcal{U},x\in M}$ such that $D\Phi_{g,x}(0) = E_x^{cu}(g)$ and

$$g^{-1}(\Phi_{g,x}(B_\rho(0))) \subset \Phi_{g,g^{-1}(x)}(\mathbb{R}^{cu}),$$

where cu stands for the dimension of E^{cu}.

Let

$$V_g(x) = \Phi_{g,x}(\mathbb{R}^{cu})$$

and for $c > 0$ let

$$D_g(x, c) = \{y \in V_g(x) : \operatorname{dist}_{V_g(x)}(x, y) < c\}$$

and write $\partial D_g(x, c)$ for $\overline{D_g(x, c)} \setminus D_g(x, c)$. For sufficiently small c (but uniform in x and g), the set $D_g(x, c)$ is a disk of radius c in the sense that

$$\operatorname{dist}_{V_g(x)}(x, y) = c$$

for every $y \in \partial D_g(x,c)$. Take $c_0 > 0$ small enough so that $D_g(x,c_0)$ is a disk of radius c_0 and also so that

$$D_g(x,c_0) \subset \Phi_{g,x}(B_\rho(0))$$

for every $(g,x) \in \mathcal{U} \times M$. Then $g^{-1}(D_g(x,c_0)) \subset V_g(g^{-1}(x))$ for every pair $(g,x) \in \mathcal{U} \times M$. The same is true if c_0 is replaced with any number smaller than c_0. In particular, given any $c_1 \le c_0$, there exists $c_2 < c_1$ such that

$$g^{-1}(D_g(x,c_2)) \subset D_g(g^{-1}(x),c_1)$$

for every pair $(g,x) \in \mathcal{U} \times M$. This can be carried out for iterates of g as well: Given any $\ell \in \mathbb{N}$ and any $c_1 < c_0$ there exists $c_2 < c_1$ such that

$$g^{-k}(D_g(x,c_2)) \subset D_g(g^{-k}(x),c_1)$$

for every pair $(g,x) \in \mathcal{U} \times M$ and every $1 \le k \le \ell$.

Proof of Theorem 3.5. Fix a mostly expanding diffeomorphism $f : M \to M$, $0 < \sigma < 1$ as in (3.8), and some number $\ell \in \mathbb{N}$. Let $\mathcal{U}$, $\{V_g(x)\}_{g\in\mathcal{U},x\in M}$ and c_0 be as in the above discussion. We shall also assume that c_0 is sufficiently small so that each $D_g(x,c_0)$ is uniformly transversal to $E^s(g)$ for every $(g,x) \in \mathcal{U} \times M$. By uniform continuity of $\log \|Df^{-\ell}|E^{cu}_x\|$, there exists $0 < c_1 < c_0$ such that

$$\|Df^{-\ell}|E^{cu}_y(f)\| < \sigma^{-1/2}\|Df^{-\ell}|E^{cu}_x(f)\|$$

whenever $\operatorname{dist}(x,y) < c_1$. Upon possibly reducing $\mathcal{U}$, we may assume that

$$\|Dg^{-\ell}|E^{cu}_y(g)\| < \sigma^{-1/2}\|Dg^{-\ell}|E^{cu}_x(g)\|$$

for every $g \in \mathcal{U}$ and every pair $x, y \in M$ such that $\operatorname{dist}(x,y) < c_1$.

Let $r = r(\ell) < c_1$ be such that

$$g^{-i}(D_g(x,r)) \subset D_g(g^{-i}(x),c_1)$$

for every $g \in \mathcal{U}$, $x \in M$, and $1 \le i \le \ell$.

Suppose in what follows that $g \in \mathcal{U}$ and that $x \in \Lambda_\ell(g)$ for some ℓ. We claim that for every $y \in D_g(x,r)$ and every $n \ge 1$ we have

$$\prod_{k=0}^{n-1} \|Dg^{-\ell}|E^{cu}_{g^{-\ell k}(y)}(g)\| \le \sigma^{\ell n/2}. \tag{3.19}$$

Let us prove this by strong induction. For $n = 1$, (3.19) follows directly from the definition of $\Lambda_\ell(g,\sigma)$. Suppose that (3.19) holds for every $1 \le n \le N$. We shall prove that it also holds for $n = N+1$.

First notice that, since (3.19) is assumed to hold for every $y \in D_g(x,r)$, it implies that

$$\operatorname{dist}_{V_g(g^{-\ell n}(x))}(g^{-\ell n}(x), g^{-\ell n}(y)) \le \sigma^{\ell n/2} \operatorname{dist}_{V_g(x)}(x,y) \tag{3.20}$$

for every $y \in D_g(x,r)$. Consequently,

$$g^{-\ell n}(D_g(x,r)) \subset D_g(g^{-\ell n}(x), r\sigma^{\ell n/2}) \subset D_g(g^{-\ell n}(x),c_1). \tag{3.21}$$

Now, (3.21) said in particular that for every $1 \le n \le N$ and for every $y \in D_g(x,r)$ we have

$$\operatorname{dist}(g^{-\ell n}(x), g^{-\ell n}(y)) \le r\sigma^{\ell n/2} < r < c_1.$$

Hence

$$\|Dg^{-\ell}|E^{cu}_{f^{-\ell n}(y)}\| \le \sigma^{-1/2}\|Df^{-\ell}|E^{cu}_{f^{-\ell n}(y)}\|$$

for every $1 \le n \le N$. It follows that

$$\prod_{k=0}^{N} \|Dg^{-\ell}|E^{cu}_{g^{-\ell k}(y)}\| \le \sigma^{-(N+1)/2} \prod_{k=0}^{N} \|Dg^{-\ell}|E^{cu}_{g^{-\ell k}(x)}\| \le \sigma^{-(N+1)/2}\sigma^{\ell(N+1)} \le \sigma^{\ell(N+1)/2},$$

which is just our inductive hypothesis for $n = N + 1$. The induction is thus complete and we conclude that (3.19) (and hence (3.20) and (3.21)) holds for every $n \geq 1$.

Let $W_r^{cu}(g, x) = D_g(x, r)$. Then (ii) is true by construction. Item (iii) is given by our inductive step and (iv) is a consequence thereof. To see why (i) holds, we argue by contradiction. Suppose that for some $g \in \mathcal{U}$ and $x \in \Lambda_\ell(g, \sigma)$ we have that $W_r^{cu}(x)$ is not tangent to $E^{cu}(g)$. Then there is some $y \in W_{r_\ell}^{cu}(x)$ and some $v \in T_y W_r^{cu}(x)$ such that $v \notin E_y^{cu}(g)$. By domination, the angle between $Dg^{-n}v$ and $E^s_{g^{-n}(y)}(g)$ then tends to zero as $n \to \infty$. But $Dg^{-n}v \in TD_g(g^{-n}(x), \delta_1)$ for every $n \geq 0$ and must therefore have angle to $E^s(g)$ which is bounded away from zero — a contradiction. We conclude that indeed $W_r^{cu}(x)$ must be tangent to $E^{cu}(g)$.

It remains to prove item (v). To this end, we first show that for $g \in \mathcal{U}$, the family $\{W_r^{cu}(x) : x \in \Lambda_\ell(g, \sigma)\}$ is *self-coherent*: given any $x, y \in \Lambda_\ell(g, \sigma)$, the intersection $W_r^{cu}(x) \cap W_r^{cu}(y)$ is an open subset of both $W_r^{cu}(x)$ and $W_r^{cu}(y)$. The argument is classic. Suppose it is not true. Then there is a point z in $\partial(W_r^{cu}(x) \cap W_r^{cu}(y))$ which is in the interior of both $W_r^{cu}(x)$ and $W_r^{cu}(y)$. Let w_1 be a point in $W_r^{cu}(x) \setminus W_r^{cu}(y)$ close to z and w_2 a point in $W_r^{cu}(y) \setminus W_r^{cu}(x)$ close to z such that w_1 and w_2 lie on the same local stable manifold. By the construction of $W_r^{cu}(x)$ and $W_r^{cu}(y)$,

$$\mathrm{dist}(f^{-n}(w_1), f^{-n}(w_2)) \leq \mathrm{dist}(f^{-n}(w_1), f^{-n}(z)) + \mathrm{dist}(f^{-n}(z), f^{-n}(w_2)) \leq 2c_1$$

for every $n \geq 0$. But w_1 and w_2 are on the same unstable manifold, so $\mathrm{dist}(f^{-n}(w_1), f^{-n}((w_2))$ must grow larger than $2c_1$.

Let $\delta = \delta(\ell) > 0$ be small enough so that if $g \in \mathcal{U}$, $x \in M$, and $y \in B_\delta(x) \cap \Lambda_\ell(g, \sigma)$, then $\partial W_r^{cu}(y) \cap B_{2\delta}(x) = \emptyset$. Now given any $y, z \in B_\delta(x) \cap \Lambda_\ell(g, \sigma)$, by the self-coherent property we have that

$$\partial(W_r^{cu}(x) \cap W_r^{cu}(y)) \subset \partial W_r^{cu}(x) \cup \partial W_r^{cu}(y).$$

Hence

$$W_r^{cu}(y) \cap B_{2\delta}(x) = W_r^{cu}(z) \cap B_{2\delta}(x)$$

as required. □

3.5. Lamination bundles and disintegration

The notion of Gibbs *cu*-states involves disintegration of a measure along Pesins' unstable manifolds. Broadly speaking, two ways to disintegrate a measure along unstable manifolds appear in the literature. One of them, used in [3] and the works influenced by it, uses a so-called foliated box. In such a box, unstable manifolds are graphs of functions from one Euclidean ball to another Euclidean ball. This approach is often practical under the presence of dominated splittings. The other approach, often used in the more general setting of non-uniform hyperbolicity, considers the union of unstable manifolds of points in the intersection of a Pesin block with a small ball. Here we use a variation of this latter approach.

Let us set up some notation for this section. Let $f : M \to M$ be a C^r-partially hyperbolic diffeomorphism, $r > 1$, and let μ be an ergodic Gibbs u-state with positive center Lyapunov exponents for f. Let $\hat{\lambda}^c(f, \mu) := \int \lambda^c(f, x) d\mu > 0$, and fix any $0 < \sigma < 1$ such that $\log \sigma^{-1} < \hat{\lambda}^c(f, \mu)$. It is well known from Pesin Theory, that μ-a.e. $x \in M$ there exists $W^{cu}(x)$ center-unstable manifolds. We denote by

$$\mathcal{W}^{cu}(f) = \{W \ : \ W \text{ is a center-unstable manifold }\}.$$

For every $\ell \geq 1$, we fix $r = r(\ell) > 0$ and $\delta = \delta(\ell) > 0$ as in Theorem 3.5. This implies that there exists a center-unstable lamination

$$\mathcal{W}_\ell^{cu}(f, \sigma) = \{W_r^{cu}(x) : x \in \Lambda_\ell(f, \sigma)\}.$$

By uniqueness of Pesin's unstable manifolds, we have that $W_r^{cu}(x) \subset W^{cu}(x)$ μ-almost everywhere. In particular, $W_r^{cu}(x)$ μ-almost everywhere of class C^r rather than just C^1, as stated in Theorem 3.5.

Given a mostly expanding diffeomorphism $f : M \to M$, we fix, for every $\ell \in \mathbb{N}$ values $r = r(\ell)$ and $\delta = \delta(\ell)$ so that the conclusions of Theorem 3.5 hold in a neighborhood of f. Then, for every $x \in M$, we write

$$\mathcal{Q}(\ell, \sigma, x, f) = \{W_r^{cu}(y) \cap B_{2\delta}(x) : y \in \Lambda_\ell(f, \sigma) \cap \overline{B_\delta(x)}\} \tag{3.22}$$

and

$$Q(\ell,\sigma,x,f) = \bigcup_{D\in\mathcal{Q}(\ell,\sigma,x,f)} D. \tag{3.23}$$

We refer to the set $Q(\ell,\sigma,x,f)$ as a *lamination bundle*. Thus whenever we talk about "the lamination bundle $Q(\ell,\sigma,x,f)$", the corresponding $r>0$ and $\delta>0$ are implicitly defined. In particular (see part v in Theorem 3.5) $\mathcal{Q}(\ell,x)$ is a partition of $Q(\ell,x)$ for every $x\in M$. This is indeed the reason why, in the definition of lamination bundle, we take intersections of unstable manifolds with $B_{2\delta}(x)$. Note also that we only consider unstable manifolds of points in the smaller ball $B_\delta(x)$. The reason for this is that by doing so we get a lower bound for the volume of the leaves in $\mathcal{Q}(\ell,\sigma,x,f)$. When it is not necessary to emphasize the dependency of one particular variable, f or σ for instance, for simplicity of notation we will omit it, writing for example $\mathcal{Q}(\ell,x)=\mathcal{Q}(\ell,\sigma,x,f)$ or $Q(\ell,x)=Q(\ell,\sigma,x,f)$ respectively.

Let μ be any Borel measure and $Q=Q(\ell,x)$ a lamination bundle. Clearly, the partition $\mathcal{Q}=\mathcal{Q}(\ell,x)$ is measurable in the sense of Rokhlin. We may therefore decompose μ_Q with respect to $\mathcal{Q}$: there exists a measurable family of probability measures $\{\mu_D : D\in\mathcal{Q}\}$ (usually called *conditional measures*) and a measure $\hat{\mu}$ on $\mathcal{Q}$ (usually called *factor measure*) with $|\hat{\mu}|=\mu(Q)$ such that

$$\int_Q \varphi\, d\mu = \int \varphi\, d\mu_Q = \int_{\mathcal{Q}} \left(\int_D \varphi(x)\, d\mu_D(x) \right) d\hat{\mu}(D) \tag{3.24}$$

for every continuous $\varphi: M\to\mathbb{R}$.

Suppose that a Gibbs u-state μ with positive central Lyapunov exponent is also a Gibbs cu-state for f. Fix $\epsilon>0$. By Lemma 3.3, there exists $\ell_0\geq 0$ such that $\mu(\Lambda_\ell(f))>1-\epsilon$ for every $\ell\geq\ell_0$. Now fix a such $\ell\geq\ell_0$ and consider any $x\in M$ such that the lamination bundle $Q=Q(\ell,x)$ has positive μ-measure. Since μ is a Gibbs cu-state, and since Pesin's unstalbe manifold of μ-almost every point x in Q coincides with $W_r^{cu}(x)$, the conditional measures μ_D are $\hat{\mu}_D$-almost everywhere absolutely continuous to m_D, i.e. there is a family of densities ρ_D such that

$$\int_Q \varphi\, d\mu = \int \varphi\, d\mu_Q = \int_{\mathcal{Q}} \left(\int_D \varphi(x)\rho_D(x)\, dm_D(x) \right), \tag{3.25}$$

for every continuous function $\varphi: M\to\mathbb{R}$. By the uniqueness of the disintegration, comparing (3.24) and (3.25) we conclude that $\hat{\mu}$-almost every $D\in\mathcal{Q}$, $\mu_D=\rho_D\cdot m_D$, so that μ_D is absolutely continuous with respect to m_D for $\hat{\mu}$-almost every $D\in\mathcal{Q}$.

Remark 3.2. A priori, μ_D is only assumed to be absolutely continuous with respect to m_D, so in principle the densities ρ_D need only be measurable and may be zero on sets of positive m_D-measure. Yet it is known *a fortiori* (see e.g. [10, Theorem 13.1.2]) that in every lamination bundle above, and $\hat{\mu}$-almost every $D\in\mathcal{Q}$, the density $\rho_D=\frac{d\nu_D}{dm_D}$ satisfies (compare with (2.1))

$$\frac{\rho_D(x)}{\rho_D(y)} = \prod_{n=0}^{\infty} \frac{\det(Df^{-1}|E^{cu}_{f^{-n}(x)})}{\det(Df^{-1}|E^{cu}_{f^{-n}(y)})}. \tag{3.26}$$

The limit (3.26) is bounded above and away from zero by constants that depend only on ℓ in a neighborhood of f. In particular, given any $\ell\geq 1$, there are a neighborhood $\mathcal{U}$ of f and $L>0$, such that for every $g\in\mathcal{U}$, every $\mu\in\mathbb{G}\mathrm{ibbs}^{\mathrm{cu}}(g)$ and every $x\in M$, we have

$$\int \varphi\, d\mu \leq L \int_{\mathcal{Q}(\ell,x)} \left(\int \varphi\, dm_D \right) d\hat{\mu} \leq \sup_{D\in\mathcal{Q}(\ell,x)} L|\mu| \int \varphi\, dm_D. \tag{3.27}$$

Reciprocally, if we consider any $x\in M$, $\ell>0$, such that the lamination bundle $Q=Q(\ell,x)$ has $\mu(Q)>0$, assuming that $\hat{\mu}$-almost every $D\in\mathcal{Q}$, $\mu_D\prec m_D$, we conclude that we can disintegrate μ_Q along the Pesin unstable manifolds (which are precisely the discs belong to $\mathcal{Q}$) and then μ has conditional measures absolutely continuous with respect to the Pesin unstable manifolds. But again, from Lemma 3.3 we conclude that the union of lamination bundles $Q = Q(\ell, x)$ as above has measure bigger that $1 - \epsilon$, where $\epsilon > 0$ is arbitrary. So μ must be a Gibbs cu-state.

The discussion above implies in particular that if μ is not a Gibbs cu-state for f, then we can find a subset $X \subseteq M$ of positive μ-measure (which is singular with respect to the Lebesgue measure along the center unstable manifolds) such that for every $x \in M$ and every lamination bundle $Q = Q(\ell, x)$, with associated partition $\mathcal{Q} = \mathcal{Q}(\ell, x)$, we have $m_D(X) = 0$ for $\hat{\mu}$-almost every $D \in \mathcal{Q}$.

4. Proof of Theorem D

4.1. A characterization of Gibbs cu-states

As we outlined in the introduction, proving statistical stability of mostly expanding diffeomorphisms involves proving that a limit of Gibbs cu-states is a Gibbs cu-state. A direct proof using lamination bundles (or foliated boxes) would be clumsy and difficult to make rigorous. Our first goal is to answer the following question: Which among the Gibbs u-states with positive Lyapunov exponents along the central direction are Gibbs cu-states? We intend to answer that question by introducing a "disintegration-free" characterization and so give a cleaner proof of Theorem D.

Theorem 4.1. *Let $f : M \to M$ be a C^r-partially hyperbolic diffeomorphism, $r > 1$, and let μ be a Gibbs u-state such that μ has positive central Lyapunov exponents. Then the following are equivalent:*

(i) μ is a Gibbs cu-state.
(ii) Given any $\epsilon > 0$ and sufficiently large ℓ, there exists $K > 0$ such that

$$\int \varphi \, d\mu < \epsilon + K \cdot \sup_{W \in \mathcal{W}_\ell^{cu}(f)} \int \varphi \, dm_W \tag{4.1}$$

for every continuous function $\varphi : M \to [0, 1]$.

Moreover, for a sufficiently small C^r neighborhood $\mathcal{U}$ of f, the constant K in (ii) depends only on ϵ and ℓ, i.e. is independent of $g \in \mathcal{U}$ and Gibbs u-state in $\mathbb{G}\text{ibbs}^{\text{u}}(g)$.

Another characterization was provided in [28, Theorem A] where the second author proved that an ergodic Gibbs u-state μ with positive central Lyapunov exponents is a Gibbs cu-state if and only if there is a local center-unstable manifold contained (Lebesgue-mod 0) in the basin of μ. Unfortunately, we are not able to apply that characterization here, as we have no means of finding such center-unstable manifold contained in the basin of μ.

Our characterization is rather subtle and logically intricate, but useful. It can be stated in symbolic form like this: $\mu \in \mathbb{G}\text{ibbs}^{\text{cu}}(f) \iff$

$$\forall \epsilon > 0, \ \exists \ell_0 > 0, \ \forall \ell \geq \ell_0, \ \exists K > 0, \ \forall \varphi \in C^0(M, [0,1]), \ \exists W \in \mathcal{W}_\ell^{cu}(f) :$$
$$\int \varphi \, d\mu < \epsilon + K \int \varphi \, dm_W.$$

It is an expression of quantifier rank equal to six and must be dealt with very carefully. We believe that it reflects an inherent intricacy of the notion of Gibbs cu-states (or SRB measures more generally) which is not always appreciated. It also explains why a carefully written proof of convergence of Gibbs cu-states is harder than one may think.

Proof that (i) implies (ii) in Theorem 4.1. Suppose that $\mu \in \mathbb{G}\text{ibbs}^{\text{cu}}(f)$. Fix some $\epsilon > 0$ and choose ℓ_0 as in Proposition 3.4. Fix $\ell \geq \ell_0$ arbitrarily and choose $x_1, \dots, x_k$, $k = k(\ell) \geq 1$, such that $\{B_{\delta(\ell)}(x_i) : i = 1, \dots, k\}$ is a cover of M. Let $\mathcal{Q}_i = \mathcal{Q}(\ell; x_i)$ and $Q_i = Q(\ell; x_i)$ for $1 \leq i \leq k$.

For every $i \in \{1, \dots, k\}$, we denote by μ_i the restriction of μ to Q_i. Denote by $\{\mu_D^i : D \in \mathcal{Q}_i\}$ the family of conditional measures of μ_i with respect to the partition $\mathcal{Q}_i$ and by $\hat{\mu}_i$ the factor measures defined on $\mathcal{Q}_i$.

Let $L = L(\ell) > 0$ be as in the Remark 3.2 such that for every $i \in \{1, \dots, k\}$ and $\hat{\mu}_i$-almost every $D \in \mathcal{Q}_i$, the density of μ_D^i with respect to m_D is bounded above by L. Then

$$\int_D \varphi \, d\mu_D^i \le L \int_D \varphi \, dm_D$$

for $\hat{\mu}_i$-almost every $D \in \mathcal{Q}_i$. Let $S = S(\ell)$ be an upper bound for $\{|\operatorname{vol}_D|^{-1} : D \in \mathcal{Q}_i, 1 \le i \le k\}$ and let $T = T(\ell)$ be an upper bound for $\{|\operatorname{vol}_W| : W \in \mathcal{W}_\ell^{cu}(f)\}$.

On the one hand we have

$$\begin{aligned}\int_{Q_i} \varphi \, d\mu &= \int_{\mathcal{Q}_i} \left(\int_D \varphi \, d\mu_D^i \right) d\hat{\mu}_i(D) \\ &\le \hat{\mu}_i(\mathcal{Q}_i) L \sup_{D \in \mathcal{Q}_i} \int_D \varphi \, dm_D \\ &= \mu(Q_i) L \sup_{D \in \mathcal{Q}_i} \int_D \varphi \, d\frac{\operatorname{vol}_D}{|\operatorname{vol}_D|} \\ &\le LS \sup_{D \in \mathcal{Q}_i} \int \varphi \, d\operatorname{vol}_D . \end{aligned} \tag{4.2}$$

On the other hand, since for every $D \in \mathcal{Q}_i$ there is a $W \in \mathcal{W}_\ell^{cu}(f)$ such that $D \subseteq W$, and the image of φ is contained in the unit interval, we have

$$\begin{aligned}\sup_{D \in \mathcal{Q}_i} \int_{Q_i} \varphi \, d\operatorname{vol}_D &\le \sup_{W \in \mathcal{W}_\ell^{cu}(f)} \int_W \varphi d\operatorname{vol}_W \\ &\le \sup_{W \in \mathcal{W}_\ell^{cu}(f)} \int_W \varphi |\operatorname{vol} W| \frac{d\operatorname{vol}_W}{|\operatorname{vol} W|} \\ &= \sup_{W \in \mathcal{W}_\ell^{cu}(f)} \int_W \varphi |\operatorname{vol} W| \, dm_W \\ &\le T \sup_{W \in \mathcal{W}_\ell^{cu}(f)} \int \varphi \, dm_W . \end{aligned} \tag{4.3}$$

Notice that $\Lambda_\ell(f) \subset Q_1 \cup \ldots \cup Q_k$ and that $\mu(\Lambda_\ell(f)) > 1 - \epsilon$. So from (4.2) and (4.3) we have we obtain,

$$\begin{aligned}\int \varphi \, d\mu &< \epsilon + \sum_{i=1}^k \int_{Q_i} \varphi \, d\mu \\ &\le \epsilon + kLS \sup_{D \in \mathcal{Q}_1 \cup \ldots \cup \mathcal{Q}_k} \int \varphi \, d\operatorname{vol}_D \\ &\le \epsilon + kLST \sup_{W \in \mathcal{W}_\ell^{cu}(f)} \int \varphi \, dm_W \end{aligned}$$

The proof follows by taking $K = kLST$.

For the second part of the statement, we note that $K = K(\ell) > 0$ depends on the cardinality $k = k(\ell) \ge 1$ of the finite covering by balls of radius $\delta(\ell, f) > 0$, and the bounds $S = S(\ell) > 0$ and $T = T(\ell)$ and the bound $L = L(\ell) > 0$ provides in Remark 3.2. All these constants can be chosen uniformly in a neighborhood $\mathcal{U}$ of f, independent of the measure $\mu \in \mathbb{G}\mathrm{ibbs}^{cu}(g)$, $g \in \mathcal{U}$, according to Lemma 3.4 and Theorem 3.5. □

The converse statement in Theorem 4.1 is harder to prove. We need an auxiliary result. Recall that if Q is a lamination bundle with associated partition $\mathcal{Q}$, such that $\mu(Q) > 0$, we define the Borel measure m_Q on Q as

$$\int \varphi\, dm_Q = \int_{\mathcal{Q}} \int_D \varphi\, dm_D d\hat{\mu}(D),$$

for every $\varphi \in C^0(M, \mathbb{R})$.

Lemma 4.2. *Let $Q = Q(\ell, x)$ be a lamination bundle with associated partition $\mathcal{Q} = \mathcal{Q}(\ell, x)$, μ be a Borel measure such that $\mu(Q) > 0$ and $\phi : M \to [0, 1]$ a continuous function. Then, given any $b > 0$ such that $b^2 > \int \phi\, dm_Q$, there exists a continuous function $\xi : M \to [0, 1]$ with $1 - \xi$ supported in $B_{2\delta}(x)$ such that*

i. $\mu(\{x : \xi(x) < 1\}) < b$*; and*
ii. $\int_D \phi \cdot \xi\, dm_D < b$*, for every* $D \in \mathcal{Q}$.

Proof. Let $\mathcal{K}$ be the set of those $D \in \mathcal{Q}$ for which $\int_D \phi\, dm_D \geq b$ and let $K = \bigcup_{D \in \mathcal{K}} D$. By Chebyshev's inequality we have

$$\mu(K) = \hat{\mu}(\mathcal{K}) \leq \frac{1}{b} \int_{\mathcal{K}} \left(\int_D \phi\, dm_D \right) d\hat{\mu}(D) \leq \frac{1}{b} \int \phi\, dm_Q < b.$$

Fix $0 < \tau < 2\delta$ large enough so that $m_D(D \setminus \overline{B_\tau(x)}) < b$ for every $D \in \mathcal{Q}$ and let $K_\tau = K \cap \overline{B_\tau(x)}$. Thus in particular

$$m_D(D \cap K_\tau^c) < b$$

for every $D \in \mathcal{K}$.

It follows from (ii) in Theorem 3.5 that the map $x \mapsto \int \phi\, dm_{D(x)}$ is continuous on Q, where $D = D(x)$ is the element of $\mathcal{Q}$ that contains x. Thus K_τ is closed. Note that $\mu(K_\tau) \leq \mu(K) < b$. Let U be an open neighborhood of K_τ with $\overline{U} \subset B_{2\delta}(x)$ such that $\mu(U) < b$. Such U can always be found due to the regularity of Borel measures. Let $\xi : M \to [0, 1]$ be a continuous function such that $\xi = 0$ on K_τ and $\xi = 1$ on the compliment of U. Then

$$\mu(\{x : \xi(x) < 1\}) \leq \mu(U) < b.$$

We claim that $\int \phi \cdot \xi\, dm_D < b$ for every $D \in \mathcal{Q}$. Indeed, if $D \notin \mathcal{K}$, then

$$\int \phi \cdot \xi\, dm_D \leq \int \phi\, dm_D < b$$

since $0 \leq \xi \leq 1$ and by the definition of $\mathcal{K}$. On the other hand, if $D \in \mathcal{K}$, then

$$\begin{aligned} \int \phi \cdot \xi\, dm_D &\leq \int \xi\, dm_D \\ &= \int_{D \cap K_\tau} \xi\, dm_D + \int_{D \cap K_\tau^c} \xi\, dm_D \\ &\leq 0 + m_D(D \cap K_\tau^c) < b, \end{aligned}$$

since $0 \leq \phi \leq 1$ and ξ vanishes at K_τ. □

Proof that (ii) implies (i) in Theorem 4.1. Suppose that μ is not a Gibbs cu-state. Then there is some measurable set $X \subset M$ with $\mu(X) > 0$ for which the following happens: For every $x \in X$ and every lamination bundle $Q = Q(\ell, \sigma, x)$, with associated partition $\mathcal{Q} = \mathcal{Q}(\ell, \sigma, x)$, we have $m_D(X) = 0$ for $\hat{\mu}$-almost every $D \in \mathcal{Q}$.

Let $\epsilon = \mu(X)/2 > 0$. We must prove that, given any integer $\ell_0 \geq 1$, there exists $\ell \geq \ell_0$ with the property that for every $K > 0$ it is possible to find a continuous function $\varphi : M \to [0, 1]$ such that

$$\int \varphi\, d\mu \geq \epsilon + K \int \varphi\, dm_W, \tag{4.4}$$

for every $W \in \mathcal{W}^{cu}_\ell(f)$. To this end, fix $\ell_0 > 0$ arbitrarily. Thereafter choose $\ell \geq \ell_0$ large enough so that $\mu(\Lambda_\ell(f,\sigma)) > 1-\epsilon$. Choose $x \in M$ such that $\mu(B_\delta(x)\cap X)) > 0$ and fix $Q' = Q(\ell, \sigma^{1/2}; x)$ and $\mathcal{Q}' = \mathcal{Q}(\ell, \sigma^{1/2}; x)$. Moreover, let $\hat{\mu}_{Q'}$ be the factor measure of $\mu_{Q'}$ with respect to the partition $\mathcal{Q}'$ and denote by $m_{Q'}$ the measure $\int_{Q'} m_D \, d\hat{\mu}_{Q'}(D)$.

Fix $K > 0$ arbitrarily and choose $0 < \alpha < \epsilon/3$ such that

$$0 < \alpha < \left(\frac{\mu(X)}{2(K+2)}\right)^2.$$

Note that $\mu_{Q'}|(X \cap B_\delta(x))$ and $m_{Q'}$ are mutually singular measures. Therefore we can find a continuous function $\phi : M \to [0,1]$ such that $\operatorname{supp}\phi \subseteq B_\delta(x)$, satisfying

$$m_{Q'}(X) = 0 \leq \int \phi \, dm_{Q'} < \alpha < \mu(X) - \alpha < \int \phi \, d\mu. \tag{4.5}$$

We can apply Lemma 4.2 to ϕ taking $b = \frac{\mu(X)}{2(K+2)} > 0$, and we find a continuous function $\xi : M \to [0,1]$ with $1-\xi$ supported in $B_{2\delta}(x)$ satisfying the following: From Lemma 4.2 item i, and since $0 \leq \phi \leq 1$ and $0 \leq \xi \leq 1$ we have,

$$\int \phi(1-\xi)\, d\mu \leq \int (1-\xi)\, d\mu \leq \mu(\{x : 1-\xi(x) > 0\}) < b.$$

Combining this last relation with (4.5) we obtain

$$\mu(X) - \alpha < \int \phi \, d\mu \leq \int \phi\xi \, d\mu + \int \phi(1-\xi)\, d\mu < \int \phi\xi\, d\mu + b$$

and since $\alpha < b$, then

$$\left(\frac{K+1}{K+2}\right)\mu(X) = \mu(X) - 2b < \mu(X) - \alpha - b < \int \phi\xi\, d\mu. \tag{4.6}$$

On the other hand, Lemma 4.2, item ii., implies immediately that for every $D \in \mathcal{Q}'$,

$$\int \phi\xi\, dm_D < \frac{\mu(X)}{2(K+2)} \tag{4.7}$$

Now, writing $\varphi = \phi\xi$ and combining (4.6) and (4.7) we have

$$K\int \varphi\, dm_D + \epsilon < K\left(\frac{\mu(X)}{2(K+2)}\right) + \frac{\mu(X)}{2} = \left(\frac{K+1}{K+2}\right)\mu(X) < \int \varphi\, d\mu. \tag{4.8}$$

Let $W \in \mathcal{W}^{cu}_\ell(f)$. If $W \cap B_\delta(x) = \emptyset$, then the right hand side of (4.4) vanishes, so there is nothing to prove. Now suppose that $W \cap B_\delta(x) \neq \emptyset$. From Theorem 3.5 item (iii), we know that $W \subset \Lambda_\ell(f, \sigma^{1/2})$. Therefore, there exists $D \subset \mathcal{Q}'$ such that $W \cap B_{2\delta}(x) \subset D$. In this case (4.4) follows from (4.8). □

4.2. *Concluding the proof of Theorem D*

Everything done so far in section 4 (briefly speaking, our Pliss-like Lemma and our characterization of Gibbs *cu*-states) have been for the purpose of proving Theorem D. The proof is based on the observation that the quantities ℓ and K in Theorem 4.1 are uniform in a neighborhood of f.

Let f be a C^r mostly expanding diffeomorphism, $r > 1$. Consider a sequence $(f_n, \mu_n) \in \mathbb{G}\mathrm{ibbs}^{\mathrm{cu}}(\mathcal{U}_{\mathcal{ME}})$ such that f_n converges in C^r to some mostly expanding diffeomorphism f and μ_n converges weakly* to some measure μ. To prove Theorem D we must establish that μ is a Gibbs *cu*-state. We do that by showing that the inequality (4.1) passes to the limit. The following straightforward lemma is useful.

Lemma 4.3. *Let $f : M \to M$ be a C^r, $r > 1$, mostly expanding diffeomorphism, $\ell \in \mathbb{N}$, and suppose that f_n is a sequence of C^r mostly expanding diffeomorphisms converging to f. Then any accumulation point of $\Lambda_\ell(f_n)$ belongs to $\Lambda_\ell(f)$. That is,*

$$\bigcap_n \overline{\bigcup_{k\geq n} \Lambda_\ell(f_k)} \subset \Lambda_\ell(f).$$

Proof. Let us fix $\ell \geq 1$ and $N \geq 1$. Then, by continuity of $(x, f) \to \|Df^{-\ell}|E^{cu}(y)\|$, for $\eta > 0$ there exist a neighborhood $\mathcal{U}_N$ of f, and $\epsilon_N > 0$ such that if $g \in \mathcal{U}_N$ and $x, y \in M$ satisfy $\operatorname{dist}(x, y) < \epsilon_N$, then

$$\prod_{j=0}^{N-1} \|Df^{-\ell}|E^{cu}_{f^{-\ell j}(x)}\| \prod_{j=0}^{N-1} \|Dg^{-\ell}|E^{cu}_{g^{-\ell j}(y)}\|^{-1} < 1+\eta. \tag{4.9}$$

Since $f_n \to f$, we can assume that there exists $n \geq 1$ such that, for every $k \geq n$, $f_k \in \mathcal{U}_N$. If we assume that $x \in \overline{\bigcup_{k\geq n} \Lambda_\ell(f_n)}$, for every $n \geq 1$, then for $\epsilon_N > 0$ there exist $k \geq n$ and $y \in \Lambda_\ell(f_k)$ such that $\operatorname{dist}(x, y) < \epsilon_N$. Since $y \in \Lambda_\ell(f_k)$,

$$\prod_{j=0}^{N-1} \|Df_k^{-\ell}|E^{cu}_{f_k^{-\ell j}(y)}\| \leq \sigma^{\ell N}. \tag{4.10}$$

On the other hand, since $\operatorname{dist}(x, y) < \epsilon_N$, it follows from (4.9) and (4.10) that

$$\prod_{j=0}^{N-1} \|Df^{-\ell}|E^{cu}_{f^{-\ell j}(x)}\| \leq (1+\eta)\sigma^{\ell N}. \tag{4.11}$$

Since $\eta > 0$ and $N \geq 1$ are arbitrary, we conclude that $x \in \Lambda_\ell(f)$ as claimed. □

Ending the proof of Theorem D. Fix $\epsilon > 0$ arbitrarily and let $\ell_0 \geq 1$ be as in Lemma 3.4. Fix $\ell \geq \ell_0$ arbitrarily. Since f_n converges to f, we have $\mu_n(\Lambda_\ell(f_n)) > 1-\epsilon$ for sufficiently large n, say $n \geq n_0$. Also, we fix $r = r(\ell) > 0$ and $\delta(\ell) > 0$ as in Theorem 3.5.

According to Theorem 4.1, it suffices to prove that there exists $K > 0$ such that, given any continuous function $\varphi : M \to [0, 1]$, we have

$$\int \varphi \, d\mu_n < \epsilon + K \sup_{W \in \mathcal{W}^{cu}_\ell(f)} \int \varphi \, dm_W.$$

For every $x \in M$, consider the sets

$$\mathcal{Q}_n(\ell, x) = \{W^{cu}_r(y) \cap B_{2\delta}(x) : y \in \Lambda_\ell(f_n) \cap \overline{B_\delta(x)}\}$$

and

$$Q_n(\ell, x) = \bigcup_{D \in \mathcal{Q}_n(\ell,x)} D.$$

Choose points $x_1, \ldots, x_k$, $k = k(\ell) > 0$, so that $M \subset B_\delta(x_1) \cup \ldots \cup B_\delta(x_k)$. Write $Q^i_n = Q^i_n(\ell, x_i)$ and $\mathcal{Q}^i_n = \mathcal{Q}^i_n(\ell, x_i)$ for $1 \leq i \leq k$ and denote by μ^i_n the restriction of μ_n to Q^i_n. Denote by $\{\mu^i_{n,D} : D \in \mathcal{Q}^i_n\}$ the family of conditional measures of μ^i_n with respect to the partition $\mathcal{Q}^i_n$ and by $\hat{\mu}^i_n$ the factor measures defined on $\mathcal{Q}^i_n$.

Let $L > 1$ be such that, for every $n \geq n_0$, every $i \in \{1, \ldots, k\}$ and $\hat{\mu}^i_n$-almost every $D \in \mathcal{Q}^i_n$, the density of $\mu^i_{n,D}$ with respect to m_D is bounded above by L. Let $K = kL$. Now consider any continuous function $\varphi : M \to [0, 1]$. For every $n > n_0$ and every $i \in \{1, \ldots, k\}$ we have

$$\int_{Q^i_n} \varphi \, d\mu_n = \int_{\mathcal{Q}^i_n} \left(\int \varphi \, d\mu^i_{n,D} \right) d\hat{\mu}^i_n(D) \leq L\mu(Q^i_n) \sup_{D \in Q^i_n} \int \varphi \, dm_D$$

Notice that $\mu_n(Q^i_n) \geq \mu_n(\Lambda_\ell(f_n) \cap B_\delta(x_i))$ for every $1 \leq i \leq k$ and every $n \geq n_0$. Since $M = B_\delta(x_1) \cup \ldots \cup B_\delta(x_k)$, this gives us the estimate $\mu_n(Q^1_n \cup \ldots \cup Q^k_n) > 1-\epsilon$. It follows that

$$\int \varphi \, d\mu_n < \epsilon + \sum_{i=1}^{k} L\mu(Q^i_n) \sup_{D \in \mathcal{Q}^i_n} \int_D \varphi \, dm_D \leq Lk \max_{1\leq i\leq k} \sup_{D \in \mathcal{Q}^i_n} \int \varphi \, dm_D.$$

For every $n > n_0$, choose a disk $D_n \in \mathcal{Q}^1_n \cup \ldots \cup \mathcal{Q}^k_n$ such that

$$\int \varphi \, dm_{D_n} = \max_{1 \le i \le k} \sup_{D \in \mathcal{Q}_n^i} \int \varphi \, dm_D.$$

Each disk D_n is contained in some $W_n \in \mathcal{W}_\ell^{cu}(f_n)$. By Lemma 4.3, W_n accumulates on some disk $W \in \mathcal{W}_\ell^{cu}(f)$. Therefore

$$\int \varphi \, d\mu \le \epsilon + kL \int \varphi \, dm_W.$$

In particular,

$$\int \varphi \, d\mu < \epsilon + K \cdot \sup_{W \in \mathcal{W}_\ell^{cu}(f)} \int \varphi \, dm_W,$$

where $K = kL + 1$ is independent of φ. □

5. Proof of main theorems

We are now in a position to prove Theorems A, B, and C.

5.1. Proof of Theorem C

Let f be mostly expanding. By Theorem 1.1 there are finitely many physical measures $\nu_1, \dots \nu_k$. By the definition of weakly statistically stability, it suffices to prove that given any sequence of diffeomorphisms f_n converging to f in the C^r topology, $r > 1$, and any sequence of physical measures μ_n of f_n, if μ is an accumulation point of μ_n, then μ is a convex combination of $\nu_1, \dots \nu_k$.

Let f_n be a sequence of diffeomorphisms converging to f in the C^r topology for some $r > 1$. We know from Theorem 1.1 that mostly expanding is an open condition and we can therefore suppose that each f_n is mostly expanding. Let μ_n be a sequence of physical measures for f_n respectively. Proposition 2.6 implies that each μ_n is a Gibbs cu-state. Upon possibly taking a subsequence, we may suppose that the sequence μ_n converges to some measure μ. Theorem D tells us that μ is a Gibbs cu-state. We know that (Proposition 2.5) Gibbs cu-states are convex combinations of ergodic Gibbs cu-states and, again by Proposition 2.6, that ergodic Gibbs cu-states are physical measures. It follows that μ is a convex combination of physical measures of f. This completes the proof of Theorem C.

5.2. Proof of Theorem B

The proof of upper semi-continuity of the number of physical measures is by contradiction. Suppose, therefore, that upper semi-continuity on the number of physical measures does not hold. That means that there exists some mostly expanding diffeomorphism f, and a sequence $f_n \to f$ of mostly expanding diffeomorphisms converging to f in the C^r topology, all of which have a number of physical measures larger than that of f. In other words, if the physical measures of f are μ^j, $j \in J$ for some finite set J, there are measures ν_n^i, $i \in I$ for some finite set I with $|I| = |J| + 1$ such that

(i) ν_n^i is a physical measure for f_n for every n,
(ii) $\nu_n^i \neq \nu_n^{i'}$ for every n and every $i, i' \in I$, with $i \neq i'$.

Upon taking an appropriate subsequence, we may also assume from Theorem C that

(iii) for each $i \in I$, there exist non-negative numbers $\alpha_{i,j}$, $j \in J$ with $\sum_{j \in J} \alpha_{i,j} = 1$, such that $\nu_n^i \to \sum_{j \in J} \alpha_{i,j} \mu^j$.

To get a contradiction, we shall prove that each column in the matrix $(\alpha_{i,j})$ can have at most one positive element. Since the number of rows is larger than the number of columns, this implies that $(\alpha_{i,j})$ must have a row of zeros, contradicting $\sum_{j \in J} \alpha_{i,j} = 1$.

To see why each column of $\alpha_{i,j}$ can have at most one positive element, let $P = \{(i, j) \in I \times J : \alpha_{i,j} > 0\}$ and $\alpha = \min\{\alpha_{i,j} : (i, j) \in P\}$. Recall that each point in the Pesin block $\Lambda_\ell(f_n)$ has an unstable manifold of a fixed size

$r = r(\ell) > 0$. Moreover, from Lemma 3.3 we know that it is possible to choose ℓ such that $\nu_n^i(\Lambda_\ell(f_n)) > 1 - \alpha/2$ for every large n.

The angle between E^s and E^{cu} is bounded away from zero in a robust manner. Hence there is some $\rho > 0$ such that for every large n, and any point $x \in \Lambda_\ell(f_n)$, the set $\Gamma(f_n, \ell, x) = \bigcup_{y \in W_r^{cu}(x)} W^s(f_n, y)$ contains the ball $B_\rho(x)$.

We cover the supports of μ^j by balls $B_{\rho/2}(x_{j,k})$, $k \in K$, where K is some finite set. For sufficiently large n we have

(iv) $\nu_n^i(B_{\rho/2}(x_{j,k})) > 0$ for every $(i, j, k) \in P \times K$, and
(v) $\nu_n^i(\Lambda_\ell(f_n) \cap B_{\rho/2}(x_{j,k})) > 0$ for every $(i, j) \in P$ and some $k \in K$.

Thus given any $i \in I$ choose $j \in J$ such that $\alpha_{i,j} > 0$ and $k \in K$ such that (v) holds. Since ν_n^i is an ergodic Gibbs cu-state, there is some $x \in B_{\rho/2}(x_{j,k})$ such that $B(\nu_n^i)$ has full leaf volume in $W_r^{cu}(x)$. Therefore, by absolute continuity of the stable foliation, $B(\nu_n^i)$ has full volume in $\Gamma(f_n, \ell, x)$. By our choice of ρ, we have $\Gamma(f_n, \ell, x) \supset B_\rho(x) \supset B_{\rho/2}(x_{j,k})$ so, in particular, $B(\nu_n^i)$ has full volume in $B_{\rho/2}(x_{i,k})$.

Now take any $i' \in I$ different from i. We claim that $(i', j) \notin P$. Indeed, if it were not so, then by (iv) we would have $\nu_n^{i'}(B_{\rho/2}(x_{j,k})) > 0$ for sufficiently large n. Therefore, there would be some ℓ' and some $x' \in B_{\rho/2}(x_{j,k})$ such that $B(\nu_n^{i'})$ has full leaf volume in $W_{r(\ell')}^{cu}$. Again, by absolute continuity of the stable foliation, that would imply that $B(\nu_n^{i'})$ has positive volume in $B_{\rho/2}(x_{j,k})$. But that is absurd, since $B(\nu_n^i)$ has full volume in $B_{\rho/2}(x_{j,k})$ and basins of distinct physical measures are disjoint. Thus we have proved that each column in the matrix $(\alpha_{i,j})$ has at most one non-zero entry and the proof of upper semi-continuity of the number of physical measures is complete.

It remains to prove statistical stability in its most general setting. To do that, we consider a sequence f_n of mostly expanding diffeomorphisms converging to a mostly expanding diffeomorphism f such that each f_n and f all have the same number of physical measures. We use the notation above, so that the physical measures of f are μ^j, $j \in J$ and those of f_n are ν_n^i, $i \in I$. The difference now is that $|I| = |J|$. By taking subsequences we may assume that $\nu_n^i \to \sum_{j\in J} \alpha_{i,j}\mu^j$ for some non-negative numbers $\alpha_{i,j}$ with $\sum_{j\in J} \alpha_{i,j} = 1$. It was proved above that in this case, each column of the $|I| \times |J|$ matrix $A = (\alpha_{i,j})_{(i,j)\in I\times J}$ has at most one positive element. Now, A is a square matrix and the sum of the entries in each row is 1. In particular each row has at least one positive entry. Therefore A must be a permutation matrix, i.e. one for which each column and each row has exactly one entry equal to 1 and all other entries are zero. Define the map $\tau : I \to J$ so that $\tau(i)$ is the unique element of A such that $\alpha_{i,j} = 1$. Then ν_i^n converges to $\mu_{\tau(i)}$ for every $i \in I$. That completes the proof of statistical stability.

5.3. *Proof of Theorem A*

Let $f : M \to M$ be as in Theorem A. Then Theorem 1.1 says that f has a finite number of physical measures, whose basin of attraction cover Lebesgue almost every point in M. Now, since f is mostly expanding, the basin of each physical measure is open, up to a zero Lebesgue measure set (see e.g. [8, Lemma 4.5]). Thus it follows from the assumption of transitivity that f has exactly one physical measure. Now, according to Theorem B, the number of physical measures varies upper semi-continuously on f. Hence there is a C^r neighborhood $\mathcal{U}$ of f such that every $g \in \mathcal{U}$ has a unique physical measure μ_g whose basin has full Lebesgue measure in M. Moreover, Theorem B implies that the map $\mathrm{Diff}^r(M) \ni g \mapsto \mu_g \in \mathbb{M}^1(M)$ is indeed continuous.

Declaration of competing interest

There is no competing interest.

Acknowledgements

We thank the anonymous referees for their thorough review and highly appreciate their comments and suggestions, which significantly contributed to improving the quality of the publication.

References

[1] Flavio Abdenur, Christian Bonatti, Sylvain Crovisier, Nonuniform hyperbolicity for c1-generic diffeomorphisms, Isr. J. Math. 183 (1) (Jun 2011) 1.
[2] José F. Alves, Strong statistical stability of non-uniformly expanding maps, Nonlinearity 17 (4) (2004) 1193–1215.
[3] José F. Alves, Christian Bonatti, Marcelo Viana, SRB measures for partially hyperbolic systems whose central direction is mostly expanding, Invent. Math. 140 (2) (2000) 351–398.
[4] José F. Alves, Maria Carvalho, Jorge Milhazes Freitas, Statistical stability and continuity of SRB entropy for systems with Gibbs-Markov structures, Commun. Math. Phys. 296 (3) (2010) 739–767.
[5] José F. Alves, Marcelo Viana, Statistical stability for robust classes of maps with non-uniform expansion, Ergod. Theory Dyn. Syst. 22 (1) (2002) 1–32.
[6] José Ferreira Alves, SRB measures for non-hyperbolic systems with multidimensional expansion, Ann. Sci. Éc. Norm. Supér. (4) 33 (1) (2000) 1–32.
[7] Martin Andersson, Robust ergodic properties in partially hyperbolic dynamics, Trans. Am. Math. Soc. 362 (4) (2010) 1831–1867.
[8] Martin Andersson, Carlos H. Vásquez, On mostly expanding diffeomorphisms, Ergod. Theory Dyn. Syst. 38 (8) (2018) 2838–2859.
[9] Artur Avila, Jairo Bochi, Nonuniform hyperbolicity, global dominated splittings and generic properties of volume-preserving diffeomorphisms, Trans. Am. Math. Soc. 364 (6) (2012) 2883–2907.
[10] Luis Barreira, Yakov Pesin, Nonuniform Hyperbolicity: Dynamics of Systems with Nonzero Lyapunov Exponents, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2007.
[11] Christian Bonatti, Lorenzo J. Díaz, Marcelo Viana, Dynamics beyond uniform hyperbolicity. A global geometric and probabilistic perspective, in: Mathematical Physics, III, in: Encyclopaedia of Mathematical Sciences, vol. 102, Springer-Verlag, Berlin, 2005.
[12] Christian Bonatti, Marcelo Viana, SRB measures for partially hyperbolic systems whose central direction is mostly contracting, Isr. J. Math. 115 (2000) 157–193.
[13] Keith Burns, Dmitry Dolgopyat, Yakov Pesin, Mark Pollicott, Stable ergodicity for partially hyperbolic attractors with negative central exponents, J. Mod. Dyn. 2 (1) (2008) 63–81.
[14] Sylvain Crovisier, Enrique R. Pujals, Essential hyperbolicity and homoclinic bifurcations: a dichotomy phenomenon/mechanism for diffeomorphisms, Invent. Math. 201 (2) (2015) 385–517.
[15] Dmitry Dolgopyat, Limit theorems for partially hyperbolic systems, Trans. Am. Math. Soc. 356 (4) (2004) 1637–1689.
[16] Dmitry Dolgopyat, On differentiability of SRB states for partially hyperbolic systems, Invent. Math. 155 (2) (2004) 389–449.
[17] Dmitry Dolgopyat, Marcelo Viana, Jiagang Yang, Geometric and measure-theoretical structures of maps with mostly contracting center, Commun. Math. Phys. 341 (3) (2016) 991–1014.
[18] Boris Hasselblatt, Yakov Pesin, Partially hyperbolic dynamical systems, in: Handbook of Dynamical Systems, Vol. 1B, Elsevier B. V., Amsterdam, 2006, pp. 1–55.
[19] M.W. Hirsch, C.C. Pugh, M. Shub, Invariant Manifolds, Lecture Notes in Mathematics, vol. 583, Springer-Verlag, Berlin-New York, 1977.
[20] Ittai Kan, Open sets of diffeomorphisms having two attractors, each with an everywhere dense basin, Bull. Am. Math. Soc. (N.S.) 31 (1) (1994) 68–74.
[21] Ricardo Mañé, Ergodic Theory and Differentiable Dynamics, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) (Results in Mathematics and Related Areas (3)), vol. 8, Springer-Verlag, Berlin, 1987, translated from the Portuguese by Silvio Levy.
[22] Zeya Mi, Yongluo Cao, Dawei Yang, A note on partially hyperbolic systems with mostly expanding centers, Proc. Am. Math. Soc. 27 (2) (2017) 589–607.
[23] Ya.B. Pesin, Ya.G. Sinaĭ, Gibbs measures for partially hyperbolic attractors, Ergod. Theory Dyn. Syst. 2 (3–4) (1982) 417–438.
[24] V.A. Pliss, On a conjecture of Smale, Differ. Uravn. 8 (1972) 268–282.
[25] Charles Pugh, Michael Shub, Ergodic attractors, Trans. Am. Math. Soc. 312 (1) (1989) 1–54.
[26] Michael Shub, Amie Wilkinson, Pathological foliations and removable zero exponents, Invent. Math. 139 (3) (2000) 495–508.
[27] Carlos H. Vásquez, Statistical stability for diffeomorphisms with dominated splitting, Ergod. Theory Dyn. Syst. 27 (1) (2007) 253–283.
[28] Carlos H. Vásquez, Stable ergodicity for partially hyperbolic attractors with positive central Lyapunov exponents, J. Mod. Dyn. 3 (2) (2009) 233–251.
[29] Marcelo Viana, Jiagang Yang, Physical measures and absolute continuity for one-dimensional center direction, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 30 (5) (2013) 845–877.
[30] Jiagang Yang, Entropy along expanding foliations, arXiv e-prints, 2018.
[31] Lai-Sang Young, Statistical properties of dynamical systems with some hyperbolicity, Ann. Math. (2) 147 (3) (1998) 585–650.
[32] Lai-Sang Young, Recurrence times and rates of mixing, Isr. J. Math. 110 (1999) 153–188.