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Fokker-Planck equations of jumping particles and mean field games of impulse control

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Abstract

This paper is interested in the description of the density of particles evolving according to some optimal policy of an impulse control problem. We first fix the sets from which the particles jump and explain how we can characterize such a density. We then investigate the coupled case in which the underlying impulse control problem depends on the density we are looking for: the mean field game of impulse control. In both cases, we give a variational characterization of the densities of jumping particles.

Keywords: Partial differential equations; Fokker-Planck equations; Impulse control; Mean field games

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1. Introduction

1.1. General introduction

This paper is the second of a series devoted to the systematic study of mean field games (MFG for short) with optimal stopping or impulse control. In [5] we developed an obstacle problem approach to solve a forward-backward system which models MFG with optimal stopping (without common noise). We here develop the same point of view for MFG with impulse control. As the definition of a Fokker-Planck equation associated with a density of players playing an impulse control problem is a difficult question in itself, it is the subject of the first part of this article. This part is independent from the MFG theory. The case of the master equation (i.e. when there is a common noise) will be treated in a subsequent work.

MFG model situations in which a continuum of indistinguishable players are playing a game in which they interact only through mean field terms. The evolution of the density of players is induced by the optimal choices the players make. In many situations, the costs involved in the game depend only on the density of players. Denoting by u the value function of a generic player and by m the density of players, a classical forward-backward MFG system during the time interval $(0, T)$ is

$$\begin{cases} -\partial_t u - \nu\Delta u + H(x, \nabla u) = f(m), \\ \partial_t m - \nu\Delta m - div(D_p H(x, \nabla u)m) = 0, \\ m(0) = m_0; u(T) = g(m(T)), \end{cases}$$

where $H(x, p)$ is the hamiltonian of a continuous optimal control problem, m_0 is the initial condition for the density of players, g and f are respectively the terminal and running costs and $\nu > 0$ characterizes the intensity of the individual noises. A solution (u, m) of this system corresponds to a Nash equilibrium for the game with an infinite number of players. This system, as well as MFG, have been introduced in [28–30] by J.-M. Lasry and P.-L. Lions. In these papers they proved general conditions under which the existence and uniqueness hold for this problem. Such games have been introduced independently in [23]. We also refer to [31,8] for more results on this system. MFG models have a wide range of applications, see [2,20,21] for examples. Many interesting questions have also been raised around this system, we can cite for example long time average [10], learning [9], the difficult problem of the convergence of the system of N players as N goes to infinity and the presence of a common noise [11]. Numerical methods are also being developed, let us cite [1,6] for instance. Let us also mention that a powerful probabilistic point of view on MFG has been developed, we refer to [14,13,26] for more details on this point of view. In this paper, we generalize the results of the existence and uniqueness of the previous system to the case in which the players face an impulse control problem. Concerning closely related works, several optimal control problems, related to the impulse control problem, have been studied in a MFG setting. Optimal stopping or obstacle problems have been studied in [5,18,32,15], singular controls in [16,22] and optimal switching in [19]. More recently, an impulse control problem in a MFG setting has been studied in [3]. The question of modeling a population impulsively controlled has also been studied in a particular setting in [33].

1.2. Impulse control problems

Impulse control problems have been studied since the 70s. We refer to the work of A. Bensoussan and J.-L. Lions (see [4]) for a complete presentation of the problem. The terminology impulse control refers to an optimization problem in which the state is driven by a stochastic ordinary differential equation with jumps:

$$\begin{cases} \forall t \in (\tau_i, \tau_{i+1}), dX_t = \sqrt{2\nu} dW_t, \\ X_{\tau_i^+} = X_{\tau_i^-} + \xi_i, \end{cases} \tag{1}$$

where $(W_t)_{t\geq 0}$ is a brownian motion under a standard probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The jump ξ_i occurs at time τ_i and is controlled by the player. The jumps are characterized by the random sequence of stopping times $(\tau_i)_{i\geq 0}$ and the random sequence of jumps $(\xi_i)_{i\geq 0}$. Those two sequences are the controls and are adapted to the brownian motion $(W_t)_{t\geq 0}$ in the sense that the sequence $(\tau_i)_{i\geq 0}$ is indeed a sequence of stopping times for this brownian motion and that (ξ_i, τ_i) is measurable with respect to the σ-algebra generated by $(W_t)_{0\leq t\leq \tau_i}$. We assume that $(\xi_i)_{i\geq 0}$ is valued in a set $K \subset \mathbb{T}^d$. Denoting by f the running cost and by $k(x, \xi)$ the cost paid to use the jump ξ while on the position x, we define the value function u by

$$u(t,x) := \inf_{(\tau_i)_i, (\xi_i)_i} \mathbb{E}\left[\int_t^T f(s, X_s) ds + \sum_{i=1}^{\#(\tau_j)_j} k(X_{\tau_i^-}, \xi_i) \right], \tag{2}$$

where the infimum is taken over all sequences $((\tau_i)_{i\geq 0}, (\xi_i)_{i\geq 0})$ which are adapted to the brownian motion in the sense prescribed above and which satisfy the fact that $(\tau_i)_{i\geq 0}$ is an increasing sequence and that $(\xi_i)_{i\geq 0}$ is valued in K. The trajectory $(X_s)_{s\geq t}$ is given by (1) with the initial condition $X_t = x$. Under some assumptions on the costs, the value function u satisfies, in the sense of quasi variational inequality (QVI), the Hamilton-Jacobi-Bellman equation:

$$\begin{cases} \max(-\partial_t u - \nu\Delta u - f, u(t,x) - \min_{\xi\in K}\{k(x,\xi) + u(t, x+\xi)\}) = 0, \\ u(T) = 0. \end{cases}$$

1.3. The density of players

In a MFG context, the main question we are addressing is how will evolve an initial density of players, if those players are facing the same impulse control problem. Intuitively, the density of players m has to satisfy (formally) at least some requirements:

- $\partial_t m - \nu\Delta m = 0$ where it is optimal for the players to wait and not to jump and where no player is arriving.
- $m = 0$ where it is strictly suboptimal not to jump.
- The flux of arriving players at x is equal to the sum over ξ of the flows of players which choose to use the jump ξ at $x - \xi$.

Let us note that, at least formally, we talk about parts of the space on which it is optimal to jump (i.e. to use a control to make the process $(X_t)_t$ jumps) because all the players being indistinguishable, if it is optimal for one player to jump, then it is optimal for all the players to jump.

The problem of finding a density m which satisfies the above requirements is not classical, mostly because there is no particular assumption on how the players use their controls. We focus on the problem of modeling the evolution of a density of jumping players in the first part of this article. We build a dual characterization of the solution of the Fokker-Planck "equation". We fix a function $V(t, x, \xi)$ which describes whether or not the players use the jump ξ at the position x and time t. Then we construct a density of players m which satisfies the required properties and thus solves a Fokker-Planck equation of jumping particles. The characterization of this density relies on the fact that we can interpret such Fokker-Planck equations as dual equations of QVI. The construction of such a solution uses a penalized version of the problem in which we can write rigorously the PDE satisfied by the density m. We then find a priori estimates which allow us to pass to the limit in this penalized version of the problem, while obtaining a characterization of the limit density. We also give results concerning the stationary case.

In the second part of this article, we present results on the uniqueness and existence for the impulse control problem in MFG. We also recall that in view of the results of [5] we expect the solutions of the MFG system to be mixed solutions, meaning that optimal strategies are random in general as the Nash equilibria of the MFG can be mixed equilibria. We end this second part by giving results on the stationary case and on the optimal control interpretation of such MFG.

We end this introduction with some notations.

- For $1 < p < \infty$, p' denotes the conjugate exponent $p' = \frac{p}{p-1}$.
- For $1 \leq p \leq \infty$, $W^{1,2,p} := \{v \in L^p((0,T) \times \mathbb{T}^d), \|\partial_t v\|_{L^p} + \|D^2_{xx} v\|_{L^p} < \infty\}$.
- For $k \geq 1$, $1 \leq p \leq \infty$, $W^{k,p}$ is the usual Sobolev space of functions on $\mathbb{T}^d$. We denote $W^{k,2}$ by H^k.
- We define $H := L^2((0,T), H^1(\mathbb{T}^d)) \cap H^1((0,T), H^{-1}(\mathbb{T}^d))$.
- The cardinal of a set E is denoted by $|E|$.

Part 1. Fokker-Planck equation of jumping particles

In this part we present the variational formulation of the Fokker-Planck equation satisfied by the density of jumping particles. We work in the d dimensional torus (denoted by $\mathbb{T}^d$) in a time dependent setting, except for the last section which is concerned with the stationary case. The positive real number T is the final time and $m_0 \in L^2(\mathbb{T}^d)$ is the initial density at time $t = 0$. The aim of this part is to construct a suitable notion to characterize densities of jumping particles. By contrast with Fokker-Planck equations of jumping processes, we do not want to model populations of particles which are driven by Poisson processes or other jump processes of the sort. We are interested in particles which are driven by diffusions and which can jump but not in particles which are driven by Poisson processes. If a unique jump ξ is possible, we are interested in building solutions of

$$\begin{cases} \partial_t m_\epsilon - \nu \Delta m_\epsilon + \frac{1}{\epsilon}\mathbb{1}_A m_\epsilon - \frac{1}{\epsilon}(\mathbb{1}_A m_\epsilon)(t, x-\xi) = 0, \\ m(0) = m_0, \end{cases} \tag{3}$$

for $\epsilon > 0$ and passing to the limit $\epsilon \to 0$. The interpretation of (3) is that the particles make a jump of ξ if they are in the set A with a probability given by an exponential law of parameter ϵ^{-1}. The interpretation of the limit m of solutions of (3) is that it describes particles evolving only along brownian trajectories in A^c and which make a jump of ξ once they reach A. If A is a smooth closed set such that the reaching time of ∂A is well defined, then the trajectory $(X_s)_{s\geq 0}$ of a generic particle is defined by

$$\begin{cases} dX_s = \sqrt{2\nu} dW_s, \forall i, \forall s \in (t_i, t_{i+1}), \\ X_{t_i^+} = X_{t_i^-} + n(t_i, X_{t_i^-})\xi, \end{cases}$$

where $n(t,x)$ is the smallest integer p such that $(t, x + p\xi) \notin A$ and where t_{i+1} is the stopping time defined by the reaching time of A by the process $(X_s)_{s \geq t_i}$. We recall that this interpretation is given in the case in which a unique jump is possible; a similar interpretation also exists in the case of a finite number of possible jumps.

Finding solutions of the penalized equation (3) does not require new techniques and is not a difficult question in itself. The majority of this part is concerned with building a priori estimates uniform in ϵ (Lemma 1.4) to pass to the limit $\epsilon \to 0$. Even though we use these uniform a priori estimates to prove existence at a penalized level, such estimates are crucial only to prove the existence of a limit as ϵ goes to 0 (Theorem 1.5).

As explained in the introduction, we shall characterize the solution of a Fokker-Planck equation (the limit density) of jumping particles with dual properties and not with a PDE. The main duality idea of this part is that a Fokker-Planck equation of jumping particle is in some sense the dual or adjoint equation of a QVI, which describes how it is optimal to jump and thus dynamics of jumping particles. QVI are crucial for the study of the density of jumping particles. Hence we start this section with a result on QVI. We then study the simpler case of a unique possible jump before addressing the case of a finite number of possible jumps. In each of these situations, we begin by constructing a penalized version of the problem and we then pass to the limit in the resulting penalized equation.

2. Quasi-variational inequalities

As stated above, QVI are the natural tools to study optimal control problems of systems driven by dynamics of the form of (1) associated with a cost of jump k and a running cost f. We refer to the book of A. Bensoussan and J.-L. Lions [4] for a more complete presentation of the problem We define here the notion of a smooth cost of jumps k. A function k is said to be a smooth cost of jumps if it satisfies:

$$\begin{cases} \forall \xi \in K, k(\cdot,\xi) \in W^{2,\infty}(\mathbb{T}^d), \\ k^* : x \to \min_{\xi \in K} k(x,\xi) \in W^{2,\infty}(\mathbb{T}^d), \\ \exists k_0 > 0 \text{ such that } k \geq k_0, \end{cases} \tag{4}$$

where K is a finite subset of $\mathbb{T}^d$. In particular let us note that we do not consider 0 as a jump. For the rest of this section, we fix such a function k. The interpretation of $k(x,\xi)$ is that it is the cost paid by the player (or the energy used by a particle) to instantaneously go from x to $x+\xi$. We also define the operator M which plays an important role in the study of QVI by:

$$M(k,u)(t,x) = \min_{\xi \in K}\{k(x,\xi) + u(t,x+\xi)\}. \tag{5}$$

When there is no ambiguity on k, we shall write only $Mu = M(k,u)$.

We now present results of existence concerning QVI for a sufficiently large class of source terms. Such results are crucial to establish some regularity for the solution of the adjoint problem we are interested in: the Fokker-Planck equation of jumping particles. Before stating our result, let us mention that in general, to establish existence result for the PDE

$$\begin{cases} \max(-\partial_t u - \nu\Delta u - f, u - M(k,u)) = 0 \text{ in } (0,T)\times\mathbb{T}^d, \\ u(T) = 0 \text{ in } \mathbb{T}^d, \end{cases} \tag{6}$$

one has to look at u being a solution of the QVI:

$$\begin{cases} u \leq Mu \text{ a.e. in } (0,T)\times\mathbb{T}^d, \\ \forall v \in L^2((0,T),H^1(\mathbb{T}^d)), v \leq Mu, v(T) = 0, \\ -\int_0^T\int_{\mathbb{T}^d}\partial_t u(v-u) + \nu\int_0^T\int_{\mathbb{T}^d}\nabla u\cdot\nabla(v-u) \geq \int_0^T\int_{\mathbb{T}^d} f(v-u), \\ u(T) = 0 \text{ in } \mathbb{T}^d. \end{cases} \tag{7}$$

For an arbitrary dimension d and $f \in L^2$, (7) is not generally solvable. This is a consequence of the fact that if f is not bounded from below, we cannot expect in general u to be bounded from below. Indeed in such a case, it is unclear in which sense the condition $u \leq Mu$ is understood. Usually a solution of (7) is built as the limit of the decreasing sequence $(u_n)_{n\in\mathbb{N}}$ defined by:

$$\begin{cases} \max(-\partial_t u_{n+1} - \nu\Delta u_{n+1} - f, u_{n+1} - Mu_n) = 0 \text{ in } (0,T)\times\mathbb{T}^d, \\ u_{n+1}(T) = 0 \text{ in } \mathbb{T}^d, \end{cases} \tag{8}$$

with the convention $u_{-1} = +\infty$. The obstacle problem (8) is understood in the sense of variational inequalities.

In the study of QVI, estimates on the sequence $(u_n)_{n\in\mathbb{N}}$ often follow from the existence of a uniform lower bound v. Let us note that if $f \geq 0$, then $u_n \geq 0$ for all $n \in \mathbb{N}$ so we can choose $v = 0$. More generally if f is bounded from below by a constant $-C$ then for all $n \in \mathbb{N}$ we deduce that $u_n(t,x) \geq -Ct$ for all $(t,x) \in (0,T)\times\mathbb{T}^d$ and $v = -Ct$ is a lower bound for $(u_n)_{n\in\mathbb{N}}$.

We now state two results concerning unbounded and irregular cost term f. Those results of existence of solutions of weak QVI for a sufficiently large class of f are crucial for the analysis of the Fokker-Planck equation of jumping particles. The first result is rather simple and, however helpful in this context, is not of particular interest in the study of QVI. (Let us recall that H is defined at the end of the introduction.)

Proposition 1.1. *There exists $C > 0$ depending only on k_0 and d such that for any $f \in L^p((0,T)\times\mathbb{T}^d)$ with $p > (1+\frac{d}{2})$, $\|f\|_{L^p} \leq C$, there exists $u \in L^2((0,T),H^1(\mathbb{T}^d))$ solution of the QVI:*

$$\begin{cases} u \le Mu, \\ \forall v \in H, v \le Mu, \\ -\int_0^T \int_{\mathbb{T}^d} \partial_t u(v-u) + \nu \int_0^T \int_{\mathbb{T}^d} \nabla u \cdot \nabla(v-u) \ge \int_0^T \int_{\mathbb{T}^d} f(v-u), \\ u(T) = 0. \end{cases} \tag{9}$$

Moreover we have the estimate

$$\|u\|_{L^\infty} + \|u\|_{L^2(H^1)} \le \tilde{C},$$

where $\tilde{C}$ only depends on k_0 and d.

Proof. Let us define u_0 by

$$\begin{cases} -\partial_t u_0 - \nu \Delta u_0 = f \text{ in } (0,T) \times \mathbb{T}^d, \\ u_0(T) = 0 \text{ in } \mathbb{T}^d. \end{cases}$$

From the integrability assumption we made on f, u_0 is continuous on the compact set $[0,T] \times \mathbb{T}^d$. Choosing f sufficiently small in L^p norm, we deduce that the maximum amplitude of u_0 is smaller than k_0. Thus choosing $u = u_0$ proves the result. □

The previous result relies only on the compactness of the domain and not on the possible jumps, in particular that they belong to a finite set or not. We now prove that under a structural assumption on the jumps, the existence result can be improved. We shall make the following assumption.

Hypothesis 1. For all ξ in the finite set K, $\xi \in \mathbb{Q}^d$, i.e. all the components of ξ are rational numbers.

Proposition 1.2. *Under Hypothesis 1 holds, for any $f \in L^2((0,T) \times \mathbb{T}^d)$ there exists $u \in L^2((0,T), H^1(\mathbb{T}^d))$ such that:*

$$\begin{cases} u \le Mu, \\ \forall v \in H, v \le Mu, \\ -\int_0^T \int_{\mathbb{T}^d} \partial_t v(v-u) + \nu \int_0^T \int_{\mathbb{T}^d} \nabla u \cdot \nabla(v-u) + \frac{1}{2}\int_{\mathbb{T}^d} |v(T)|^2 \ge \int_0^T \int_{\mathbb{T}^d} f(v-u). \end{cases} \tag{10}$$

Moreover we have the estimate

$$\|u\|_{L^\infty(L^1)} \le C(\|f\|_{L^1}),$$

where C only depends on K.

Proof. To prove existence of such a function u, we want to build a lower bound for the impulse control problem by looking at the best possible running cost. From Hypothesis 1 and the fact that the domain is the torus, we know that starting from a given point, there is only a finite number of points we can reach only by jumping. We now make use of this remark. Because Hypothesis 1 is satisfied and because of periodicity, there exists $\theta \in \mathbb{N}^*$ such that for all $\xi \in K$:

$$\xi \in \left\{ (\frac{n_1}{\theta}, ..., \frac{n_d}{\theta}); 0 \le n_1, ..., n_d \le \theta - 1 \right\} =: \mathcal{K}.$$

We now define $\tilde{f}$ by

$$\tilde{f} = \min_{\xi \in \mathcal{K}} f_\xi,$$

where f_ξ is the translation of f in space following the vector ξ and where the minimum is understood pointwise. The function $\tilde{f} \in L^2((0,T) \times \mathbb{T}^d)$ is well defined and satisfies

$$\|\tilde{f}\|_{L^1} \le |\mathcal{K}| \|f\|_{L^1}.$$

We define $\tilde{u}$ by:

$$\begin{cases} -\partial_t \tilde{u} - \nu \Delta \tilde{u} = \tilde{f} \text{ in } (0,T) \times \mathbb{T}^d, \\ \tilde{u}(T) = 0 \text{ in } \mathbb{T}^d. \end{cases}$$

Let us note that $\tilde{u} \in H$. As already mentioned above, an existence result for QVI usually comes from the existence of a lower bound for an approximating sequence. The function $\tilde{u}$ plays the role of a lower bound for the sequence $(u_n)_{n\in\mathbb{N}}$ that we now define. We denote by $u_0 \in H$ the unique solution of

$$\begin{cases} -\partial_t u_0 - \nu \Delta u_0 = f \text{ in } (0,T) \times \mathbb{T}^d, \\ u_0(T) = 0 \text{ in } \mathbb{T}^d. \end{cases}$$

We then define for all $n \in \mathbb{N}$, $u_n \in L^2((0,T), H^1(\mathbb{T}^d))$ by: $u_{n+1} \in L^2((0,T), H^1(\mathbb{T}^d))$ is a solution of the weak variational inequality:

$$\begin{cases} u_{n+1} \leq M u_n, \\ \forall v \in H, v \leq M u_n, \\ -\int_0^T \int_{\mathbb{T}^d} \partial_t v (v - u_{n+1}) + \nu \int_0^T \int_{\mathbb{T}^d} \nabla u_{n+1} \cdot \nabla (v - u_{n+1}) + \\ \qquad + \frac{1}{2} \int_{\mathbb{T}^d} |v(T)|^2 \geq \int_0^T \int_{\mathbb{T}^d} f(v - u_{n+1}). \end{cases} \tag{11}$$

Let us remark that by the maximum principle,

$$\tilde{u} \leq u_0;$$

and by construction $\tilde{u}$ is invariant by any translation by a vector of $\mathcal{K}$, hence

$$\tilde{u} \leq M\tilde{u}.$$

We deduce iteratively that for every $n \in \mathbb{N}$, $\tilde{u} \leq M u_n$, $(u_{n+1})_{n\in\mathbb{N}}$ is well defined, $u_{n+1} \leq u_n$. Evaluating the second line of (11) with $v = \tilde{u}$, we deduce:

$$\nu \int_0^T \int_{\mathbb{T}^d} |\nabla u_n|^2 \leq -\int_0^T (-\partial_t \tilde{u} - f, \tilde{u} - u_n)_{H^{-1}\times H^1} + \nu \int_0^T \int_{\mathbb{T}^d} \nabla u_n \cdot \nabla \tilde{u}. \tag{12}$$

Thus, $(u_n)_{n\in\mathbb{T}^d}$ is a bounded sequence of $L^2((0,T), H^1(\mathbb{T}^d))$. Because, it is also a decreasing sequence, it converges weakly in $L^2((0,T), H^1(\mathbb{T}^d))$ to a limit $u \in L^2((0,T), H^1(\mathbb{T}^d))$. It follows that

$$\begin{cases} u \leq Mu, \\ \forall v \in H, v \leq Mu, \\ -\int_0^T \int_{\mathbb{T}^d} \partial_t v(v-u) + \nu \int_0^T \int_{\mathbb{T}^d} \nabla u \cdot \nabla (v-u) + \frac{1}{2}\int_{\mathbb{T}^d} |v(T)|^2 \geq \int_0^T \int_{\mathbb{T}^d} f(v-u). \end{cases}$$

Moreover, because

$$\tilde{u} \leq u \leq u_0,$$

we deduce that

$$\|u\|_{L^\infty(L^1)} \leq C\|f\|_{L^1},$$

where C depends only on K. □

We now state the following result:

Proposition 1.3. *Under the assumptions of Proposition 1.2, the penalized problem:*

$$\begin{cases} -\partial_t u_\epsilon - \nu \Delta u_\epsilon + \frac{1}{\epsilon}(u_\epsilon - Mu_\epsilon)^+ = f \textit{ in } (0,T)\times\mathbb{T}^d, \\ u_\epsilon(T) = 0 \textit{ in } \mathbb{T}^d, \end{cases} \tag{13}$$

admits a unique solution $u_\epsilon \in L^2((0,T), H^2(\mathbb{T}^d)) \cap H^1((0,T), L^2(\mathbb{T}^d))$. The sequence $(u_\epsilon)_\epsilon$ converges toward the maximal solution u of the QVI (10) weakly in $L^2((0,T), H^1(\mathbb{T}^d))$ as $\epsilon \to 0$, moreover $u_\epsilon \downarrow_{\epsilon\to 0} u$.

We do not prove this result as it came following the same argument as in [4].

Remark 1. Let us comment on Hypothesis 1. This assumption is, together with the fact that the set K is finite and the domain periodic, used to build the function $\tilde{u}$ in the previous proof. The key argument is that we are able to bound, for any given point x, the set of reachable (using only jumps) points from x. For instance in the previous proof, this property is useful to construct $\tilde{f}$ as the minimum of a finite number of functions. Without such an assumption, because we are expecting the solution u of the QVI to be unbounded from below, there is in general no solution of the QVI. Indeed, let us recall ([4]) that considering the sequence $(u_n)_{n\in\mathbb{N}}$ built in the previous proof, for any solution u of the QVI one has for all $n \geq 0$, $u_n \geq u$. Thus if u_0 is not bounded from below, we expect in general that $u_n \to -\infty$ as n goes to infinity. Another type of assumptions for which we can establish lower bounds are assumptions which limit the sets from which the jumps are feasible to, for instance, sets from which it is optimal to jump for another impulse control problem, for which the value function is bounded. This is equivalent to imposing the constraint $u \leq Mu$ only on a subset of $(0,T)\times\mathbb{T}^d$. Such results, thus depending on the geometry on the sets on which jumps are feasible, can be proven quite easily on a case by case basis.

3. The case of a unique possible jump

We study here a density of particles in the case in which a unique jump ξ is possible. We assume that there is a measurable set A from which the particles jump. The first part of this section is devoted to the study of a penalized equation. We then prove the existence and uniqueness of the limit density under certain assumptions on the set A. The study of the penalized equation is simple however we warn the reader that the estimates of Lemma 1.2 that we use at a penalized level are crucial to study the limit case. An important feature of our model is that some assumption has to be made on the set from which the particles jump. This assumption shows in some sense the limit of this model. It can be formulated in the following way.

Hypothesis 2. The set $A \subset (0,T)\times\mathbb{T}^d$ is such that there exists k satisfying (4), and $w \in W^{1,2,\infty}$ such that

$$\begin{cases} w(t,x) = k(x,\xi) + w(t,x+\xi) \text{ for } x \in A, \\ w(T) = 0. \end{cases}$$

Formally, this assumption restricts the situations which we are able to model to a case in which the jumps the particles are making, are connected to a certain "value function" w. Indeed if we interpret w as the value function of some impulse control problem, then we are assuming that it is optimal to use the controls on the set A. For instance if $A = (0,T)\times\mathbb{T}^d$ or if $A = (0,T)\times([x_1,x_2] + vect(\xi))/\mathbb{Z}^d$ for some $x_1, x_2 \in \mathbb{T}^d$ (i.e. A is a strip of "direction" ξ), then the hypothesis does not hold. On the other hand, if $A \subset (t_1,t_2)\times B(x_0,\alpha)$ for $0 \leq t_1 < t_2 < T$, $x_0 \in \mathbb{T}^d$, $\alpha < |\xi|/2$, then A satisfies the hypothesis (the construction of a function w satisfying the requirements is in this case not trivial but fairly simple and we do not detail it here). Let us also note that this hypothesis does not cover the case in which particles jump at the terminal time. To tackle such situations, without loss of generality we can look at the parabolic problem on a slightly larger time interval, so that the hypothesis may be verified. Moreover, this assumption is not useful for the study of the forthcoming penalized equation, only to pass to the limit.

3.1. A penalized version of the problem

In order to understand how the density of jumping particles behaves we first introduce a smoother version of the problem. We here assume that the particles do not simply jump when they are in A but that they have a given uniform probability of jumping in this set. This method allows us to work with a PDE. We naturally work with the equation:

$$\begin{cases} \partial_t m_\epsilon(t,x) - \nu\Delta m_\epsilon(t,x) + \frac{1}{\epsilon}\mathbb{1}_A m_\epsilon(t,x) - \frac{1}{\epsilon}(\mathbb{1}_A m_\epsilon)(t,x-\xi) = 0 \text{ in } (0,T)\times\mathbb{T}^d, \\ m(0,x) = m_0(x) \text{ in } \mathbb{T}^d, \end{cases}$$

where $\epsilon > 0$ is a real number which describes the probability of jumping from A. The term $\frac{1}{\epsilon}\mathbb{1}_A m_\epsilon(t,x)$ stands for the leaving rate of particles which jump from A. The term $-\frac{1}{\epsilon}(\mathbb{1}_A m_\epsilon)(t,x-\xi)$ stands for the arriving rate of particles which jump from $(t,x-\xi)$ and thus which arrive at (t,x). As $\epsilon \to 0$, the probability of jumping becomes more and more important. Thus finding the limit as ϵ goes to 0 gives the desired density of particles.

We begin by showing how we can find estimates on such a penalized equation. The following lemma will be useful to establish a priori estimates on m (let us recall that H is defined at the end of the introduction).

Lemma 1.1. *Let $m \in L^2((0,T),H^2(\mathbb{T}^d))$ be a solution of (3) and k satisfy (4). If $m \geq 0$, $u \in H$ and $u \leq M(k,u)$ almost everywhere on A, then*

$$\int_0^T\int_{\mathbb{T}^d}(\partial_t m - \nu\Delta m)u + \frac{1}{\epsilon}\int_A km \geq 0.$$

Moreover, equality holds if $u = M(k,u)$ almost everywhere on A.

Proof. We multiply (3) by u and we integrate, we then obtain after a change of variable:

$$\int_0^T\int_{\mathbb{T}^d}(\partial_t m - \nu\Delta m)u + \frac{1}{\epsilon}\int_A mu = \frac{1}{\epsilon}\int_A m(t,x)u(t,x+\xi)dtdx.$$

Using the fact that $u \leq M(k,u)$ on A we deduce the desired result. □

The previous result suggests to work with the set $\mathcal{H}(k)$ for some k where $\mathcal{H}(k)$ is defined by:

$$\mathcal{H}(k) := \{m \in L^2, D(k,m) > -\infty\},$$

in which $D(k,m)$ is defined by

$$D(k,m) := \inf\left\{\int_0^T\int_{\mathbb{T}^d}(-\partial_t u - \nu\Delta u)m - \int_{\mathbb{T}^d} u(0)m_0 | u \in W^{1,2,2}, u \leq M(k,u), u(T) = 0\right\}. \tag{14}$$

We recall that M is defined in (5).

For any k, the set $\mathcal{H}(k)$ has to be interpreted as a set of admissible solutions of the Fokker-Planck equation. Indeed when particles leave (or jump) we should have exactly the same arriving particles ξ further. We recall that, $\partial_t m - \nu\Delta m \leq 0$ is interpreted as particles leaving (locally) and $\partial_t m - \nu\Delta m \geq 0$ as particles arriving (locally) from other causes than the diffusions. Thus it is natural to measure the variation of $\partial_t m - \nu\Delta m$ with functions u such that $u(x) - u(x+\xi) \leq k$. This "test" quantifies the fact that some negativity for $\partial_t m - \nu\Delta m$ has to be compensated by some positivity of this quantity ξ further. We now prove the following lemma, which states that in some sense, the quantity $D(k,m)$ is of interest to bound m in some function space (this lemma is crucial to study the limit $\epsilon \to 0$).

Lemma 1.2. *Let k be such that it satisfies (4). There exists C depending only on k and d, such that for any $m \in H$, $m \geq 0$, for $p < 1 + \frac{2}{d} = (1 + \frac{d}{2})'$:*

$$\|m\|_{L^p} \le C(-D(k,m) + \|m(0)\|_{L^1}).$$

Moreover, if Hypothesis 1 is satisfied and $\Delta m \in L^2((0,T)\times\mathbb{T}^d)$, there exists C_1 depending on d and on ξ such that

$$\|m\|^2_{L^2((0,T),H^1)} \le -D(k,m) + C_1(\|\Delta m\|_{L^1})\|m(0)\|_{L^\infty}.$$

Let us remark that the previous estimates are established even in the case in which the right hand side is infinite, case in which they are trivially correct.

Proof. Let us recall that $D(k,m)$ is defined in (14). Because $m \in H$, we deduce that

$$D(k,m) = \inf\left\{ \int_0^T (\partial_t m - \nu\Delta m, v)_{H^{-1}\times H^1} | v \in L^2((0,T),H^1(\mathbb{T}^d)), v \le M(k,v) \right\}.$$

This follows easily from the fact that $\{v \in W^{1,2,2}, v(T)=0\}$ is dense in $L^2((0,T),H^1(\mathbb{T}^d))$. We begin by proving the first estimate. Let us take $f \in L^{p'}$, $\|f\|_{L^{p'}} \le C$ where C is given by Proposition 1.1 and $u_1 \in L^2((0,T),H^1(\mathbb{T}^d))$ the solution of the associated QVI which is also given by:

$$\begin{cases} -\partial_t u_1 - \nu\Delta u_1 = f \text{ in } (0,T)\times\mathbb{T}^d, \\ u_1(T) = 0 \text{ in } \mathbb{T}^d. \end{cases}$$

Multiplying this equation by m and integrating, we deduce that

$$\int_0^T (\partial_t m - \nu\Delta m, u_1)_{H^{-1}\times H^1} \le -\int_{\mathbb{T}^d} u_1(0)m(0) - \int_0^T\int_{\mathbb{T}^d} f\, m.$$

Because $u_1 \le Mu_1$, we obtain that

$$\sup_{\|f\|_{L^{p'}}\le C} \int_0^T\int_{\mathbb{T}^d} f\, m \le -D(k,m) - \int_{\mathbb{T}^d} u_1(0)m(0),$$

from which we easily deduce the result thanks to Proposition 1.1.

We now prove the second estimate. From Proposition 1.2, we know that there exists $u_2 \in L^2((0,T),H^1(\mathbb{T}^d))$ such that:

$$\begin{cases} u_2 \le M(k,u_2), \\ \forall v \in H, v \le M(k,u_2), \\ -\int_0^T\int_{\mathbb{T}^d} \partial_t v(v-u_2) + \nu\int_0^T\int_{\mathbb{T}^d} \nabla u_2\cdot\nabla(v-u_2) + \frac{1}{2}\int_{\mathbb{T}^d}|v(T)|^2 \ge \int_0^T\int_{\mathbb{T}^d}(\Delta m)(v-u_2). \end{cases}$$

We define u_ϵ, the unique solution of:

$$\begin{cases} -\partial_t u_\epsilon - \nu\Delta u_\epsilon + \frac{1}{\epsilon}(u_\epsilon - Mu_\epsilon)^+ = \Delta m \text{ in } (0,T)\times\mathbb{T}^d, \\ u_\epsilon(T) = 0 \text{ in } \mathbb{T}^d. \end{cases}$$

Multiplying the PDE satisfied by u_ϵ by m and integrating, we deduce that:

$$\int_0^T (\partial_t m - \nu\Delta m, u_\epsilon)_{H^{-1}\times H^1} = -\int_{\mathbb{T}^d} u_\epsilon(0)m(0) + \int_0^T\int_{\mathbb{T}^d}\left(\Delta m - \frac{1}{\epsilon}(u_\epsilon - Mu_\epsilon)^+\right)m.$$

Recalling Proposition 1.3 (in particular $u_\epsilon \downarrow_{\epsilon\to 0} u_2$), we obtain that

$$\int_0^T (\partial_t m - \nu\Delta m, u_2)_{H^{-1}\times H^1} \le \|u_2\|_{L^\infty(L^1)}\|m(0)\|_{L^\infty} - \int_0^T\int_{\mathbb{T}^d} |\nabla m|^2.$$

Once again, we obtain the estimate from Proposition 1.2. □

This lemma suggests to find uniform in ϵ a priori estimates for solutions of (3) by looking at the quantity $D(k, m)$ for some k. We begin by presenting a result of existence and uniqueness for solutions of the penalized equation.

Theorem 1.1. *For any $m_0 \in L^2(\mathbb{T}^d)$, $m_0 \geq 0$ and $\lambda \in L^\infty((0,T)\times\mathbb{T}^d)$, $\lambda \geq 0$ there exists a unique $m \in H$ such that*

$$\begin{cases} \partial_t m(t,x) - \nu\Delta m(t,x) + \lambda(t,x)m(t,x) - \lambda(t,x-\xi)m(t,x-\xi) = 0 \text{ in } (0,T)\times\mathbb{T}^d, \\ m(0) = m_0 \text{ in } \mathbb{T}^d. \end{cases} \tag{15}$$

Moreover, $m \geq 0$.

Proof. We begin this proof with the uniqueness of such solutions.

For any m_1 and m_2 solutions of (3), we denote by $\delta m = m_1 - m_2$ the difference of these solutions. The function δm satisfies

$$\begin{cases} \partial_t \delta m(t,x) - \nu\Delta \delta m(t,x) + \lambda(t,x)\delta m(t,x) - \lambda(t,x-\xi)\delta m(t,x-\xi) = 0 \text{ in } (0,T)\times\mathbb{T}^d, \\ \delta m(0) = 0 \text{ in } \mathbb{T}^d. \end{cases}$$

Multiplying this equation by δm and integrating in space, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d}(\delta m)^2 + \nu\int_{\mathbb{T}^d}|\nabla\delta m|^2 + \int_{\mathbb{T}^d}\lambda(\delta m)^2 = \int_{\mathbb{T}^d}(\lambda\delta m)(t,x)(\delta m)(t,x+\xi)dx,$$
$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d}(\delta m)^2 \leq \|\lambda\|_{L^\infty}\|\delta m(t)\|^2_{L^2(\mathbb{T}^d)}.$$

Finally, we deduce, using Gronwall's lemma, that $\delta m = 0$ and thus that there is at most one solution of (3).

We now pass to the proof of the positivity. This part of the proof is a generic use of a maximum principle. We assume first that λ and m_0 are smooth functions and that $m_0 \geq \alpha > 0$ for a real number α. We first show that the result holds true for small T, more precisely, for T which satisfies

$$\|\lambda\|_{L^\infty} < \frac{1}{T}. \tag{16}$$

We recall that by classical parabolic estimates, m is smooth (C^1 in time and C^2 in space) because of the smoothness of λ and m_0. If there exists $(t_0, x_0) \in (0,T)\times\mathbb{T}^d$ such that $m(t_0,x_0) < 0$, then there exists $\delta > 0$ such that $m(t_0,x_0) + \delta t_0 < 0$. We define μ by:

$$\mu(t,x) = m(t,x) + \delta t, \forall (t,x) \in (0,T)\times\mathbb{T}^d.$$

For any $x \in \mathbb{T}^d$, $\mu(0,x) > 0$ and $\mu(t_0,x_0) < 0$, thus there exists $(t_1,x_1) \in (0,T)\times\mathbb{T}^d$ such that

$$\begin{cases} \mu(t) \geq 0, \forall t \leq t_1, \\ \mu(t_1,x_1) = 0, \\ \partial_t\mu(t_1,x_1) \leq 0, \\ \Delta\mu(t_1,x_1) \geq 0. \end{cases}$$

Let us remark that

$$\begin{aligned} \partial_t\mu &= \partial_t m + \delta, \\ &= \delta + \nu\Delta m - \lambda m + (\lambda m)(\cdot - \xi). \end{aligned}$$

Evaluating this last expression at (t_1,x_1) we obtain that

$$\partial_t\mu(t_1,x_1) \geq \delta + \delta t_1(\lambda(t_1,x_1) - \lambda(t_1,x_1-\xi)) > 0,$$

which is impossible. Now let us remark that the propagation of positivity for small time holds and that the assumption on T does not depend on m_0. Thus by repeating the same argument with $m_0 = m(T)$, we are able to state that if λ and m_0 are smooth, then the results holds true.

Because of the uniqueness of solutions of (15) this result extends to non smooth λ and m_0 with only $m_0 \geq 0$.

We now pass to the proof of existence. We begin by showing an a priori estimate for the solutions of (15). Let $m \in W^{1,2,2}$ be a solution of

$$\partial_t m(t,x) - \nu \Delta m(t,x) + \lambda(t,x)m(t,x) - \lambda(t,x-\xi)m(t,x-\xi) = 0 \text{ in } (0,T) \times \mathbb{T}^d.$$

By Lemma 1.2, we deduce that there exists $C > 0$ such that for $1 \leq p < 1 + \frac{2}{d}$.

$$\begin{aligned}\|m\|_{L^p} &\leq C\|m(0)\|_{L^1} + C\int_0^T\int_{\mathbb{T}^d} \lambda m,\\ &\leq C\|m(0)\|_{L^1} + C\|\lambda\|_{L^\infty}\|m\|_{L^1}.\end{aligned}$$

Let us now remark that, because of the PDE satisfied by m, $\|m(t)\|_{L^1} = \|m(0)\|_{L^1}$. We then deduce from parabolic regularity that we can bound the norm $W^{1,2,2}$ of m by a number depending only on $\|\lambda\|_{L^\infty}$ and $\|m(0)\|_{L^1}$. Using this estimate, a classical Galerkin method immediately yields existence of a solution of (15). □

3.2. Existence of a limit density

We show here how we can pass to the limit in the equation (3) and hence obtain a characterization of the density of jumping particles. We describe the behavior of the solutions of (3) as ϵ goes to 0. To pass to the limit, we make a crucial use of Hypothesis 2 to bound the quantity $D(k,m)$ and use Lemma 1.2.

Theorem 1.2. *Take $m_0 \in L^2(\mathbb{T}^d)$ and $\epsilon > 0$, we denote by m_ϵ the unique solution of (3). Under Hypothesis 2, there exists $m \in L^p$, for $1 < p < 1 + \frac{2}{d}$, such that, extracting a subsequence if necessary, $(m_\epsilon)_{\epsilon>0}$ converges weakly in L^p toward m which satisfies*

- *$m = 0$ a.e. in A.*
- *For all $v \in W^{1,2,p'}$ such that $v \leq M(k,v)$ on A and $v(T) = 0$,*

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}^d} (-\partial_t v - \nu\Delta v)m - \int_{\mathbb{T}^d} v(0)m_0\\ &\geq \int_0^T (-\partial_t u - \nu\Delta u)m - \int_{\mathbb{T}^d} u(0)m_0,\end{aligned}$$

for any $u \in W^{1,2,p'}$ which satisfies $u = M(k,u)$ on A, $u(T) = 0$.

If $m_0 \in L^\infty$ and Hypothesis 1 holds, then $m \in L^2((0,T), H^1(\mathbb{T}^d))$ and $(m_\epsilon)_{\epsilon>0}$ converges weakly in $L^2((0,T), H^1(\mathbb{T}^d))$ toward m. Moreover in this case, m satisfies:

For all $v \in H$ such that $v \leq M(k,v)$ on A and $v(T) = 0$,

$$\begin{aligned}&\int_0^T (-\partial_t v - \nu\Delta v, m)_{H^{-1}\times H^1} - \int_{\mathbb{T}^d} v(0)m_0\\ &\geq \int_0^T (-\partial_t u - \nu\Delta u, m)_{H^{-1}\times H^1} - \int_{\mathbb{T}^d} u(0)m_0,\end{aligned}$$

for any $u \in H$ which satisfies $u = M(k,u)$ on A, $u(T) = 0$.

Proof. We denote by m_ϵ for all $\epsilon > 0$ the unique solution of

$$\begin{cases} \partial_t m_\epsilon - \nu\Delta m_\epsilon + \frac{1}{\epsilon}\mathbb{1}_A(t,x)m_\epsilon(t,x) - \frac{1}{\epsilon}\mathbb{1}_A(t,x-\xi)m_\epsilon(t,x-\xi) = 0 \text{ in } (0,T)\times\mathbb{T}^d, \\ m_\epsilon(0) = m_0 \text{ in } \mathbb{T}^d. \end{cases}$$

Let us denote by (k,w) a couple which satisfies Hypothesis 2. For all $\epsilon > 0$, in view of Lemma 1.1, we can observe that

$$-\infty < D(k,m_\epsilon) = \int_0^T\int_{\mathbb{T}^d}(-\partial_t w - \nu\Delta w)m_\epsilon - \int_{\mathbb{T}^d} w(0)m_0.$$

Let us observe that $\|m_\epsilon\|_{L^1} = T\|m_0\|_{L^1}$, thus from the regularity of w we deduce that $(D(k,m_\epsilon))_{\epsilon>0}$ is bounded. From Lemma 1.2:

$$\|m_\epsilon\|_{L^p} \le C(-D(k,m_\epsilon) + \|m_0\|_{L^1}).$$

This yields that $(m_\epsilon)_{\epsilon>0}$ is a bounded sequence of L^p. Thus, extracting a subsequence if necessary, $(m_\epsilon)_{\epsilon>0}$ converges weakly toward a limit $m \in L^p$. Now take any $v, w \in W^{1,2,p'}$ such that $v \le M(k,v)$, $w = M(k,w)$ on A and $v(T) = w(T) = 0$. For all $\epsilon > 0$ we deduce from Lemma 1.1:

$$\int_0^T\int_{\mathbb{T}^d}(-\partial_t w - \nu\Delta w)m_\epsilon - \int_{\mathbb{T}^d} w(0)m_0 \le \int_0^T\int_{\mathbb{T}^d}(-\partial_t v - \nu\Delta v)m_\epsilon - \int_{\mathbb{T}^d} v(0)m_0.$$

Passing to the limit ϵ goes to 0 we deduce:

$$\int_0^T\int_{\mathbb{T}^d}(-\partial_t w - \nu\Delta w)m - \int_{\mathbb{T}^d} w(0)m_0 \le \int_0^T\int_{\mathbb{T}^d}(-\partial_t v - \nu\Delta v)m - \int_{\mathbb{T}^d} v(0)m_0.$$

Let us note that for all $\epsilon > 0$

$$D(k,m_\epsilon) = -\frac{1}{\epsilon}\int_A k m_\epsilon \le -\frac{1}{\epsilon}\int_A k_0 m_\epsilon.$$

Thus, $m = 0$ almost everywhere on A because $(D(k,m_\epsilon))_{\epsilon>0}$ is bounded, which concludes the proof of the first part of the result. Let us now assume that $m_0 \in L^\infty$ and Hypothesis 1 is satisfied. Using the same argument as in the first part of the proof, we deduce from Lemma 1.2 that

$$\|m_\epsilon\|^2_{L^2(H^1)} \le -\int_0^T(-\partial_t w - \nu\Delta w, m_\epsilon)_{H^{-1},H^1} + \int_{\mathbb{T}^d} w(0)m_0 + C(\|\Delta m_\epsilon\|_{L^1})\|m_0\|_{L^\infty}.$$

Let us remark that, because $(D(k,m_\epsilon))_{\epsilon>0}$ is bounded, $(\frac{1}{\epsilon}\mathbb{1}_A m_\epsilon)_{\epsilon>0}$ is bounded in L^1. Thus from the PDE satisfied by m_ϵ, we obtain that $(\Delta m_\epsilon)_{\epsilon>0}$ is bounded in L^1. The rest of the proof follows the same type of argument as the first part of this proof. □

Remark 2. In the case in which Hypothesis 1 is satisfied, Hypothesis 2 can be weakened, namely w can be chosen in H.

3.3. Interpretation of the limit density

From a variational point of view, the properties of the limit density m given in this theorem are what we expect. Indeed, as we mentioned earlier, $D(k,m) > -\infty$ formally stands for the fact that m represents a density of jumping particles. The condition $m = 0$ on A stands for the fact that m is a density of particles which are actually jumping from A because otherwise there will be particles on A. The condition:

$$\begin{cases} \forall u \in H, u = Mu \text{ on } A, u(T) = 0: \\ D(m) = \int_0^T (-\partial_t u - \nu\Delta u, m)_{H^{-1}\times H^1} - \int_{\mathbb{T}^d} u(0)m_0, \end{cases} \tag{17}$$

stands formally from the fact that those particles are not jumping elsewhere than form A. Indeed at a penalized level we know that

$$D(k, m_\epsilon) = -\frac{1}{\epsilon}\int_A km_\epsilon.$$

Thus the quantity $D(k,m)$ is closely related to the set A from which the particles actually jump. It appears that the quantity $D(k,m)$ measures the aggregate costs "paid" by all the particles which jump. To illustrate, let us look at an example: we take B a measurable subset of $(0,T)\times\mathbb{T}^d$ which satisfies hypothesis 1 and such that $A \subset B$. We define μ_ϵ by:

$$\begin{cases} \partial_t\mu_\epsilon - \nu\Delta\mu_\epsilon + \frac{1}{\epsilon}\mathbb{1}_B\mu_\epsilon - \frac{1}{\epsilon}(\mathbb{1}_B\mu_\epsilon)(t, x-\xi) = 0, \\ \mu(0) = m_0. \end{cases}$$

Letting ϵ go to 0, recalling the previous theorem, μ satisfies $D(k,\mu) > -\infty$ and $\mu = 0$ on A because $\mu = 0$ on B. What is differentiating μ from m is that, for $v \in H$, $v \le M(k,v)$, $v = M(k,v)$ on A, we do not necessary have an equality in:

$$D(k,\mu) \le \int_0^T (\partial_t\mu, v)_{H^{-1}\times H^1} + \nu\int_0^T\int_{\mathbb{T}^d} \nabla\mu\nabla v,$$

if $v \ne M(k,v)$ on B.

Thus we interpret (17) as the characterization of the fact that particles are not jumping from elsewhere than from A. Finally, let us note that the initial condition $m(0) = m_0$ may not be satisfied. This is a consequence of the fact that if A is not negligible near $\{t = 0\}$, then particles are jumping instantaneously. We cannot expect in such a case for the initial condition to be satisfied. However the variational relation satisfied by m is sufficient to "remember" that the density starts from m_0. This last remark leads us to think that in general, there is no time regularity for the solution m. This discussion leads to the following definition:

Definition 1. Let $m_0 \in L^p(\mathbb{T}^d)$ be a positive density and A a measurable subset of $(0,T)\times\mathbb{T}^d$, $m \in L^p$ is called a solution of the Fokker-Planck equation of particles jumping from A if for some function k satisfying (4):

- $m = 0$ a.e. on A,
- $$\forall v, u \in W^{1,2,p'}, v \le M(k,v) \text{ on } A, u = M(k,u) \text{ on } A, v(T) = 0, u(T) = 0:$$ $$\int_0^T\int_{\mathbb{T}^d} (-\partial_t(u-v) - \nu\Delta(u-v))m - \int_{\mathbb{T}^d} (u-v)(0)m_0 \le 0.$$

We explained formally why this definition is legitimate, the following section on the uniqueness, even though the assumptions on A will be a bit more restrictive, also justifies this choice of definition.

3.4. Uniqueness of the limit density

We now discuss the uniqueness of such solutions. We state that the uniqueness holds under certain assumptions on the set A. Let us note that it is classical to have some assumptions on the domain in the study of parabolic PDE in time dependent domain, see [17,7]. We make here the following assumption:

Hypothesis 3. The set A is either a closed set with Lipschitz boundary such that $(\{T\}\times\mathbb{T}^d)\cap A$ is a closed set with Lipschitz boundary, or it is non-decreasing in time (with respect to the inclusion).

Our main argument is that the uniqueness of solutions of the Fokker-Planck equation can be deduced from an existence result for an "adjoint" equation. We start with this existence result:

Lemma 1.3. *Assume Hypothesis 2 holds with $w(T) = 0$ replaced by $w(T) \leq M(k, w)(T)$ where (k, w) is the couple in this hypothesis. Assume Hypothesis 3 holds. For any $f \in L^\infty((0, T) \times \mathbb{T}^d)$, there exists $v \in H$ such that, for all $\mu \in L^2((0, T), H^1(\mathbb{T}^d))$, $\mu = 0$ on A:*

$$\begin{cases} \int_0^T (-\partial_t v - \nu\Delta v, \mu)_{H^{-1}\times H^1} = \int_0^T \int_{\mathbb{T}^d} (-\partial_t w - \nu\Delta w + f, \mu)_{H^{-1}\times H^1}, \\ v = M(k, v) \text{ on } A, \\ v(T) = w(T). \end{cases}$$

Proof. We fix $f \in L^\infty((0, T) \times \mathbb{T}^d)$ and we define $\tilde{f} := -\partial_t w - \nu\Delta w + f$. In the two cases in Hypothesis 3, the following operator is well defined:

$$\begin{aligned} \mathcal{T} :& H \to H \\ & v \to \mathcal{T}(v) \end{aligned}$$

where $\mathcal{T}(v)$ is the unique solution of

$$\begin{cases} -\partial_t \mathcal{T}(v) - \nu\Delta \mathcal{T}(v) = \tilde{f} \text{ in } A^c, \\ \mathcal{T}(v) = M(k, v) \text{ a.e. in } A, \\ v(T) = w(T). \end{cases}$$

For the case of a Lipschitz boundary we refer to [7] and to [17] for the case in which A is increasing. Moreover the map $\mathcal{T}$ is monotone (order preserving). Next we claim using classical results on QVI (see [4]) that there exists $v_1, \tilde{v}_2 \in H$, respectively solutions of the two following QVI:

$$\begin{cases} v_1 \leq M v_1, \\ \forall \varphi \in H, \varphi \leq M v_1 : \\ \int_0^T \int_{\mathbb{T}^d} (-\partial_t v_1 - \nu\Delta v_1 - \tilde{f})(\varphi - v_1) \geq 0, \\ v_1(T) = w(T). \end{cases}$$

$$\begin{cases} \tilde{v}_2 \leq M \tilde{v}_2, \\ \forall \varphi \in H, \varphi \leq M \tilde{v}_2 : \\ \int_0^T \int_{\mathbb{T}^d} (-\partial_t \tilde{v}_2 - \nu\Delta \tilde{v}_2 + \tilde{f} - 2(-\partial_t w - \nu\Delta w))(\varphi - \tilde{v}_2) \geq 0, \\ \tilde{v}_2(T) = w(T). \end{cases}$$

We define $v_2 = 2w - \tilde{v}_2$. Because $\mathcal{T}$ is monotone, the set $\mathcal{J} := \{v \in H, v_1 \leq v \leq v_2\}$ is invariant under $\mathcal{T}$. Moreover this set is non-empty because $v_1 \leq v_2$. Thus $\mathcal{T}$ has a fixed point $u \in \mathcal{J}$. This fixed point satisfies the properties. □

We are now able to prove the following result:

Theorem 1.3. *Under the assumptions of Lemma 1.3, for any non negative $m_0 \in L^2(\mathbb{T}^d)$ there exists at most one $m \in L^2((0, T), H^1(\mathbb{T}^d))$ solution of the problem in the sense of Definition 1.*

Proof. We denote by m_1 and m_2 two solutions. We construct for $i = 1, 2$, $v_i \in H$ as in Lemma 1.3 with respective source terms $f = f_1$ and $f = f_2$, where $\|f_1\|_{L^\infty}, \|f_2\|_{L^\infty} < +\infty$. We then evaluate

$$\int_0^T (-\partial_t (v_1 - v_2) - \nu\Delta(v_1 - v_2), m_1 - m_2)_{H^{-1}\times H^1}.$$

Because m_1 and m_2 are solutions, the following holds:

$$D(m_j) = \int_0^T (-\partial_t(v_i) - \nu\Delta(v_i), m_j)_{H^{-1}\times H^1} - \int_{\mathbb{T}^d} v_i(0)m_0.$$

Hence,

$$\int_0^T (-\partial_t(v_1 - v_2) - \nu\Delta(v_1 - v_2), m_1 - m_2)_{H^{-1}\times H^1} = 0. \tag{18}$$

On the other hand, using Lemma 1.3, we derive that

$$\int_0^T (-\partial_t(v_1 - v_2) - \nu\Delta(v_1 - v_2), m_1 - m_2)_{H^{-1}\times H^1} = \int_0^T \int_{\mathbb{T}^d} (m_1 - m_2)(f_1 - f_2).$$

Recalling (18), we obtain that

$$\int_0^T \int_{\mathbb{T}^d} (m_1 - m_2)(f_1 - f_2) = 0.$$

Thus we deduce that $m_1 = m_2$. □

3.5. A remark on the hypothesis for uniqueness

Even though we are only able to prove the uniqueness of solutions of the Fokker-Planck equation under Hypothesis 3, we conjecture that uniqueness is a more general property for this equation. Furthermore, as we shall see in the next part, we do not need such an assumption in the MFG setting.

In any case, we hope that the range of applications of Hypothesis 3 is large enough to convince the reader that the notion of solution of the Fokker-Planck equation is the correct one.

4. The case of a finite number of possible jumps

We now address a more general model as we look at situations in which different jumps can occur. As we shall see, all the results of the case of a single jump extend to the case of a finite number of jumps. Because there are more notations, we advise not to read this section before the previous one. We denote by $K \subset \mathbb{T}^d$ the finite set of possible jumps. In this setting a single set A is no longer sufficient to describe all the jumps. We introduce $V(\xi, t, x)$, the proportion of particles which use the jump ξ at (t, x). We assume the following:

$$\begin{cases} V \in L^\infty(K, (0, T), \mathbb{T}^d), \\ V \geq 0, \\ \sum_{\xi\in K} V(\xi, t, x) \leq 1. \end{cases} \tag{19}$$

We also define the sets:

$$\forall \xi \in K, A_\xi := \{V(\xi, \cdot, \cdot) > 0\},$$
$$A = \cup_{\xi\in K} A_\xi.$$

As in the case of a single jump, an assumption on the sets from which the particles jump is still needed. We make the following assumption:

Hypothesis 4. There exists k satisfying (4) and $w \in W^{1,2,\infty}$ such that

$$\begin{cases} w \leq M(k, w) \text{ in } (0, T) \times \mathbb{T}^d, \\ w(T) = 0, \\ \forall \xi \in K, V(\xi, t, x)(k(x, \xi) + w(t, x + \xi) - w(t, x)) = 0 \text{ a.e. in } (0, T) \times \mathbb{T}^d. \end{cases} \tag{20}$$

This hypothesis is slightly more sophisticated than hypothesis 1. In some sense, we still assume that the sets from which the particles jump are given as a result of an impulse control problem.

4.1. The penalized equation

We introduce first a penalized version of the problem. We recall that this penalization models situations in which the particles have a certain probability to jump from the prescribed sets, and that the limit as ϵ goes to 0 corresponds to the probability of jumping going to 1. At this penalized level, we expect the density of particles m to satisfy:

$$\begin{cases} \partial_t m_\epsilon - \nu\Delta m_\epsilon + \frac{1}{\epsilon} m_\epsilon(t,x)\big(\sum_{\xi\in K} V(\xi,t,x)\big) - \frac{1}{\epsilon}\sum_{\xi\in K} m_\epsilon(t,x-\xi)V(\xi,t,x-\xi) = 0 \\ \text{in } (0,T)\times\mathbb{T}^d, \\ m_\epsilon(0) = m_0 \text{ in } \mathbb{T}^d. \end{cases} \tag{21}$$

Let us assume that m is a smooth solution of the previous PDE. Then, as in the case of a single jump, for any $u \in L^2((0,T), H^1(\mathbb{T}^d))$, $u \le M(k,u)$, after a simple change of variable we obtain that:

$$\begin{aligned} \int_0^T (\partial_t m - \nu\Delta m, u)_{H^{-1}\times H^1} &= -\frac{1}{\epsilon}\int_0^T\int_{\mathbb{T}^d} m(t,x)\Big(\sum_{\xi\in K} V(\xi,t,x)(u(t,x) - u(t,x+\xi))\Big)dtdx, \\ &\ge -\frac{1}{\epsilon}\int_0^T\int_{\mathbb{T}^d} m(t,x)\Big(\sum_{\xi\in K} V(\xi,t,x)k(x,\xi)\Big)dtdx. \end{aligned}$$

This leads us to define the meaningful quantity:

$$D(k,m) := \inf\left\{ \int_0^T\int_{\mathbb{T}^d} (-\partial_t u - \nu\Delta u)m - \int_{\mathbb{T}^d} m_0 u(0) \,|\, u \in W^{1,2,2}, u \le Mu, u(T) = 0 \right\}. \tag{22}$$

Let us remark that for any w satisfying (20):

$$D(k,m_\epsilon) = \int_0^T (\partial_t m_\epsilon - \nu\Delta m_\epsilon, w)_{H^{-1}\times H^1}.$$

The proof of the following lemma is the exact analogous of the proof we did earlier in the case of a single jump so we do not present it here.

Lemma 1.4. *Let k be such that it satisfies (4). For $1 \le p < 1 + \frac{2}{d}$, there exists C depending on K, k and d, such that for any $m \in H$, $m \ge 0$:*

$$\|m\|_{L^p} \le C(-D(k,m) + \|m(0)\|_{L^1}).$$

Moreover, if Hypothesis 1 is satisfied, there exists C_1 depending only on K and d such that

$$\|m\|^2_{L^2((0,T),H^1)} \le -D(k,m) + C_1\|\Delta m\|_{L^1}\|m(0)\|_{L^\infty}.$$

Furthermore, as in the case of a single jump, we can prove the following result.

Theorem 1.4. *For all $m_0 \in L^2(\mathbb{T}^d)$ there exists a unique solution $m_\epsilon \in H$, $m_\epsilon \ge 0$ of the penalized equation (21).*

4.2. The limit density

We now address the passage to the limit in (21). As in the case of a single jump, the existence follows from Lemma 1.4 and we prove uniqueness under a more restrictive assumption in order to avoid technical difficulties.

Theorem 1.5. *For $\epsilon > 0$, we denote by m_ϵ the solution of (21). Assume that Hypothesis 4 holds and denote by (k, w) a couple which satisfies this hypothesis. Then, there exists $m \in L^p$, $1 < p < 1 + \frac{2}{d}$ such that, extracting a subsequence if necessary, $(m_\epsilon)_{\epsilon>0}$ converges weakly in L^p toward m which satisfies*

- *$m = 0$ a.e. in A.*
- *For all $v \in W^{1,2,p'}$ such that $v \leq Mv$ and $v(T) = 0$,*

$$\int_0^T \int_{\mathbb{T}^d} (-\partial_t v - \nu \Delta v)m - \int_{\mathbb{T}^d} v(0)m_0 \geq \int_0^T \int_{\mathbb{T}^d} (-\partial_t u - \nu \Delta u)m - \int_{\mathbb{T}^d} u(0)m_0,$$

for any $u \in W^{1,2,p'}$ which satisfies (20).

If $m_0 \in L^\infty$ and Hypothesis 1 holds, then there exists $m \in L^2((0,T), H^1(\mathbb{T}^d))$ and, extracting a subsequence if necessary, $(m_\epsilon)_{\epsilon>0}$ converges weakly in $L^2((0,T), H^1(\mathbb{T}^d))$ toward m. Moreover in this case, m satisfies:
For all $v \in H$ such that $v \leq Mv$ and $v(T) = 0$,

$$\int_0^T (-\partial_t v - \nu \Delta v, m)_{H^{-1}\times H^1} - \int_{\mathbb{T}^d} v(0)m_0 \geq \int_0^T (-\partial_t u - \nu \Delta u, m)_{H^{-1}\times H^1} - \int_{\mathbb{T}^d} u(0)m_0,$$

for any $u \in H$ which satisfies (20).

We recall that A is defined by

$$A = \cup_{\xi \in K} \{V(\xi, \cdot, \cdot) > 0\}.$$

We do not present the proof of this result as it is the same as the one we presented in the case of a unique jump. Like we did in the case of a single jump, we give the following definition:

Definition 2. For any $m_0 \in L^2(\mathbb{T}^d)$, V measurable function satisfying (19), $m \in L^p((0,T) \times \mathbb{T}^d)$ is called a solution of the Fokker-Planck equation of particles jumping with jumps described by V if for some k satisfying (4)

- $m = 0$ on A,
- $$\begin{cases} \forall v \in W^{1,2,p'}, v \leq M(k, v), v(T) = 0, \forall u \in W^{1,2,p'} \text{ satisfying (20)}: \\ \int_0^T \int_{\mathbb{T}^d} (-\partial_t v - \nu \Delta v)m - \int_{\mathbb{T}^d} v(0)m_0 \geq \int_0^T \int_{\mathbb{T}^d} (-\partial_t u - \nu \Delta u)m - \int_{\mathbb{T}^d} u(0)m_0. \end{cases}$$

We now turn to the question of the uniqueness of such solutions. As in the case of a single jump, uniqueness is a consequence of an existence result for an adjoint equation. Similarly as we proceeded in the particular case of a single jump, we are going to make an assumption on the function V.

Hypothesis 5. The set A is either a closed set with Lipschitz boundary such that $(\{T\} \times \mathbb{T}^d) \cap A$ is a closed set with Lipschitz boundary, or it is non-decreasing in time (with respect to the inclusion). Moreover, there exists (k, w) satisfying Hypothesis 4 except for the fact that $w(T) = 0$, such that:

$$\begin{cases}\forall \xi \in K, x \in A_\xi \Rightarrow (k(x,\xi) + w(t,x+\xi) - w(t,x)) = 0,\\ \forall \xi \in K, x \in A_\xi \Rightarrow \forall \xi' \neq \xi, w(t,x) < k(x,\xi') + w(t,x+\xi').\end{cases} \tag{23}$$

Let us remark that we have to specify that only one jump is optimal on A_ξ. We are now able to state the uniqueness of solutions of Fokker-Planck equation of jumping particles in the case of a finite number of possible jumps for a set A verifying Hypothesis 5. As in the case of a single possible jump, we start with a lemma.

Lemma 1.5. *Assume Hypothesis 5 holds, we denote by (k, w) a couple which satisfies this hypothesis. There exists $\epsilon > 0$, such that for any $f \in L^\infty((0,T) \times \mathbb{T}^d)$, $\|f\|_{L^\infty} \leq \epsilon$, there exists $v \in H$ such that, for all $\mu \in L^2((0,T), H^1(\mathbb{T}^d))$, $\mu = 0$ on A:*

$$\begin{cases}\int_0^T (-\partial_t v - \nu\Delta v, \mu)_{H^{-1}\times H^1} = \int_0^T \int_{\mathbb{T}^d} (-\partial_t w - \nu\Delta w + f, \mu)_{H^{-1}\times H^1},\\ v(T) = w(T),\end{cases}$$

and v satisfies (20).

Proof. As in the previous case, the following operator is well defined:

$$\begin{aligned}\mathcal{T} : &H \to H\\ &v \to \mathcal{T}(v)\end{aligned}$$

where $\mathcal{T}(v)$ is the unique solution of

$$\begin{cases}-\partial_t \mathcal{T}(v) - \nu\Delta\mathcal{T}(v) = \tilde{f} \text{ in } A^c,\\ \mathcal{T}(v) = M(k,v) \text{ a.e. in } A,\\ v(T) = w(T).\end{cases}$$

As in the previous case, $\mathcal{T}$ has a fixed point u. Let us remark that u does not satisfy (20) but only $u = M(k,u)$ on A. However, as $\|f\|_{L^\infty}$ goes to 0, $\|u - w\|_{L^\infty} \to 0$. Thus because, w satisfies (23), if $\|f\|_{L^\infty}$ is small enough, then u also satisfies (20). Thus the result is proved. □

Theorem 1.6. *Under Hypothesis 5, for any positive $m_0 \in L^2(\mathbb{T}^d)$, there exists at most one $m \in L^2((0,T), H^1(\mathbb{T}^d))$ solution of the Fokker-Planck equation of particles jumping with jumps described by V.*

Proof. We denote by m_1 and m_2 two solutions. We construct for $i = 1, 2$, $v_i \in H$ as in Lemma 1.5 with respective source terms $f = f_1$ and $f = f_2$, where $\|f_1\|_{L^\infty}, \|f_2\|_{L^\infty} \leq \epsilon$, for ϵ sufficiently small (given by Lemma 1.5). We then evaluate

$$\int_0^T (-\partial_t(v_1 - v_2) - \nu\Delta(v_1 - v_2), m_1 - m_2)_{H^{-1}\times H^1}.$$

Because m_1 and m_2 are solutions, the following holds:

$$D(m_j) = \int_0^T (-\partial_t(v_i) - \nu\Delta(v_i), m_j)_{H^{-1}\times H^1} - \int_{\mathbb{T}^d} v_i(0) m_0.$$

Hence,

$$\int_0^T (-\partial_t(v_1 - v_2) - \nu\Delta(v_1 - v_2), m_1 - m_2)_{H^{-1}\times H^1} = 0. \tag{24}$$

On the other hand, using Lemma 1.5, we derive that

$$\int_0^T (-\partial_t(v_1 - v_2) - \nu\Delta(v_1 - v_2), m_1 - m_2)_{H^{-1}\times H^1} = \int_0^T\int_{\mathbb{T}^d} (m_1 - m_2)(f_1 - f_2).$$

Recalling (24), we obtain that

$$\int_0^T\int_{\mathbb{T}^d} (m_1 - m_2)(f_1 - f_2) = 0.$$

Because of the generality of f_1 and f_2, we easily deduce that $m_1 = m_2$. □

Remark 3. Let us insist on the fact that the two previous results rely on Hypothesis 5 which, in particular, states that at maximum one jump is feasible at a given location.

5. The stationary case

We now turn to the stationary case. Most of the arguments of the proofs of the results below follow the ones from the time dependent setting. Thus, we only present the arguments which differ from the ones in the time dependent case.

We assume that there exists V such that:

$$\begin{cases} V \in L^\infty(K, \mathbb{T}^d), \\ V \geq 0, \\ \sum_{\xi\in K} V(\xi, x) \leq 1, \end{cases}$$

where $K \subset \mathbb{T}^d$ is the finite set of possible jumps. We assume the following.

Hypothesis 6. There exists $w \in W^{2,\infty}(\mathbb{T}^d)$ such that

$$\forall \xi \in K, V(\xi, x)\,(k(x, \xi) + u(x + \xi) - u(x)) = 0 \text{ a.e. in } \mathbb{T}^d,$$

We study a stationary Fokker-Planck equation in which there is a fixed leaving rate of players $\delta > 0$ and a constant entry of players $\rho \in L^\infty(\mathbb{T}^d)$, $\rho \geq 0$. Namely, at a penalized level, we are interested in:

$$-\nu\Delta m_\epsilon + \delta m_\epsilon + \frac{1}{\epsilon} m_\epsilon(x)\Big(\sum_{\xi\in K} V(\xi, x)\Big) - \frac{1}{\epsilon}\sum_{\xi\in K} m_\epsilon(x - \xi)V(\xi, x - \xi) = \rho \text{ in } \mathbb{T}^d. \tag{25}$$

This section is organized as follows: we first show the existence and uniqueness of solutions of the penalized equation (25). We then show the existence of a limit as ϵ goes to 0. This limit satisfies the Fokker-Planck equation of jumping particles in a weak sense. We then prove the uniqueness of such limits.

5.1. The penalized equation

We begin this section by showing a general uniqueness result for equations of the type of (25).

Proposition 1.4. *For $(\lambda_1, .., \lambda_n) \in L^\infty(\mathbb{T}^d)^n$ with $\lambda_i \geq 0$ for all i. For any $\rho \in L^2(\mathbb{T}^d)$, there exists at most one solution $m \in H^2(\mathbb{T}^d)$ of:*

$$-\nu\Delta m + \delta m + m(x)\Big(\sum_{i=1}^n \lambda_i(x)\Big) - \sum_{i=1}^n m(x - \xi)\lambda_i(x - \xi) = \rho \text{ in } \mathbb{T}^d. \tag{26}$$

Proof. Let us assume that there exist two such solutions m_1 and m_2. Then if we note $\mu = m_1 - m_2$, it satisfies:

$$-\nu\Delta\mu + \delta\mu + \mu(x)\Big(\sum_{i=1}^n \lambda_i(x)\Big) - \sum_{i=1}^n \mu(x - \xi)\lambda_i(x - \xi) = 0 \text{ in } \mathbb{T}^d.$$

Let us define the operator $\mathcal{T}$ from $L^1(\mathbb{T}^d)$ into itself as follows: for any $m \in L^1(\mathbb{T}^d)$, $\mathcal{T}(m)$ is the unique solution of

$$-\nu\Delta\mathcal{T}(m) + \delta\mathcal{T}(m) + \mathcal{T}(m)(x)\big(\sum_{i=1}^{n}\lambda_i(x)\big) = \sum_{i=1}^{n} m(x-\xi)\lambda_i(x-\xi) \text{ in } \mathbb{T}^d.$$

This operator is well defined. Indeed it is well defined for $m \in L^2$ and can be easily extended to L^1. Let us assume $\mu \neq 0$, then the operator $\mathcal{T}$ has a spectral radius $r \geq 1$. The operator $\mathcal{T}$ is compact and

$$\mathcal{T}(\{m \in L^1(\mathbb{T}^d), m \geq 0\}) \subset \{m \in L^1(\mathbb{T}^d), m \geq 0\}.$$

Thus by the Krein-Rutman theorem [25], there exists $r \geq 1$, $w \in \{m \in L^1(\mathbb{T}^d), m \geq 0\}$, $w \neq 0$ such that:

$$-\nu\Delta w + \delta w + w\big(\sum_{i=1}^{n}\lambda_i(x)\big) = \frac{1}{r}\sum_{i=1}^{n} w(x-\xi)\lambda_i(x-\xi) \text{ in } \mathbb{T}^d.$$

Integrating in space, we obtain that

$$\int_{\mathbb{T}^d}\delta w + \int_{\mathbb{T}^d}\left(1-\frac{1}{r}\right)(\sum_{i=1}^{n}\lambda_i)w = 0.$$

The two terms of the left hand side are positive so we deduce that $w = 0$, which is a contradiction. Thus $\mu = 0$ and there is at most one solution of this PDE. □

We now show that there exists a solution of (25).

Proposition 1.5. *For any $\epsilon > 0$, there exists a unique solution $m \in L^1(\mathbb{T}^d)$ of (25). This solution m is positive.*

Proof. We define the application $\mathcal{T}$ from $L^1(\mathbb{T}^d)$ into itself by: for any $m \in L^1(\mathbb{T}^d)$, $\mathcal{T}(m)$ is the unique solution of

$$-\nu\Delta\mathcal{T}(m) + \delta\mathcal{T}(m) + \epsilon^{-1}\mathcal{T}(m)(x)\big(\sum_{\xi\in K} V(\xi,x)\big) = \epsilon^{-1}\sum_{\xi\in K} m(x-\xi)V(\xi,x-\xi) + \rho \text{ in } \mathbb{T}^d.$$

Let us observe that if $m \geq 0$, then by the maximum principle $\mathcal{T}(m) \geq 0$. Moreover, if $m \in \Omega$ defined by:

$$\Omega := \left\{ m \in L^1(\mathbb{T}^d), m \geq 0, \int_{\mathbb{T}^d} m(x)\left(\sum_{\xi\in K} V(\xi,x)\right)dx \leq \delta^{-1}\int_{\mathbb{T}^d}\rho \right\},$$

then $\mathcal{T}(m) \in \Omega$. Indeed, integrating in space the equation which defines $\mathcal{T}(m)$, we obtain that

$$\delta\int_{\mathbb{T}^d}\mathcal{T}(m) + \frac{1}{\epsilon}\int_{\mathbb{T}^d}\mathcal{T}(m)\left(\sum_{\xi\in K}V(\xi)\right) = \int_{\mathbb{T}^d}\rho + \frac{1}{\epsilon}\int_{\mathbb{T}^d} m\left(\sum_{\xi\in K}V(\xi)\right),$$

$$\left(\delta+\frac{1}{\epsilon}\right)\int_{\mathbb{T}^d}\mathcal{T}(m)\left(\sum_{\xi\in K}V(\xi)\right) \leq \int_{\mathbb{T}^d}\rho + \frac{1}{\delta\epsilon}\int_{\mathbb{T}^d}\rho.$$

Thus, Ω is stable under $\mathcal{T}$. Applying Schauder's fixed point theorem, we deduce that $\mathcal{T}$ has a fixed point. The uniqueness is given by the previous proposition. □

Remark 4. This result does not depend on the sets from which the particles jump, i.e. it does not depend on the function V except for the fact that $V \geq 0$ and $\sum_{\xi\in K} V(\xi) \leq 1$.

5.2. Existence and uniqueness of the solution of the stationary Fokker-Planck equation

We now turn to the existence of a limit density as ϵ goes to 0. A crucial result to pass to the limit is the following:

Lemma 1.6. *For $m \in H^1(\mathbb{T}^d)$ such that:*

$$\begin{cases} m \geq 0, \\ \delta \int_{\mathbb{T}^d} m = \int_{\mathbb{T}^d} \rho. \end{cases}$$

For $1 \leq p < 1 + \frac{2}{d}$ and for any k satisfying (4), there exists C depending on K, k and d such that:

$$\|m\|_{L^p} \leq C\left(-D(k,m) + \|\rho\|_{L^2}\right),$$

where $D(k,m)$ is defined by:

$$D(k,m) = \inf\left\{ \int_{\mathbb{T}^d} (-\nu\Delta v + \delta v)m - \int_{\mathbb{T}^d} \rho v | v \in H^1, v \leq M(k,v) \right\}.$$

If Hypothesis 1 holds, then there is C_1 depending only on K and d such that

$$\|m\|_{H^1}^2 \leq C_1\left(-D(k,m) + \|\Delta m\|_{L^1}\|\rho\|_{L^\infty}\right).$$

The following result holds true:

Theorem 1.7. *Under Hypothesis 6, there exists $m \in L^p(\mathbb{T}^d)$ for $1 < p < 1 + \frac{2}{d}$ such that:*

- $\delta \int_{\mathbb{T}^d} m = \int_{\mathbb{T}^d} \rho$.
- $\forall \xi \in K : V(\xi, x)m(x) = 0$ *almost everywhere in* $\mathbb{T}^d$.
- *For any $u \in W^{2,p'}(\mathbb{T}^d)$ which satisfies*

$$\forall \xi \in K, V(\xi,x)\left(k(x,\xi) + u(x+\xi) - u(x)\right) = 0 \text{ a.e. in } \mathbb{T}^d, \tag{27}$$

  the following holds:

$$\forall v \in W^{2,p'}, v \leq M(k,v): \quad \int_{\mathbb{T}^d} (-\nu\Delta(v-u) + \delta(v-u))m \geq \int_{\mathbb{T}^d} \rho(v-u).$$

Moreover, if Hypothesis 1 is satisfied, then we can choose $m \in H^1(\mathbb{T}^d)$ and such a function is unique.

Proof. The proof of both existence and uniqueness of the two previous results are the analogous of the ones in the time dependent case. □

Remark 5. Let us note that the question of the uniqueness of solutions is much simpler in the stationary case. Indeed the time regularity is no longer a problem and the existence of solutions for the "adjoint" problem is easily proved following the argument of the time-dependent case.

6. A remark on the generality of this method

Before using this notion of solution of Fokker-Planck equation in a MFG of impulse control, we indicate some natural generalizations of the results of the previous part. First working on the torus $\mathbb{T}^d$ does not play any role but to spare some technical difficulties. Thus those results generalize to more complex domain and boundary conditions. Secondly, the cost of jumps k can be allowed to depend on the time variable. If this dependence is smooth, this does not change our results. Also more general densities can be modeled with this kind of method, like for instance densities of populations governed by optimal switching policies, or by optimal stopping time type policies, where instead of leaving, "stopping" the trajectories restarts it at the origin (or at any given point). Both those two problems can be formulated in terms of QVI. As the regularity of the densities may vary, depending on the problem we are modeling, the exact definition of weak solution may also vary. The important point is that with the QVI comes a notion of "admissible" solutions for the QVI (in this article being admissible is satisfying $u \le Mu$). With this notion comes the notion of admissible density of particles which is, in this article, the fact that $D(m) > -\infty$. In general, a priori estimates on the solutions of the Fokker-Planck equation are then available using the same technique as we used and allow to develop such notions of solutions.

Let us end this section by commenting on the assumption on the fact that the number of possible jumps is finite. Such an assumption is crucial to establish an existence theory in some L^p space. However, it seems that in the case in which an infinite number of jumps occur, existence results in some measure space can be proven. The main difficulty is to write down the penalized equation, even though this can be done in a more case by case approach.

Part 2. Mean field games of impulse control through quasi-variational inequalities

We present in this part an application of the notion of solutions of a Fokker-Planck equation of jumping particles. We study a MFG of impulse control, where the density of players is naturally a solution of this kind of equation. We work here in the case of a finite number of possible jumps. We denote by K the set of jumps. We denote by k satisfying (4) the cost of the different jumps depending on the position. For any $v \in L^2((0,T), H^1(\mathbb{T}^d))$, we define Mv by

$$Mv(t,x) = \min_{\xi \in K}\{k(x,\xi) + v(t, x+\xi)\}.$$

We denote by f the running cost of the problem. The function f depends on space, time and on the distribution of the players (i.e. $f = f(t,x,m)$). The variables (t,x) will be suppressed whenever f appears. We make the following assumptions on f:

- f is continuous from $L^p((0,T)\times\mathbb{T}^d)$ endowed with its weak topology, for some $1 < p < 1 + \frac{2}{d}$ to $L^2((0,T), H^{-1}(\mathbb{T}^d))$.
- f is uniformly bounded from below by a constant $-C$ (where $C > 0$) on the positive elements of $L^1((0,T)\times\mathbb{T}^d)$
- Either
  - For any $C > 0$, f maps $\{m \in L^1((0,T)\times\mathbb{T}^d) | (\int_{\mathbb{T}^d}|m|)(t) \le C, \forall 0 \le t \le T\}$ into a bounded set of $L^\infty((0,T)\times\mathbb{T}^d)$.
  - or Hypothesis 1 holds and f maps bounded subsets of $L^2((0,T), H^1(\mathbb{T}^d))$ into bounded subsets of $L^q((0,T)\times\mathbb{T}^d)$ for $q > \frac{d}{2} + 1$ and $\|f(m)\|_{L^q}(1 + \|m\|_{L^2(H^1)})^{-1} \to 0$ as $\|m\|_{L^2(H^1)} \to \infty$.

Those assumptions are uniform in (t,x). For instance those assumptions are satisfied by f if it is built using a convolution of m with a smooth positive function.

As in the previous part, we work on the d dimensional torus to simplify the notations but all the following results can be extended to more complex situations. The problem we are interested in, is the existence of $(u,m) \in W^{1,2,p'} \times L^p((0,T)\times\mathbb{T}^d)$ such that:

$$\begin{cases} \max(-\partial_t u - \nu\Delta u - f(m), u - Mu) = 0 \text{ in } (0,T)\times\mathbb{T}^d, \\ u(T) = 0 \text{ in } \mathbb{T}^d, \\ \forall v \in W^{1,2,p'}, v \le Mv, v(T) = 0: \\ \int_0^T (-\partial_t(v-u) - \nu\Delta(v-u), m)_{H^{-1}\times H^1} - \int_{\mathbb{T}^d}(v-u)(0)m_0 \ge 0, \\ \int_0^T \int_{\mathbb{T}^d}(-\partial_t u - \nu\Delta u - f(m))m = 0. \end{cases} \tag{28}$$

The function u stands for the value function of the impulse control problem for a generic player of the MFG and m is the density of players. The first two lines have to be taken in the sense that u is the solution of the associated (strong) QVI. Formally it solves the impulse control problem for the generic player in which the running cost is $f(m)$ and k is the cost for the jumps. Once again we refer to [4] for a complete study of QVI and impulse control problems. In view of the previous part, m is a solution of a Fokker -Planck equation which models the density of (jumping) players of the game. Let us note that because there is no constraint such that $m = 0$ on $\{u = Mu\}$, m is not necessarily the solution of a limit problem as in the previous part with some V (describing the jumps) well chosen depending on u. Indeed, in view of [5], we do not expect the existence of solutions if we impose such strong conditions which are assimilated with Nash equilibria in pure strategies for the MFG. That is why we do not impose the condition $m = 0$ on $\{u = Mu\}$ but the integral relation of the last line. It is the formulation of the fact that (u, m) is a mixed solution of the MFG, i.e. that this system characterizes Nash equilibria in mixed strategies. Let us recall the interpretation of such a relation.

Formally, a natural requirement for the solution of such a MFG is to impose that $m = 0$ on $\{u = Mu\}$ which is the set where it is optimal to use an impulse control. The integral formulation in this system requires that $m = 0$ on $\{-\partial_t u - \Delta u < f(m)\}$ which is the set where it is strictly optimal to use an impulse control. The difference here is that on $\{u = Mu\}$ one can still have $-\partial_t u - \Delta u = f(m)$ and thus that it is both optimal to stay and to use a control. We do not impose that m vanish in such a situation. Such a relaxation makes the problem more convex and allows us to prove an existence result while still conserving a uniqueness property.

The methodology to study (28) is the following: we first introduce a penalized version of this problem and then we show how we can pass to the limit to obtain the existence of solutions of (28). Later on we prove a result of uniqueness for such solutions.

7. The penalized problem

We introduce here the penalized problem:

$$\begin{cases} \max(-\partial_t u - \nu\Delta u - f(m), u - Mu) = 0 \text{ in } (0,T)\times\mathbb{T}^d, \\ u(T) = 0 \text{ in } \mathbb{T}^d, \\ \partial_t m - \nu\Delta m = \sum_{\xi\in K} V(t, x-\xi, \xi)\frac{1}{\epsilon}\mathbb{1}_{\{u=Mu\}}(t, x-\xi)m(t, x-\xi) \\ \qquad\qquad - \frac{1}{\epsilon}\left(\sum_{\xi\in K} V\right)\mathbb{1}_{\{u=Mu\}} m \text{ in } (0,T)\times\mathbb{T}^d, \\ m(0) = m_0 \text{ in } \mathbb{T}^d, \\ \sum_{\xi\in K} V \geq 1 \text{ on } \{-\partial_t u - \nu\Delta u < f(m)\}, \\ \forall \xi\in K, u(t,x) \neq k(t,x,\xi) + u(t, x+\xi) \Rightarrow V(t,x,\xi) = 0, \\ \forall\xi\in K, \forall(t,x)\in(0,T)\times\mathbb{T}^d, 0\leq V(t,x,\xi)\leq 1. \end{cases} \tag{29}$$

Let us mention that because of the condition

$$\forall\xi\in K, u(t,x)\neq k(t,x,\xi) + u(t,x+\xi) \Rightarrow V(t,x,\xi) = 0,$$

the function u plays the role of the "adjoint state" for the Fokker-Planck equation. Let us remark that u plays the role of the function which we assumed exists in the previous part, with different hypotheses depending on the context. In consequence, we do not make any additional assumption here. Let us remark that we do not impose that the sum over $\xi \in K$ of the functions V is bounded by 1. Up to a change of the constant ϵ, this does not play any role in the following. The important fact is that this sum is bounded from below by a non negative constant on the set $\{-\partial_t u - \nu\Delta u < f(m)\}$. This means that players are trying to use their control when it is strictly suboptimal to stay where they are.

Recalling the previous part, it is natural to study first such a penalized system, in order to pass to the limit $\epsilon \to 0$. Indeed the equation satisfied by m cannot be easily written in terms of a partial differential equation whereas it can at a penalized level. The potential V gives at each point (t, x) the jump used by the players at this point. If $V(t, x, \xi) \neq 0$ then some players use the jump ξ at (t, x). The way V is defined adds convexity to the problem and makes the existence of a solution possible; it is a way of expressing at a penalized level that we are looking for Nash equilibria of the MFG in mixed strategies. The technique is inspired from [5] where it is shown that such a system, for variational inequalities instead of quasi variational inequalities, leads to a solution of the MFG system for optimal stopping. Before proving the main result concerning the penalized MFG system, we prove the following preliminary result:

Proposition 2.1. *Let us take any sequence $(f_n)_{n\geq 0}$ and a constant $C > 0$, such that for all $n \in \mathbb{N}$, $f_n \in L^2((0,T)\times\mathbb{T}^d)$ and $f_n \geq -C$. If $(f_n)_n$ is bounded in $L^2((0,T)\times\mathbb{T}^d)$ and converges toward $f \in L^2((0,T)\times\mathbb{T}^d)$ in $L^2((0,T),H^{-1}(\mathbb{T}^d))$ with $f \geq -C$, then the sequence $(u_n)_n$ of solutions of $QVI(f_n,k)$ converges toward the solution u of $QVI(f,k)$ in $L^2((0,T),H^1(\mathbb{T}^d))$.*

Proof. The sequence $(\|f_n\|_{L^2})_n$ is bounded and for all $n \geq 0$, $f_n \geq -C$. Hence $(u_n)_n$ is a bounded sequence of $W^{1,2,2}$. Extracting a subsequence if necessary, it converges to a limit $u^* \in L^2((0,T),H^2(\mathbb{T}^d))$ in $L^2((0,T),H^1(\mathbb{T}^d))$. The limit u^* satisfies $u^* \leq Mu^*$ almost everywhere. Let us take $v \in L^2((0,T),H^1(\mathbb{T}^d))$ such that $v \leq Mu^*$. Obviously the following holds:

$$v_n := v - Mu^* + Mu_n \leq Mu_n.$$

Thus because of $QVI(f_n,k)$, we obtain

$$-\int_0^T\int_{\mathbb{T}^d} \partial_t u_n(v_n-u_n) + \nu\int_0^T\int_{\mathbb{T}^d}\nabla u_n\cdot\nabla(v_n-u_n) \geq \int_0^T\int_{\mathbb{T}^d} f_n(v_n-u_n).$$

Re arranging this inequality leads to

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^d} f_n(v-u_n) \leq &-\int_0^T\int_{\mathbb{T}^d}\partial_t u_n(v-u_n) + \nu\int_0^T\int_{\mathbb{T}^d}\nabla u_n\cdot\nabla(v-u_n),\\ &-\int_0^T\int_{\mathbb{T}^d}(\partial_t u_n + \nu\Delta u_n - f_n)(Mu_n - Mu^*).\end{aligned}$$

Let us remark that $(\|Mu_n - Mu^*\|_{L^2})_n$ converges to 0 as $n \to \infty$. Thus, because $(\partial_t u_n + \Delta u_n - f_n)_n$ is bounded in L^2, passing to the limit in the previous equation we obtain

$$-\int_0^T\int_{\mathbb{T}^d}\partial_t u^*(v-u^*) + \nu\int_0^T\int_{\mathbb{T}^d}\nabla u^*\cdot\nabla(v-u^*) \geq \int_0^T\int_{\mathbb{T}^d} f(v-u^*).$$

We conclude by the uniqueness of solutions of QVI [27], that $u^* = u$, the unique solution of this QVI. □

We are now able to prove the following.

Theorem 2.1. *There exists a solution (u,m,V) of (29) such that $(u,m,V) \in (W^{1,2,2})^2 \times L^\infty(K\times(0,T)\times\mathbb{T}^d)$ and the equations are satisfied in L^2.*

Proof. Let us fix $1 < p < 1+\frac{2}{d}$ such that f is continuous from $L^p((0,T)\times\mathbb{T}^d)$ endowed with its weak topology to $L^2((0,T),H^{-1}(\mathbb{T}^d))$. We define an application $\mathcal{F}_1$ from $L^2((0,T),H^1(\mathbb{T}^d))$ (endowed with its weak topology) into itself (endowed with its strong topology) by: for any $m \in L^2((0,T),H^1(\mathbb{T}^d))$, $\mathcal{F}_1(m)$ is the unique solution of the QVI with costs $f(m)$ and k. $\mathcal{F}_1$ is well defined and continuous from Proposition 2.1. Then, we define the set-valued function $\mathcal{F}_2$ from $L^2((0,T)\times\mathbb{T}^d)$ into itself by: for any $\mu \in L^2$, (we define $u = \mathcal{F}_1(\mu)$):

$$\mathcal{F}_2(\mu) := \{m \in L^2((0,T)\times\mathbb{T}^d), \exists V \in L^\infty, (m,V) \text{ solves } (29)(u)\},$$

where (29)(u) is

$$(29)(u)\begin{cases}\partial_t m - \nu\Delta m - \sum_{\xi\in K} V(t,x-\xi,\xi)\frac{1}{\epsilon}\mathbb{1}_{\{u=Mu\}}(t,x-\xi)m(t,x-\xi)+\\ \qquad +\frac{1}{\epsilon}(\sum_{\xi\in K} V)\mathbb{1}_{\{u=Mu\}}m = 0 \text{ in } (0,T)\times\mathbb{T}^d,\\ m(0)=m_0 \text{ in } \mathbb{T}^d,\\ \sum_{\xi\in K} V \geq 1 \text{ on } \{-\partial_t u - \nu\Delta u < f(\mu)\},\\ \forall \xi\in K, u(t,x)\neq k(x,\xi)+u(t,x+\xi) \Rightarrow V(t,x,\xi)=0,\\ 0\leq V(t,x,\xi)\leq 1.\end{cases}$$

Let us note that for any $\mu \in L^2$, $\mathcal{F}_2(\mu)\neq\emptyset$ because of the results of the first part. As we want to apply Kakutani's fixed point theorem [24] on $\mathcal{F}_2$, we need to prove that $\mathcal{F}_2$ has a closed graph and that it is valued in the set of convex and closed subsets of $L^2((0,T)\times\mathbb{T}^d)$. The last point comes easily from the linearity of the equation in m. We focus in this proof on the closed graph property. We thus take two converging sequences $(\mu_n)_{n\geq 0}$ and $(m_n)_{n\geq 0}$ valued in $L^2((0,T)\times\mathbb{T}^d)$ which converges to respectively μ and m and such that for all $n\geq 0$, $m_n\in\mathcal{F}_2(\mu_n)$. We want to show that $m\in\mathcal{F}_2(\mu)$. For all $n\geq 0$, we define $u_n=\mathcal{F}_1(\mu_n)$ and we denote by V_n the function such that m_n is the solution of (29)(u) associated to u_n, V_n and μ_n. From the assumption we made on f, we know that $(u_n)_{n\geq 0}$ is a bounded sequence of $W^{1,2,q}$ for some $q>1+\frac{d}{2}$. Thus it uniformly converges toward its limit u given by Proposition 2.1. Extracting a subsequence if necessary, $(V_n)_{n\geq 0}$ weakly converges toward some function V in L^∞. It is now sufficient to show that V satisfies

$$\begin{cases}\sum_{\xi\in K} V \geq 1 \text{ on } \{-\partial_t u - \nu\Delta u < f(\mu)\},\\ \forall \xi\in K, u(t,x)\neq k(x,\xi)+u(t,x+\xi) \Rightarrow V(t,x,\xi)=0.\end{cases} \tag{30}$$

The second point is immediately verified as it is true all along the sequence (i.e. for u_n and V_n) and because $(u_n)_{n\geq 0}$ uniformly converges toward the continuous function u. It remains to show the first point of (30). From the regularity assumptions we made on the coupling f, the sequence of positive functions $(\partial_t u_n+\nu\Delta u_n + f(\mu_n))_{n\geq 0}$ is bounded in $L^2((0,T)\times\mathbb{T}^d)$ and thus weakly converges toward $\partial_t u+\nu\Delta u+f(\mu)$. From this we deduce that

$$\limsup_{n\to\infty}\mathbb{1}_{\{-\partial_t u_n-\nu\Delta u_n = f(\mu_n)\}}\leq \mathbb{1}_{\{-\partial_t u-\nu\Delta u=f(\mu)\}}.$$

Thus the inequality

$$\left(\sum_{\xi\in K} V_n\right)+\mathbb{1}_{\{-\partial_t u_n-\nu\Delta u_n=f(\mu_n)\}}\geq 1,$$

yields at the limit

$$\liminf_{n\to\infty}\left(\sum_{\xi\in K} V_n\right)+\mathbb{1}_{\{-\partial_t u-\nu\Delta u_n=f(\mu)\}}\geq 1.$$

Thus V satisfies (30). From this we obtain that (u,m,V,μ) is indeed a solution of (29)(u) and thus that $\mathcal{F}_2$ has closed graph, and thus a fixed point. □

8. Existence of solutions of the MFG system

In this section we discuss the existence of solutions of the MFG system (28). The proof of this result consists in passing to the limit in the penalized system. Let us remark that in the first part of this article, we use either Hypothesis 2 or 4 to pass to the limit $\epsilon\to 0$ in a Fokker-Planck equation of jumping particles. Here such an assumption is no longer required, as the jumps of the players are by definition optimal for a certain optimization problem. Obviously this problem is the optimization problem the players have to solve. We now state the main result of this paper.

Theorem 2.2. *For $m_0\in L^2(\mathbb{T}^d)$, there exists a solution $(u,m)\in W^{1,2,p'}\times L^p((0,T)\times\mathbb{T}^d)$ of (28), where p is given by the assumption on f. Moreover if Hypothesis 1 is satisfied $m_0\in L^\infty$, then $m\in L^2((0,T),H^1(\mathbb{T}^d))$.*

Proof. For $\epsilon > 0$ we denote by $(u_\epsilon, m_\epsilon, V_\epsilon)$ a solution of the penalized system (29). We start with the case in which Hypothesis 1 is not verified. We first show some compactness for the sequence $(m_\epsilon)_{\epsilon>0}$. Let us remark that

$$\begin{aligned} D(m_\epsilon) &= \int_0^T \int_{\mathbb{T}^d} (-\partial_t u_\epsilon - \nu\Delta u_\epsilon) m_\epsilon - \int_{\mathbb{T}^d} u_\epsilon(0) m_0, \\ &= -\frac{1}{\epsilon} \int_0^T \int_{\mathbb{T}^d} \left(\sum_{\xi\in K} V_\epsilon(t,x,\xi) k(x,\xi) \right) m_\epsilon(t,x) dx dt. \end{aligned}$$

Furthermore, because of Lemma 1.4,

$$C(-D(m_\epsilon) + \|m_0\|_{L^1}) \geq \|m_\epsilon\|_{L^p}.$$

We then deduce that:

$$\begin{aligned} \|m_\epsilon\|_{L^p} &\leq C \left(-\int_0^T \int_{\mathbb{T}^d} (-\partial_t u_\epsilon - \nu\Delta u_\epsilon) m_\epsilon + \int_{\mathbb{T}^d} u_\epsilon(0) m_0 + C\|m_0\|_{L^1} \right), \\ &\leq C\|u_\epsilon\|_{W^{1,2,\infty}} \|m_\epsilon\|_{L^1} + C \int_{\mathbb{T}^d} u_\epsilon(0) m_0 + C\|m_0\|_{L^1}. \end{aligned}$$

Because $\|m_\epsilon\|_{L^1} = T\|m_0\|_{L^1}$, we deduce that $(m_\epsilon)_{\epsilon>0}$ is a bounded sequence of $L^p((0,T)\times\mathbb{T}^d)$. So, extracting a subsequence if necessary, $(m_\epsilon)_\epsilon$ weakly converges toward m in $L^p((0,T)\times\mathbb{T}^d)$. Because f is continuous for the weak topology, we deduce from Lemma 2.1, that $(u_\epsilon)_\epsilon$ converges toward u solution of the quasi variational inequality associated to

$$\begin{cases} \max(-\partial_t u - \nu\Delta u - f(m), u - Mu) = 0 \text{ in } (0,T)\times\mathbb{T}^d, \\ u(T) = 0 \text{ in } \mathbb{T}^d. \end{cases}$$

Now let us remark that:

$$\begin{aligned} \int_0^T \int_{\mathbb{T}^d} (-\partial_t u - \nu\Delta u) m - \int_{\mathbb{T}^d} u(0) m_0 &= \lim_{\epsilon\to 0} \int_0^T \int_{\mathbb{T}^d} (-\partial_t u - \nu\Delta u) m_\epsilon - \int_{\mathbb{T}^d} u(0) m_0, \\ &= \lim_{\epsilon\to 0} -\frac{1}{\epsilon} \int_0^T \int_{\mathbb{T}^d} \left(\sum_{\xi\in K} V_\epsilon(t,x,\xi)(u(t,x) - u(t,x+\xi)) \right) m_\epsilon(t,x) dx dt, \\ &= \lim_{\epsilon\to 0} -\frac{1}{\epsilon} \int_0^T \int_{\mathbb{T}^d} \left(\sum_{\xi\in K} V_\epsilon(t,x,\xi)(u(t,x) - k(x,\xi) - u(t,x+\xi)) \right) m_\epsilon(t,x) dx dt \\ &\quad + D(m_\epsilon). \end{aligned}$$

Because of the uniform estimates on $(u_\epsilon)_{\epsilon>0}$, we deduce that the convergence toward u is uniform. Thus there exists a continuous function $c : \mathbb{R} \to \mathbb{R}$ such that $c(0) = 0$ and

$$\int_0^T \int_{\mathbb{T}^d} (-\partial_t u - \nu\Delta u) m - \int_{\mathbb{T}^d} u(0) m_0 = \lim_{\epsilon\to 0} -\frac{c(\epsilon)}{\epsilon} \int_0^T \int_{\mathbb{T}^d} \left(\sum_{\xi\in K} V_\epsilon(t,x,\xi) \right) m_\epsilon(t,x) dx dt + D(m_\epsilon).$$

From the uniform bounds assumed on k, we deduce that the first term in the limit is negligible in front of $D(m_\epsilon)$ as ϵ goes to 0. Thus we obtain first that $(D(m_\epsilon))_{\epsilon>0}$ is bounded, and then that the first term in the limit converges to 0 as ϵ goes to 0. Hence

$$\int_0^T \int_{\mathbb{T}^d} (-\partial_t u - \nu \Delta u) m - \int_{\mathbb{T}^d} u(0) m_0 = \lim_{\epsilon \to 0} D(m_\epsilon).$$

Thus we deduce easily

$$\begin{cases} \forall v \in W^{1,2,p'}, v \leq Mv, v(T) = 0: \\ \int_0^T \int_{\mathbb{T}^d} (-\partial_t v - \nu \Delta v) m - \int_{\mathbb{T}^d} v(0) m_0 \geq \int_0^T \int_{\mathbb{T}^d} (-\partial_t u - \nu \Delta u) m - \int_{\mathbb{T}^d} u(0) m_0. \end{cases}$$

Finally, let us compute

$$\begin{aligned} -D(m_\epsilon) &= \frac{1}{\epsilon} \int_0^T \int_{\mathbb{T}^d} (\sum_{\xi \in K} V_\epsilon(t,x,\xi) k(x,\xi)) m_\epsilon(t,x) dx dt, \\ &\geq \frac{1}{\epsilon} \int_0^T \int_{\mathbb{T}^d} (\sum_{\xi \in K} V_\epsilon(t,x,\xi)) k_0 m_\epsilon(t,x) \mathbb{1}_{\{-\partial_t u_\epsilon - \nu \Delta u_\epsilon < f(m_\epsilon)\}}(t,x) dx dt, \\ &\geq \frac{k_0}{\epsilon} \int_0^T \int_{\mathbb{T}^d} m_\epsilon \mathbb{1}_{\{-\partial_t u_\epsilon - \nu \Delta u_\epsilon < f(m_\epsilon)\}}, \end{aligned}$$

where we have used the assumption that $\sum_{\xi \in K}(V_\epsilon) \geq 1$ on $\{-\partial_t u_\epsilon - \nu \Delta u_\epsilon < f(m_\epsilon)\}$. Since $(D(m_\epsilon))_{\epsilon > 0}$ is a bounded sequence, we deduce that

$$\left(\frac{k_0}{\epsilon} \int_0^T \int_{\mathbb{T}^d} m_\epsilon \mathbb{1}_{\{-\partial_t u_\epsilon - \nu \Delta u_\epsilon < f(m_\epsilon)\}} \right)_{\epsilon > 0}$$

is also a bounded sequence and thus that:

$$\int_0^T \int_{\mathbb{T}^d} (-\partial_t u - \nu \Delta u - f(m)) m = 0.$$

This ends the proof of the fact that (u, m) is a solution of (28).

In the case in which Hypothesis 1 is satisfied we use the following: starting from Lemma 1.4, we know that

$$\|m_\epsilon\|^2_{L^2((0,T),H^1)} \leq - \int_0^T \int_{\mathbb{T}^d} (-\partial_t u_\epsilon - \nu \Delta u_\epsilon) m_\epsilon + \int_{\mathbb{T}^d} u_\epsilon(0) m_0 + C \|\Delta m_\epsilon\|_{L^1} \|m(0)\|_{L^\infty}.$$

In this case, because the assumption on f are weaker, we have not yet proven that $(D(k, m_\epsilon))_{\epsilon > 0}$ is bounded. However we have the following estimate

$$\|\Delta m_\epsilon\|_{L^1} \leq -C D(k, m_\epsilon),$$

where C does not depend on ϵ. This estimate comes from the fact that $D(k, m_\epsilon)$ controls the terms outside $\partial_t m_\epsilon - \nu \Delta m_\epsilon$ in the PDE satisfied by m_ϵ. Thus we deduce that

$$\|m_\epsilon\|^2_{L^2((0,T),H^1)} \leq C(1 + \|m(0)\|_{L^\infty}) \left(- \int_0^T \int_{\mathbb{T}^d} (-\partial_t u_\epsilon - \nu \Delta u_\epsilon) m_\epsilon + \int_{\mathbb{T}^d} u_\epsilon(0) m_0 \right).$$

From the growth assumption made on f, we deduce that $(m_\epsilon)_{\epsilon > 0}$ is a bounded sequence of $L^2((0,T), H^1(\mathbb{T}^d))$. The rest of the proofs follows the same argument as in the previous case. □

9. Uniqueness of solutions of the MFG system

We now turn to the question of the uniqueness of solutions of (28). As it is the case in MFG of continuous control [30], uniqueness does not hold in general. However it does under an assumption on the monotonicity of the costs (i.e. the coupling). In our model the density of players appears only in the running cost f and thus only an assumption on f is required for uniqueness to hold. We recall that f is said to be strictly monotone if:

$$\int_0^T \int_{\mathbb{T}^d} (f(m_1) - f(m_2))(m_1 - m_2) > 0 \text{ if } m_1 \neq m_2.$$

For instance, if $f(m)(t,x) = g(m(t,x))$ and g is a strictly increasing function from $\mathbb{R}_+$ to $\mathbb{R}$ then f is strictly monotone.

Theorem 2.3. *Assume that f is strictly monotone, then there exists at most one solution of* (28).

Proof. The proof of this statement is directly inspired from the original proof of uniqueness in MFG of continuous control [30]. Let us take (u_1, m_1) and (u_2, m_2) two solutions of (28). We note by u and m the differences $u_1 - u_2$ and $m_1 - m_2$. Let us observe that because of the optimality of u_1 in the variational relation satisfied by m_1, and similarly for u_2 and m_2, we obtain

$$\begin{aligned}\int_0^T \int_{\mathbb{T}^d} (-\partial_t u - \nu\Delta u)m &= \int_0^T \int_{\mathbb{T}^d} (-\partial_t(u_1 - u_2) - \nu\Delta(u_1 - u_2))m_1 \\ &\quad + \int_0^T \int_{\mathbb{T}^d} (-\partial_t(u_2 - u_1) - \nu\Delta(u_2 - u_1))m_2, \\ &\leq 0.\end{aligned}$$

On the other hand, because $m_2 \geq 0$, and

$$\int_0^T \int_{\mathbb{T}^d} (-\partial_t u_1 - \nu\Delta u_1 - f(m_1))m_1 = 0,$$

we deduce that

$$\begin{aligned}\int_0^T \int_{\mathbb{T}^d} (-\partial_t u_1 - \nu\Delta u_1)m &= \int_0^T \int_{\mathbb{T}^d} f(m_1)m_1 + m_2(\partial_t u_1 + \nu\Delta u_1), \\ &\geq \int_0^T \int_{\mathbb{T}^d} f(m_1)(m_1 - m_2).\end{aligned}$$

We have used the fact that $-\partial_t u_1 - \nu\Delta u_1 \leq f(m_1)$. Obviously we have the analogous for u_2. Putting the pieces together we finally obtain

$$\int_0^T \int_{\mathbb{T}^d} (f(m_1) - f(m_2))(m_1 - m_2) \leq \int_0^T \int_{\mathbb{T}^d} (-\partial_t u - \nu\Delta u)m \leq 0.$$

Using the strict monotonicity of f, we have just proven that $m_1 = m_2$ and thus that there exists at most one solution of (28). □

10. The stationary setting

In this section, we present a stationary setting for a MFG of impulse control. We denote by k the cost of jumps and we assume that it satisfies (4). Let f denotes the running cost of the players. We assume that

- f is continuous from $L^p(\mathbb{T}^d)$ endowed with its the weak topology, for some $1 < p < 1 + \frac{2}{d}$, to $H^{-1}(\mathbb{T}^d)$.
- f is uniformly bounded from below by a constant $-C$ (where $C > 0$) on the positive elements of $L^1(\mathbb{T}^d)$.
- Either
  - Hypothesis 1 holds and f maps bounded subsets of $H^1(\mathbb{T}^d)$ into a bounded subset of $L^q(\mathbb{T}^d)$ for $q > \frac{d}{2}$ and f grows sublinearly with respect to those spaces; i.e. $\|f(m)\|_{L^q}(1 + \|m\|_{H^1})^{-1} \to 0$ as $\|m\|_{H^1} \to \infty$.
  - or f maps bounded subsets of $L^1(\mathbb{T}^d)$ into bounded subsets of $L^\infty(\mathbb{T}^d)$.

Let $\delta > 0$ be the death rate of the players and $\lambda > 0$ their discount rate. Let $\rho \in L^2(\mathbb{T}^d)$, $\rho \geq 0$ be the entry rate of the players. The jump operator M is defined by:

$$Mu(x) = \min_{\xi \in K}\{k(x,\xi) + u(x+\xi)\}.$$

We are interested in a solution $(u,m) \in W^{2,p'} \times L^p(\mathbb{T}^d)$ of the following MFG system:

$$\begin{cases} \max(-\nu\Delta u + \lambda u - f(m), u - Mu) = 0 \text{ in } \mathbb{T}^d, \\ \forall v \in W^{2,p'}(\mathbb{T}^d), v \leq Mv: \\ \int_{\mathbb{T}^d}(-\nu\Delta(v-u) + \delta(v-u))m \geq \int_{\mathbb{T}^d}\rho(v-u), \\ \int_{\mathbb{T}^d}(-\nu\Delta u + \lambda u - f(m))m = 0. \end{cases} \tag{31}$$

The following result holds true:

Theorem 2.4. *There exists a solution $(u,m) \in W^{2,\infty}(\mathbb{T}^d) \times L^p(\mathbb{T}^d)$ of (31), where p is given by the assumption on f. If Hypothesis 1 is satisfied, then $m \in H^1(\mathbb{T}^d)$. The solution of (31) is unique under the assumption that f is strictly monotone.*

We do not detail the proof of this result because its argument follows step by step the one of the time dependent case. Indeed we pass to the limit in the following penalized system:

$$\begin{cases} \max(-\nu\Delta u + \lambda u - f(m), u - Mu) = 0 \text{ in } \mathbb{T}^d, \\ -\nu\Delta m + \delta m + \frac{1}{\epsilon}\left(\sum_{\xi\in K} V\right)\mathbb{1}_{\{u=Mu\}}m - \sum_{\xi \in K} V(x-\xi,\xi)\frac{1}{\epsilon}\mathbb{1}_{\{u=Mu\}}(x-\xi)m(x-\xi) = \rho \text{ in } \mathbb{T}^d, \\ \sum_{\xi\in K} V \geq 1 \text{ on } \{-\nu\Delta u + \lambda u < f(m)\}, \\ \forall \xi \in K, u(x) \neq k(x,\xi) + u(x+\xi) \Rightarrow V(x,\xi) = 0, \\ \forall \xi \in K, \forall (x) \in (0,T)\times\mathbb{T}^d, 0 \leq V(x,\xi) \leq 1, \end{cases}$$

using the estimate in Lemma 1.6.

11. The optimal control interpretation

In this section, we present an optimal control interpretation of a MFG of impulse control. We indicate that a certain optimization problem has a solution, and that the solution of the MFG, for which we have proven the existence in the previous part, is the solution of this optimization problem. A natural interpretation of this phenomenon is that in this variational setting, a MFG equilibrium is also an optimal point for the problem of a social planner.

To make this section simpler we work only in the stationary setting. The case of the time dependent problem is mentioned at the end of this section. Let us note that the optimal control interpretation of MFG has been introduced in [30] and that it can be used to show the existence of weak solutions for certain MFG system, see for instance [12].

Let $\mathcal{F}$ be a strictly convex function from $L^p(\mathbb{T}^d)$ to $\mathbb{R}$, bounded from below. We assume that there exists f, satisfying the requirements of the previous section for p, such that for any $m, m' \in L^p(\mathbb{T}^d)$:

$$\lim_{\theta \to 0} \frac{\mathcal{F}((1-\theta)m + \theta m') - \mathcal{F}(m)}{\theta} = \int_{\mathbb{T}^d} f(m)(m' - m).$$

We define the following application from $L^p(\mathbb{T}^d)$ to $\mathbb{R} \cup \{-\infty\}$:

$$D(m) = \inf \left\{ \int_{\mathbb{T}^d} (-\nu \Delta v + \delta v)m - \int_{\mathbb{T}^d} \rho v | v \in W^{2,p'}(\mathbb{T}^d), v \leq Mv \right\}.$$

The optimal control interpretation of the MFG of impulse control leads to the following optimization problem:

$$\inf_{m \in \mathcal{H}} \mathcal{F}(m) - D(m) \tag{32}$$

where $\mathcal{H} = \{m \in L^p(\mathbb{T}^d), m \geq 0\}$. We now establish the following result:

Proposition 2.2. *Under the previous assumptions on $\mathcal{F}$, the problem (32) admits a unique minimizer $m^* \in \mathcal{H}$.*

Proof. We first prove that the function $D(\cdot)$ is concave and upper semi continuous on $\mathcal{H}$. Assume that the sequence $(m_n)_{n \geq 0}$ converges weakly toward m in L^p. For any $v \in W^{2,p'}(\mathbb{T}^d)$, $v \leq Mv$ we obtain

$$\int_{\mathbb{T}^d} (-\nu \Delta v + \delta v)(m_n - m) \to_{n \to \infty} 0.$$

Thus taking a minimizing sequence $(v_p)_{p \geq 0}$ in $W^{2,p'}(\mathbb{T}^d)$ for $D(m)$

$$\begin{aligned} D(m_n) &\leq \int_{\mathbb{T}^d} (-\nu \Delta v_p + \delta v_p)m_n - \int_{\mathbb{T}^d} \rho v_p, \\ &\to_{n \to \infty} \int_{\mathbb{T}^d} (-\nu \Delta v_p + \delta v_p)m - \int_{\mathbb{T}^d} \rho v_p. \end{aligned}$$

Thus, because $(v_p)_{p \geq 0}$ is a minimizing sequence, we deduce that $D(\cdot)$ is upper semi continuous, and it is concave because it is the infimum of linear functions. Now let $(m_n)_{n \geq 0}$ be a minimizing sequence of (32). If we denote by μ the solution of

$$-\nu \Delta \mu + \delta \mu = \rho,$$

then we observe that $\mu \in \mathcal{H}$ and $D(\mu) = 0$. Recalling that D is negative because we can always choose $v = 0$ in the infimum, we deduce that

$$0 \leq -D(m_n) \leq \mathcal{F}(\mu) - \inf \mathcal{F} + o(1).$$

Recalling Lemma 1.6, $(m_n)_{n \geq 0}$ is thus a bounded sequence of $L^p(\mathbb{T}^d)$. Thus it converges weakly to $m^* \in L^p(\mathbb{T}^d)$. Because $\mathcal{F}$ is weakly sequentially lower semi continuous (it is continuous and convex) and $D(\cdot)$ is weakly sequentially upper semi continuous, we deduce that m^* is a minimizer of (32). This minimizer is unique because $\mathcal{F}$ is strictly convex and D is concave. □

Now let us remark that the solution (u, m) of the MFG system (31) (with $\lambda = \delta$) given by Theorem 2.4 satisfies the following system of variational inequalities:

$$\begin{cases} \forall \mu \in \mathcal{H} : \\ \int_{\mathbb{T}^d} (-\nu \Delta u + \delta u - f(m))(\mu - m) \leq 0, \\ \forall v \in W^{2,p'}(\mathbb{T}^d), v \leq Mv : \\ \int_{\mathbb{T}^d} (-\nu \Delta (v - u) + \delta (v - u))m \geq \int_{\mathbb{T}^d} \rho (v - u). \end{cases} \tag{33}$$

This system of variational inequalities is almost the characterization of a saddle point of (32). From this observation we deduce the following:

Theorem 2.5. *Under the assumptions of the previous result, the unique minimizer of (32) is the density of players m of the MFG of impulse control. Thus if (u,m) is the solution of (31) given by Theorem 2.4, then m is the unique minimizer of (32).*

Proof. Let (u,m) be the unique solution of (31). For any $m' \in \mathcal{H}$, $0<\theta<1$, using the second variational inequality of (33) we deduce that:

$$\begin{aligned}
&\mathcal{F}\left((1-\theta)m+\theta m'\right) - D\left((1-\theta)m+\theta m'\right) - \mathcal{F}(m) + D(m)\\
&= \mathcal{F}\left((1-\theta)m+\theta m'\right) - D\left((1-\theta)m+\theta m'\right) - \mathcal{F}(m)\\
&\qquad + \int_{\mathbb{T}^d} (-\nu\Delta u + \delta u)m - \int_{\mathbb{T}^d} \rho u.
\end{aligned}$$

Using the definition of $D(\cdot)$, we now obtain that:

$$\begin{aligned}
&\mathcal{F}\left((1-\theta)m+\theta m'\right) - D\left((1-\theta)m+\theta m'\right) - \mathcal{F}(m) + D(m)\\
&\quad \geq \mathcal{F}\left((1-\theta)m+\theta m'\right) - \mathcal{F}(m) + \int_{\mathbb{T}^d} (-\nu\Delta u + \delta u)\left((1-\theta)m+\theta m'\right)\\
&\qquad + \int_{\mathbb{T}^d} \rho u + \int_{\mathbb{T}^d} (-\nu\Delta u + \delta u)m - \int_{\mathbb{T}^d} \rho u,\\
&\quad = \mathcal{F}\left((1-\theta)m+\theta m'\right) - \mathcal{F}(m) - \theta \int_{\mathbb{T}^d} (-\nu\Delta u + \delta u)(m'-m).
\end{aligned}$$

Hence, using the first variational inequality of (33), we obtain that:

$$\lim_{\theta\to 0^+} \frac{\mathcal{F}\left((1-\theta)m+\theta m'\right) - D\left((1-\theta)m+\theta m'\right) - \mathcal{F}(m) + D(m)}{\theta} \geq 0.$$

We deduce from the previous line that m is a local minimum of the functional $\mathcal{F}(\cdot) - D(\cdot)$, because this functional is strictly convex, m is the unique minimizer of (32). □

11.1. The time dependent case

We only indicate the optimal control interpretation in the time dependent case. We do not give any proofs or precise statements. In a time dependent setting with time horizon $T>0$ and initial condition $m_0 \in L^2(\mathbb{T}^d)$, $m_0 \geq 0$, let us assume that there exists $\mathcal{F}$ such that for some $1<p<1+\frac{2}{d}$, for any $m, m' \in L^p((0,T)\times\mathbb{T}^d)$:

$$\lim_{\theta\to 0} \frac{\mathcal{F}((1-\theta)m+\theta m') - \mathcal{F}(m)}{\theta} = \int_0^T \int_{\mathbb{T}^d} f(m)(m'-m).$$

Then the optimal control problem associated with (28) is:

$$\inf_{m\in\mathcal{H}} \mathcal{F}(m) - D(m),$$

where $\mathcal{H} := \{m \in L^p((0,T)\times\mathbb{T}^d), m \geq 0\}$ and $D(m)$ is defined by

$$D(m) = \inf\left\{ \int_0^T \int_{\mathbb{T}^d} (-\partial_t v - \nu\Delta v)m - \int_{\mathbb{T}^d} v(0)m_0 \middle| v \in W^{1,2,p'}, v \leq Mv, v(T) = 0 \right\}.$$

12. Extensions and other developments

We conclude this paper by indicating some natural extensions of this work. First of all, the d dimensional torus or the particular form of the diffusion which drive the trajectories of the players do not play a strong role in this study and the framework we developed here can be extended to other cases almost immediately. The same holds for MFG of impulse control problem where the players can also control their velocity, i.e. in the case in which a Hamiltonian appears in the QVI, for a certain class of Hamiltonians (uniformly Lipschitz for instance). We refer to [4] for details on QVI involving such Hamiltonians.

Then, as we mentioned at the end of the first part, MFG of other optimal control problems involving QVI can be modeled and studied using this framework, for instance this is the case for optimal switching problems. Concerning the case of impulse control problems where all the jumps are possible, except the proof of existence using the penalized system, most of the study we presented can be exactly transpose in such a case, that is the definition of Nash equilibria of the MFG, the uniqueness result and the optimal control interpretation. In such a case, the optimal control interpretation may be a way to prove existence of solutions to the MFG system. Let us mention that in order the weaken the rather strong assumptions we made on the coupling f, we could also use an optimal control approach (or more generally a monotonicity assumption on f) to prove existence of solutions.

Declaration of competing interest

The author declares that he has no conflict of interest.

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