Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN 37 (2020) 1271–1297

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Global existence of weak solutions for the anisotropic compressible Stokes system

D. Bresch [a,*], C. Burtea [b]

[a] *Univ. Grenoble Alpes, Univ. Savoie Mont-Blanc, CNRS, LAMA, Chambéry, France*
[b] *Université de Paris and Institut de mathématiques de Jussieu-Paris Rive Gauche, Bâtiment Sophie Germain, Bureau 727, 8 place Aurélie Nemours, 75013 Paris, France*

Received 20 July 2019; received in revised form 10 April 2020; accepted 15 April 2020
Available online 24 April 2020

Dedicated to the memory of Geneviève Raugel

Abstract

In this paper, we study the problem of global existence of weak solutions for the quasi-stationary compressible Stokes equations with an anisotropic viscous tensor. The key idea is a new identity that we obtain by comparing the limit of the equations of the energies associated to a sequence of weak-solutions with the energy equation associated to the system verified by the limit of the sequence of weak-solutions. In the context of stability of weak solutions, this allows us to construct a defect measure which is used to prove compactness for the density and therefore allowing us to identify the pressure in the limiting model. By doing so we avoid the use of the so-called effective flux. Using this new tool, we solve an open problem namely global existence of solutions à la Leray for such a system without assuming any restriction on the anisotropy amplitude. This provides a flexible and natural method to treat compressible quasilinear Stokes systems which are important for instance in biology, porous media, supra-conductivity or other applications in the low Reynolds number regime.

MSC: 35Q35; 35B25; 76T20

Keywords: Compressible quasi-stationary Stokes equations; Anisotropic viscous tensor; Global weak solutions

1. Introduction

1.1. Presentation of the main result

As explained in [15], Chapter 8, there are various motivations for the study of the quasi-stationary Stokes problem. On the one hand such a study may be used to try to understand how to construct solutions of the compressible Navier-Stokes system which exhibit persistent oscillations. On the other hand this system naturally arises either when dealing

* Corresponding author.
E-mail addresses: didier.bresch@univ-smb.fr (D. Bresch), cburtea@math.univ-paris-diderot.fr (C. Burtea).

https://doi.org/10.1016/j.anihpc.2020.04.003

with flows in the low Reynolds number regime, which it typically the case in porous media or biology either as a mean field model for the motion of vortices in a superconductor in the Ginzburg–Landau theory. There is a rather rich literature regarding the mathematical study assuming isotropic diffusion: see for instance [5], [12], [13], [14], [17], [18], or [21] for constant viscosity coefficients or [2] for density dependent viscosity coefficients. More complicated versions of the quasi-stationary compressible Stokes system have been also analyzed in [6], [7], [11] and [10] in the multi-fluid setting.

Global existence of weak solutions for general anisotropic viscosities for non-stationary compressible barotropic Navier-Stokes equations or even quasi-stationary Stokes equations are open problems. Only recently a positive result has been obtained by D. Bresch and P.-E. Jabin in [4] assuming some restrictions on the shear and bulk viscosities. The result is not straightforward to prove because the anisotropy introduces non-locality effects in the compactness characterization process. This explains in some sense the new method introduced by the authors in order to conclude compactness: propagation of a non-local L^p-compactness module introducing appropriate time-evolving weights. In our paper, we consider a very general form of the quasi-stationary compressible Stokes equations:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ -\operatorname{div} \tau + a\nabla \rho^\gamma = f, \end{cases} \tag{1.1}$$

completed with an initial density distribution

$$\rho|_{t=0} = \rho_0 \geq 0. \tag{1.2}$$

Above, u stands for the fluid velocity field, ρ is the fluid density and τ represents the viscous stress tensor which is given by[1]

$$\tau_{ij}(t,x,D(u)) = A_{ijkl}(t,x)[D(u)]_{kl}, \tag{1.3}$$

where $D(u) = (\nabla u + {}^t\nabla u)/2$ is the strain tensor and

$$A_{ijkl} = A_{ijkl}(t,x) \in W^{1,\infty}((0,T)\times \mathbb{T}^3), \tag{1.4}$$

are given coefficients. Also, $a > 0$ is a given constant. The classical isotropic case is obtained by choosing

$$\begin{cases} A_{iiii} = \mu + \lambda, \\ A_{iijj} = \lambda \text{ for } i \neq j, \\ A_{ijij} = A_{ijji} = \frac{\mu}{2} \text{ for } i \neq j, \\ A_{ijkl} = 0 \text{ otherwise.} \end{cases}$$

The simplest example of anisotropic viscous stress tensor is obtained for

$$\begin{cases} A_{1111} = \mu_1, A_{2222} = \mu_2, \ A_{3333} = \mu_3, \\ A_{ijkl} = 0 \text{ otherwise,} \end{cases} \tag{1.5}$$

case in which we have

$$\operatorname{div} \tau = \partial_{11} u + \partial_{22} u + \mu \partial_{33} u = \Delta_\mu u.$$

The aim of this paper is to present a new defect measure for the pressure which allows to control the oscillation of an approximating sequence of solutions of system (1.1)–(1.2). Of course, the key point that allows to account for anisotropy is that we are able to control this defect measure without using the effective flux. This provides a new procedure to ensure nonlinear weak stability for the semi-stationary Stokes system. For the reader's convenience we will present a sketch of the proof in the next section in the case of the viscous tensor given by (1.5). In order to obtain a satisfactory mathematical theory we need to further assume the following hypothesis on the stress tensor τ:

[1] We use the convention of summation over repeated indices.

- for all $i, j, k, l,$ we impose $A_{ijkl} = A_{ijlk}$ which allows us to write that

$$\tau(t,x,D(u)) : \nabla u = \frac{1}{2}\tau(t,x,D(u)) : D(u). \tag{1.6}$$

- The application $D(u) \longmapsto \tau(t,x,D(u)) : D(u)$ is weakly lower semi-continuous. (1.7)
- There exists $c > 0$ such that

$$E = \int_{\mathbb{T}^3} \tau(t,x,D(u)) : \nabla u \geq c \int_{\mathbb{T}^3} |\nabla u|^2. \tag{1.8}$$

- The application $\mathcal{A} : v \mapsto -\operatorname{div} \tau(t,x,D(v))$ is a second order invertible elliptic operator such that $\mathcal{A}^{-1}\nabla\operatorname{div}$ is a bounded operator from $L^{\frac{3}{2}-\delta}\left(\mathbb{T}^3\right)$ into $L^{\frac{3}{2}-\delta}\left(\mathbb{T}^3\right)$ for some $\delta \in (0, 1/2)$. (1.9)

We are now in the position of announcing our main result:

Theorem 1.1. *Let us assume $f \in W^{1,2}(0,T; L^{\frac{6}{5}}\left(\mathbb{T}^3\right))$ and let the initial data ρ_0 satisfy*

$$\rho_0 \geq 0, \qquad 0 < M_0 = \int_{\mathbb{T}^3} \rho_0 < +\infty, \qquad E_0 = \int_{\mathbb{T}^3} \rho_0^\gamma \, dx < +\infty, \qquad \int_{\mathbb{T}^3} f(t)\, dx = 0,$$

where $\gamma > 1$ and assume that the viscous stress tensor τ given by (1.3) satisfies (1.6)–(1.9). Then there exists a global weak solution (ρ, u) of the system (1.1) and (1.2) with

$$\rho \in \mathcal{C}([0,T]; L^\gamma_{weak}(\mathbb{T}^3)) \cap L^{2\gamma}((0,T) \times \mathbb{T}^3), \quad u \in L^2(0,T; H^1(\mathbb{T}^3)) \text{ with } \int_{\mathbb{T}^3} u = 0.$$

A similar result can be obtained for the case of a bounded domain with Dirichlet boundary condition: we have chosen periodic boundary conditions to simplify the presentation. One of the most delicate points in proving Theorem 1.1 is the nonlinear weak stability of weak-solutions namely, given a sequence of solutions $\{(\rho^\varepsilon, u^\varepsilon)\}_\varepsilon$ of (1.1) verifying uniformly the energy estimates and therefore, at least on a subsequence, weakly converging to some (ρ, u), show that (ρ, u) is also a solution for (1.1). Of course, the most difficult part is to identify the pressure term in the limit namely to prove that $\lim_{\varepsilon \to 0} (\rho^\varepsilon)^\gamma = \rho^\gamma$. The case $\gamma = 1$ being different, we choose to focus only on the case $\gamma > 1$.

Remark 1.2. As explained in [15], one cannot expect that (1.1) to be well-posed when considering an arbitrary force term in the momentum equation of the form ρg with $g \in L^\infty_{t,x}$. This comes from the compatibility condition

$$\int_{\mathbb{T}^3} (\rho g + f) = 0.$$

Thus, if g is a vector with positive components this would imply that $\rho = 0$ for all times and this independently of the initial data.

One limitation of our work seems to be the choice of the pressure function: we cannot consider more general convex pressure laws other than $p(\rho) = a\rho^\gamma$, see Remark 1.5. Also, it seems difficult to adapt the method presented in this paper to the non-stationary Navier-Stokes system for a compressible fluid. Note that actually only one result exists for this system in the case of anisotropic diffusion, see [4]. Loosely speaking, the authors require that the "quantity of anisotropy" that they allow in the system should be small compared to the total viscosity $2\mu + \lambda$. Observe that we do not impose such a restriction for the quasi-stationary Stokes system. However we are able to treat a stationary system that can be interpreted as an implicit discretization of the full Navier-Stokes system, see Section 4.

The rest of the paper is organized as follows:

- Section 1.2 is dedicated to present the new defect measure associated to the pressure and to show how it is possible to control it if this is the case initially. Our result uses in a crucial manner compactness properties of the velocity field in $L^2((0,T)\times\mathbb{T}^3)$. For the readers' convenience, we recall the classical approach due to P.-L. Lions and latter refined by E. Feireisl-A. Novotný-H. Petzeltová. In particular, we explain why the anisotropic case seems to fall completely out of such strategy (see also [4] for further discussions).

The rest of the paper is devoted to the proof of Theorem 1.1. As it is accustomed when dealing with the existence of weak solutions, the proof is divided into two parts.

- In Section 2 we investigate the stability of a sequence of finite-energy weak-solutions of the system (1.1). In Section 2.1 we recall the basic nonlinear analysis tools that allow us to render rigorous the formal computations presented in Section 1.2. In Section 2.2 we prove that finite energy-weak-solutions enjoy extra-integrability and time regularity properties, with respect to the basic energy estimates, of course. More precisely it turns out that $\rho^\gamma \in L^2_{t,x}$ and that $\partial_t u \in L^1\left(0,T;L^r\left(\mathbb{T}^3\right)\right)$ for some $r\in(1,3/2)$. In Section 2.3 we investigate the stability of a sequence of finite energy weak-solutions $\{(\rho^\varepsilon,u^\varepsilon)\}_\varepsilon$ satisfying uniformly the energy estimates. It turns out that comparing the limit of the energy associated to each solution $(\rho^\varepsilon,u^\varepsilon)$ with the energy of the system verified by $(\rho,u)=\lim(\rho^\varepsilon,u^\varepsilon)$ we obtain an identity that involves a defect measure associated to the pressure. The stability result, interesting in itself is formalized in Theorem 2.10 and it can be adapted to construct solutions for the system (1.1).
- In Section 3 we construct weak-solutions for the system (1.1). More precisely, we propose an approximate model that depends on two parameters such that, at least formally, system (1.1) is obtained by a limit process by making the parameters tend to zero. We show that we can construct solutions by a classical fixed-point argument for the approximate system. Moreover, we show that the solutions verify uniform bounds with respect to the parameters introduced such that we are able to pass to the limit in a sequence of solutions and show that the limiting object is a solution of for the system (1.1) and thus achieving the proof of Theorem 1.1.
- Finally, in Section 4 we discuss some extents of our method of proof to other systems.

1.2. Formal approach to control the defect measure associated to the pressure in a simplified case

In the following lines we explain, in the case of the simple viscous stress tensor (1.5), why the classical approach to control defect measures fails to apply in the anisotropic case. Next, we present a new identity which provides a flexible method in order to treat Stokes-type systems. More precisely, let us consider $\{(\rho^\varepsilon,u^\varepsilon)\}_\varepsilon$ a sequence of solutions for the following system.

$$\begin{cases} \partial_t\rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ -\Delta_\mu u^\varepsilon + \nabla((\rho^\varepsilon)^\gamma) = f, \end{cases} \tag{1.10}$$

where

$$\Delta_\mu = \mu_1\partial_{11} + \mu_2\partial_{22} + \mu_3\partial_{33},$$

with $\mu_1,\mu_2,\mu_3>0$ which may be different. Assume

$$\|u^\varepsilon\|_{L^2(0,T;H^1(\mathbb{T}^3))} + \|\rho^\varepsilon\|_{L^{2\gamma}((0,T)\times\mathbb{T}^3)} + \|\rho^\varepsilon\|_{L^\infty(0,T;L^\gamma(\mathbb{T}^3))} \le C < +\infty,$$

where C does not depend on ε and assume that

$$\{u^\varepsilon\}_\varepsilon \text{ is compact in } L^2((0,T)\times\mathbb{T}^3).$$

We denote (ρ,u) the weak limit and, using classical functional analysis arguments it is not hard to see that we have

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ -\Delta_\mu u + \nabla(\overline{\rho^\gamma}) = f, \end{cases} \tag{1.11}$$

for some function $\overline{\rho^\gamma}\in L^2((0,T)\times\mathbb{T}^3)$. Of course, the main difficulty is to prove that $\overline{\rho^\gamma}=\rho^\gamma$ and therefore to be able to characterize the possible defect measures.

Remark 1.3. Throughout the paper we denote the weak limit of a sequence $\{a^\varepsilon\}_\varepsilon$ by $\bar{a}$.

Classical approach to control defect measures. As mentioned in [4], the usual method for isotropic viscosities (namely $\mu_1 = \mu_2 = \mu_3 = \mu$) is based on the careful analysis of the defect measures

$$\mathrm{dft}[\rho^\varepsilon - \rho](t) = \int_{\mathbb{T}^3} \left((\overline{\rho \log \rho})(t) - (\rho \log \rho)(t) \right) dx.$$

More precisely, we can write the two equations

$$\partial_t(\rho \log \rho) + \mathrm{div}(\rho \log \rho u) + \rho \mathrm{div} u = 0 \tag{1.12}$$

and

$$\partial_t(\overline{\rho \log \rho}) + \mathrm{div}(\overline{\rho \log \rho u}) + \overline{\rho \mathrm{div} u} = 0. \tag{1.13}$$

Note that if $\rho \in L^2((0,T) \times \mathbb{T}^3)$ then using the uniform bound on $u \in L^2(0,T; H^1(\mathbb{T}^3))$, we have $\rho\, \mathrm{div} u \in L^1((0,T) \times \mathbb{T}^3)$ and therefore the third quantity is well defined. At this level comes the so called effective flux comes into play. More precisely, Lions [16] in ′93 (see also D. Serre [22] for the $1d$ case) observes that the following quantity

$$F^\varepsilon = p(\rho^\varepsilon) - \mu \mathrm{div} u^\varepsilon$$

enjoys the following compactness property:

$$\lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{T}^3} (p(\rho^\varepsilon) - \mu \mathrm{div} u^\varepsilon) b(\rho^\varepsilon) \varphi = \int_0^T \int_{\mathbb{T}^3} (\overline{p(\rho)} - \mu \mathrm{div} u) \overline{b(\rho)} \varphi. \tag{1.14}$$

This is important as it allows us to express $\overline{\rho \mathrm{div} u}$ in terms of $\rho \mathrm{div} u$ and an extra term which is signed. Subtracting the two equations (1.12) and (1.13) and using the important property of the effective flux (1.14), one gets that

$$\partial_t(\overline{\rho \log \rho} - \rho \log \rho) + \mathrm{div}((\overline{\rho \log \rho} - \rho \log \rho) u) = \frac{1}{\mu}(\overline{p(\rho)}\rho - \overline{p(\rho)\rho})$$

and using the monotonicity of the pressure, one may deduce that

$$\mathrm{dft}[\rho^\varepsilon - \rho](t) \leq \mathrm{dft}[\rho^\varepsilon - \rho](0).$$

On the other hand, the strict convexity of the function $s \mapsto s \log s$ with $s \geq 0$ implies that $\mathrm{dft}[\rho^\varepsilon - \rho](t) \geq 0$. If initially this quantity vanishes, it then vanishes at every time. The commutation of the weak convergence with a strictly convex function yields compactness of $\{\rho^\varepsilon\}_\varepsilon$ in $L^1((0,T) \times \mathbb{T}^3)$.

Assuming anisotropic viscosities $\mu_1 = \mu_2 \neq \mu_3$, the effective flux property reads

$$\overline{\rho \mathrm{div} u} - \rho \mathrm{div} u = \frac{1}{\mu_1}[\rho \overline{A_\mu \rho^\gamma} - \overline{\rho A_\mu \rho^\gamma}]$$

with some non-local anisotropic operator $A_\mu = (\Delta - (\mu_3 - \mu_1)\partial_z^2)^{-1}\partial_z^2$ where Δ is the total Laplacian in terms of (X, z) with variables $X = (x, y)$ and z. Unfortunately, we are loosing the sign of the right-hand side. This explains why the anisotropic case seems to fall completely out the theory developed by P.-L. Lions [15] and E. Feireisl, A. Novotný and H. Petzeltová [9]. The first result in this direction has been obtained by D. Bresch and P.-E. Jabin in [4] for the compressible Navier-Stokes equations where they develop a more systematic way to characterize compactness in space for the density: it involves a non-local compactness criterion with the introduction of appropriate weights. This allows them to obtain the existence of weak solutions assuming the viscosity coefficient μ_1, μ_2, μ_3 are close enough.

New approach to control defect measures in the Stokes regime. Our new approach is based on the careful analysis of the defect measures

$$\mathrm{dft}[\rho^\varepsilon - \rho](t) = \int_{\mathbb{T}^3} \left((\overline{\rho^\gamma})(t) - \rho^\gamma(t) \right)^{1/\gamma} dx.$$

The main idea here is to write the equation related to the energy which will not use the effective flux expression but is related to the viscous dissipation in the Stokes regime. More precisely, let us observe that the pressure verifies the following equation:

$$\partial_t \left(\rho^\varepsilon\right)^\gamma + \operatorname{div}\left(\left(\rho^\varepsilon\right)^\gamma u^\varepsilon\right) + (\gamma-1)\left(\rho^\varepsilon\right)^\gamma \operatorname{div} u^\varepsilon = 0,$$

which rewrites

$$\partial_t \left(\rho^\varepsilon\right)^\gamma + \gamma \operatorname{div}\left(\left(\rho^\varepsilon\right)^\gamma u^\varepsilon\right) - (\gamma-1)\, u^\varepsilon \nabla \left(\rho^\varepsilon\right)^\gamma = 0.$$

We observe that with the aid of the second equation of (1.10) we may write that

$$\partial_t \left(\rho^\varepsilon\right)^\gamma + \gamma \operatorname{div}\left(\left(\rho^\varepsilon\right)^\gamma u^\varepsilon\right) - (\gamma-1)\, u^\varepsilon \Delta_\mu u^\varepsilon = (\gamma-1) u^\varepsilon f,$$

which can be put under the following form

$$\begin{aligned}\partial_t \left(\rho^\varepsilon\right)^\gamma &+ \gamma \operatorname{div}\left(\left(\rho^\varepsilon\right)^\gamma u^\varepsilon\right) - (\gamma-1)\, \Delta_\mu \left(\frac{|u^\varepsilon|^2}{2}\right) \\ &= -(\gamma-1)\, \nabla_\mu u^\varepsilon : \nabla_\mu u^\varepsilon + (\gamma-1) u^\varepsilon f, \end{aligned} \tag{1.15}$$

where we use the notation

$$\nabla_\mu = \left(\mu_1^{\frac{1}{2}} \partial_1, \mu_2^{\frac{1}{2}} \partial_2, \mu_3^{\frac{1}{2}} \partial_3 \right).$$

Of course, we used that

$$\partial_{jj} u_i u_i = \partial_{jj} \left(\frac{(u_i)^2}{2} \right) - (\partial_j u_i)^2$$

Assuming that

$$\left\{u^\varepsilon\right\}_{\varepsilon>0} \text{ is compact in } L^2((0,T)\times\mathbb{T}^3)$$

and passing to the limit in (1.15) we obtain that

$$\begin{aligned}\partial_t \overline{\rho^\gamma} &+ \gamma \operatorname{div}\left(\overline{\rho^\gamma} u\right) - (\gamma-1)\, \Delta_\mu \left(\frac{|u|^2}{2}\right) \\ &= -(\gamma-1)\, \overline{\nabla_\mu u : \nabla_\mu u} + (\gamma-1) f u. \end{aligned} \tag{1.16}$$

In the following we will apply the same recipe to the limiting function (ρ, u). Indeed, from (1.11) one can deduce that

$$\begin{aligned}\partial_t \rho^\gamma + \gamma \operatorname{div}\left(\rho^\gamma u\right) &= (\gamma-1)\, u \cdot \nabla \rho^\gamma \\ &= (\gamma-1)\, u \cdot \nabla(\rho^\gamma - \overline{\rho^\gamma}) - (\gamma-1)\, u \cdot \nabla \overline{\rho^\gamma} \\ &= -(\gamma-1)\, u \cdot \nabla(\overline{\rho^\gamma} - \rho^\gamma) - (\gamma-1)\, u \cdot (\Delta_\mu u + f)\end{aligned}$$

which rewrites

$$\begin{aligned}&\partial_t \rho^\gamma + \gamma \operatorname{div}\left(\rho^\gamma u\right) + (\gamma-1)\, u \nabla(\overline{\rho^\gamma} - \rho^\gamma) - (\gamma-1)\, \Delta_\mu \left(\frac{|u|^2}{2}\right) \\ &= -(\gamma-1)\, \nabla_\mu u : \nabla_\mu u + (\gamma-1) f u. \end{aligned} \tag{1.17}$$

Let us consider the difference between (1.16) and (1.17) in order to write that

$$\partial_t\left(\overline{\rho^\gamma}-\rho^\gamma\right)+\gamma\operatorname{div}\left(\left(\overline{\rho^\gamma}-\rho^\gamma\right)u\right)-(\gamma-1)\,u\nabla\left(\overline{\rho^\gamma}-\rho^\gamma\right)$$
$$=-(\gamma-1)\left(\overline{\nabla_\mu u:\nabla_\mu u}-\nabla_\mu u:\nabla_\mu u\right),$$

which we put under the form

$$\partial_t\left(\overline{\rho^\gamma}-\rho^\gamma\right)+\operatorname{div}\left(\left(\overline{\rho^\gamma}-\rho^\gamma\right)u\right)+(\gamma-1)\left(\overline{\rho^\gamma}-\rho^\gamma\right)\operatorname{div}u \tag{1.18}$$
$$=-(\gamma-1)\left(\overline{\nabla_\mu u:\nabla_\mu u}-\nabla_\mu u:\nabla_\mu u\right).$$

At this point we observe that owing to the convexity of the pressure function, we have that

$$\overline{\rho^\gamma}\geq\rho^\gamma \text{ a.e.}$$

and

$$\overline{\nabla_\mu u:\nabla_\mu u}-\nabla_\mu u:\nabla_\mu u\geq 0, \tag{1.19}$$

in the sense of measures. By multiplying (1.18) with $\frac{1}{\gamma}\left(\overline{\rho^\gamma}-\rho^\gamma\right)^{\frac{1}{\gamma}-1}$ and using (1.19), we get that

$$\partial_t\left(\overline{\rho^\gamma}-\rho^\gamma\right)^{\frac{1}{\gamma}}+\operatorname{div}\left(\left(\overline{\rho^\gamma}-\rho^\gamma\right)^{\frac{1}{\gamma}}u\right)\leq 0. \tag{1.20}$$

Integrating in time, we end up with

$$\sup_{t\in[0,T]}\int_{\mathbb{T}^3}\left(\overline{\rho^\gamma}-\rho^\gamma\right)^{\frac{1}{\gamma}}(t)\leq\int_{\mathbb{T}^3}\left(\overline{\rho^\gamma}-\rho^\gamma\right)^{\frac{1}{\gamma}}_{|t=0}.$$

Therefore if we have compactness initially, we get compactness of the sequence $\{\rho_\varepsilon\}_{\varepsilon>0}$. Of course all the previous formal calculations have to be justified because of the weak regularity of the functions involved: this will be the subject of Subsection 2.3.

Remark 1.4. It is interesting to note that our new approach to characterize the defect measure of the pressure sequence is related to the energy equation and strongly uses the energy dissipation. We speculate that it has a physical meaning in some sense.

Remark 1.5. Even though our method allows us to treat very general viscous stress tensors, it does not seem to apply to general monotone convex pressure laws $p(\rho)$. If we let $H(\rho)$ be the potential energy which is defined via

$$\rho H'(\rho)-H(\rho)=p(\rho),$$

then, we still have the identity

$$\partial_t\left(\overline{H(\rho)}-H(\rho)\right)+\operatorname{div}\left(\left(\overline{H(\rho)}-H(\rho)\right)u\right)+\left(\overline{p(\rho)}-p(\rho)\right)\operatorname{div}u$$
$$=-\left(\overline{\tau:\nabla u}-\tau:\nabla u\right)\leq 0,$$

but by multiplication with $H^{-1}\left(\overline{H(\rho)}-H(\rho)\right)$ or $p^{-1}\left(\overline{H(\rho)}-H(\rho)\right)$ the left-hand side cannot be written in conservative form.

2. Nonlinear weak stability of sequences of global weak solutions

2.1. *Classical functional analysis tools*

This section is devoted to a quick recall of the main results from functional analysis that we need in order to justify the computations done above. Let $g\in L^q(0,T;L^p(\mathbb{T}^3))$ with $p,q\geq 1$, introduce a new function

$$g_\varepsilon=g*\omega_\varepsilon(x)\qquad\text{with}\qquad\omega_\varepsilon=\frac{1}{\varepsilon^d}\omega\left(\frac{x}{\varepsilon}\right) \tag{2.1}$$

with ω a smooth, nonnegative, even function compactly supported in the unit ball centered at the origin and with integral equal to 1. We recall the following classical analysis result

$$\lim_{\varepsilon\to 0}\|g_\varepsilon - g\|_{L^q(0,T;L^p(\mathbb{T}^3))} = 0.$$

Next let us recall the following commutator estimate which was obtained for the first time by DiPerna and Lions:

Proposition 2.1. *Consider $\beta \in (1,\infty)$ and (a,b) such that $a \in L^\beta\left((0,T)\times\mathbb{T}^3\right)$ and $b, \nabla b \in L^p\left((0,T)\times\mathbb{T}^3\right)$ where $\frac{1}{s} = \frac{1}{\beta} + \frac{1}{p} \leq 1$. Then, we have*

$$\lim r_\varepsilon(a,b) = 0 \text{ in } L^s\left((0,T)\times\mathbb{T}^3\right),$$

where

$$r_\varepsilon(a,b) = \partial_i(a_\varepsilon b) - \partial_i\left((ab)_\varepsilon\right). \tag{2.2}$$

Whenever we have a *regular solution* for the transport equation

$$\partial_t\rho + \operatorname{div}(\rho u) = 0, \tag{2.3}$$

then, multiplying the former equation with $b'(\rho)$ gives

$$\partial_t b(\rho) + \operatorname{div}(b(\rho)u) + \left\{\rho b'(\rho) - b(\rho)\right\}\operatorname{div} u = 0. \tag{2.4}$$

The following proposition gives us a framework for justifying these computations when ρ is just a Lebesgue function.

Proposition 2.2. *Consider $2 \leq \beta < \infty$ and λ_0, λ_1 such that $\lambda_0 < 1$ and $-1 \leq \lambda_1 \leq \beta/2 - 1$. Also, consider $\rho \in L^\beta\left((0,T)\times\mathbb{T}^3\right)$, $\rho \geq 0$ a.e. and $u, \nabla u \in L^2\left((0,T)\times\mathbb{T}^3\right)$ verifying the transport equation (2.3) in the sense of distributions. Then, for any function $b \in C^0([0,\infty)) \cap C^1((0,\infty))$ such that*

$$\begin{cases} b'(t) \leq ct^{-\lambda_0} \text{ for } t \in (0,1], \\ \left|b'(t)\right| \leq ct^{\lambda_1} \text{ for } t \geq 1. \end{cases}$$

Then, equation (2.4) holds in the sense of distributions.

The proof of the above results follows by adapting in a straightforward manner lemmas 6.7 and 6.9 from the book of A. Novotný-I. Straškraba [19], pages 304–308.

2.2. *Estimates for finite-energy weak solutions*

Let us begin this section by recalling the basic a priori estimates for (regular) solutions for the system

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ -\operatorname{div}\tau + \nabla\rho^\gamma = f, \\ \rho_{|t=0} = \rho_0, \end{cases} \tag{2.5}$$

with $\tau_{ij} = A_{ijkl}(t,x)D_{kl}(u)$ and

$$\int_{\mathbb{T}^3} u(t) = \int_{\mathbb{T}^3} f(t) = 0.$$

We set the adiabatic constant $a = 1$ for the sake of simplicity in the computations that follow.

First, of course, we have the mass conservation identity:

$$\int_{\mathbb{T}^3}\rho(t) = \int_{\mathbb{T}^3}\rho|_{t=0} = \int_{\mathbb{T}^3}\rho_0, \tag{2.6}$$

for all $t > 0$ which follows by integrating the first equation of (2.5). Next, by multiplying the velocity equation with u and integrating in space and time we get that

$$\int_{\mathbb{T}^3} \rho^\gamma(t) + \int_0^t \int_{\mathbb{T}^3} \tau : \nabla u \le \int_{\mathbb{T}^3} \rho_0^\gamma + \int_0^t \int_{\mathbb{T}^3} u \cdot f \tag{2.7}$$

$$\le \int_{\mathbb{T}^3} \rho_0^\gamma + \|u\|_{L^2(0,T;L^6(\mathbb{T}^3))} \|f\|_{L^2(0,T;L^{\frac{6}{5}}(\mathbb{T}^3))}. \tag{2.8}$$

The coercivity hypothesis (1.9)

$$c \int_{\mathbb{T}^3} |\nabla u|^2 \le \int_{\mathbb{T}^3} \tau : \nabla u,$$

with $c > 0$, the zero mean value on u, the Körn inequality and Sobolev embedding allows us to conclude that

$$\rho \in L^\infty\left(0, T; L^\gamma\left(\mathbb{T}^3\right)\right), \qquad u \in L^2(0, T; H^1\left(\mathbb{T}^3\right)),$$

with

$$\int_{\mathbb{T}^3} \rho^\gamma(t) + \int_0^t \int_{\mathbb{T}^3} |\nabla u|^2 \le C(c) \left(\|\rho_0\|^\gamma_{L^\gamma(\mathbb{T}^3)} + \int_0^t \|f(\tau)\|^2_{L^{\frac{6}{5}}(\mathbb{T}^3)} d\tau \right), \tag{2.9}$$

for all $t \ge 0$ where $C(c)$ is a constant depending only on the coercivity constant appearing in (1.9).

Of course, the previous computations hold for regular solutions. It is to be expected however that any reasonably physical solution to (2.5) would verify the mass conservation and the energy inequality. Thus, we introduce the following

Definition 2.3. A pair $(\rho, u) \in L^\infty\left(0, T; L^\gamma\left(\mathbb{T}^3\right)\right) \times L^2\left(0, T; H^1\left(\mathbb{T}^3\right)\right)$ is called a finite energy weak-solution for (2.5) if it is a solution in the sense of distributions for (2.5) which moreover verifies the mass conservation identity (2.6) along with the energy inequality (2.7).

Of course, a finite energy weak-solution for (2.5) also verifies (2.9). It turns out that finite energy weak-solutions verify some extra integrability properties. More precisely, we have

Proposition 2.4. *Consider $(\rho, u) \in L^\infty\left(0, T; L^\gamma\left(\mathbb{T}^3\right)\right) \times L^2\left(0, T; H^1\left(\mathbb{T}^3\right)\right)$ a finite energy weak-solution for (2.5). Then, we have that*

$$\begin{aligned}
&\|\rho^\gamma\|_{L^2((0,T)\times\mathbb{T}^3)} \le C(c,\gamma)\left(\sqrt{t} + \max\{1, \|A\|_{L^\infty}\}\right)\left(\|\rho_0\|^{\frac{\gamma}{2}}_{L^\gamma(\mathbb{T}^3)} + \|f\|_{L^2(0,T;L^{\frac{6}{5}}(\mathbb{T}^3))}\right), \\
&\|\partial_t u\|_{L^1(0,T;L^{\frac{3}{2}-\delta}(\mathbb{T}^3))} \\
&\le C(c,\gamma)\left(\sqrt{t} + \max\left\{1, \|A\|_{L^\infty((0,T)\times\Omega)}\right\}\right)\left(\|\rho_0\|^\gamma_{L^\gamma(\mathbb{T}^3)} + \|f\|^2_{L^2(0,T;L^{\frac{6}{5}}(\mathbb{T}^3))}\right) \\
&\qquad + C(c,\gamma)\sqrt{t}\left(1 + \|\partial_t A\|_{L^\infty((0,T)\times\mathbb{T}^3)}\right)\left(\|\rho_0\|^{\frac{\gamma}{2}}_{L^\gamma(\mathbb{T}^3)} + \|(f, \partial_t f)\|_{L^2(0,T;L^{\frac{6}{5}}(\mathbb{T}^3))}\right)
\end{aligned} \tag{2.10}$$

where $C(c,\gamma)$ depends only on c and γ and $\delta \in (0, 1/2)$ is the constant appearing in (1.9).

Proof of Proposition 2.4. The integrability assumptions on the weak solution (ρ, u) ensure that for all $\psi \in \left[L^2\left(0, T; H^1\left(\mathbb{T}^3\right)\right)\right]^3$ we have that

$$\int_0^t \int_{\mathbb{T}^3} \rho^\gamma \operatorname{div} \psi = \int_0^t \int_{\mathbb{T}^3} \tau : \nabla \psi + \int_0^t \int_{\mathbb{T}^3} f \cdot \psi$$

Take $\phi \in L^2\left((0,T)\times\mathbb{T}^3\right)$ and consider ψ such that

$$\Delta\psi = \nabla\phi \qquad \text{with} \qquad \int_{\mathbb{T}^3} \psi = 0.$$

We obtain that

$$\operatorname{div}\psi = \phi - \int_{\mathbb{T}^3} \phi,$$

such that using ψ as a test function in weak formulation of the velocity equation and using that $A(t,x) \in W^{1,\infty}((0,T)\times\mathbb{T}^3))^{3\times 3}$ along with the energy estimate (2.9), we get

$$\begin{aligned}\int_0^t\int_{\mathbb{T}^3} \rho^\gamma\phi &= \int_0^t\int_{\mathbb{T}^3}\phi\int_{\mathbb{T}^3}\rho^\gamma + \int_0^t\int_{\mathbb{T}^3}\tau:\nabla\psi + \int_0^t\int_{\mathbb{T}^3} f\cdot\psi \\ &\le C(c,\gamma)\left(\sqrt{t}+\max\left\{1,\|A\|_{L^\infty((0,T)\times\mathbb{T}^3)}\right\}\right)\left(\|\rho_0\|^{\frac{\gamma}{2}}_{L^\gamma(\mathbb{T}^3)} + \|f\|_{L^2(0,T;L^{\frac{6}{5}}(\mathbb{T}^3))}\right)\|\phi\|_{L^2_{t,x}}\end{aligned}$$

and thus we get that

$$\rho^\gamma \in L^2\left((0,T)\times\mathbb{T}^3\right), \tag{2.11}$$

verifies the uniform bound announced in the first relation of (2.10).

We prove now the estimate for the time derivative of $\partial_t u$. We can recover time regularity for u by proceeding in the following way. We write that

$$\begin{aligned}-\mathcal{A}\partial_t u &= \operatorname{div}(\partial_t A(t,x)D(u)) + \partial_t f - \nabla\partial_t\rho^\gamma \\ &= \operatorname{div}(\partial_t A(t,x)D(u)) + \partial_t f \\ &\quad + \nabla\operatorname{div}\left(\rho^\gamma u - \int_{\mathbb{T}^3}\rho^\gamma u\right) + (\gamma-1)\nabla\left(\rho^\gamma\operatorname{div}u - \int_{\mathbb{T}^3}\rho^\gamma\operatorname{div}u\right),\end{aligned}$$

where the passage from the second line to the third is justified by Proposition 2.2 which of course, can be applied owing to the fact that we recover (2.11). Above, the first two terms behave better and thus taking advantage of the linearity of the operator $-\mathcal{A}$ it is more convenient to separate $\partial_t u$ in two parts and estimate them separately. To this end, consider ϕ with $\int_{\mathbb{T}^3}\phi = 0$, such that

$$-\mathcal{A}\phi = \operatorname{div}(\partial_t A(t,x)D(u)) + \partial_t f.$$

Multiplying by ϕ we get that

$$\begin{aligned}c\int_0^t\int_{\mathbb{T}^3}|\nabla\phi|^2 &\le -\int_0^t\int_{\mathbb{T}^3}\phi\cdot\mathcal{A}\phi = -\int_0^t\int_{\mathbb{T}^3}\partial_t A(t,x)D(u):\nabla\phi + \int_0^t\int_{\mathbb{T}^3}\partial_t f\cdot\phi \\ &\le \frac{1}{8c}\int_0^t\int_{\mathbb{T}^3}|\partial_t A(t,x)D(u)|^2 + \frac{C^2}{8c}\int_0^t\|\partial_t f\|^2_{L^{\frac{6}{5}}(\mathbb{T}^3)} + \frac{c}{2}\int_0^t\int_{\mathbb{T}^3}|\nabla\phi|^2,\end{aligned}$$

where C is the constant appearing in the Sobolev inequality and thus, we get that

$$\frac{c}{2}\int_0^t\int_{\mathbb{T}^3}|\nabla\phi|^2 \le \frac{1}{8c}\int_0^t\int_{\mathbb{T}^3}|\partial_t A(t,x)D(u)|^2 + \frac{C^2}{8c}\int_0^t\|\partial_t f\|^2_{L^{\frac{6}{5}}(\mathbb{T}^3)}. \tag{2.12}$$

It remains to estimate $\partial_t u - \phi$ which verifies

$$\mathcal{A}(\partial_t u - \phi) = -\nabla \operatorname{div}\left(\rho^\gamma u - \int_{\mathbb{T}^3} \rho^\gamma u\right) - (\gamma - 1)\nabla\left(\rho^\gamma \operatorname{div} u - \int_{\mathbb{T}^3} \rho^\gamma \operatorname{div} u\right).$$

We will use a periodic variant of the following result due to Stampacchia and for more general second order elliptic equation to Boccardo-Gallouët that can be found for instance in [20], Proposition 5.1, page 77. Let ψ be the solution of

$$-\Delta\psi = f \text{ with } \psi_{|\partial\Omega} = 0,$$

where $f \in L^1(\Omega)$ with Ω a smooth bounded domain then we have that

$$\|\nabla\psi\|_{L^r(\Omega)} \le C_\delta \|f\|_{L^1(\Omega)}, \tag{2.13}$$

for all $r \in [1, 3/2)$. The periodic version reads as follows: let ψ a solution of

$$-\Delta\psi = f \text{ with } f \in L^1(\mathbb{T}^3) \text{ and } \int_{\mathbb{T}^3} f = 0,$$

then (2.13) is satisfied, see Theorem A.3 from the Appendix for a proof. As $\rho^\gamma \operatorname{div} u \in L^1((0,T) \times \mathbb{T}^3)$, let us consider ψ the solution of

$$-\Delta\psi(\rho, u) = \rho^\gamma \operatorname{div} u - \int_{\mathbb{T}^3} \rho^\gamma \operatorname{div} u,$$

which verifies that

$$\|\nabla\psi(\rho,u)\|_{L^1(0,T:L^{\frac{3}{2}-\delta}(\mathbb{T}^3))} \le C_\delta \left\|\rho^\gamma \operatorname{div} u\right\|_{L^1(0,T;L^1(\mathbb{T}^3))} \le C_\delta \left\|\rho^\gamma\right\|_{L^2((0,T)\times\mathbb{T}^3)} \|\operatorname{div} u\|_{L^2((0,T)\times\mathbb{T}^3)},$$

where $\delta \in (0, 1/2)$ is the constant appearing in (1.9). But then, we may write that

$$\begin{aligned}\mathcal{A}(\partial_t u - \phi) &= -\nabla \operatorname{div}\left(\rho^\gamma u\right) - (\gamma-1)\nabla\left(\rho^\gamma \operatorname{div} u\right)\\ &= \nabla \operatorname{div}\left(\rho^\gamma u\right) + (\gamma-1)\nabla \operatorname{div}\nabla\psi(\rho,u)\end{aligned}$$

and using hypothesis (1.9) we get

$$\begin{aligned}\|(\partial_t u - \phi)\|_{L^1(0,T;L^{\frac{3}{2}-\delta}(\mathbb{T}^3))} &\le \left\|\rho^\gamma u - \int_{\mathbb{T}^3}\rho^\gamma u\right\|_{L^1(0,T;L^{\frac{3}{2}}(\mathbb{T}^3))} + \|\nabla\psi(\rho,u)\|_{L^1(0,T;L^{\frac{3}{2}-\delta}(\mathbb{T}^3))}\\ &\le \left\|\rho^\gamma\right\|_{L^2((0,T)\times\mathbb{T}^3)}\|u\|_{L^2(0,T;L^6(\mathbb{T}^3))} + \left\|\rho^\gamma\right\|_{L^2((0,T)\times\mathbb{T}^3)}\|\operatorname{div} u\|_{L^2((0,T)\times\mathbb{T}^3)}\\ &\le \left\|\rho^\gamma\right\|_{L^2((0,T)\times\mathbb{T}^3)}\|\nabla u\|_{L^2((0,T)\times\mathbb{T}^3)}.\end{aligned} \tag{2.14}$$

We get a uniform bound for $\partial_t u$ in $L^1\left(0,T; L^{3/2-}(\mathbb{T}^3)\right)$ by combining estimates (2.12) and (2.14) in the following manner

$$\begin{aligned}\|\partial_t u\|_{L^1(0,T;L^{\frac{3}{2}-\delta}(\mathbb{T}^3))} &\le \|(\partial_t u - \phi)\|_{L^1(0,T;L^{\frac{3}{2}-\delta}(\mathbb{T}^3))} + \|\phi\|_{L^1(0,T;L^{\frac{3}{2}-\delta}(\mathbb{T}^3))}\\ &\le \left\|\rho^\gamma\right\|_{L^2((0,T)\times\mathbb{T}^3)}\|\nabla u\|_{L^2((0,T)\times\mathbb{T}^3)} + \sqrt{t}\,\|\phi\|_{L^2(0,T;L^6(\mathbb{T}^3))}\\ &\le \left\|\rho^\gamma\right\|_{L^2((0,T)\times\mathbb{T}^3)}\|\nabla u\|_{L^2((0,T)\times\mathbb{T}^3)} + \sqrt{t}\,\|\nabla\phi\|_{L^2((0,T)\times\mathbb{T}^3)}\\ &\le C(c,\gamma)\left(\sqrt{t} + \max\left\{1, \|A\|_{L^\infty((0,T)\times\mathbb{T}^3)}\right\}\right)\left(\|\rho_0\|^\gamma_{L^\gamma(\mathbb{T}^3)} + \|f\|^2_{L^2(0,T:L^{\frac{6}{5}}(\mathbb{T}^3))}\right)\\ &\quad + C(c,\gamma)\sqrt{t}\left(1 + \|\partial_t A\|_{L^\infty((0,T)\times\mathbb{T}^3)}\right)\left(\|\rho_0\|^{\frac{\gamma}{2}}_{L^\gamma(\mathbb{T}^3)} + \|(f,\partial_t f)\|_{L^2(0,T;L^{\frac{6}{5}}(\mathbb{T}^3))}\right),\end{aligned}$$

which is exactly the estimate (2.10). Of course combining this information with the energy inequality (2.7) we obtain a uniform bound for

$$u \in L^2(0,T;H^1(\mathbb{T}^3)) \cap W^{1,1}(0,T;L^{3/2-\delta}(\mathbb{T}^3)).$$

This ends the proof of Proposition 2.4. □

Remark 2.5. Also, for later purposes it is convenient to observe that we actually proved that if

$$-\mathcal{A}u = \operatorname{div} F, \tag{2.15}$$

then hypothesis (1.9) made on the operator $\mathcal{A}$ implies that there exists some constant C such that

$$\|\nabla u\|_{L^{\frac{3}{2}-\delta}(\mathbb{T}^3)} \leq C \|F\|_{L^1(\mathbb{T}^3)}, \tag{2.16}$$

for any u, F verifying (2.15).

Remark 2.6. The previous estimates are not available in the case of the full compressible Navier-Stokes system. For instance we do not have control on the time derivative of the velocity and ρ^γ is not square integrable: we control only $\partial_t(\rho u)$ in $L^1(0,T;H^{-1}(\mathbb{T}^3))$ allowing to get compactness on $\sqrt{\rho}u$ in $L^2((0,T)\times\mathbb{T}^3))$ and we gain extra integrability $\rho^{\gamma+\theta} \in L^1((0,T)\times\mathbb{T}^3)$ for $0<\theta<2\gamma/3-1$.

2.3. *Weak stability of solutions of (1.10)*

The aim of this section is to provide the arguments that render rigorous the formal computations presented in Section 1.2. Let us temporarily include an extra potential source term in the system:

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ -\operatorname{div}\tau + \nabla\rho^\gamma = \nabla g + f. \end{cases} \tag{2.17}$$

As we saw in Section 2.1 under certain integrability conditions one may conclude that ρ^γ verifies the following equation:

$$\partial_t\rho^\gamma + \operatorname{div}\left(\rho^\gamma u\right) + (\gamma-1)\,\rho^\gamma \operatorname{div} u = 0.$$

Of course, the result of Proposition 2.2 that allows us to write the above equation does not take in account the structure of the system (2.17). In the following, we propose a more accurate result taking in consideration the equation of the velocity.

Proposition 2.7. *Consider $f \in L^2(0,T;L^{\frac{6}{5}}(\mathbb{T}^3))$, $g \in L^2\left((0,T)\times\mathbb{T}^3\right)$ and (ρ,u) a finite energy weak-solution of (2.17). Then, we obtain*

$$\frac{1}{\gamma-1}\left\{\partial_t\rho^\gamma + \gamma\operatorname{div}\left(\rho^\gamma u\right)\right\} = \operatorname{div}(\tau:u) - \tau:\nabla u + u\cdot f + \operatorname{div}(ug) - g\operatorname{div} u, \tag{2.18}$$

in the sense of distributions.

Remark 2.8. In order to prove Proposition 2.7 we do not require regularity on the time derivative of f as it is needed in order to obtain the a priori estimates for $\partial_t u$, see Proposition 2.4.

Remark 2.9. Proposition 2.7 is valid for all tensor fields $\tau \in L^2\left((0,T)\times\mathbb{T}^3\right)$.

Proof of 2.7. The proof uses the regularizing techniques introduced by Lions in [15], see also the book of A. Novotný and I. Straškraba, [19]. Recall the notation introduced in (2.1) and (2.2) and let us write

$$\partial_t\rho_\varepsilon + \operatorname{div}(\rho_\varepsilon u) = r_\varepsilon(\rho,u),$$

which by multiplying with $\gamma(\rho_\varepsilon)^{\gamma-1}$ yields

$$\partial_t (\rho_\varepsilon)^\gamma + \operatorname{div}\left((\rho_\varepsilon)^\gamma u\right) + (\gamma-1)(\rho_\varepsilon)^\gamma \operatorname{div} u = \gamma\, r_\varepsilon(\rho,u)\,(\rho_\varepsilon)^{\gamma-1}.$$

Let us rewrite the above equation in the following manner:

$$\begin{aligned}&\partial_t (\rho_\varepsilon)^\gamma + \operatorname{div}\left((\rho_\varepsilon)^\gamma u\right) + (\gamma-1)\left\{(\rho_\varepsilon)^\gamma - (\rho^\gamma)_{\varepsilon'}\right\}\operatorname{div} u + (\gamma-1)(\rho^\gamma)_{\varepsilon'}\{\operatorname{div} u - \operatorname{div} u_{\varepsilon'}\}\\ &+(\gamma-1)(\rho^\gamma)_{\varepsilon'}\operatorname{div} u_{\varepsilon'} = \gamma r_\varepsilon(\rho,u)(\rho_\varepsilon)^{\gamma-1}.\end{aligned}$$

Next, we observe that owing to the second equation of (2.17) we get by convolution through a sequence of kernel $\{\omega_{\varepsilon'}\}_{\varepsilon'}$ that

$$\begin{aligned}(\rho^\gamma)_{\varepsilon'}\operatorname{div} u_{\varepsilon'} &= \operatorname{div}\left((\rho^\gamma)_{\varepsilon'}u_{\varepsilon'}\right) - u_{\varepsilon'}\cdot\nabla(\rho^\gamma)_{\varepsilon'}\\ &= \operatorname{div}\left((\rho^\gamma)_{\varepsilon'}u_{\varepsilon'}\right) - u_{\varepsilon'}\cdot\operatorname{div}\tau_{\varepsilon'} - u_{\varepsilon'}\cdot\nabla g_{\varepsilon'} - u_{\varepsilon'}\cdot f_{\varepsilon'}\\ &= \operatorname{div}\left((\rho^\gamma)_{\varepsilon'}u_{\varepsilon'}\right) - \operatorname{div}(\tau_{\varepsilon'}u_{\varepsilon'}) + \tau_{\varepsilon'}:\nabla u_{\varepsilon'} - \operatorname{div}(u_{\varepsilon'}g_{\varepsilon'}) + g_{\varepsilon'}\operatorname{div} u_{\varepsilon'} - u_{\varepsilon'}\cdot f_{\varepsilon'}\end{aligned}$$

and thus, we may write that

$$\begin{aligned}&\tfrac{1}{\gamma-1}\left\{\partial_t(\rho_\varepsilon)^\gamma + \operatorname{div}\left((\rho_\varepsilon)^\gamma u\right)\right\} + \left\{(\rho_\varepsilon)^\gamma - (\rho^\gamma)_{\varepsilon'}\right\}\operatorname{div} u + (\rho^\gamma)_{\varepsilon'}\{\operatorname{div} u - \operatorname{div} u_{\varepsilon'}\}\\ &+\operatorname{div}\left((\rho^\gamma)_{\varepsilon'}u_{\varepsilon'}\right) - \operatorname{div}(\tau_{\varepsilon'}:u_{\varepsilon'}) + \tau_{\varepsilon'}:\nabla u_{\varepsilon'} - \operatorname{div}(u_{\varepsilon'}g_{\varepsilon'}) + g_{\varepsilon'}\operatorname{div} u_{\varepsilon'} - u_{\varepsilon'}\cdot f_{\varepsilon'}\\ &= \tfrac{\gamma}{\gamma-1} r_\varepsilon(\rho,u)(\rho_\varepsilon)^{\gamma-1}.\end{aligned}$$

Using the strong convergence properties of the convolution, Proposition 2.1 along with the fact that finite energy weak-solutions also satisfy $\rho\in L^{2\gamma}\left((0,T)\times\mathbb{T}^3\right)$ we get that

$$\begin{cases}(\rho_\varepsilon)^\gamma \to \rho^\gamma \text{ in } L^2\left((0,T)\times\mathbb{T}^3\right) \text{ for } \varepsilon\to 0,\\ (\rho_\varepsilon)^\gamma u \to \rho^\gamma u \text{ in } L^1\left((0,T)\times\mathbb{T}^3\right) \text{ for } \varepsilon\to 0,\\ (\rho^\gamma)_{\varepsilon'}\{\operatorname{div} u - \operatorname{div} u_{\varepsilon'}\} \to 0 \text{ in } L^1\left((0,T)\times\mathbb{T}^3\right) \text{ for } \varepsilon'\to 0\\ (\rho^\gamma)_{\varepsilon'}\operatorname{div} u_{\varepsilon'} \to \rho^\gamma\operatorname{div} u \text{ in } L^1\left((0,T)\times\mathbb{T}^3\right) \text{ for } \varepsilon'\to 0,\\ \tau_{\varepsilon'}:u_{\varepsilon'}\to\tau:u \text{ and } \tau_{\varepsilon'}:\nabla u_{\varepsilon'}\to\tau:u \text{ in } L^1\left((0,T)\times\mathbb{T}^3\right) \text{ for } \varepsilon'\to 0,\\ u_{\varepsilon'}f_{\varepsilon'}\to uf \text{ in } L^1\left((0,T)\times\mathbb{T}^3\right) \text{ for } \varepsilon'\to 0,\\ u_{\varepsilon'}g_{\varepsilon'}\to ug \text{ in } L^1\left((0,T)\times\mathbb{T}^3\right) \text{ for } \varepsilon'\to 0,\\ g_{\varepsilon'}\operatorname{div} u_{\varepsilon'}\to g\operatorname{div} u \text{ in } L^1\left((0,T)\times\mathbb{T}^3\right) \text{ for } \varepsilon'\to 0,\\ r_\varepsilon(\rho,u)(\rho_\varepsilon)^{\gamma-1}\to 0 \text{ in } L^1\left((0,T)\times\mathbb{T}^3\right) \text{ for } \varepsilon\to 0.\end{cases}$$

Consequently, we get that

$$\tfrac{1}{\gamma-1}\left\{\partial_t\rho^\gamma + \gamma\operatorname{div}\left(\rho^\gamma u\right)\right\} = \operatorname{div}(\tau u) - \tau:\nabla u + f\cdot u + \operatorname{div}(gu) - g\operatorname{div} u.$$

This ends the proof of Proposition 2.7. □

Next, we investigate the weak stability of a sequence of solutions of system (2.17). Our main result reads

Theorem 2.10. *Consider a sequence of finite energy weak-solutions* $\{(\rho^\varepsilon,u^\varepsilon)\}_\varepsilon$ *for (2.17) with initial data* $\left\{\rho_0^\varepsilon\right\}_{\varepsilon>0}\subset L^\gamma\left(\mathbb{T}^3\right)$*, i.e.*

$$\begin{cases}\partial_t\rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0,\\ -\operatorname{div}\tau^\varepsilon + \nabla(\rho^\varepsilon)^\gamma = f^\varepsilon,\\ \rho^\varepsilon_{|t=0} = \rho_0^\varepsilon,\end{cases} \tag{2.19}$$

with

$$\tau^\varepsilon_{ij} = A^\varepsilon_{ijkl}(t,x)D_{kl}(u^\varepsilon),$$

where

$$\begin{cases} \rho_0^\varepsilon \to \rho_0 \text{ in } L^\gamma\left(\mathbb{T}^3\right), \\ A^\varepsilon(t,x) \to A(t,x) \text{ in } W^{1,\infty}((0,T)\times\mathbb{T}^3), \\ f^\varepsilon \to f \text{ in } L^2(0,T;L^{\frac{6}{5}}(\mathbb{T}^3))). \end{cases} \tag{2.20}$$

Then, there exists $(\rho,u) \in L^{2\gamma}\left((0,T)\times\mathbb{T}^3\right)\times\left[L^2(0,T;H^1(\mathbb{T}^3))\right]^3$ *such that modulo a subsequence we have*

$$\begin{cases} \rho^\varepsilon \rightharpoonup \rho \text{ weakly in } L^{2\gamma}\left((0,T)\times\mathbb{T}^3\right), \\ \rho^\varepsilon \to \rho \text{ in } L^q\left((0,T)\times\mathbb{T}^3\right) \text{ with } q<2\gamma, \\ u^\varepsilon \rightharpoonup u \text{ weakly in } L^2(0,T;H^1(\mathbb{T}^3)), \\ u^\varepsilon \to u \text{ in } L^2((0,T)\times\mathbb{T}^3)), \end{cases} \tag{2.21}$$

where (ρ,u) *verifies*

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ -\operatorname{div}\tau + \nabla\rho^\gamma = \nabla f, \\ \rho_{|t=0} = \rho_0, \end{cases} \tag{2.22}$$

with

$$\tau_{ij} = A_{ijkl}(t,x)D_{kl}(u).$$

Proof of Theorem 2.10. The information on the initial data (2.20) along with Proposition 2.11 ensures that

$$\|\rho^\varepsilon\|_{L^\infty(0,T;L^\gamma(\mathbb{T}^3))\cap L^{2\gamma}((0,T)\times\mathbb{T}^3)} + \|u^\varepsilon\|_{L^2(0,T;H^1(\mathbb{T}^3))\cap W^{1,1}(0,T;L^{3/2-\delta}(\mathbb{T}^d))} \le C\left(1+\sqrt{T}\right),$$

for all $T>0$. The assumptions allow us to conclude that there exist three functions $\left(\rho,u,\overline{\rho^\gamma}\right)$ such that up to a subsequence we have the following:

$$\begin{cases} \rho^\varepsilon \rightharpoonup \rho \text{ weakly in } L^{2\gamma}\left((0,T)\times\mathbb{T}^3\right), \\ (\rho^\varepsilon)^\gamma \rightharpoonup \overline{\rho^\gamma} \text{ weakly in } L^2\left((0,T)\times\mathbb{T}^3\right), \\ \nabla u^\varepsilon \rightharpoonup \nabla u \text{ weakly in } L^2\left((0,T)\times\mathbb{T}^3\right), \\ u^\varepsilon \to u \text{ strongly in } L^2\left((0,T)\times\mathbb{T}^3\right). \end{cases} \tag{2.23}$$

Moreover, we may take the above subsequence such as

$$\begin{cases} \tau^\varepsilon : \nabla u^\varepsilon \rightharpoonup \overline{\tau:\nabla u} \text{ in } \mathcal{M}\left((0,T)\times\mathbb{T}^3\right) \text{ and} \\ \tau:\nabla u \le \overline{\tau:\nabla u} \text{ in the sense of measures.} \end{cases} \tag{2.24}$$

The last inequality follows from the weak lower semi-continuity of the viscous work: see hypothesis (1.7). All the above information allows us to conclude that

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ -\operatorname{div}\tau + \nabla\overline{\rho^\gamma} = f, \end{cases} \tag{2.25}$$

with

$$\tau_{ij} = A_{ijkl}(t,x)D_{kl}(u).$$

In order to finish the proof we still have to identify $\overline{\rho^\gamma}$ with ρ^γ. Let us observe that for any $\varepsilon>0$, $(\rho^\varepsilon,u^\varepsilon)$ verifies the hypothesis of Proposition 2.7 and thus we infer that

$$\tfrac{1}{\gamma-1}\left\{\partial_t(\rho^\varepsilon)^\gamma + \gamma\operatorname{div}\left((\rho^\varepsilon)^\gamma u^\varepsilon\right)\right\} = \operatorname{div}(\tau^\varepsilon : u^\varepsilon) - \tau^\varepsilon:\nabla u^\varepsilon + f^\varepsilon u^\varepsilon. \tag{2.26}$$

Moreover, using the information of relation (2.23) we may pass to the limit in (2.26) such as to obtain

$$\tfrac{1}{\gamma-1}\left\{\partial_t\overline{\rho^\gamma} + \gamma\operatorname{div}\left((\overline{\rho^\gamma}u)\right)\right\} = \operatorname{div}(\tau:u) - \overline{\tau:\nabla u} + fu. \tag{2.27}$$

Observing that we may put the system (2.25) under the form

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ -\operatorname{div}\tau + \nabla \rho^\gamma = \nabla(\rho^\gamma - \overline{\rho^\gamma}) + f, \end{cases} \tag{2.28}$$

with $\tau_{ij} = A_{ijkl}(t,x)D_{kl}(u)$ and using Proposition 2.7 we write that

$$\begin{aligned} &\tfrac{1}{\gamma-1}\left\{\partial_t \rho^\gamma + \gamma \operatorname{div}\left(\rho^\gamma u\right)\right\} - \operatorname{div}\left(u\left(\rho^\gamma - \overline{\rho^\gamma}\right)\right) + \left(\rho^\gamma - \overline{\rho^\gamma}\right)\operatorname{div} u \\ &= \operatorname{div}(\tau : u) - (\gamma-1)\,\tau : \nabla u + (\gamma-1)\, f\cdot u. \end{aligned} \tag{2.29}$$

Next, we take the difference between (2.29) and (2.27) we get that

$$\begin{aligned} &\partial_t\left(\overline{\rho^\gamma} - \rho^\gamma\right) + \operatorname{div}\left(\left(\overline{\rho^\gamma} - \rho^\gamma\right)u\right) + (\gamma-1)\left(\overline{\rho^\gamma} - \rho^\gamma\right)\operatorname{div} u \\ &= -(\gamma-1)\left\{\overline{\tau : \nabla u} - \tau : \nabla u\right\}. \end{aligned} \tag{2.30}$$

We denote by

$$\theta \overset{not.}{=} \overline{\rho^\gamma} - \rho^\gamma \qquad \zeta \overset{not.}{=} \overline{\tau : \nabla u} - \tau : \nabla u,$$

and thus (2.30) rewrites as

$$\partial_t \theta + \operatorname{div}(\theta u) + (\gamma-1)\,\theta \operatorname{div} u = -(\gamma-1)\,\zeta.$$

We regularize the above equation in order to obtain (again recall the notations introduced in (2.1) and (2.2))

$$\partial_t \theta_{\varepsilon'} + \operatorname{div}(\theta_{\varepsilon'} u) + (\gamma-1)\,(\theta \operatorname{div} u)_{\varepsilon'} = r_{\varepsilon'}(\theta, u) - (\gamma-1)\,\zeta_{\varepsilon'}.$$

We multiply with $\frac{1}{\gamma}(h+\theta_{\varepsilon'})^{\frac{1}{\gamma}-1}$ where h is a fixed positive constant. We end up with

$$\begin{aligned} &\partial_t (h+\theta_{\varepsilon'})^{\frac{1}{\gamma}} + \operatorname{div}\left((h+\theta_{\varepsilon'})^{\frac{1}{\gamma}} u\right) + (h+\theta_{\varepsilon'})^{\frac{1}{\gamma}-1}[\left(\frac{1}{\gamma}-1\right)\theta_{\varepsilon'} - h]\operatorname{div} u \\ &+ \left(1-\frac{1}{\gamma}\right)(h+\theta_{\varepsilon'})^{\frac{1}{\gamma}-1}(\theta \operatorname{div} u)_{\varepsilon'} \\ &= \frac{1}{\gamma}(h+\theta_{\varepsilon'})^{\frac{1}{\gamma}-1} r_{\varepsilon'}(\theta,u) - \frac{1}{\gamma}(h+\theta_{\varepsilon'})^{\frac{1}{\gamma}-1}(\gamma-1)\,\zeta_{\varepsilon'}. \end{aligned}$$

Let us integrate the above relation in order to get that

$$\begin{aligned} &\int_{\mathbb{T}^3} (h+\theta_{\varepsilon'})^{\frac{1}{\gamma}}(t) \\ &= \int_{\mathbb{T}^3} (h+\theta_{\varepsilon'})^{\frac{1}{\gamma}}(0) + \int_0^T\int_{\mathbb{T}^3}\left[\frac{1}{\gamma}(h+\theta_{\varepsilon'})^{\frac{1}{\gamma}-1} r_{\varepsilon'}(\theta,u) - \frac{1}{\gamma}(h+\theta_{\varepsilon'})^{\frac{1}{\gamma}-1}(\gamma-1)\,\zeta_{\varepsilon'}\right] \\ &\leq \int_{\mathbb{T}^3} (h+\theta_{\varepsilon'})^{\frac{1}{\gamma}}(0) + \int_0^T\int_{\mathbb{T}^3} \frac{1}{\gamma}(h+\theta_{\varepsilon'})^{\frac{1}{\gamma}-1} r_{\varepsilon'}(\theta,u) - \int_0^T\int_{\mathbb{T}^3} R_{\varepsilon'}, \end{aligned}$$

with

$$R_{\varepsilon'} = (h+\theta_{\varepsilon'})^{\frac{1}{\gamma}-1}\left[\left(\frac{1}{\gamma}-1\right)\left(\theta_{\varepsilon'}\operatorname{div} u - (\theta \operatorname{div} u)_{\varepsilon'}\right) - h \operatorname{div} u\right] \tag{2.31}$$

The last inequality is justified by combining the positiveness of the measure ζ (which is obtained using the lower semi-continuity assumption (1.7)) along with the fact that the convolution kernel is positive. We integrate the above relation in time in order to recover that

$$\int_0^T \int_{\mathbb{T}^3} (h+\theta_{\varepsilon'})^{\frac{1}{\gamma}}(t)$$

$$\leq T\int_{\mathbb{T}^3} (h+\theta_{\varepsilon'})^{\frac{1}{\gamma}}(0) + T\int_0^T\int_{\mathbb{T}^3} \frac{1}{\gamma}(h+\theta_{\varepsilon'})^{\frac{1}{\gamma}-1} r_{\varepsilon'}(\theta,u) - \int_0^T\int_{\mathbb{T}^3} R_{\varepsilon'},$$

with $R_{\varepsilon'}$ given by (2.31). Thanks to Proposition 2.1, we get that

$$r_{\varepsilon'}(\theta,u) \to 0 \text{ in } L^1\left((0,T)\times\mathbb{T}^3\right).$$

Thus observing that $(h+\theta_{\varepsilon'})^{1/\gamma-1} \leq h^{1/\gamma-1}$ (because $\gamma>1$ and $\theta_{\varepsilon'}\geq 0$), we have

$$\int_0^T\int_{\mathbb{T}^3} (h+\theta_{\varepsilon'})^{\frac{1}{\gamma}-1} r_{\varepsilon'}(\theta,u) \leq h^{\frac{1}{\gamma}-1}\int_0^T\int_{\mathbb{T}^3} |r_{\varepsilon'}(\theta,u)|$$

and we conclude that

$$|R_{\varepsilon'}| \leq \left(1-\frac{1}{\gamma}\right) h^{\frac{1}{\gamma}-1}\,|r_{\varepsilon'}(\theta,u)| + h^{\frac{1}{\gamma}}\,|\operatorname{div} u|.$$

Taking in account the last observations, by making $\varepsilon'\to 0$ we get that

$$\int_0^T\int_{\mathbb{T}^3} \left(\overline{\rho^\gamma}-\rho^\gamma+h\right)^{\frac{1}{\gamma}} \leq T\int_{\mathbb{T}^3}\left(\overline{\rho^\gamma}-\rho^\gamma+h\right)^{\frac{1}{\gamma}}_{|t=0} + h^{1/\gamma}\int_0^T\int_{\mathbb{T}^3} |\operatorname{div} u|.$$

Letting h go to zero and using the strong convergence at initial time shows that the term in the RHS of the above equation is 0 and the conclusion is that

$$\overline{\rho^\gamma} = \rho^\gamma \text{ a.e. on } (0,T)\times\mathbb{T}^3.$$

This ends the proof of Theorem 2.10. □

3. Construction of solutions

In this section, we propose a regularized system with diffusion and drag terms in the density equation for which we prove global existence and uniqueness of strong solution on $(0,T)$ using a fixed point procedure. Then passing to the limit with respect to the regularization parameter provides a global solution of the quasi-stationary compressible Stokes system with diffusion on the density and drag terms on the density. It remains to show that these extra terms do not perturb the stability procedure, we explained in subsection 2.3, to prove Theorem 2.10.

3.1. The approximate system

Let us be more precise. For any fixed strictly positive parameter ε,δ we are able to construct a global solution of the following regularized version of the original system:

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho\omega_\delta * u) = \varepsilon\Delta\rho - \varepsilon\rho^{2\gamma} - \varepsilon\rho^{2\gamma+1} - \varepsilon\rho^3, \\ \mathcal{A}u + \nabla\omega_\delta * \rho^\gamma = f, \\ \rho_{|t=0} = \rho_0^{reg}, \end{cases} \tag{S_ε,δ}$$

with ω_δ the standard regularizing kernel see (2.1). The function ρ_0^{reg} is supposed to be regular enough as to ensure existence of solutions for the transport equation with regular velocity and initial data ρ_0^{reg}. The construction of solutions for (S_ε,δ) is achieved by a classical fixed point argument.

In a second time, we show that a sequence of solutions $\{(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta})\}_\varepsilon$ of (S_ε, δ) tends, when we let δ go to zero, to $(\rho^\varepsilon, u^\varepsilon)$ which is a solution of the system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = \varepsilon \Delta \rho - \varepsilon \rho^{2\gamma} - \varepsilon \rho^{2\gamma+1} - \varepsilon \rho^3, \\ \mathcal{A}u + \nabla \rho^\gamma = f, \\ \rho_{|t=0} = \rho_0, \end{cases} \tag{S_ε}$$

which, moreover, verifies the following estimates, uniformly in ε (we skip the ε script in the inequalities below as to ease the reading):

$$\begin{cases} \int_{\mathbb{T}^3} \rho(t) + \varepsilon \int_0^t \int_{\mathbb{T}^3} \rho^{2\gamma} + \varepsilon \int_0^t \int_{\mathbb{T}^3} \rho^{2\gamma+1} + \varepsilon \int_0^t \int_{\mathbb{T}^3} \rho^3 = \int_{\mathbb{T}^3} \rho_0, \\ \int_{\mathbb{T}^3} \rho^\gamma(t) + \frac{c(\gamma-1)}{2} \int_0^t \int_{\mathbb{T}^3} |\nabla u|^2 \\ \qquad + \varepsilon\gamma \int_0^t \int_{\mathbb{T}^3} \rho^{3\gamma-1} + \varepsilon\gamma \int_0^t \int_{\mathbb{T}^3} \rho^{3\gamma} + \varepsilon\gamma \int_0^t \int_{\mathbb{T}^3} \rho^{\gamma+2} \\ \qquad + 4\varepsilon[1 - \frac{1}{\gamma}] \int_0^t \int_{\mathbb{T}^3} \left|\nabla \rho^{\frac{\gamma}{2}}\right|^2 \le C(c,\gamma) \left(\int_{\mathbb{T}^3} \rho_0^\gamma + \|f\|^2_{L^2(0,T;L^{\frac{6}{5}}(\mathbb{T}^3))} \right), \\ \|\rho^\gamma\|_{L^2((0,T)\times\mathbb{T}^3)} \\ \qquad \le C(c,\gamma) \left(\sqrt{t} + \max\left\{1, \|A\|_{L^\infty((0,T)\times\mathbb{T}^3)}\right\}\right) \left(\|\rho_0\|^{\frac{\gamma}{2}}_{L^\gamma(\mathbb{T}^3)} + \|f\|_{L^2(0,T;L^{\frac{6}{5}}(\mathbb{T}^3))} \right), \end{cases} \tag{3.1}$$

with c defined by (1.9) and $C(c,\gamma)$ a constant depending only on c and γ.

Finally, we show that we can adapt the proof of Theorem 2.10 in order to pass to the limit $\varepsilon \to 0$ and thus obtaining a solution for the compressible Stokes system.

3.2. *Construction of solutions for the regularized system* (S_ε, δ)

We consider $T > 0$ to be precise later and we denote by

$$L^2(0,T;\dot{H}^1(\mathbb{T}^3)) = \left\{ u \in L^2(0,T;H^1(\mathbb{T}^3)) : \int_{\mathbb{T}^3} u(t) = 0 \text{ a.e. } t \in (0,T) \right\}.$$

Consider

$$B : L^2(0,T;\dot{H}^1(\mathbb{T}^3)) \to L^2(0,T;\dot{H}^1(\mathbb{T}^3)),$$

defined as

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \omega_\delta * v) = \varepsilon \Delta \rho - \varepsilon \rho^{2\gamma} - \varepsilon \rho^{2\gamma+1} - \varepsilon \rho^3, \\ \mathcal{A}B(v) + \nabla \omega_\delta * \rho^\gamma = f, \\ \rho_{|t=0} = \rho_0^{reg}. \end{cases} \tag{3.2}$$

Obviously if $v \in L^2(0,T;\dot{H}^1(\mathbb{T}^3))$ then $\omega_\delta * v \in L^2(0,T;C^\infty(\mathbb{T}^3))$ such that the existence of a regular *positive* solution for the first equation of system (3.2) follows by classical arguments. Also, $B(v)$ is well-defined as an element of $L^2(0,T;\dot{H}^1(\mathbb{T}^3))$ and

$$\int_0^T \int_{\mathbb{T}^3} A(t,x) D(B(v)) : D(B(v)) = \int_0^T \int_{\mathbb{T}^3} \omega_\delta * \rho^\gamma \operatorname{div} B(v) + \int_0^T \int_{\mathbb{T}^3} f \cdot u,$$

which provides

$$\|\nabla B(v)\|_{L^2((0,T)\times\mathbb{T}^3)} \le C\left\|\omega_\delta * \rho^\gamma\right\|_{L^2((0,T)\times\mathbb{T}^3)} + C\|f\|_{L^2(0,T;L^{\frac{6}{5}}(\mathbb{T}^3))}, \tag{3.3}$$

with C depending only on the dissipation operator. Let us integrate the equation defining ρ in order to see that

$$\int_{\mathbb{T}^3}\rho(t) + \varepsilon\int_0^t\int_{\mathbb{T}^3}\rho^{2\gamma} + \varepsilon\int_0^t\int_{\mathbb{T}^3}\rho^{2\gamma+1} + \varepsilon\int_0^t\int_{\mathbb{T}^3}\rho^3 = \int_{\mathbb{T}^3}\rho_0^{reg},$$

which, enables us to conclude, that

$$\|\nabla B(v)\|_{L^2((0,T)\times\mathbb{T}^3)} \le \tilde{C}(c,\gamma)\left(\frac{1}{\varepsilon}\int_{\mathbb{T}^3}\rho_0^{reg} + \|f\|^2_{L^2(0,T;L^{\frac{6}{5}}(\mathbb{T}^3))}\right)^{\frac{1}{2}}. \tag{3.4}$$

Thus, we conclude that for any $T>0$, the operator B (trivially) maps E_T into itself where

$$E_T = \left\{v\in L^2_T(\dot{H}^1(\mathbb{T}^3)) : \|\nabla v\|_{L^2((0,T)\times\mathbb{T}^3)} \le \tilde{C}(c,\gamma)\left(\frac{1}{\varepsilon}\int_{\mathbb{T}^3}\rho_0^{reg} + \|f\|^2_{L^2(0,T;L^{\frac{6}{5}}(\mathbb{T}^3))}\right)\right\}.$$

In the following, we aim at showing that B is a contraction on E_T. The first observation that we make in towards this direction is that using a maximum principle we get

$$\begin{aligned}\|\rho\|_{L^\infty((0,t)\times\mathbb{T}^3)} &\le \left\|\rho_0^{reg}\right\|_{L^\infty(\mathbb{T}^3)}\exp\left(\int_0^t\|\operatorname{div}\omega_\delta * v\|_{L^\infty(\mathbb{T}^3)}\right)\\ &\le \left\|\rho_0^{reg}\right\|_{L^\infty(\mathbb{T}^3)}\exp\left(\sqrt{t}C_{\varepsilon,\delta}\right).\end{aligned} \tag{3.5}$$

Next, let us multiply the first equation of (3.2) with ρ and integrate in order to obtain that

$$\frac{1}{2}\int_{\mathbb{T}^3}\rho^2 + \varepsilon\int_0^t\int_{\mathbb{T}^3}|\nabla\rho|^2 + \varepsilon\int_0^t\int_{\mathbb{T}^3}\rho^{2\gamma+1} + \varepsilon\int_0^t\int_{\mathbb{T}^3}\rho^{2\gamma+2} + \varepsilon\int_0^t\int_{\mathbb{T}^3}\rho^4 = \varepsilon\int_{\mathbb{T}^3}\rho^2\operatorname{div}(\omega_\delta * v)$$

and thus by Grönwall's lemma we get that

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{T}^3}\rho^2 + \varepsilon\int_0^t\int_{\mathbb{T}^3}|\nabla\rho|^2 + \varepsilon\int_0^t\int_{\mathbb{T}^3}\rho^{2\gamma+1} + \varepsilon\int_0^t\int_{\mathbb{T}^3}\rho^{2\gamma+2} + \varepsilon\int_0^t\int_{\mathbb{T}^3}\rho^4\\ &\le \frac{1}{2}\int_{\mathbb{T}^3}\left(\rho_0^{reg}\right)^2\exp\left(\int_0^t\|\operatorname{div}(\omega_\delta * v)\|_{L^\infty(\mathbb{T}^3)}\right)\\ &\le \frac{1}{2}\int_{\mathbb{T}^3}\left(\rho_0^{reg}\right)^2\exp\left(tC_\delta\int_0^t\|\nabla v\|^2_{L^2(\mathbb{T}^3)}\right)\\ &\le \frac{1}{2}\int_{\mathbb{T}^3}\left(\rho_0^{reg}\right)^2\exp\left(tC_\delta\left(\frac{1}{\varepsilon}\int_{\mathbb{T}^3}\rho_0^{reg} + \|f\|^2_{L^2(0,T;L^{\frac{6}{5}}(\mathbb{T}^3))}\right)^{\frac{1}{2}}\right).\end{aligned} \tag{3.6}$$

Let us consider $v_1, v_2 \in E_T$ and let us consider

$$\begin{cases} \partial_t \rho_i + \operatorname{div}(\rho_i \omega_\delta * v_i) = \varepsilon \Delta \rho_i - \varepsilon \rho_i^{2\gamma} - \varepsilon \rho_i^{2\gamma+1} - \varepsilon \rho_i^3, \\ \mathcal{A}B(v_i) + \nabla \omega_\delta * \rho_i^\gamma = 0, \\ \rho_{i|t=0} = \rho_0^{reg} \end{cases}$$

with $i \in 1, 2$. Of course, ρ_1 and ρ_2 verify the estimate (3.6). We denote by $r = \rho_1 - \rho_2$ and $w = v_1 - v_2$. We infer that

$$\begin{cases} \partial_t r + \operatorname{div}(r \omega_\delta * v_1) = \varepsilon \Delta r - \varepsilon \left(\rho_1^{2\gamma} + \rho_1^{2\gamma+1} + \rho_1^3 - \rho_2^{2\gamma} - \rho_2^{2\gamma+1} - \rho_2^3\right) - \operatorname{div}(\rho_2 V_\delta * w), \\ \mathcal{A}(B(v_1) - B(v_2)) + \nabla \omega_\delta * \left(\rho_1^\gamma - \rho_2^\gamma\right) = 0, \\ r_{|t=0} = 0 \end{cases}$$

By multiplying the first equation with r we get that

$$\begin{aligned}
&\int_{\mathbb{T}^3} \frac{r^2(t)}{2} + \varepsilon \int_0^t \int_{\mathbb{T}^3} |\nabla r|^2 + \varepsilon \int_0^t \int_{\mathbb{T}^3} \left(\rho_1^{2\gamma} + \rho_1^{2\gamma+1} + \rho_1^3 - \rho_2^{2\gamma} - \rho_2^{2\gamma+1} - \rho_2^3\right) r \\
&\leq \int_0^t \int_{\mathbb{T}^3} r^2 \operatorname{div} \omega_\delta * v_1 + \int_0^t \int_{\mathbb{T}^3} \operatorname{div}(\rho_2 \omega_\delta * w)\, r \\
&\leq \int_0^t \int r^2 \|\operatorname{div} \omega_\delta * v_1\|_{L^\infty(\mathbb{T}^3)} + \frac{1}{2\varepsilon} \int_0^t \|\rho_2\|_{L^2(\mathbb{T}^3)}^2 \|\omega_\delta * \delta v\|_{L^\infty(\mathbb{T}^3)}^2 + \frac{\varepsilon}{2} \int_0^t \int_{\mathbb{T}^3} |\nabla r|^2 \\
&\leq \int_0^t \int_{\mathbb{T}^3} r^2 \|\operatorname{div} \omega_\delta * v_1\|_{L^\infty(\mathbb{T}^3)} + C_{\delta,\varepsilon} \exp\left(t C_{\delta,\varepsilon} \int \rho_0^{reg}\right) \int_0^t \|\delta v\|_{L^6(\mathbb{T}^3)}^2 + \frac{\varepsilon}{2} \int_0^t \int_{\mathbb{T}^3} |\nabla r|^2 \\
&\leq \int_0^t \int_{\mathbb{T}^3} r^2 \|\operatorname{div} \omega_\delta * v_1\|_{L^\infty(\mathbb{T}^3)} + C_{\delta,\varepsilon} \exp\left(t C_{\delta,\varepsilon} \int \rho_0^{reg}\right) \int_0^t \|\nabla \delta v\|_{L^2(\mathbb{T}^3)}^2 + \frac{\varepsilon}{2} \int_0^t \int_{\mathbb{T}^3} |\nabla r|^2
\end{aligned} \tag{3.7}$$

and thus using Grönwall's lemma we get that

$$\begin{aligned}
&\int_{\mathbb{T}^3} \frac{r^2(t)}{2} + \frac{\varepsilon}{2} \int_0^t \int_{\mathbb{T}^3} |\nabla r|^2 + \varepsilon \int_0^t \int_{\mathbb{T}^3} \left(\rho_1^{2\gamma} - \rho_2^{2\gamma}\right) r + \varepsilon \int_0^t \int_{\mathbb{T}^3} \left(\rho_1^{2\gamma+1} - \rho_2^{2\gamma+1}\right) r + \varepsilon \int_0^t \int_{\mathbb{T}^3} \left(\rho_1^3 - \rho_2^3\right) r \\
&\leq C_{\delta,\varepsilon} \exp\left(t C_{\delta,\varepsilon} \left(\int_{\mathbb{T}^3} \rho_0^{reg} + \|f\|_{L^2(0,T;L^{\frac{6}{5}}(\mathbb{T}^3))}^2\right)^{\frac{1}{2}}\right) \int_0^t \|\nabla w\|_{L^2(\mathbb{T}^3)}^2 \exp\left(\int_0^t \int_{\mathbb{T}^3} \|\operatorname{div} \omega_\delta * v_1\|_{L^\infty(\mathbb{T}^3)}\right) \\
&\leq C_{\delta,\varepsilon} \exp\left(C_{\delta,\varepsilon} t\right) \int_0^t \|\nabla w\|_{L^2(\mathbb{T}^3)}^2 = C_{\delta,\varepsilon} \exp\left(C_{\delta,\varepsilon} t\right) \int_0^t \|\nabla v_1 - \nabla v_2\|_{L^2(\mathbb{T}^3)}^2 .
\end{aligned} \tag{3.8}$$

Finally, recalling that

$$\mathcal{A}(B(v_1) - B(v_2)) + \nabla \omega_\delta * \left(\rho_1^\gamma - \rho_2^\gamma\right) = 0,$$

we infer that

$$\|\nabla(B(v_1) - B(v_2))\|_{L^2((0,t)\times\mathbb{T}^3)} \leq C t^{\frac{1}{2}} \left\|\rho_1^\gamma - \rho_2^\gamma\right\|_{L^\infty(0,t;L^2(\mathbb{T}^3))}. \tag{3.9}$$

We use the intermediate value theorem and estimate (3.5) in order to assess that

$$\begin{aligned}\left|\rho_1^\gamma-\rho_2^\gamma\right| &\le \gamma\left|\rho_1-\rho_2\right|\max\left\{\|\rho_1\|_{L^\infty((0,t)\times\mathbb{T}^3)}^{\gamma-1},\|\rho_2\|_{L^\infty((0,t)\times\mathbb{T}^3)}^{\gamma-1}\right\}\\ &\le \gamma\left|\rho_1-\rho_2\right|\left\|\rho_0^{reg}\right\|_{L^\infty(\mathbb{T}^3)}^{\gamma-1}\exp\left(\sqrt{t}C_{\delta,\varepsilon}\right),\end{aligned} \tag{3.10}$$

which, in turn implies that

$$\left\|\rho_1^\gamma-\rho_2^\gamma\right\|_{L^\infty(0,t;L^2(\mathbb{T}^3))}\le\gamma\left\|\rho_0^{reg}\right\|_{L^\infty(\mathbb{T}^3))}^{\gamma-1}\exp\left(\sqrt{t}C_{\delta,\varepsilon}\right)\|r\|_{L^\infty(0,t;L^2(\mathbb{T}^3))}.$$

This last estimate along with (3.8) gives us

$$\|\nabla\left(B\left(v_1\right)-B(v_2)\right)\|_{L^2((0,t)\times\mathbb{T}^3)}\le t^{\frac12}C_{\delta,\varepsilon}\exp\left(\left(1+t\right)C_{\delta,\varepsilon}\right)\|\nabla v_1-\nabla v_2\|_{L^2((0,t)\times\mathbb{T}^3)}.$$

We conclude that for a small $T^\star$ the operator has a fixed point $u\in E_{T^\star}$ which verifies (S_ε,δ). As the pair (ρ,u) solution of the above system verifies by integration of the first equation

$$\int_{\mathbb{T}^3}\rho\left(t\right)+\varepsilon\int_0^t\int_{\mathbb{T}^3}\rho^{2\gamma}+\varepsilon\int_0^t\int_{\mathbb{T}^3}\rho^{2\gamma+1}+\varepsilon\int_0^t\int_{\mathbb{T}^3}\rho^3=\int_{\mathbb{T}^3}\rho_0^{reg},$$

using the second equation of (S_ε,δ) we see that the last relation implies that

$$\|\nabla u\|_{L^2((0,T^\star)\times\mathbb{T}^3)}\le\tilde{C}\left(c,\gamma\right)\left(\frac{1}{\varepsilon}\int_{\mathbb{T}^3}\rho_0^{reg}+\|f\|^2_{L^2(0,T^\star;L^{\frac65}(\mathbb{T}^3))}\right)^{\frac12},$$

with the same $\tilde{C}\left(c,\gamma\right)$ appearing in (3.4). Thus, we may re-iterate the fixed point argument. This implies that the solution (ρ,u) of (S_ε,δ) is global.

3.3. The limit $\delta\to 0$

We consider $\{(\rho^\delta,u^\delta)\}_\varepsilon$ a sequence of solutions to

$$\begin{cases}\partial_t\rho^\delta+\operatorname{div}\left(\rho^\delta\omega_\delta*u^\delta\right)=\varepsilon\Delta\rho^\delta-\varepsilon(\rho^\delta)^{2\gamma}-\varepsilon(\rho^\delta)^{2\gamma+1}-\varepsilon\left(\rho^\delta\right)^3,\\ \mathcal{A}u^\delta+\nabla\omega_\delta*\left(\rho^\delta\right)^\gamma=f,\\ \rho_{|t=0}=\omega_\delta*\rho_0.\end{cases} \tag{S_ε,δ}$$

The sequence verifies the following estimates uniformly in δ:

$$\begin{cases}\displaystyle\int_{\mathbb{T}^3}\rho^\delta\left(t\right)+\varepsilon\int_0^t\int_{\mathbb{T}^3}\left(\rho^\delta\right)^{2\gamma}+\varepsilon\int_0^t\int_{\mathbb{T}^3}\left(\rho^\delta\right)^{2\gamma+1}+\varepsilon\int_0^t\int_{\mathbb{T}^3}\left(\rho^\delta\right)^3=\int_{\mathbb{T}^3}\omega_\delta*\rho_0\le\int_{\mathbb{T}^3}\rho_0,\\ \displaystyle\int_{\mathbb{T}^3}\left(\rho^\delta\right)^\gamma(t)+\frac{c\left(\gamma-1\right)}{2}\int_0^t\int_{\mathbb{T}^3}\left|\nabla u^\delta\right|^2\\ \displaystyle\qquad+\varepsilon\gamma\int_0^t\int_{\mathbb{T}^3}\left(\rho^\delta\right)^{3\gamma-1}+\varepsilon\gamma\int_0^t\int_{\mathbb{T}^3}\left(\rho^\delta\right)^{3\gamma}+\varepsilon\gamma\int_0^t\int_{\mathbb{T}^3}\left(\rho^\delta\right)^{\gamma+2}\\ \displaystyle\qquad\qquad+4\varepsilon[1-\frac{1}{\gamma}]\int_0^t\int_{\mathbb{T}^3}\left|\nabla\left(\rho^\delta\right)^{\frac{\gamma}{2}}\right|\le C\left(c,\gamma\right)\left(\int_{\mathbb{T}^3}\rho_0^\gamma+\|f\|^2_{L^2(0,T;L^{\frac65}(\mathbb{T}^3))}\right),\\ \displaystyle\left\|\omega_\delta*\left(\rho^\delta\right)^\gamma\right\|_{L^2((0,T)\times\mathbb{T}^3)}\le\sqrt{t}\int_{\mathbb{T}^3}\rho^\gamma+\left\|\Delta^{-1}\operatorname{div}\mathcal{A}u^\delta\right\|_{L^2((0,T)\times\mathbb{T}^3)}+\left\|\Delta^{-1}\operatorname{div}f\right\|_{L^2((0,T)\times\mathbb{T}^3)}\\ \displaystyle\le C\left(\gamma,c\right)\left(\sqrt{t}+\max\left\{1,\|A\|_{L^\infty((0,T)\times\mathbb{T}^3)}\right\}\right)\left(\int_{\mathbb{T}^3}\rho_0^\gamma+\|f\|^2_{L^2(0,T;L^{\frac65}(\mathbb{T}^3))}\right)^{\frac12}.\end{cases} \tag{3.11}$$

Moreover, we have that

$$\frac{1}{2}\int_{\mathbb{T}^3}\left(\rho^\delta\right)^2+\varepsilon\int_0^t\int_{\mathbb{T}^3}\left|\nabla\rho^\delta\right|^2$$

$$+\varepsilon\int_0^t\int_{\mathbb{T}^3}\left(\rho^\delta\right)^{2\gamma+1}+\varepsilon\int_0^t\int_{\mathbb{T}^3}\left(\rho^\delta\right)^{2\gamma+2}+\varepsilon\int_0^t\int_{\mathbb{T}^3}\left(\rho^\delta\right)^4=\gamma\int_0^t\int_{\mathbb{T}^3}\left(\rho^\delta\right)^2\operatorname{div}\left(\omega_\delta*u^\delta\right)$$

$$\leq\frac{\varepsilon}{2}\int_0^t\int_{\mathbb{T}^3}\left(\rho^\delta\right)^4+\frac{\gamma^2}{2\varepsilon}\int_0^t\int_{\mathbb{T}^3}\left(\omega_\delta*\operatorname{div}u^\delta\right)^2$$

and owing to the uniform bound on ∇u^δ ensured by the estimates (3.11) we get that

$$\frac{1}{2}\int_{\mathbb{T}^3}\left(\rho^\delta\right)^2+\varepsilon\int_0^t\int_{\mathbb{T}^3}\left|\nabla\rho^\delta\right|^2+\varepsilon\int_0^t\int_{\mathbb{T}^3}\left(\rho^\delta\right)^{2\gamma+1}+\varepsilon\int_0^t\int_{\mathbb{T}^3}\left(\rho^\delta\right)^{2\gamma+2}+\frac{\varepsilon}{2}\int_0^t\int_{\mathbb{T}^3}\left(\rho^\delta\right)^4$$
$$\leq\frac{C\left(\gamma,\|A\|_{L^\infty}\right)}{\varepsilon}\left(\int_{\mathbb{T}^3}\rho_0^\gamma+\|f\|^2_{L^2(0,T;L^{\frac{6}{5}}(\mathbb{T}^3))}\right).\tag{3.12}$$

Moreover, we have that

$$\partial_t\rho^\delta \text{ is bounded uniformly in } W^{-1,1}\left((0,T)\times L^1\left(\mathbb{T}^3\right)\right)+L^1\left((0,T)\times\mathbb{T}^3\right).\tag{3.13}$$

The estimates (3.11), (3.12) and (3.13) are enough in order to pass to the limit when $\delta\to0$ such that we obtain the existence of a solution of system

$$\begin{cases}\partial_t\rho+\operatorname{div}(\rho u)=\varepsilon\Delta\rho-\varepsilon\rho^{2\gamma}-\varepsilon\rho^{2\gamma+1}-\varepsilon\rho^3,\\ \mathcal{A}u+\nabla\rho^\gamma=f,\\ \rho_{|t=0}=\rho_0,\end{cases}$$

which verifies the following bounds

$$\begin{cases}\int_{\mathbb{T}^3}\rho(t)+\varepsilon\int_0^t\int_{\mathbb{T}^3}\rho^{2\gamma}+\varepsilon\int_0^t\int_{\mathbb{T}^3}\rho^{2\gamma+1}+\varepsilon\int_0^t\int_{\mathbb{T}^3}\rho^3=\int_{\mathbb{T}^3}\rho_0,\\ \int_{\mathbb{T}^3}\rho(t)+\frac{c(\gamma-1)}{2}\int_0^t\int_{\mathbb{T}^3}|u|^2\\ \qquad+\varepsilon\gamma\int_0^t\int_{\mathbb{T}^3}\rho^{3\gamma-1}+\varepsilon\gamma\int_0^t\int_{\mathbb{T}^3}\rho^{3\gamma}+\varepsilon\gamma\int_0^t\int_{\mathbb{T}^3}\rho^{\gamma+2}\\ \qquad+4\varepsilon[1-\frac{1}{\gamma}]\int_0^t\int_{\mathbb{T}^3}\left|\nabla\rho^{\frac{\gamma}{2}}\right|\leq C(c,\gamma)\left(\int_{\mathbb{T}^3}\rho_0^\gamma+\|f\|^2_{L^2(0,T;L^{\frac{6}{5}}(\mathbb{T}^3))}\right),\\ \left\|\rho^\gamma\right\|_{L^2((0,T)\times\mathbb{T}^3)}\\ \qquad\leq C(\gamma,c)\left(\sqrt{t}+\max\left\{1,\|A\|_{L^\infty((0,T)\times\mathbb{T}^3)}\right\}\right)\left(\int_{\mathbb{T}^3}\rho_0^\gamma+\|f\|^2_{L^2(0,T:L^{\frac{6}{5}}\mathbb{T}^3))}\right)^{\frac{1}{2}}.\end{cases}\tag{3.14}$$

3.4. Weak stability result for the perturbed system with diffusion and drag terms

In view of what was proved in the last section, let us consider a sequence $\{(\rho^\varepsilon, u^\varepsilon)\}_\varepsilon$ of solutions of

$$\begin{cases} \partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = \varepsilon \Delta \rho^\varepsilon - \varepsilon (\rho^\varepsilon)^{2\gamma} - \varepsilon (\rho^\varepsilon)^3, \\ \mathcal{A} u^\varepsilon + \nabla (\rho^\varepsilon)^\gamma = f, \\ \rho_{|t=0} = \rho_0, \end{cases} \tag{S_ε}$$

which verifies the following estimates uniformly in ε

$$\begin{cases} \int_{\mathbb{T}^3} \rho^\varepsilon (t) + \varepsilon \int_0^t \int_{\mathbb{T}^3} (\rho^\varepsilon)^{2\gamma} + \varepsilon \int_0^t \int_{\mathbb{T}^3} (\rho^\varepsilon)^{2\gamma+1} + \varepsilon \int_0^t \int_{\mathbb{T}^3} (\rho^\varepsilon)^3 = \int_{\mathbb{T}^3} \rho_0, \\ \int_{\mathbb{T}^3} (\rho^\varepsilon)^\gamma (t) + (\gamma - 1) \int_0^t \int_{\mathbb{T}^3} \tau^\varepsilon : \nabla u^\varepsilon \\ \qquad + \varepsilon\gamma \int_0^t \int_{\mathbb{T}^3} (\rho^\varepsilon)^{3\gamma-1} + \varepsilon\gamma \int_0^t \int_{\mathbb{T}^3} (\rho^\varepsilon)^{3\gamma} + \varepsilon\gamma \int_0^t \int_{\mathbb{T}^3} (\rho^\varepsilon)^{\gamma+2} \\ \qquad + 4\varepsilon[1 - \frac{1}{\gamma}] \int_0^t \int_{\mathbb{T}^3} \left| \nabla (\rho^\varepsilon)^{\frac{\gamma}{2}} \right|^2 \leq C(\gamma, c) \left(\int_{\mathbb{T}^3} \rho_0^\gamma + \| f \|^2_{L^2(0,T;L^{\frac{6}{5}}(\mathbb{T}^3))} \right), \\ \| (\rho^\varepsilon)^\gamma \|_{L^2((0,T)\times\mathbb{T}^3)} \\ \qquad \leq C(\gamma, c) \left(\sqrt{t} + \max\left\{1, \|A\|_{L^\infty((0,T)\times\mathbb{T}^3)}\right\} \right) \left(\int_{\mathbb{T}^3} \rho_0^\gamma + \| f \|^2_{L^2(0,T;L^{\frac{6}{5}}(\mathbb{T}^3))} \right)^{\frac{1}{2}}. \end{cases} \tag{3.15}$$

In the following we show that it is possible to slightly modify the proof of Theorem 2.10 in order to show that the limiting function (ρ, u) is a solution of the semi-stationary Stokes system. Indeed, let us observe that

$$\begin{aligned} &\gamma \left(h + \omega_{\varepsilon'} * (\rho^\varepsilon)\right)^{\gamma-1} \Delta \omega_{\varepsilon'} * (\rho^\varepsilon) \\ &= \Delta \left(\left(h + \omega_{\varepsilon'} * (\rho^\varepsilon)\right)^\gamma \right) - \nabla \left(h + \omega_{\varepsilon'} * (\rho^\varepsilon)\right)^{\gamma-1} \nabla \omega_{\varepsilon'} * (\rho^\varepsilon) \\ &= \Delta \left(\left(h + \omega_{\varepsilon'} * (\rho^\varepsilon)\right)^\gamma \right) - (\gamma - 1) \left(h + \omega_{\varepsilon'} * (\rho^\varepsilon)\right)^{\gamma-2} \nabla \omega_{\varepsilon'} * (\rho^\varepsilon) \nabla \omega_{\varepsilon'} * (\rho^\varepsilon) \\ &= \Delta \left(\left(h + \omega_{\varepsilon'} * (\rho^\varepsilon)\right)^\gamma \right) - \gamma \frac{(\gamma - 1)}{\left(\frac{\gamma}{2}\right)^2} \nabla \left(h + \omega_{\varepsilon'} * (\rho^\varepsilon)\right)^{\frac{\gamma}{2}} \nabla \left(h + \omega_{\varepsilon'} * (\rho^\varepsilon)\right)^{\frac{\gamma}{2}}. \end{aligned}$$

Thus, in the sense of distributions, we get that

$$\gamma \left(h + \omega_{\varepsilon'} * (\rho^\varepsilon)\right)^{\gamma-1} \Delta \omega_{\varepsilon'} * (\rho^\varepsilon) \underset{\varepsilon', h \to 0}{\to} \Delta (\rho^\varepsilon)^\gamma - 4[1 - \frac{1}{\gamma}] \left| \nabla (\rho^\varepsilon)^{\frac{\gamma}{2}} \right|^2.$$

Also, we have that

$$\begin{cases} (h + \omega_{\varepsilon'} * (\rho^\varepsilon))^{\gamma-1} \omega_{\varepsilon'} * (\rho^\varepsilon)^{2\gamma} \underset{\varepsilon', h \to 0}{\to} (\rho^\varepsilon)^{3\gamma-1} \text{ in } L^1_{t,x}, \\ (h + \omega_{\varepsilon'} * (\rho^\varepsilon))^{\gamma-1} \omega_{\varepsilon'} * (\rho^\varepsilon)^{2\gamma+1} \underset{\varepsilon', h \to 0}{\to} (\rho^\varepsilon)^{3\gamma} \text{ in } L^1_{t,x}, \\ (h + \omega_{\varepsilon'} * (\rho^\varepsilon))^{\gamma-1} \omega_{\varepsilon'} * (\rho^\varepsilon)^3 \underset{\varepsilon', h \to 0}{\to} (\rho^\varepsilon)^{\gamma+2} \text{ in } L^1_{t,x}. \end{cases}$$

We may thus write the renormalized equation for $(\rho^\varepsilon)^\gamma$ in two ways. Firstly, we have that

$$\begin{aligned} &\partial_t (\rho^\varepsilon)^\gamma + \operatorname{div}\left((\rho^\varepsilon)^\gamma u^\varepsilon\right) + (\gamma - 1) (\rho^\varepsilon)^\gamma \operatorname{div} u^\varepsilon \\ &= \varepsilon \Delta (\rho^\varepsilon)^\gamma - 4\varepsilon [1 - \frac{1}{\gamma}] \left| \nabla (\rho^\varepsilon)^{\frac{\gamma}{2}} \right|^2 - \varepsilon (\rho^\varepsilon)^{3\gamma-1} - \varepsilon (\rho^\varepsilon)^{3\gamma} - \varepsilon (\rho^\varepsilon)^{\gamma+2}, \end{aligned}$$

which we will use to obtain uniform bounds for $(\partial_t u^\varepsilon)_{\varepsilon>0}$. Secondly, we have that

$$\begin{aligned}
&\partial_t(\rho^\varepsilon)^\gamma + \gamma \operatorname{div}\left((\rho^\varepsilon)^\gamma u^\varepsilon\right) \\
&= (\gamma-1)\operatorname{div}(u^\varepsilon \tau^\varepsilon) - (\gamma-1)\,\tau^\varepsilon : \nabla u^\varepsilon + u^\varepsilon f \\
&+ \varepsilon\Delta\left(\rho^\varepsilon\right)^\gamma - 4\varepsilon\,[1-\frac{1}{\gamma}]\left|\nabla\left(\rho^\varepsilon\right)^{\frac{\gamma}{2}}\right|^2 - \varepsilon\,(\rho^\varepsilon)^{3\gamma-1} - \varepsilon\,(\rho^\varepsilon)^{3\gamma} - \varepsilon\,(\rho^\varepsilon)^{\gamma+2},
\end{aligned}$$

which is used for the compactness argument.

Let us observe that the time derivative of u verifies

$$\begin{aligned}
\mathcal{A}\partial_t u^\varepsilon &= \operatorname{div}(\partial_t A(t,x) D(u^\varepsilon)) + \partial_t f - \nabla\partial_t(\rho^\varepsilon)^\gamma \\
&= \operatorname{div}(\partial_t A(t,x) D(u^\varepsilon)) + \partial_t f \\
&\quad - \nabla\operatorname{div}(\left(\rho^\varepsilon\right)^\gamma u^\varepsilon\big) - (\gamma-1)\,\nabla\left((\rho^\varepsilon)^\gamma \operatorname{div} u^\varepsilon\right) - \varepsilon\nabla\Delta\left(\rho^\varepsilon\right)^\gamma \\
&\quad + 4\varepsilon\,[1-\frac{1}{\gamma}]\nabla\left|\nabla\left(\rho^\varepsilon\right)^{\frac{\gamma}{2}}\right|^2 + \varepsilon\nabla\,(\rho^\varepsilon)^{3\gamma-1} + \varepsilon\nabla\,(\rho^\varepsilon)^{3\gamma} + \varepsilon\nabla\,(\rho^\varepsilon)^{\gamma+2}.
\end{aligned}$$

Also, we have that

$$\varepsilon\nabla\left(\rho^\varepsilon\right)^\gamma = 2\varepsilon\left(\rho^\varepsilon\right)^{\frac{\gamma}{2}}\nabla\left(\rho^\varepsilon\right)^{\frac{\gamma}{2}},$$

such that we obtain

$$\begin{aligned}
\varepsilon\int_0^t \left\|\nabla\left(\rho^\varepsilon\right)^\gamma\right\|_{L^{\frac{3}{2}}(\mathbb{T}^3)} &\le \varepsilon\int_0^t\left(\int_{\mathbb{T}^3}\left(\rho^\varepsilon\right)^{3\gamma}\right)^{\frac{1}{3}}\left\|\nabla\left(\rho^\varepsilon\right)^{\frac{\gamma}{2}}\right\|_{L^2(\mathbb{T}^3)} \\
&\le C\left(t^{\frac{1}{6}} + \varepsilon\int_0^t\int_{\mathbb{T}^3}\left(\rho^\varepsilon\right)^{3\gamma} + \varepsilon\int_0^t\int_{\mathbb{T}^3}\left|\nabla\left(\rho^\varepsilon\right)^{\frac{\gamma}{2}}\right|^2\right)
\end{aligned}$$

and we see that $\left\{\nabla\left(\rho^\varepsilon\right)^\gamma\right\}_{\varepsilon>0}$ is uniformly bounded in $L^1(0,T;L^{\frac{3}{2}}\left(\mathbb{T}^3\right))$. It remains to write that

$$\mathcal{A}\partial_t u^\varepsilon = \mathcal{A}\phi_1^\varepsilon + \mathcal{A}\phi_2^\varepsilon + \mathcal{A}\phi_3^\varepsilon,$$

with

$$\begin{cases}
\mathcal{A}\phi_1^\varepsilon = \operatorname{div}(\partial_t A(t,x) D(u^\varepsilon)), \\
\mathcal{A}\phi_2^\varepsilon = -\nabla\left\{\operatorname{div}((\rho^\varepsilon)^\gamma u^\varepsilon) + (\gamma-1)\,(\rho^\varepsilon)^\gamma \operatorname{div} u^\varepsilon + \varepsilon\Delta\,(\rho^\varepsilon)^\gamma\right\}, \\
\mathcal{A}\phi_3^\varepsilon = \nabla\left\{4\varepsilon\,[1-\frac{1}{\gamma}]\left|\nabla\,(\rho^\varepsilon)^{\frac{\gamma}{2}}\right|^2 + \varepsilon(\rho^\varepsilon)^{3\gamma-1} + \varepsilon\nabla\,(\rho^\varepsilon)^{3\gamma} + \varepsilon(\rho^\varepsilon)^{\gamma+2}\right\}.
\end{cases}$$

Proceeding as in Proposition 2.11 we obtain an uniform bound for $(\partial_t u^\varepsilon)_{\varepsilon>0}$ in $L_t^1(L^{\frac{3}{2}-}\left(\mathbb{T}^3\right))$. Taking in consideration the renormalized equation for ρ, we conclude that

$$\begin{aligned}
&\partial_t\left(\overline{\rho^\gamma} - \rho^\gamma\right) + \operatorname{div}\left(\left(\overline{\rho^\gamma} - \rho^\gamma\right)u\right) + (\gamma-1)\left(\overline{\rho^\gamma} - \rho^\gamma\right)\operatorname{div} u \\
&= -(\gamma-1)\left\{\overline{\tau:\nabla u} - \tau:\nabla u\right\} - \nu,
\end{aligned} \tag{3.16}$$

where ν is a positive measure i.e.

$$\nu = \lim_{\varepsilon\to 0}\Big(4\varepsilon[1-\frac{1}{\gamma}]\left|\nabla\left(\rho^\varepsilon\right)^{\frac{\gamma}{2}}\right|^2 + \varepsilon(\rho^\varepsilon)^{3\gamma-1} + \varepsilon(\rho^\varepsilon)^{3\gamma} + (\rho^\varepsilon)^{\gamma+2}\Big).$$

Arguing along the same lines as in Subsection 2.3 we obtain that $\overline{\rho^\gamma} = \rho^\gamma$. This concludes the proof of the existence part of Theorem 1.1.

4. Applications to other systems

The objective of this paper is to give a proof à la Lions for the problem of existence of weak solutions for the Quasi-Stationary Stokes system. In the presentation, we choose to keep the model as simple as possible in order to avoid technical difficulties that would hinder the main idea to obtain compactness for the density: comparing the limit of the energy associated to a sequence of weak-solutions with the energy associated to the system verified by the limit. The objective of this section is to briefly discuss some further extensions of our work that require only slight modifications of the arguments presented above in order to be formally proved.

First of all our results apply to any perturbation of system (1.1) in the form:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ -\operatorname{div}\tau + a\nabla\rho^\gamma + Lu = f, \end{cases} \tag{4.1}$$

where $L : \left[L^2\left(\mathbb{T}^3\right)\right]^3 \to \left[L^2\left(\mathbb{T}^3\right)\right]^3$ is a linear bounded operator such that

$$\int_{\mathbb{T}^3} Lu \cdot u \geq 0,\ \partial_t (Lu) = L\partial_t u,$$

for simplicity. Of course, one has to further suppose that $L + \mathcal{A}$ verifies (1.9) in order to be able to recover estimates for the time derivative. An interesting choice that fits in this framework is

$$(Lu)^i = \partial_j \left(\kappa * (Du)_{ij} - \eta * \operatorname{div} u \delta_{ij}\right),$$

where κ, η are some smooth convolution kernels which amounts in changing the stress tensor into

$$\tau_{ij} = \tau_{ij}^{loc} + \tau_{ij}^{nonloc} = A_{ijkl}\,[D(u)]_{kl} + \kappa * (Du)_{ij} - \eta * \operatorname{div} u\, \delta_{ij}.$$

Of course, one has to assume appropriate conditions such as to ensure coercivity. Then existence of weak-solutions for system (4.1) follows without any significant modifications. Nonlocal effects are important in micro-fluidics where one is interested in fluids flowing within thin domains see for instance [8]. Another common choice for the operator L, see [15], modeling the effect of an electromagnetic field on the fluid is

$$Lu = B \times (B \times u),$$

where $B \in L^\infty\left(\mathbb{T}^3\right)$ with B non-constant, case in which we can incorporate also a force term of the type ρg.

Another situation where the weak-stability part of our result can be adapted without too much of an effort is given by the following stationary system

$$\begin{cases} \alpha\rho + \operatorname{div}(\rho u) = f, \\ \beta\rho u + \operatorname{div}(\rho u \otimes u) - \operatorname{div}\tau + a\nabla\rho^\gamma = g, \end{cases}$$

where $a, \alpha, \beta > 0,\ f \geq 0$ and τ is as above. This later system can be viewed as an implicit time discretization of the Navier-Stokes system. Obviously, one may add nonlocality into the model. Note however that our results do not apply to the case $\alpha = \beta = 0$ corresponding to the stationary Navier-Stokes system. This will be the object of a forthcoming paper [3].

Declaration of competing interest

There is no competing interest.

Acknowledgements

D. Bresch and C. Burtea are supported by the SingFlows project, grant ANR-18-CE40-0027 and D. Bresch is also supported by the Fraise project, grant ANR-16-CE06-0011 of the French National Research Agency (ANR). The authors thank the referees for their important suggestions allowing to improve the presentation of the paper.

Appendix A. Fourier analysis on the torus and elliptic estimates

In the following lines we present some results from Fourier analysis in the periodic setting. The proofs are essentially the same as those in the whole space presented in the book by H. Bahouri, J.-Y. Chemin, R. Danchin [1], Chapter 2, pages 52–53. To simplify the presentation, assume that $u \in L^1(\mathbb{T}^d)$. We start by reminding the definition and properties of Fourier coefficients of u:

$$\hat{u}_\eta = \int_{\mathbb{T}^n} \exp(-2\pi y \cdot \eta) u(y)\, dy.$$

We recall the existence of two positive functions $(\chi, \phi) \in \mathcal{D}(\mathbb{R}^d)$ such that $\operatorname{Supp} \chi \subset B\left(0, \frac{2}{3}\right)$, $\operatorname{Supp} \phi \subset \left\{x : \frac{3}{4} \leq |x| \leq \frac{8}{3}\right\}$ with the property that

$$\chi(\eta) + \sum_{j \geq -1} \phi\left(2^{-j}\eta\right) = 1 \; \forall \eta \in \mathbb{T}^d.$$

Next, for any $u \in L^1(\mathbb{T}^d)$, we introduce the jth-dyadic block operator defined as

$$\Delta_j^{\text{per}} u(x) = \sum_{\eta \in \mathbb{Z}^d} \phi\left(2^{-j}\eta\right) \hat{u}_\eta \exp(2\pi x \cdot \eta).$$

This operator localizes u near its frequencies of magnitude 2^j. Using the Poisson summation formula we see that

$$\Delta_j^{\text{per}} u(x) = \int_{\mathbb{R}^d} 2^{jd} h\left(2^j(x-y)\right) u(y)\, dy$$

where h is the Fourier inverse of ϕ. This last identity is useful to show that Δ_j^{per} maps all $L^p(\mathbb{T}^d)$ into $L^p(\mathbb{T}^d)$ with norm independent of j and p. For all $u \in L^1(\mathbb{T}^d)$ we have that

$$u = \int_{\mathbb{T}^d} u + \sum_{j \geq -1} \Delta_j^{\text{per}} u$$

at least in the sense of distributions. We infer that for any $u \in L^p(\mathbb{T}^d)$ with $\int_{\mathbb{T}^d} u = 0$ we have that

$$\|u\|_{L^p(\mathbb{T}^d)} \leq \sum_{j \geq -1} \left\|\Delta_j^{\text{per}} u\right\|_{L^p(\mathbb{T}^d)}. \tag{A.1}$$

Next, let us recall the celebrated Bernstein lemma.

Lemma A.1. *Consider any nonnegative integer k, a couple $p, q \in [1,\infty]^2$ with $p \leq q$ and a function $u \in L^1(\mathbb{T}^d)$. Then, there exists a constant C such that the following inequalities hold true:*

$$\sup_{|\alpha|=k} \left\|\partial^\alpha \Delta_j^{\text{per}} u\right\|_{L^q(\mathbb{T}^d)} \leq C^{k+1} 2^{jk + j\left(\frac{d}{p} - \frac{d}{q}\right)} \left\|\Delta_j^{\text{per}} u\right\|_{L^p(\mathbb{T}^d)}, \tag{A.2}$$

and

$$C^{-k-1} 2^{jk} \left\|\Delta_j^{\text{per}} u\right\|_{L^p(\mathbb{T}^d)} \leq \sup_{|\alpha|=k} \left\|\partial^\alpha \Delta_j^{\text{per}} u\right\|_{L^p(\mathbb{T}^d)} \leq C^{k+1} 2^{jk} \left\|\Delta_j^{\text{per}} u\right\|_{L^p(\mathbb{T}^d)}. \tag{A.3}$$

The following proposition will be very useful in establishing estimates for the Poisson problem.

Proposition A.2. *Consider $m \in \mathbb{R}$ and a smooth function $\sigma : \mathbb{R}^d \backslash \{0\} \to \mathbb{R}$ such that for all multi-index α with $|\alpha| \leq 2 + 2[d/2]$, there exists a constant C_α such that:*

$$\forall \xi \in \mathbb{R}^d \backslash \{0\} : \left|\partial^\alpha \sigma(\xi)\right| \leq C_\alpha |\xi|^{m-|\alpha|}.$$

Then for any $p \in [1, \infty]$ we have that

$$\left\| \sigma(D) \Delta_j^{\text{per}} v \right\|_{L^p(\mathbb{T}^d)} \leq 2^{jm} \left\| \Delta_j^{\text{per}} v \right\|_{L^p(\mathbb{T}^d)}$$

where

$$\sigma(D) \Delta_j^{\text{per}} v = \sum_{\eta \in \mathbb{Z}^d} \phi\left(2^{-j}\eta\right) \sigma(\eta) \hat{u}_\eta \exp(2\pi x \cdot \eta).$$

Finally, we use the Littlewood-Paley apparatus in order to prove the following $3D$ estimate for the Poisson problem.

Theorem A.3. *Consider $f \in L^1(\mathbb{T}^3)$ such that $\int_{\mathbb{T}^3} f = 0$ and ψ solution to the Poisson problem*

$$\begin{cases} -\Delta\psi = f, \\ \int_{\mathbb{T}^3} \psi = 0 \end{cases}$$

Then there exists a constant C such that for any $p \in [1, \frac{3}{2})$ we have

$$\|\nabla\psi\|_{L^p(\mathbb{T}^3)} \leq C \|f\|_{L^1(\mathbb{T}^3)}.$$

Proof. For any $l \in \overline{1,3}$ let observe that the function $\sigma_l : \mathbb{R}^3 \backslash \{0\} \to \mathbb{R}$ defined as

$$\sigma_l(\xi) = \frac{i\xi_l}{|\xi|^2},$$

verifies the hypothesis of Proposition A.2. Next, we see that for any $\eta \in \mathbb{Z}^d \backslash \{0\}$ and any $l = 1, 2, 3$ we have that

$$\widehat{\partial_l \psi}(\eta) = i\eta_l \hat{\psi}(\eta) = \frac{i\eta_l}{|\eta|^2} \hat{f}(\eta) = \sigma_l(\eta) \hat{f}(\eta)$$

such that

$$\Delta_j^{\text{per}}(\partial_l \psi) = \sigma_l(D) \Delta_j^{\text{per}} f$$

Let $p \in [1, \frac{3}{2})$. As, $\int \partial_l \psi = 0$ using (A.1), Proposition A.2 and Bernstein's inequality, we infer that

$$\begin{aligned} \|\partial_l \psi\|_{L^p(\mathbb{T}^3)} &\leq \sum_{j \geq -1} \left\| \Delta_j^{\text{per}} \partial_l \psi \right\|_{L^p(\mathbb{T}^3)} = \sum_{j \geq -1} \left\| \sigma_l(D) \Delta_j^{\text{per}} f \right\|_{L^p(\mathbb{T}^3)} \leq 2^{-j} \sum_{j \geq -1} \left\| \Delta_j^{\text{per}} f \right\|_{L^p(\mathbb{T}^3)} \\ &\leq \sum_{j \geq -1} 2^{j\left(2-\frac{3}{p}\right)} \left\| \Delta_j^{\text{per}} f \right\|_{L^1(\mathbb{T}^3)} \leq \|f\|_{L^1(\mathbb{T}^3)} \sum_{j \geq -1} 2^{j\left(2-\frac{3}{p}\right)}, \end{aligned}$$

where, of course the fact that $p \in [1, 3/2)$ ensures the convergence of the series $\sum_{j \geq -1} 2^{j\left(2-\frac{3}{p}\right)}$. With this remark we conclude the proof of Theorem A.3. □

Remark A.4. In fact, a more careful analysis of the proof of Theorem A.3 yields the following refined estimate

$$\|\nabla\psi\|_{L^p(\mathbb{T}^3)} \lesssim \|\nabla\psi\|_{B^0_{p,1}(\mathbb{T}^3)} \lesssim \|f\|_{B^0_{1,\infty}(\mathbb{T}^3)}$$

which is stronger than the classical result as the space of bounded measures is continuously included in $B^0_{1,\infty}(\mathbb{T}^3)$.

References

[1] H. Bahouri, J.-Y. Chemin, R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, vol. 343, Springer Science & Business Media, 2011.
[2] C. Bernardi, O. Pironneau, On the shallow water equations at low Reynolds number, Commun. Partial Differ. Equ. 16 (1) (1991) 59–104.
[3] D. Bresch, C. Burtea, Existence of weak solutions for the anisotropic stationary compressible Navier-Stokes system, https://arxiv.org/abs/2003.04587.
[4] D. Bresch, P.-E. Jabin, Global existence of weak solutions for compressible Navier–Stokes equations: thermodynamically unstable pressure and anisotropic viscous stress tensor, Ann. Math. 188 (2) (2018) 577–684.
[5] D. Bresch, P.-E. Jabin, Global weak solutions of PDEs for compressible media: a compactness criterion to cover new physical situations, in: Shocks, Singularities and Oscillations in Nonlinear Optics and Fluid Mechanics, in: Springer INdAM, vol. 17, 2017, pp. 33–54.
[6] D. Bresch, P. Mucha, E. Zatorska, Finite-energy solutions for compressible two-fluid Stokes system, Arch. Ration. Mech. Anal. 232 (2) (2019) 987–1029.
[7] D. Bresch, Š. Nečasová, C. Perrin, Compression effects in heterogeneous media, J. Éc. Polytech. Math. 6 (2019) 433–467.
[8] A.C. Eringen, On nonlocal fluid mechanics, Int. J. Eng. Sci. 10 (6) (1972) 561–575.
[9] E. Feireisl, A. Novotný, H. Petzeltová, On the existence of globally defined weak solutions to the Navier-Stokes equations, J. Math. Fluid Mech. 3 (4) (2001) 358–392.
[10] J. Frehse, S. Goj, J. Málek, On a Stokes-like system for mixtures of fluids, SIAM J. Math. Anal. 36 (2005) 1259–1281.
[11] J. Frehse, W. Weigant, On quasi-stationary models of mixtures of compressible fluids, Appl. Math. (4) (2008) 319–345.
[12] A.V. Kazhikhov, The equations of potential flow of compressible viscous fluid at low Reynolds number, Acta Appl. Math. 37 (1994) 71–77.
[13] A.V. Kazhikhov, The equation of potential flows of a compressible viscous fluid at small Reynolds numbers: existence, uniqueness, and stabilization of solutions, Sib. Mat. Zh. 34 (3) (1993) 70–80, Sib. Math. J. 34 (3) (1993) 457–467.
[14] A.V. Kazhikov, Some new statement for initial boundary value problems for Navier-Stokes equations of viscous gas, in: G.P. Galdi, J. Malek, J. Necas (Eds.), Progress in Theoretical and Computational Fluid Mechanics, Winter School, Paseky, 1993.
[15] P.-L. Lions, Mathematical Topics in Fluid Mechanics: vol. 2: Compressible Models, vol. 2, Oxford University Press, 1998.
[16] P.-L. Lions, Compacité des solutions des équations de Navier-Stokes compressible isentropiques, C. R. Acad. Sci. Paris 317 (1993) 115–120.
[17] A.E. Mamontov, Well-posedness of a quasistationary model of a viscous compressible fluid, Sib. Math. J. 37 (5) (1996) 983–996.
[18] N. Masmoudi, P. Zhang, Global solutions to vortex density equations arising from sup-conductivity, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 22 (4) (2005) 441–458.
[19] A. Novotný, I. Straškraba, Introduction to the Mathematical Theory of Compressible Flow, vol. 27, Oxford University Press, 2004.
[20] Augusto C. Ponce, Elliptic PDEs, Measures and Capacities, Tracts in Mathematics, vol. 23, 2016.
[21] V.A. Vaigant, A.V. Kazhikhov, Global solutions of equations of potential flows of a compressible viscous fluid for small Reynolds numbers, Differ. Uravn. 30 (1994) 1010–1022.
[22] D. Serre, Variations de grande amplitude pour la densité d'un fluide visqueux compressible, Phys. D: Nonlinear Phenom. 48 (1) (1991) 113–128.