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Existence of solutions for a higher-order semilinear parabolic equation with singular initial data

Kazuhiro Ishige [a,*], Tatsuki Kawakami [b], Shinya Okabe [c]

[a] *Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan*
[b] *Applied Mathematics and Informatics Course, Faculty of Advanced Science and Technology, Ryukoku University, 1-5 Yokotani, Seta Oe-cho, Otsu, Shiga 520-2194, Japan*
[c] *Mathematical Institute, Tohoku University, Aoba, Sendai 980-8578, Japan*

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Abstract

We establish the existence of solutions of the Cauchy problem for a higher-order semilinear parabolic equation by introducing a new majorizing kernel. We also study necessary conditions on the initial data for the existence of local-in-time solutions and identify the strongest singularity of the initial data for the solvability of the Cauchy problem.

Keywords: Higher-order semilinear parabolic equation; Majorizing kernel; Singular initial data; Solvability

1. Introduction

Consider the Cauchy problem for a higher-order nonlinear parabolic equation

$$\begin{cases} \partial_t u + (-\Delta)^m u = |u|^p, & x \in \mathbf{R}^N,\ t > 0, \\ u(x,0) = \mu(x) \ge 0, & x \in \mathbf{R}^N, \end{cases} \tag{1.1}$$

where $m = 2, 3, \dots,$ $p > 1$ and μ is a nonnegative measurable function in $\mathbf{R}^N$ or a nonnegative Radon measure in $\mathbf{R}^N$. Problem (1.1) is one of the simplest evolution problems for higher-order nonlinear parabolic equations. In this paper we establish the existence of solutions of problem (1.1) by introducing a new majorizing kernel to the higher order parabolic equation

$$\partial_t u + (-\Delta)^m u = 0 \quad \text{in} \quad \mathbf{R}^N \times (0,\infty). \tag{1.2}$$

* Corresponding author.
E-mail addresses: ishige@ms.u-tokyo.ac.jp (K. Ishige), kawakami@math.ryukoku.ac.jp (T. Kawakami), shinya.okabe@tohoku.ac.jp (S. Okabe).

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We also study necessary conditions on the initial data for the existence of local-in-time solutions of (1.1). Furthermore, we find a nonnegative smooth function f in $\mathbf{R}^N \setminus \{0\}$ with the following properties:

(i) There exists $\lambda^* > 0$ such that problem (1.1) possesses no local-in-time solutions if μ has a stronger singularity at 0 than $\lambda^* f$, that is, $\mu(x) \geq \lambda^* f(x)$ in a neighborhood of the origin 0;
(ii) Problem (1.1) possesses a local-in-time solution with $\mu = \lambda f$ for some $\lambda > 0$.

Then we call the singularity of f at $x = 0$ the *strongest singularity* of the initial data for the solvability of problem (1.1).

Before considering problem (1.1), we recall some results on the Cauchy problem for a semilinear parabolic equation

$$\begin{cases} \partial_t u - \Delta u = u^p, & x \in \mathbf{R}^N,\ t > 0, \\ u(x,0) = \mu(x) \geq 0, & x \in \mathbf{R}^N, \end{cases} \tag{1.3}$$

where $p > 1$. The necessary conditions for the existence of local-in-time solutions of problem (1.3) were studied in the papers [3] and [16], which ensure the following result. See [16, Theorem 1.1].

Theorem 1.1. *Let u be a nonnegative solution of problem (1.3) in $\mathbf{R}^N \times [0, T)$ for some $T > 0$, where $p > 1$ and μ is a nonnegative Radon measure in $\mathbf{R}^N$. Then there exists $c = c(N, p) > 0$ such that*

$$\sup_{x \in \mathbf{R}^N} \mu(B(x,\sigma)) \leq c\sigma^{N - \frac{2}{p-1}} \quad \text{for all } \sigma \in (0, T^{\frac{1}{2}}]. \tag{1.4}$$

In particular, in the case of $p = p_1 := 1 + 2/N$, there exists $c' = c'(N) > 0$ such that

$$\sup_{x \in \mathbf{R}^N} \mu(B(x,\sigma)) \leq c'\left[\log\left(e + \frac{T^{\frac{1}{2}}}{\sigma}\right)\right]^{-\frac{N}{2}} \quad \text{for all } \sigma \in (0, T^{\frac{1}{2}}].$$

We remark that, if $1 < p < p_1$, then problem (1.4) is equivalent to

$$\sup_{x \in \mathbf{R}^N} \mu(B(x, T^{1/2})) \leq cT^{\frac{N}{2} - \frac{1}{p-1}}. \tag{1.5}$$

By Theorem 1.1 we have:

(a) There exists $c_1 = c_1(N, p) > 0$ such that, if μ is a nonnegative measurable function in $\mathbf{R}^N$ satisfying

$$\mu(x) \geq c_1 |x|^{-N}\left[\log\left(e + \frac{1}{|x|}\right)\right]^{-\frac{N}{2}-1} \quad \text{if} \quad p = p_1,$$

$$\mu(x) \geq c_1 |x|^{-\frac{2}{p-1}} \quad \text{if} \quad p > p_1,$$

in a neighborhood of the origin, then problem (1.3) possesses no local-in-time solutions.

Sufficient conditions for the existence of solutions of problem (1.3) have been studied in many papers since the pioneering work due to [28]. See e.g. [1,2,6,12,16,19,23–26,29] and references therein. Among others, by [16, Corollary 1.2] and [25, Theorem 3] we have:

(b) Let $1 < p < p_1$. Then there exists $c_2 = c_2(N, p) > 0$ such that, if

$$\sup_{x \in \mathbf{R}^N} \mu(B(x, T^{\frac{1}{2}})) \leq c_2 T^{\frac{N}{2} - \frac{1}{p-1}}$$

for some $T > 0$, then problem (1.3) possesses a solution in $\mathbf{R}^N \times [0, T)$.

(c) Let $p \ge p_1$. Then there exists $c_3 = c_3(N, p) > 0$ such that, if

$$0 \le \mu(x) \le c_3|x|^{-N}\left[\log\left(e + \frac{1}{|x|}\right)\right]^{-\frac{N}{2}-1} + c_3 \qquad \text{if} \quad p = p_1,$$
$$0 \le \mu(x) \le c_3|x|^{-\frac{2}{p-1}} + c_3 \qquad \text{if} \quad p > p_1,$$

in $\mathbf{R}^N$, then problem (1.3) possesses a local-in-time solution.

By assertions (a) and (c) we can identify the strongest singularity of the initial data for the existence of solutions of (1.3) with $p \ge p_1$. Assertions (b) and (c) are proved by the construction of suitable supersolutions of (1.3) and the order-preserving property and the semigroup property of the heat operator are crucial in the proofs.

The operator $\partial_t + (-\Delta)^m$ is not order-preserving and the study of the solvability of problem (1.1) is more delicate than that of problem (1.3). Indeed, the fundamental solution $G_m = G_m(x,t)$ of (1.2) changes its sign for $t > 0$. In the study of higher-order parabolic equations it is crucial to find a suitable majorizing kernel associated with $\partial_t + (-\Delta)^m$. Galaktionov and Pohozaev [14] found a majorizing kernel of the form

$$\overline{G}_m(x,t) := Dt^{-\frac{N}{2m}} \exp\left(-d\eta^{\frac{2m}{2m-1}}\right) \quad \text{with} \quad \eta = \eta(x,t) = t^{-\frac{1}{2m}}|x|, \tag{1.6}$$

where D and d are positive constants (see Section 2.1), and proved the existence of global-in-time solutions of (1.1) for any sufficiently small initial data in $L^1 \cap L^\infty$ in the case of $p > p_m := 1 + 2m/N$. They also proved nonexistence of global-in-time solutions of (1.1) provided that $1 < p \le p_m$ and $\mu(x) \ge 0$ ($\not\equiv 0$) in $\mathbf{R}^N$. Subsequently, the existence and the asymptotic behavior of global-in-time solutions with bounded initial data have been studied in several papers under suitable assumptions on the decay of the initial data at the space infinity. See e.g. [14,19,20]. (See also [10,13].) On the other hand, it does not seem enough to study sufficient conditions for the existence of local-in-time solutions of problem (1.1) with singular initial data, although the results in [8] are available. As far as we know, there are no results related to the identification of the strongest singularity of the initial data for the existence of solutions of (1.1). One of the difficulties is that the integral operator associated with $\overline{G}_m$ does not have the semigroup property. Indeed, we can not apply the arguments in [16,25,28] with the majorizing kernel $\overline{G}_m$ to problem (1.1).

In this paper, by use of the fundamental solution of

$$\partial_t u + (-\Delta)^{\frac{\theta}{2}} u = 0 \quad \text{in} \quad \mathbf{R}^N \times (0,\infty), \tag{1.7}$$

where $0 < \theta < 2$, we introduce a new majorizing kernel $K = K(x,t)$ satisfying

$$\begin{aligned} &|G_m(x,t)| \le C_1 K(x,t), \\ &\int_{\mathbf{R}^N} K(x-y, t-s) K(y,s)\, dy \le C_2 K(x,t), \end{aligned} \tag{1.8}$$

for all $x \in \mathbf{R}^N$ and $0 < s < t$. Here C_1 and C_2 are positive constants. Applying the arguments in [16,27] with an integral operator associated with K, we establish the existence of solutions of problem (1.1). Furthermore, we modify the arguments in [7,18] to study necessary conditions on the initial data for the existence of local-in-time solutions of (1.1). Then we can identify the strongest singularity of the initial data for the existence of local-in-time solutions of problem (1.1).

We formulate a definition of solutions of problem (1.1).

Definition 1.1. Let $N \ge 1$, $m = 2, 3, \dots$, $p > 1$ and μ be a nonnegative Radon measure in $\mathbf{R}^\mathbf{N}$. Let $u \in L^p_{\mathrm{loc}}(\mathbf{R}^N \times [0,T))$. Then we say that u is a solution of (1.1) in $\mathbf{R}^N \times [0,T)$ if u satisfies

$$-\int_{\mathbf{R}^N} \varphi(x,0)\, d\mu(x) + \int_0^T \int_{\mathbf{R}^N} \left[-u\partial_t\varphi + u(-\Delta)^m\varphi\right] dx\, dt = \int_0^T \int_{\mathbf{R}^N} |u|^p \varphi\, dx\, dt \tag{1.9}$$

for all $\varphi \in C_c^\infty(\mathbf{R}^N \times [0,T))$.

We state our main results of this paper. The first theorem concerns necessary conditions for the solvability of problem (1.1) and it corresponds to Theorem 1.1.

Theorem 1.2. *Let $N \ge 1$, $m = 2, 3, \dots$ and $p > 1$. Let u be a solution of (1.1) in $\mathbf{R}^N \times [0, T)$ for some $T > 0$, where μ is a nonnegative Radon measure in $\mathbf{R}^N$. Then there exists $\gamma = \gamma(N, m, p) > 0$ such that*

$$\sup_{x\in\mathbf{R}^N} \mu(B(x,\sigma)) \le \gamma\sigma^{N-\frac{2m}{p-1}} \quad \text{for all } \sigma \in (0, T^{\frac{1}{2m}}]. \tag{1.10}$$

In particular, if $p = p_m := 1 + 2m/N$, then there exists $\gamma' = \gamma'(N, m)$ such that

$$\sup_{x\in\mathbf{R}^N} \mu(B(x,\sigma)) \le \gamma'\left[\log\left(e + \frac{T^{\frac{1}{2m}}}{\sigma}\right)\right]^{-\frac{N}{2m}} \quad \text{for all } \sigma \in (0, T^{\frac{1}{2m}}]. \tag{1.11}$$

Similarly to (1.5), if $1 < p < p_m$, then (1.10) is equivalent to

$$\sup_{x\in\mathbf{R}^N} \mu(B(x, T^{\frac{1}{2m}})) \le \gamma T^{\frac{N}{2m}-\frac{1}{p-1}}.$$

As a corollary of Theorem 1.2, we have:

Corollary 1.1. *Let $N \ge 1$, $m = 2, 3, \dots$ and $p \ge p_m$. Then there exists $\gamma_1 = \gamma_1(N, m, p) > 0$ such that, if μ is a nonnegative measurable function in $\mathbf{R}^N$ satisfying*

$$\mu(x) \ge \gamma_1|x|^{-N}\left[\log\left(e + \frac{1}{|x|}\right)\right]^{-\frac{N}{2m}-1} \quad \text{if} \quad p = p_m,$$

$$\mu(x) \ge \gamma_1|x|^{-\frac{2m}{p-1}} \quad \text{if} \quad p > p_m,$$

in a neighborhood of the origin, then problem (1.1) possesses no local-in-time solutions.

Corollary 1.1 corresponds to assertion (a). Next we state results on sufficient conditions for the existence of solutions of problem (1.1).

Theorem 1.3. *Let $N \ge 1$, $m = 2, 3, \dots$ and $1 < p < p_m$. Let μ be a nonnegative Radon measure in $\mathbf{R}^N$. Then there exists $\gamma_2 = \gamma_2(N, m, p) > 0$ such that, if*

$$\sup_{x\in\mathbf{R}^N} \mu(B(x, T^{\frac{1}{2m}})) \le \gamma_2 T^{\frac{N}{2m}-\frac{1}{p-1}} \tag{1.12}$$

for some $T > 0$, then problem (1.1) possesses a solution in $\mathbf{R}^N \times [0, T)$.

Theorem 1.4. *Let $N \ge 1$, $m = 2, 3, \dots$ and $p \ge p_m$. Then there exists $\gamma_3 = \gamma_3(N, m, p) > 0$ such that, if*

$$0 \le \mu(x) \le \gamma_3|x|^{-N}\left[\log\left(e + \frac{1}{|x|}\right)\right]^{-\frac{N}{2m}-1} + \gamma_3 \quad \text{if} \quad p = p_m, \tag{1.13}$$

$$0 \le \mu(x) \le \gamma_3|x|^{-\frac{2m}{p-1}} + \gamma_3 \quad \text{if} \quad p > p_m, \tag{1.14}$$

in $\mathbf{R}^N$, then problem (1.1) possesses a local-in-time solution. Furthermore, there exists $\gamma_4 = \gamma_4(N, m, p) > 0$ such that, if $p > p_m$ and

$$0 \le \mu(x) \le \gamma_4|x|^{-\frac{2m}{p-1}} \quad \text{in} \quad \mathbf{R}^N, \tag{1.15}$$

then problem (1.1) possesses a global-in-time solution.

Theorems 1.3 and 1.4 correspond to assertions (b) and (c), respectively. Theorem 1.4 is a direct consequence of Theorems 5.2 and 5.3. (See also Remarks 5.1 and 5.2.) Furthermore, as a corollary of Theorems 1.2 and 1.3, we have:

Corollary 1.2. *Let δ be the Dirac measure in $\mathbf{R}^N$. Then problem (1.1) possesses a local-in-time solution with $\mu = D\delta$ for some $D > 0$ if and only if $1 < p < p_m$.*

The rest of this paper is organized as follows. In Section 2 we collect preliminary results on the operator $\partial_t + (-\Delta)^m$ ($m = 2, 3, \dots$) and its associated semigroup. We also formulate the definition of solutions of an integral equation associated with problem (1.1) and prove some properties of the solutions. In Section 3 we modify the arguments in [7,18] to prove Theorem 1.2. In Section 4 we collect preliminary results on the operator $\partial_t + (-\Delta)^{\theta/2}$ ($0 < \theta < 2$) and its associated semigroup. Furthermore, we introduce a majorizing kernel $K = K(x, t)$ associated with $\partial_t + (-\Delta)^m$ and prove inequality (1.8). In Section 5 we establish the existence of solutions of problem (1.1).

2. Preliminaries

This section is divided into two subsections. In Section 2.1 we recall some preliminary results on the operators $\partial_t + (-\Delta)^m$ ($m = 2, 3, \dots$). In Section 2.2 we introduce an integral equation associated with problem (1.1) and prove some properties of the solutions.

We introduce some notation. For any $1 \le r \le \infty$, we denote by $\|\cdot\|_r$ the usual norm of $L^r := L^r(\mathbf{R}^N)$. For any $x \in \mathbf{R}^N$ and $R > 0$, we set $B(x, R) := \{y \in \mathbf{R}^N : |x - y| < R\}$. For any multi-index $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbf{M} := (\mathbf{N} \cup \{0\})^N$, we write

$$|\alpha| := \sum_{i=1}^N \alpha_i, \qquad \partial_x^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}.$$

By the letter C we denote generic positive constants and they may have different values also within the same line.

2.1. *Fundamental solutions to $\partial_t + (-\Delta)^m$ ($m = 2, 3, \dots$)*

Let $G_m = G_m(x, t)$ ($m = 2, 3, \dots$) be the fundamental solution of parabolic equation (1.2). Then G_m is represented by

$$G_m(x, t) = (2\pi)^{-\frac{N}{2}} \int_{\mathbf{R}^N} e^{ix\cdot\xi} e^{-t|\xi|^{2m}} \, d\xi. \tag{2.1}$$

The function $G_m(\cdot, t)$ changes its sign for any $t > 0$. Let $\overline{G}_m$ be as in (1.6). Then, under a suitable choice of D and d, it follows that

$$|G_m(x, t)| \le \overline{G}_m(x, t), \quad x \in \mathbf{R}^N, \ t > 0. \tag{2.2}$$

(See [14, Proposition 2.1].) Furthermore, G_m satisfies

$$G_m(x, t) = t^{-\frac{N}{2m}} G_m(t^{-\frac{1}{2m}} x, 1), \tag{2.3}$$

$$G_m(0, t) > 0, \tag{2.4}$$

$$|\partial_x^\alpha G_m(x, t)| \le C_\alpha t^{-\frac{N+|\alpha|}{2m}} \exp\left(-C_\alpha^{-1} \eta^{\frac{2m}{2m-1}}\right) \quad \text{with} \quad \eta = t^{-\frac{1}{2m}}|x|, \tag{2.5}$$

for all $x \in \mathbf{R}^N$, $t > 0$ and $\alpha \in \mathbf{M}$, where C_α is a positive constant. Properties (2.3) and (2.4) immediately follow from (2.1). For property (2.5), see e.g. [8, Section 3] and [10]. Moreover, for any $t > 0$,

$$\int_{\mathbf{R}^N} G_m(x, t) \, dx = 1, \tag{2.6}$$

and it follows that

$$G_m(x, t) = \int_{\mathbf{R}^N} G_m(x - y, t - s) G_m(y, s) \, dy \quad \text{for all } x \in \mathbf{R}^N, 0 < s < t. \tag{2.7}$$

(See e.g. [14, Section 2].)

We define an integral operator associated with G_m. For any (signed) Radon measure μ in $\mathbf{R}^N$, we set

$$[S_m(t)\mu](x) := \int_{\mathbf{R}^N} G_m(x-y,t)\,d\mu(y), \quad x \in \mathbf{R}^N,\ t>0. \tag{2.8}$$

Similarly, for any measurable function ϕ in $\mathbf{R}^N$, we set

$$[S_m(t)\phi](x) := \int_{\mathbf{R}^N} G_m(x-y,t)\phi(y)\,dy, \quad x \in \mathbf{R}^N,\ t>0. \tag{2.9}$$

Since G_m changes its sign, the integral operator $S_m(t)$ is not order-preserving.

Let $j = 0, 1, 2, \dots$. The Young inequality together with property (2.5) implies that

$$\|\partial_x^\alpha S_m(t)\phi\|_q \le C_m t^{-\frac{N}{2m}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{j}{2m}}\|\phi\|_p, \quad t>0, \tag{2.10}$$

for all $\phi \in L^p$ and $\alpha \in \mathbf{M}$ with $|\alpha| = j$, where $1 \le p \le q \le \infty$ and C_m is a positive constant independent of p and q. (See also [8, Section 2].) Furthermore, it follows from properties (2.5) and (2.6) that

$$\lim_{t\to+0}\|S_m(t)\phi-\phi\|_\infty = 0 \tag{2.11}$$

for any $\phi \in C_c(\mathbf{R}^N)$. (See e.g. [15, Section 4.2.3].) The convergence rate depends on the modulus of continuity of ϕ.

2.2. *Integral equation associated with problem (1.1)*

We consider the following integral equation associated with problem (1.1)

$$u(x,t) = \int_{\mathbf{R}^N} G_m(x-y,t)\,d\mu(y) + \int_0^t\int_{\mathbf{R}^N} G_m(x-y,t-s)|u(y,s)|^p\,dy\,ds. \tag{I}$$

We formulate a definition of solutions of (I) as follows.

Definition 2.1. Let $N \ge 1$, $m = 2, 3, \dots$, $p > 1$ and μ be a nonnegative Radon measure in $\mathbf{R}^N$. Let u be a continuous function in $\mathbf{R}^N \times (0,T)$ for some $T > 0$ and set

$$\begin{aligned}\overline{u}_1(x,t) &:= \int_{\mathbf{R}^N} |G_m(x-y,t)|\,d\mu(y),\\ \overline{u}_2(x,t) &:= \int_0^t\int_{\mathbf{R}^N} |G_m(x-y,t-s)||u(y,s)|^p\,dy\,ds.\end{aligned} \tag{2.12}$$

We say that u is a solution of integral equation (I) in $\mathbf{R}^N \times [0,T)$ if

$$\sup_{\tau\le t<T}\|\overline{u}_1(t)\|_\infty + \sup_{\tau\le t<T}\|\overline{u}_2(t)\|_\infty < \infty \quad \text{for all } \tau \in (0,T) \tag{2.13}$$

and u satisfies integral equation (I) for all $(x,t) \in \mathbf{R}^N \times (0,T)$.

In the rest of this subsection we show that the solution of integral equation (I) is a solution of (1.1).

Proposition 2.1. *Let u be a solution of integral equation (I) in $\mathbf{R}^N \times [0,T)$ for some $T > 0$.*

(a) *For any $\tau \in (0,T)$, u_τ defined by $u_\tau(x,t) := u(x,t+\tau)$ is a solution of integral equation (I) in $\mathbf{R}^N \times [0,T-\tau)$ with the initial data $u(\tau)$.*
(b) *Let $\alpha \in \mathbf{M}$ and $i \in \{0,1\}$ be such that $|\alpha| + 4i \le 2m$. Then $\partial_t^i\partial_x^\alpha u \in BC(\mathbf{R}^N \times [\tau,T))$ for all $\tau \in (0,T)$.*

(c) *u satisfies*

$$\partial_t u + (-\Delta)^m u = |u|^p \quad \textit{for all } (x,t) \in \mathbf{R}^N \times (0,T) \tag{2.14}$$

in the classical sense.

Furthermore, u is a solution of problem (1.1) *in* $\mathbf{R}^N \times [0,T)$.

Proof of assertions (a), (b) and (c). Let u be a solution of integral equation (I) in $\mathbf{R}^N \times [0,T)$ for some $T > 0$. By (1.6), (2.2) and (2.13) we see that

$$\int_{\mathbf{R}^N} |G_m(x-y,t-\tau)| \left[\int_{\mathbf{R}^N} |G_m(y-z,\tau)|\, d\mu(z) \right] dy < \infty,$$

$$\int_{\mathbf{R}^N} |G_m(x-y,t-\tau)| \left[\int_0^\tau \int_{\mathbf{R}^N} |G_m(y-z,\tau-s)| |u(z,s)|^p\, dz\, ds \right] dy < \infty,$$

for all $x \in \mathbf{R}^N$ and $0 < \tau < t$. It follows from the Fubini theorem with property (2.7) that

$$\begin{aligned}
&\int_{\mathbf{R}^N} G_m(x-y,t-\tau) u(y,\tau)\, dy \\
&= \int_{\mathbf{R}^N} G_m(x-y,t-\tau) \\
&\qquad \times \left[\int_{\mathbf{R}^N} G_m(y-z,\tau)\, d\mu(z) + \int_0^\tau \int_{\mathbf{R}^N} G_m(y-z,\tau-s)|u(z,s)|^p\, dz\, ds \right] dy \\
&= \int_{\mathbf{R}^N} \left(\int_{\mathbf{R}^N} G_m(x-y,t-\tau) G_m(y-z,\tau)\, dy \right) d\mu(z) \\
&\qquad + \int_0^\tau \int_{\mathbf{R}^N} \left(\int_{\mathbf{R}^N} G_m(x-y,t-\tau) G_m(y-z,\tau-s)\, dy \right) |u(z,s)|^p\, dz\, ds \\
&= \int_{\mathbf{R}^N} G_m(x-z,t)\, d\mu(z) + \int_0^\tau \int_{\mathbf{R}^N} G_m(x-z,t-s)|u(z,s)|^p\, dz\, ds
\end{aligned}$$

for all $x \in \mathbf{R}^N$ and $0 < \tau < t$. This together with Definition 2.1 implies that

$$\sup_{\tau \le t < T} \|u(t)\|_\infty < \infty \quad \text{for all } \tau \in (0,T), \tag{2.15}$$

$$\begin{aligned}
u(x,t) &= \int_{\mathbf{R}^N} G_m(x-y,t-\tau) u(y,\tau)\, dy \\
&\qquad + \int_\tau^t \int_{\mathbf{R}^N} G_m(x-y,t-s)|u(y,s)|^p\, dy\, ds \quad \text{for all } x \in \mathbf{R}^N \text{ and } 0 < \tau < T,
\end{aligned}$$

and assertion (a) holds. By (2.15) we apply similar arguments in regularity theorems for second order parabolic equations (see e.g. [11, Chapter 1]) to integral equation (I), and obtain assertions (b) and (c). See also [21, Section 2]. □

It remains to prove that u is a solution of problem (1.1). For this aim, we modify the arguments in [16] to prepare the following two lemmas.

Lemma 2.1. *Let u be a solution of integral equation* (I) *in* $\mathbf{R}^N \times [0,T)$ *for some* $T>0$. *Then*

$$\lim_{t\to+0} \int_{\mathbf{R}^N\setminus B(0,R)} \overline{G}_m(\lambda x,t)\,d\mu(x) = 0, \tag{2.16}$$

$$\lim_{t\to+0} \int_0^t \int_{\mathbf{R}^N\setminus B(0,R)} \overline{G}_m(\lambda x,t-s)|u(x,s)|^p\,dx\,ds = 0, \tag{2.17}$$

for all $R>0$ *and* $\lambda>0$.

Proof. By (2.4) we find $R_*>0$ and $c_*>0$ such that

$$\inf_{x\in B(0,R_*)} G_m(x,1) \ge c_* > 0.$$

Then it follows from property (2.3) that

$$G_m(x-y,t) \ge t^{-\frac{N}{2m}} c_* \quad \text{for all } x-y \in B(0,R_* t^{\frac{1}{2m}}).$$

This together with (2.12) and (2.13) implies that

$$\begin{aligned}
\infty > \|\overline{u}_1(T_\epsilon)\|_\infty \ge \overline{u}_1(x,T_\epsilon) &\ge \int_{B(x,T^{\frac{1}{2m}}R_*)} |G_m(x-y,T_\epsilon)|\,d\mu(y) \\
&\ge c_* T^{-\frac{N}{2m}} \mu(B(x,R_* T_\epsilon^{\frac{1}{2m}})), \\
\infty > \|\overline{u}_2(T_\epsilon)\|_\infty \ge \overline{u}_2(x,T_\epsilon) &\ge \int_0^{T_{2\epsilon}} \int_{B(x,R_*(T_\epsilon-s)^{\frac{1}{2m}})} |G_m(x-y,T_\epsilon-s)|u(y,s)|^p\,dy\,ds \\
&\ge c_* \int_0^{T_{2\epsilon}} (T_\epsilon-s)^{-\frac{N}{2m}} \int_{B(x,R_*(T_\epsilon-s)^{\frac{1}{2m}})} |u(y,s)|^p\,dy\,ds \\
&\ge c_* \epsilon^{-\frac{N}{2m}} \int_0^{T_{2\epsilon}} \int_{B(x,\epsilon^{\frac{1}{2m}}R_*)} |u(y,s)|^p\,dy\,ds,
\end{aligned}$$

for all $x\in\mathbf{R}^N$, where $T_\epsilon := T-\epsilon$, $T_{2\epsilon} = T-2\epsilon$ and $0<\epsilon<T/2$. Since $x\in\mathbf{R}^N$ is arbitrary, we deduce that

$$\begin{aligned}
&\sup_{x\in\mathbf{R}^N} \mu(B(x,R)) < \infty, \\
&\sup_{x\in\mathbf{R}^N} \int_0^{T_{2\epsilon}} \int_{B(x,R)} |u(y,s)|^p\,dy\,ds < \infty,
\end{aligned} \tag{2.18}$$

for all $R>0$ and $0<\epsilon<T/2$. (See [22, Lemma 2.1].)

Let $0<R<\infty$ and set $R' := \min\{R/2, 1/2\}$. By the Besicovitch covering lemma (see e.g. [9, Section 1.5.2]) we can find an integer n_* depending only on N and a set $\{x_{k,i}\}_{k=1,\dots,n_*,\,i\in\mathbf{N}} \subset \mathbf{R}^N \setminus B(0,R)$ such that

$$\begin{aligned}
&\overline{B(x_{k,i},R')} \cap \overline{B(x_{k,j},R')} = \emptyset \quad \text{if} \quad i\ne j, \\
&\mathbf{R}^N\setminus B(0,R) \subset \bigcup_{k=1}^{n_*}\bigcup_{i=1}^{\infty} \overline{B(x_{k,i},R')} \subset \mathbf{R}^N \setminus B(0,R/2).
\end{aligned} \tag{2.19}$$

Then we have

$$\begin{aligned}\int\limits_{\mathbf{R}^N\setminus B(0,R)} \overline{G}_m(\lambda x,t)\,d\mu(x) &\le \sum_{k=1}^{n_*}\sum_{i=1}^{\infty}\int\limits_{B(x_{k,i},R')} \overline{G}_m(\lambda x,t)\,d\mu(x)\\ &\le \sum_{k=1}^{n_*}\sum_{i=1}^{\infty}\mu(\overline{B(x_{k,i},R')})\sup_{x\in B(x_{k,i},R')}\overline{G}_m(\lambda x,t)\\ &\le \sup_{x\in\mathbf{R}^N}\mu(B(x,1))\sum_{k=1}^{n_*}\sum_{i=1}^{\infty}\sup_{x\in B(x_{k,i},R')}\overline{G}_m(\lambda x,t).\end{aligned}\tag{2.20}$$

Let $\epsilon>0$ be such that $2(1-\epsilon)>1+\epsilon$. For $k=1,\dots,n_*$ and $i\in\mathbf{N}$, since $x_{k,i}\notin B(0,R)$ and $R'\le R/2$, we have

$$\frac{|x_{k,i}|}{R'}\ge\frac{R}{R'}\ge 2>\frac{1+\epsilon}{1-\epsilon},$$

which implies that $|x_{k,i}|-R'\ge\epsilon(|x_{k,i}|+R')$. Then it holds that

$$|y|\ge|x_{k,i}|-R'\ge\epsilon(|x_{k,i}|+R')\ge\epsilon|z|$$

for all $y,z\in\overline{B(x_{k,i},R')}$, $k=1,\dots,n_*$ and $i\in\mathbf{N}$. Therefore we observe from (1.6) that

$$\sup_{x\in B(x_{k,i},R')}\overline{G}_m(\lambda x,t)\le\inf_{x\in B(x_{k,i},R')}\overline{G}_m(\lambda\epsilon x,t)\le\frac{1}{|B(0,R')|}\int\limits_{B(x_{k,i},R')}\overline{G}_m(\lambda\epsilon z,t)\,dz$$

for all $k=1,\dots,n_*$ and $i\in\mathbf{N}$, where $|B(0,R')|$ is the volume of $B(0,R')$. This together with (1.6) and (2.19) implies that

$$\begin{aligned}\sum_{k=1}^{n_*}\sum_{i=1}^{\infty}\sup_{x\in B(x_{k,i},R')}\overline{G}_m(\lambda x,t) &\le Cn_*R'^{-N}\int\limits_{\mathbf{R}^N\setminus B(0,R/2)}\overline{G}_m(\lambda\epsilon z,t)\,dz\\ &\le Cn_*R'^{-N}\int\limits_{|y|\ge R/(2t^{1/2m})}\exp\left(-d(\lambda\epsilon y)^{\frac{2m}{2m-1}}\right)dy\to 0\end{aligned}\tag{2.21}$$

as $t\to+0$. Combining (2.20) and (2.21), we obtain relation (2.16).

Since

$$\begin{aligned}\overline{G}_m(\lambda x,t-s) &\le C(t-s)^{-\frac{N}{2m}}\exp\left(-C^{-1}\eta(\lambda x,t-s)^{\frac{2m}{2m-1}}\right)\\ &\le C\exp\left(-(2C)^{-1}\eta(\lambda x,t-s)^{\frac{2m}{2m-1}}\right)\\ &\le C\exp\left(-(2C)^{-1}\eta(\lambda x,t)^{\frac{2m}{2m-1}}\right)=:\hat{G}_m(\lambda x,t)\end{aligned}$$

for all $x\in\mathbf{R}^N\setminus B(0,R)$ and $0<s<t$, we have

$$
\begin{aligned}
&\int_0^t \int_{\mathbf{R}^N\setminus B(0,R)} \overline{G}_m(\lambda x, t-s)|u(x,s)|^p\,dx\,ds \\
&\le \sum_{k=1}^{n_*}\sum_{i=1}^{\infty}\int_0^t \int_{B(x_{k,i},R')} \overline{G}_m(\lambda x, t-s)|u(x,s)|^p\,dx\,ds \\
&\le \sum_{k=1}^{n_*}\sum_{i=1}^{\infty}\sup_{x\in B(x_{k,i},R')} \hat{G}_m(\lambda x, t)\int_0^t \int_{B(x_{k,i},R')} |u(x,s)|^p\,dx\,ds \\
&\le \sup_{x\in\mathbf{R}^N}\int_0^{T/2}\int_{B(x,1)} |u(x,s)|^p\,dx\,ds \sum_{k=1}^{n_*}\sum_{i=1}^{\infty}\sup_{x\in B(x_{k,i},R')} \hat{G}_m(\lambda x, t)
\end{aligned}
\tag{2.22}
$$

for all $0 < t \le T/2$. Similarly to (2.21), we observe that

$$
\sum_{k=1}^{n_*}\sum_{i=1}^{\infty}\sup_{x\in B(x_{k,i},R')} \hat{G}_m(\lambda x, t) \le C n_* R'^{-N} \int_{\mathbf{R}^N\setminus B(0,R/2)} \hat{G}_m(\lambda\epsilon z, t)\,dz \to 0 \tag{2.23}
$$

as $t \to +0$. Combining (2.22) and (2.23), we see that

$$
\lim_{t\to+0}\int_0^t \int_{\mathbf{R}^N\setminus B(0,R)} \overline{G}_m(\lambda x, t-s)|u(x,s)|^p\,dx\,ds = 0,
$$

which implies relation (2.17). Thus Lemma 2.1 follows. □

Lemma 2.2. *Let u be a solution of integral equation (I) in $\mathbf{R}^N \times [0,T)$ for some $T > 0$. Then*

$$
\lim_{t\to+0}\int_{\mathbf{R}^N}\int_{\mathbf{R}^N} \varphi(x,t)G_m(x-y,t)\,d\mu(y)\,dx = \int_{\mathbf{R}^N} \varphi(y,0)\,d\mu(y), \tag{2.24}
$$

$$
\lim_{t\to+0}\int_{\mathbf{R}^N}\int_0^t\int_{\mathbf{R}^N} G_m(x-y,t-s)\varphi(x,t)|u(y,s)|^p\,dy\,ds\,dx = 0, \tag{2.25}
$$

for all $\varphi \in C_c^\infty(\mathbf{R}^N \times [0,T))$.

Proof. Let $\varphi \in C_c^\infty(\mathbf{R}^N \times [0,T))$. Then we can take $R > 0$ and $\epsilon \in (0,T)$ such that $\operatorname{supp}\varphi \subset B(0,R) \times [0, T-\epsilon]$. Set

$$
\Phi(x,t:\tau) := [S_m(t)\varphi(\tau)](x) = \int_{\mathbf{R}^N} G_m(x-y,t)\varphi(y,\tau)\,dy, \quad x \in \mathbf{R}^N,\ t > 0,\ \tau \in (0,T).
$$

By (2.10) we have

$$
\|\Phi(t:\tau)\|_\infty \le C\|\varphi(\tau)\|_\infty \le C\|\varphi\|_{L^\infty(\mathbf{R}^N\times(0,T))} \tag{2.26}
$$

for all $t > 0$ and $\tau \in (0,T)$. On the other hand, it follows from the Fubini theorem that

$$
\begin{aligned}
&\int_{\mathbf{R}^N}\int_{\mathbf{R}^N} \varphi(x,t)G_m(x-y,t)\,d\mu(y)\,dx \\
&= \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} \varphi(x,t)G_m(x-y,t)\,dx\,d\mu(y) = \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} \varphi(x,t)G_m(y-x,t)\,dx\,d\mu(y) \\
&= \int_{\mathbf{R}^N} \Phi(y,t:t)\,d\mu(y) = \int_{\mathbf{R}^N} \varphi(y,0)\,d\mu(y) + \int_{\mathbf{R}^N} [\Phi(y,t:t)-\varphi(y,0)]\,d\mu(y).
\end{aligned}
\tag{2.27}
$$

Since $|x-y| \ge |x|/2$ for $x \in \mathbf{R}^N \setminus B(0,2R)$ and $y \in B(0,R)$, by (2.2) we can find $\lambda > 0$ such that

$$
\begin{aligned}
|\Phi(x,t:\tau)| &\le \|\varphi\|_{L^\infty(\mathbf{R}^N\times(0,T))} \int_{B(0,R)} |G_m(x-y,t)|\,dy \\
&\le C\|\varphi\|_{L^\infty(\mathbf{R}^N\times(0,T))}\overline{G}_m(\lambda x,t)
\end{aligned}
\tag{2.28}
$$

for all $x \in \mathbf{R}^N \setminus B(0,2R)$, $t > 0$ and $\tau \in (0,T)$. Furthermore, by the uniform continuity of φ in $\overline{B(0,2R)} \times [0, T-\epsilon]$ and (2.11) we observe that

$$
\begin{aligned}
&\sup_{x\in B(0,2R)} |\Phi(x,t:t)-\varphi(x,0)| \\
&\le \sup_{x\in B(0,2R)} |\Phi(x,t:t)-\varphi(x,t)| + \sup_{x\in B(0,2R)} |\varphi(x,t)-\varphi(x,0)| \to 0
\end{aligned}
\tag{2.29}
$$

as $t \to +0$. Therefore, by (2.16), (2.26) and (2.29) we have

$$
\begin{aligned}
&\left|\int_{\mathbf{R}^N} [\Phi(y,t:t)-\varphi(y,0)]\,d\mu(y)\right| \\
&\le \int_{B(0,2R)} |\Phi(y,t:t)-\varphi(y,0)|\,d\mu(y) + \int_{\mathbf{R}^N\setminus B(0,2R))} |\Phi(y,t:t)|\,d\mu(y) \\
&\le \sup_{x\in B(0,2R)} |\Phi(x,t:t)-\varphi(x,0)\,|\mu(B(0,2R)) \\
&\qquad + C\|\varphi\|_{L^\infty(\mathbf{R}^N\times(0,T))} \int_{\mathbf{R}^N\setminus B(0,2R)} \overline{G}_m(\lambda y,t)\,d\mu(y) \to 0
\end{aligned}
\tag{2.30}
$$

as $t \to +0$. Combining (2.27) and (2.30), we have relation (2.24). Furthermore, by (2.17), (2.18), (2.26) and (2.28) we have

$$
\begin{aligned}
&\left|\int_{\mathbf{R}^N}\int_0^t\int_{\mathbf{R}^N} G_m(x-y,t-s)\varphi(x,t)|u(y,s)|^p\,dy\,ds\,dx\right| \\
&= \left|\int_0^t\int_{\mathbf{R}^N} \Phi(y,t-s:t)|u(y,s)|^p\,dy\,ds\right| \\
&\le C\|\varphi\|_{L^\infty(\mathbf{R}^N\times(0,T))}\int_0^t\int_{B(0,2R)} |u(y,s)|^p\,dy\,ds \\
&\qquad + C\|\varphi\|_{L^\infty(\mathbf{R}^N\times(0,T))}\int_0^t\int_{\mathbf{R}^N\setminus B(0,2R)} \overline{G}_m(\lambda y,t-s)|u(y,s)|^p\,dy\,ds \to 0
\end{aligned}
$$

as $t \to +0$. This implies (2.25). Thus Lemma 2.2 follows. □

Now we are ready to complete the proof of Proposition 2.1.

Proof of Proposition 2.1. Let u be a solution of integral equation (I) in $\mathbf{R}^N \times [0,T)$ for some $T>0$. By (2.18) we see that $u \in L^p_{\rm loc}(\mathbf{R}^N \times (0,T))$. Thus it suffices to prove that u satisfies relation (1.9).

Let $\varphi \in C^\infty(\mathbf{R}^N \times [0,T))$ be such that $\operatorname{supp}\varphi \subset B(0,R) \times [0, T-\epsilon]$ for some $R>0$ and $\epsilon \in (0,T)$. Then it follows from Definition 2.1 and Lemma 2.2 that

$$\begin{aligned}
&\int_{\mathbf{R}^N} u(x,t)\varphi(x,t)\,dx \\
&= \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} \varphi(x,t) G_m(x-y,t)\,d\mu(y)\,dx \\
&+ \int_{\mathbf{R}^N}\int_0^t\int_{\mathbf{R}^N} \varphi(x,t)G_m(x-y,t-s)|u(y,s)|^p\,dy\,ds\,dx \to \int_{\mathbf{R}^N}\varphi(x,0)\,d\mu(x)
\end{aligned} \tag{2.31}$$

as $t \to +0$. On the other hand, by (2.14) we see that

$$-\int_{\mathbf{R}^N}\varphi(x,\tau)u(x,\tau)\,dx + \int_\tau^T\int_{\mathbf{R}^N}\left[-u\partial_t\varphi + u(-\Delta)^m\varphi\right]dx\,dt = \int_\tau^T\int_{\mathbf{R}^N}|u|^p\varphi\,dx\,dt.$$

Letting $\tau \to +0$, by (2.18) and (2.31) we have

$$-\int_{\mathbf{R}^N}\varphi(x,0)\,d\mu(x) + \int_0^T\int_{\mathbf{R}^N}\left[-u\partial_t\varphi + u(-\Delta)^m\varphi\right]dx\,dt = \int_0^T\int_{\mathbf{R}^N}|u|^p\varphi\,dx\,dt.$$

This means that u is a solution of problem (1.1) in $\mathbf{R}^N \times [0,T)$. Thus Proposition 2.1 follows. □

3. Proof of Theorem 1.2

In this section we modify the arguments in [18] (see also [7]) to prove Theorem 1.2.

Proof of Theorem 1.2. Let u be a solution of problem (1.1) in $\mathbf{R}^N \times [0,T)$ for some $T>0$. Set

$$u_T(x,t) := T^{\frac{1}{p-1}} u(T^{\frac{1}{2m}}x, Tt), \qquad \mu_T(x) := T^{\frac{1}{p-1}}\mu(T^{\frac{1}{2m}}x). \tag{3.1}$$

Then u_T is a solution of problem (1.1) in $\mathbf{R}^N \times [0,1)$ with the initial data μ_T. Due to similar transformation (3.1), it suffices to consider the case of $T=1$ for the proof of Theorem 1.2.

Let

$$f(s) := e^{-\frac{1}{s}} \quad \text{if} \quad s>0, \qquad f(s)=0 \quad \text{if} \quad s \le 0.$$

Set

$$\eta(s) := \frac{f(2-s)}{f(2-s)+f(s-1)}.$$

Then $\eta \in C^\infty([0,\infty))$ and

$$\begin{aligned}
&\eta'(s) = \frac{-f'(2-s)f(s-1) - f(2-s)f'(s-1)}{[f(2-s)+f(s-1)]^2} \le 0 \quad \text{on} \quad [0,\infty), \\
&\eta(s) = 1 \quad \text{on} \quad [0,1], \qquad \eta(s)=0 \quad \text{on} \quad [2,\infty).
\end{aligned}$$

Set

$$\eta^*(s) = 0 \quad \text{on} \quad [0,1), \qquad \eta^*(s) = \eta(s) \quad \text{on} \quad [1,\infty).$$

Since $p > 1$, for any $k \in \mathbf{N}$, we find $C_k > 0$ such that

$$|\eta^{(k)}(s)| \le C_k \eta^*(s)^{\frac{1}{p}} \quad \text{for all } s \ge 1. \tag{3.2}$$

This follows from the fact that $|f^{(k)}(s)| \le f(s)^{1/p}$ for all sufficiently small $s > 0$.

Let u be a solution of problem (1.1) in $\mathbf{R}^N \times [0,1)$. Let $x_0 \in \mathbf{R}^N$ and $0 < r_* < 1/3$ be such that

$$\mu\left(B\left(x_0, (r_*/3)^{\frac{1}{2m}}\right)\right) > 0.$$

For any $R \in (0,1]$, we set

$$\psi_R(x,t) := \eta\Big(3\frac{|x-x_0|^{2m}+t}{R}\Big), \qquad \psi_R^*(x,t) := \eta^*\Big(3\frac{|x-x_0|^{2m}+t}{R}\Big).$$

By (3.2), for any $k \in \mathbf{N}$, we have

$$|\partial_t \psi_R(x,t)| \le CR^{-1}\psi_R^*(x,t)^{\frac{1}{p}}, \qquad |\nabla_x^k \psi_R(x,t)| \le CC_k R^{-\frac{k}{2m}} \psi_R^*(x,t)^{\frac{1}{p}}, \tag{3.3}$$

for all $x \in \mathbf{R}^N$ and $0 < t \le 1$. Indeed, since $\eta' = 0$ for all $s \in (0,1) \cup (2,\infty)$, we observe from (3.2) that

$$|\nabla_x \psi_R(x,t)| \le \frac{C}{R}|x-x_0|^{2m-1}\left|\eta'\Big(3\frac{|x-x_0|^{2m}+t}{R}\Big)\right| \le CC_1 R^{-\frac{1}{2m}}\psi_R^*(x,t)^{\frac{1}{p}},$$

which implies (3.3) with $k = 1$. Repeating this argument, we see that inequality (3.3) holds.

It follows from (3.3) that

$$\begin{aligned}
&\int_{\mathbf{R}^N} \psi_R(x,0)\,d\mu + \int_0^R \int_{\mathbf{R}^N} |u(x,t)|^p \psi_R(x,t)\,dx\,dt \\
&= \int_0^R \int_{\mathbf{R}^N} u(x,t)(-\partial_t + (-\Delta)^m)\psi_R(x,t)\,dx\,dt \\
&\le CR^{-1} \int_0^R \int_{\mathbf{R}^N} |u(x,t)|\psi_R^*(x,t)^{\frac{1}{p}}\,dx\,dt \\
&\le CR^{-1}\left(\int_0^R \int_{\mathbf{R}^N} \chi_{\{\psi_R^*(x,t)>0\}}\,dx\,dt\right)^{1-\frac{1}{p}} \left(\int_0^R \int_{\mathbf{R}^N} |u(x,t)|^p \psi_R^*(x,t)\,dx\,dt\right)^{\frac{1}{p}}
\end{aligned} \tag{3.4}$$

for all $0 < R \le 1$. On the other hand, it follows that

$$\int_0^R \int_{\mathbf{R}^N} \chi_{\{\psi_R^*(x,t)>0\}}\,dx\,dt = R^{\frac{N}{2m}+1} \int_0^1 \int_{\mathbf{R}^N} \chi_{\{\psi_1^*(x,t)>0\}}\,dx\,dt.$$

This together with (3.4) implies that

$$\begin{aligned}
&m_R + \int_0^R \int_{\mathbf{R}^N} |u(x,t)|^p \psi_R(x,t)\,dx\,dt \\
&\le CR^{\frac{1}{p}\left(\frac{N(p-1)}{2m}-1\right)} \left(\int_0^R \int_{\mathbf{R}^N} |u(x,t)|^p \psi_R^*(x,t)\,dx\,dt\right)^{\frac{1}{p}}
\end{aligned} \tag{3.5}$$

for all $0 < R \le 1$, where

$$m_R := \mu\Big(B\Big(x_0, (R/3)^{\frac{1}{2m}}\Big)\Big).$$

Let ϵ be a sufficiently small positive constant. For any $0 < r \le R \le 1$, set

$$z(r) := \int_0^R \int_{\mathbf{R}^N} |u(x,t)|^p \psi_r^*(x,t)\,dx\,dt, \quad Z(R) := \int_0^R z(r)\min\{r^{-1}, \epsilon^{-1}\}\,dr. \tag{3.6}$$

Since η^* is decreasing on $[1,\infty)$ and $\operatorname{supp}\eta^* \subset [1,2]$, for any $(x,t) \in \mathbf{R}^N \times (0,1)$ with $3(|x-x_0|^{2m}+t) \ge R$, we have

$$\begin{aligned} \int_0^R \psi_r^*(x,t)\min\{r^{-1},\epsilon^{-1}\}\,dr &= \int_0^R \eta^*\Big(3\frac{|x-x_0|^{2m}+t}{r}\Big)r^{-1}\,dr \\ &= \int_{3(|x-x_0|^{2m}+t)/R}^{\infty} \eta^*(s)s^{-1}\,ds \\ &\le \eta^*\Big(3\frac{|x-x_0|^{2m}+t}{R}\Big)\int_1^2 s^{-1}\,ds \le C\psi_R^*(x,t). \end{aligned} \tag{3.7}$$

Since $\psi_R^*(x,t) = 0$ if $3(|x-x_0|^{2m}+t) < R$, by (3.6) and (3.7) we obtain

$$\begin{aligned} \int_0^R \int_{\mathbf{R}^N} |u(x,t)|^p \psi_R(x,t)\,dx\,dt &\ge \int_0^R \int_{\mathbf{R}^N} |u(x,t)|^p \psi_R^*(x,t)\,dx\,dt \\ &\ge C^{-1}\int_0^R \int_{\mathbf{R}^N} |u(x,t)|^p \left(\int_0^R \psi_r^*(x,t)\min\{r^{-1},\epsilon^{-1}\}\,dr\right)dx\,dt \\ &= C^{-1}\int_0^R\int_0^R\int_{\mathbf{R}^N} |u(x,t)|^p \psi_r^*(x,t)\min\{r^{-1},\epsilon^{-1}\}\,dx\,dt\,dr = C^{-1}Z(R). \end{aligned} \tag{3.8}$$

Therefore we deduce from (3.5), (3.6) and (3.8) that

$$m_R + C^{-1}Z(R) \le C R^{\frac{1}{p}\left(\frac{N(p-1)}{2m}-1\right)}(\max\{R,\epsilon\}Z'(R))^{\frac{1}{p}}. \tag{3.9}$$

Since $m_R \ge m_r \ge m_{r_*} > 0$ for $r \ge r_*$, it follows from (3.9) that

$$[m_{r_*} + Z(R)]^{-p}Z'(R) \ge C^{-1}R^{-\left(\frac{N(p-1)}{2m}-1\right)}(\max\{R,\epsilon\})^{-1}$$

for all $R \in [r_*, 1]$. Therefore we have

$$\int_{Z(r)}^{Z(1)} [m_{r_*}+s]^{-p}\,ds \ge C^{-1}\int_r^1 R^{-\left(\frac{N(p-1)}{2m}-1\right)}(\max\{R,\epsilon\})^{-1}\,dR \tag{3.10}$$

for all $r \in [r_*, 1)$. Since

$$\int_{Z(r)}^{Z(1)} [m_{r_*}+s]^{-p}\,ds \le \frac{1}{p-1}(Z(r)+m_{r_*})^{-p+1} \le \frac{1}{p-1}m_{r_*}^{-p+1},$$

by (3.10) we obtain

$$\frac{1}{p-1} m_{r_*}^{-p+1} \geq C^{-1} \int_r^1 R^{-\left(\frac{N(p-1)}{2m}-1\right)} (\max\{R,\epsilon\})^{-1}\, dR$$

for all $r \in [r_*, 1]$. Letting $\epsilon \to +0$, we see that

$$\frac{1}{p-1} m_{r_*}^{-p+1} \geq C^{-1} \int_r^1 R^{-\frac{N(p-1)}{2m}}\, dR$$

for all $r \in [r_*, 1)$. This implies that

$$\begin{aligned} \mu\left(B\left(x_0, (r_*/3)^{\frac{1}{2m}}\right)\right) = m_{r_*} &\leq C \left(\int_r^1 R^{-\frac{N(p-1)}{2m}}\, dR \right)^{-\frac{1}{p-1}} \\ &\leq C \left(\int_r^{3r} R^{-\frac{N(p-1)}{2m}}\, dR \right)^{-\frac{1}{p-1}} \leq C r^{\frac{N}{2m}-\frac{1}{p-1}} \end{aligned} \tag{3.11}$$

for all $r \in [r_*, 1/3)$. Set $\sigma = (r/3)^{1/2m} = (r_*/3)^{1/2m} \in (0, 9^{-1/m})$. Since $x_0 \in \mathbf{R}^N$ is arbitrary, we deduce from (3.11) that

$$\sup_{x \in \mathbf{R}^N} \mu(B(x,\sigma)) \leq C \sigma^{N - \frac{2m}{p-1}} \quad \text{for all } 0 < \sigma < 9^{-1/m}. \tag{3.12}$$

On the other hand, for any $\lambda \geq 1$, we find $C_\lambda > 0$ such that

$$\sup_{x \in \mathbf{R}^N} \mu(B(x,\lambda\eta)) \leq C_\lambda \sup_{x \in \mathbf{R}^N} \mu(B(x,\eta)) \tag{3.13}$$

for all $\eta > 0$ (see e.g. [22, Lemma 2.1]). Here C_λ is independent of η. This together with (3.12) implies inequality (1.10).

It remains to prove inequality (1.11). Let $p = p_m$. By (3.11) we have

$$\mu\left(B\left(x_0, (r_*/3)^{\frac{1}{2m}}\right)\right) \leq C \left(\int_r^1 R^{-\frac{N(p-1)}{2m}}\, dR \right)^{-\frac{1}{p-1}} \leq C|\log r|^{-\frac{N}{2m}} \leq C\left|\log \frac{r}{3}\right|^{-\frac{N}{2m}}$$

for all $r \in [r_*, 1/3)$. Then, similarly to (3.12), we have

$$\sup_{x \in \mathbf{R}^N} \mu(B(x,\sigma)) \leq C\left|\log \frac{r}{3}\right|^{-\frac{N}{2m}} \leq C\left[\log\left(e + \frac{1}{\sigma}\right)\right]^{-\frac{N}{2m}} \quad \text{for all } 0 < \sigma < 9^{-1/m}.$$

This together with (3.13) implies inequality (1.11). Thus Theorem 1.2 follows. □

4. Majorizing kernel

Let $0 < \theta < 2$. Let $G_\theta = G_\theta(x,t)$ be the fundamental solution of parabolic equation (1.7), that is,

$$G_\theta(x,t) = (2\pi)^{-\frac{N}{2}} \int_{\mathbf{R}^N} e^{ix\cdot\xi} e^{-t|\xi|^\theta}\, d\xi.$$

Then $G_\theta = G_\theta(x,t)$ is a positive, smooth and radially symmetric function in $\mathbf{R}^N \times (0,\infty)$ and satisfies the following properties (see [4,5]):

$$G_\theta(x,t)=t^{-\frac{N}{\theta}}G_\theta(t^{-\frac{1}{\theta}}x,1), \tag{4.1}$$
$$|(\partial_x^\alpha G_\theta)(x,t)|\le C_\alpha t^{-\frac{N+|\alpha|}{\theta}}\left(1+t^{-\frac{1}{\theta}}|x|\right)^{-N-\theta-|\alpha|}, \tag{4.2}$$
$$G_\theta(x,t)\ge Ct^{-\frac{N}{\theta}}\left(1+t^{-\frac{1}{\theta}}|x|\right)^{-N-\theta}, \tag{4.3}$$

for all $x\in\mathbf{R}^N$, $t>0$ and $\alpha\in\mathbf{M}$, where C_α is a positive constant. Furthermore, it follows that

$$G_\theta(x,t)=\int_{\mathbf{R}^N}G_\theta(x-y,t-s)\,G_\theta(y,s)\,dy \quad \text{for all } x\in\mathbf{R}^N,\ 0<s<t. \tag{4.4}$$

Similarly to (2.8) and (2.9), for any (signed) Radon measure μ in $\mathbf{R}^N$ and measurable function ϕ in $\mathbf{R}^N$, we set

$$[S_\theta(t)\mu](x):=\int_{\mathbf{R}^N}G_\theta(x-y,t)\,d\mu(y), \quad [S_\theta(t)\phi](x):=\int_{\mathbf{R}^N}G_\theta(x-y,t)\phi(y)\,dy.$$

Then, for any $j=0,1,2,\dots$, by the Young inequality and (4.2) we find $C_j>0$ such that

$$\|\partial_x^\alpha S_\theta(t)\phi\|_q\le C_j t^{-\frac{N}{\theta}(\frac{1}{p}-\frac{1}{q})-\frac{j}{\theta}}\|\phi\|_p$$

for all $t>0$, $\phi\in L^q$, $1\le p\le q\le\infty$ and $\alpha\in\mathbf{M}$ with $|\alpha|=j$. See e.g. [19, Section 2]. Moreover, we recall the following lemma on the decay of $\|S_\theta(t)\mu\|_\infty$ (see [16, Lemma 2.1]).

Lemma 4.1. *Let μ be a nonnegative Radon measure in $\mathbf{R}^N$ and $0<\theta<2$. Then there exists $C=C(N,\theta)>0$ such that*

$$\|S_\theta(t)\mu\|_\infty\le Ct^{-\frac{N}{\theta}}\sup_{x\in\mathbf{R}^N}\mu(B(x,t^{\frac{1}{\theta}})) \quad \textit{for all } t>0.$$

Define

$$K(x,t):=G_\theta\left(x,t^{\frac{\theta}{2m}}\right), \quad x\in\mathbf{R}^N,\ t>0. \tag{4.5}$$

Similarly to (2.8) and (2.9), we define an integral operator $S_K(t)$ by

$$[S_K(t)\mu](x):=\int_{\mathbf{R}^N}K(x-y,t)\,d\mu(y), \quad [S_K(t)\phi](x):=\int_{\mathbf{R}^N}K(x-y,t)\phi(y)\,dy,$$

for (signed) Radon measure μ and measurable function ϕ in $\mathbf{R}^N$. The aim of this section is to prove the following theorem, which is one of the main ingredients of this paper.

Theorem 4.1. *Let $N\ge 1$, $m=2,3\dots$ and $\theta\in(0,2)$. Let G_m and K be as in (1.2) and (4.5), respectively. Then $K=K(x,t)>0$ in $\mathbf{R}^N\times(0,\infty)$ and the following properties hold.*

(a) *For any $j=0,1,2,\dots$, there exist $d_j>0$ and $d_j'>0$ such that*

$$|\partial_x^\alpha G_m(x,t)|\le d_j t^{-\frac{j}{2m}}K(x,t)\le d_j' t^{-\frac{N}{2m}-\frac{j}{2m}}$$

for all $x\in\mathbf{R}^N$, $t>0$ and $\alpha\in\mathbf{M}$ with $|\alpha|=j$.

(b) *There exists $d''>0$ such that*

$$\|S_K(t)\mu\|_\infty\le d''t^{-\frac{N}{2m}}\sup_{x\in\mathbf{R}^N}\mu(B(x,t^{\frac{1}{2m}})), \quad t>0,$$

for all nonnegative Radon measure μ in $\mathbf{R}^N$.

(c) *There exists* $d_* > 0$ *such that*

$$\int_{\mathbf{R}^N} K(x-y,t-s)K(y,s)\,dy \le d_* K(x,t)$$

for all $x \in \mathbf{R}^N$ *and* $t > s > 0$.

Proof. The positivity of K follows from the positivity of G_θ (see Section 2.2). Let $j = 0, 1, 2, \dots$ and $\alpha \in \mathbf{M}$ with $|\alpha| = j$. By (2.5) we find $c_j > 0$ such that

$$|\partial_x^\alpha G_m(x,t)| \le c_j t^{-\frac{N}{2m}-\frac{j}{2m}} \exp\left(-c_j^{-1}\eta^{\frac{2m}{2m-1}}\right) \quad \text{with} \quad \eta = t^{-\frac{1}{2m}}|x| \tag{4.6}$$

for all $(x,t) \in \mathbf{R}^N \times (0,\infty)$. On the other hand, it follows from (4.2) and (4.3) that

$$C_1^{-1}(1+|x|)^{-N-\theta} \le G_\theta(x,1) \le C_1(1+|x|)^{-N-\theta} \quad \text{in } \mathbf{R}^N$$

for some $C_1 > 0$. Then we find $C_2 > 0$ such that

$$\exp\left(-c_j^{-1}|x|^{\frac{2m}{2m-1}}\right) \le C_2 G_\theta(x,1) \quad \text{in } \mathbf{R}^N. \tag{4.7}$$

Let $\tau := t^{\theta/2m}$. By (4.1), (4.6) and (4.7) we obtain

$$\begin{aligned}|\partial_x^\alpha G_m(x,t)| &\le c_j C_2 t^{-\frac{N}{2m}-\frac{j}{2m}} G_\theta\left(t^{-\frac{1}{2m}}x,1\right)\\ &= c_j' t^{-\frac{N}{2m}-\frac{j}{2m}} G_\theta\left(\tau^{-\frac{1}{\theta}}x,1\right)\\ &= c_j' t^{-\frac{N}{2m}-\frac{j}{2m}} \tau^{\frac{N}{\theta}} G_\theta(x,\tau) = c_j' t^{-\frac{j}{2m}} G_\theta\left(x, t^{\frac{\theta}{2m}}\right) = c_j' t^{-\frac{j}{2m}} K(x,t)\end{aligned}$$

for all $(x,t) \in \mathbf{R}^N \times (0,\infty)$, where $c_j' = c_j C_2$. This together with (4.1) implies assertion (a). On the other hand, by Lemma 4.1 and (4.5) we have

$$\begin{aligned}\|S_K(t)\mu\|_\infty &= \left\|S_\theta\big(t^{\frac{\theta}{2m}}\big)\mu\right\|_\infty \le C\big(t^{\frac{\theta}{2m}}\big)^{-\frac{N}{\theta}} \sup_{x\in\mathbf{R}^N} \mu\left(B(x, \big(t^{\frac{\theta}{2m}}\big)^{\frac{1}{\theta}}\right)\\ &= Ct^{-\frac{N}{2m}} \sup_{x\in\mathbf{R}^N} \mu(B(x, t^{\frac{1}{2m}}))\end{aligned}$$

for all $t > 0$ and nonnegative Radon measure μ in $\mathbf{R}^N$. This implies assertion (b).

We prove assertion (c). For any $0 < s < t$, set

$$\omega_{t,s} := (t-s)^{\frac{\theta}{2m}} + s^{\frac{\theta}{2m}}.$$

It follows from $\theta/2m \in (0,1)$ that

$$t^{\frac{\theta}{2m}} \le \omega_{t,s} = (t-s)^{\frac{\theta}{2m}} + s^{\frac{\theta}{2m}} \le 2t^{\frac{\theta}{2m}}. \tag{4.8}$$

Then, by (4.4) we have

$$\begin{aligned}\int_{\mathbf{R}^N} K(x-y,t-s)K(y,s)\,dy &= \int_{\mathbf{R}^N} G_\theta(x-y,(t-s)^{\frac{\theta}{2m}})G_\theta(y,s^{\frac{\theta}{2m}})\,dy\\ &= G_\theta(x,\omega_{t,s}) = \omega_{t,s}^{-\frac{N}{\theta}} G_\theta\left(\omega_{t,s}^{-\frac{1}{\theta}}x,1\right) \le t^{-\frac{N}{2m}} G_\theta\left(\omega_{t,s}^{-\frac{1}{\theta}}x,1\right)\end{aligned} \tag{4.9}$$

for all $x \in \mathbf{R}^N$ and $0 < s < t$. Furthermore, we observe from (4.2), (4.3) and (4.8) that

$$\begin{aligned}G_\theta\left(\omega_{t,s}^{-\frac{1}{\theta}}x,1\right) &\le C\left(1+\omega_{t,s}^{-\frac{1}{\theta}}|x|\right)^{-N-\theta} \le C\left(1+2^{-\frac{1}{\theta}}t^{-\frac{1}{2m}}|x|\right)^{-N-\theta}\\ &\le C\left(1+t^{-\frac{1}{2m}}|x|\right)^{-N-\theta} \le CG_\theta\left(t^{-\frac{1}{2m}}x,1\right).\end{aligned} \tag{4.10}$$

Combining (4.9) and (4.10), we obtain

$$\int_{\mathbf{R}^N} K(x-y,t-s)K(y,s)\,dy \le Ct^{-\frac{N}{2m}} G_\theta\left(t^{-\frac{1}{2m}}x,1\right) = CG_\theta\left(x,t^{\frac{\theta}{2m}}\right) = CK(x,t)$$

for all $x \in \mathbf{R}^N$ and $0 < s < t$. This implies assertion (c). Thus Theorem 4.1 follows. □

5. Sufficient conditions on the solvability

In this section, by use of the majorizing kernel K we establish the existence of solutions of problem (1.1).

5.1. *Existence of solutions of integral equation* (I)

We modify the argument in [27, Theorem 2.3] to obtain sufficient conditions on the existence of solutions of integral equation (I) (see Section 2.2). Let $T > 0$ and

$$X := \left\{ f \in C(\mathbf{R}^N \times (0,T)) : \sup_{\tau \le t < T} \|f(t)\|_\infty < \infty \quad \text{for all } \tau \in (0,T) \right\}.$$

Let K be as in Theorem 4.1. Let $U \in X$ be such that

$$d_* U(x,t) \ge \int_{\mathbf{R}^N} K(x-y,t-s)U(y,s)\,dy > 0, \quad x \in \mathbf{R}^N,\ 0 < s < t < T, \tag{5.1}$$

where d_* is as in Theorem 4.1. Let Ψ be a positive continuous function in $(0,\infty)$ and set $V = \Psi(U)$. Assume that

$$D_* := \sup_{0<t<T} \left\| \frac{U(t)}{\Psi(U(t))} \right\|_\infty \int_0^t \left\| \frac{\Psi(U(s))^p}{U(s)} \right\|_\infty ds < \infty. \tag{5.2}$$

Define

$$X_V := \{ f \in X : |||f||| < \infty \} \quad \text{with} \quad |||f||| := \sup_{0<t<T} \sup_{x \in \mathbf{R}^N} \frac{|f(x,t)|}{V(x,t)}.$$

Then the set X_V is a Banach space equipped with the norm $|||\cdot|||$ (see [27, Section 2]). Note that

$$|f(x,t)| \le |||f||| V(x,t) \quad \text{for all } x \in \mathbf{R}^N,\ t \in (0,T).$$

We apply the fixed point theorem in X_V to prove the existence of solutions of integral equation (I).

Theorem 5.1. *Let $T > 0$, $m = 2, 3, \dots$, $p > 1$. Assume relations* (5.1) *and* (5.2)*. Let $\delta > 0$ and $M > 0$ be such that*

$$\delta + d_0 d_* D_* M^p \le M, \qquad p d_0 d_* D_* M^{p-1} < 1, \tag{5.3}$$

where d_0 and d_* are as in Theorem 4.1. Assume that $u_0(t) := S_m(t)\mu \in X$ and $|||u_0||| \le \delta$. Then there exists a unique solution $u \in X_V$ with $|||u||| \le M$ of integral equation (I) *in $\mathbf{R}^N \times [0,T)$.*

Proof. Set

$$B_M := \{ u \in X_V : |||u||| \le M \}.$$

For any $u \in B_M$, we define

$$\mathcal{F}u(t) := u_0(t) + \mathcal{N}(t), \qquad \mathcal{N}(t) := \int_0^t S_m(t-s)|u(s)|^p\,ds.$$

Then

$$|\mathcal{F}u(t)| \le \delta V(t) + |\mathcal{N}(t)|, \quad |\mathcal{N}(t)| \le d_0 M^p \int_0^t S_K(t-s)V(s)^p\, ds, \tag{5.4}$$

for all $t \in (0, T)$. Since

$$V(x,t)^p = \frac{\Psi(U(x,t))^p}{U(x,t)} U(x,t) \le \left\| \frac{\Psi(U(t))^p}{U(t)} \right\|_\infty U(x,t),$$
$$U(x,t) \le \frac{U(x,t)}{\Psi(U(x,t))} \Psi(U(x,t)) \le \left\| \frac{U(t)}{\Psi(U(t))} \right\|_\infty V(x,t),$$

for all $(x,t) \in \mathbf{R}^N \times (0, T)$, by (5.1) we have

$$\begin{aligned} \int_0^t S_K(t-s)V(s)^p\, ds &\le \int_0^t \left\| \frac{\Psi(U(s))^p}{U(s)} \right\|_\infty S_K(t-s)U(s)\, ds \\ &\le d_* U(t) \int_0^t \left\| \frac{\Psi(U(s))^p}{U(s)} \right\|_\infty ds \\ &\le d_* \left\| \frac{U(t)}{\Psi(U(t))} \right\|_\infty V(t) \int_0^t \left\| \frac{\Psi(U(s))^p}{U(s)} \right\|_\infty ds \le d_* D_* V(t) \end{aligned} \tag{5.5}$$

for all $0 < t < T$. It follows from (5.3), (5.4) and (5.5) that

$$|||\mathcal{F}u||| \le \delta + d_0 d_* D_* M^p \le M \quad \text{for all } u \in B_M, \tag{5.6}$$

which means that $\mathcal{F}$ is a mapping from B_M to itself. On the other hand, for any $u_1, u_2 \in B_M$, by (5.3) and (5.5) we find $\nu \in (0, 1)$ such that

$$\begin{aligned} &|\mathcal{F}u_1(t) - \mathcal{F}u_2(t)| \\ &\le d_0 \int_0^t S_K(t-s)\big||u_1|^p - |u_2|^p\big|\, ds \\ &\le p d_0 \int_0^t S_K(t-s) \max\{|u_1(s)|^{p-1}, |u_2(s)|^{p-1}\} V(s) \frac{|u_1(s) - u_2(s)|}{V(s)}\, ds \\ &\le p d_0 M^{p-1} |||u_1 - u_2||| \int_0^t S_K(t-s)V(s)^p\, ds \\ &\le p d_0 d_* D_* M^{p-1} V(t)\, |||u_1 - u_2||| \le \nu V(t)\, |||u_1 - u_2|||. \end{aligned}$$

This implies that

$$|||\mathcal{F}u_1 - \mathcal{F}u_2||| \le \nu |||u_1 - u_2|||. \tag{5.7}$$

This means that $\mathcal{F}$ is a contraction mapping on B_M. Therefore, by (5.6) and (5.7) we apply the Banach fixed point theorem to find $u_* \in B_M$ uniquely such that $\mathcal{F}u_* = u_*$ in X_V. This implies that $u_* \in C(\mathbf{R}^N \times (0, T))$ and u_* satisfies

$$u_*(x,t) = u_0(x,t) + \int_0^t S_m(t-s)|u_*(s)|^p\, ds$$

for all $(x,t) \in \mathbf{R}^N \times (0, T)$. Furthermore, by (5.4) and (5.5) we have

$$\sup_{\tau\le t<T}\|u_0(t)\|_\infty \le \delta \sup_{\tau\le t<T} V(t)<\infty,$$
$$\sup_{\tau\le t<T}\left\|\int_0^t S_m(t-s)|u_*(s)|^p\,ds\right\|_\infty \le d_0 d_* M^p D_* \sup_{\tau\le t<T} V(t)<\infty, \tag{5.8}$$

for all $\tau\in(0,T)$. Therefore we see that u_* is a solution of integral equation (I) in $\mathbf{R}^N\times(0,T)$. Thus Theorem 5.1 follows. □

5.2. Sufficient conditions for solvability

We obtain sufficient conditions for the existence of solutions of problem (1.1) by combining Theorem 5.1 and the arguments in [16], [25] and [28]. (See also [17].) We prove Theorem 1.3.

Proof of Theorem 1.3. By similar transformation (3.1) and Proposition 2.1 it suffices to show the existence of solutions of integral equation (I) in $\mathbf{R}^N\times[0,1)$.

We assume relation (1.12) with $T=1$ and show the existence of solutions of integral equation (I) in $\mathbf{R}^N\times[0,1)$. Let K be as in Theorem 4.1, that is,

$$K(x,t)=G_\theta\left(x,t^{\frac{\theta}{2m}}\right)\quad\text{with}\quad 0<\theta<2.$$

Set $U(x,t):=2d_0[S_K(t)\mu](x)$ and $u_0(x,t):=[S_m(t)\mu](x)$. Then it follows from assertion (c) of Theorem 4.1 that

$$\begin{aligned}\int_{\mathbf{R}^N} K(x-y,t-s)U(y,s)\,dy &= 2d_0\int_{\mathbf{R}^N}\int_{\mathbf{R}^N} K(x-y,t-s)K(y-z,s)\,dy\,d\mu(z)\\ &\le 2d_0 d_*\int_{\mathbf{R}^N} K(x-z,t)\,d\mu(z)=d_*U(x,t)\end{aligned} \tag{5.9}$$

for all $x\in\mathbf{R}^N$ and $0<s<t$, that is, U satisfies inequality (5.1). Furthermore, it follows from assertion (a) of Theorem 4.1 with $j=0$ that

$$|u_0(x,t)|\le d_0\int_{\mathbf{R}^N} K(x-y,t)\,d\mu(y)=\frac{1}{2}U(x,t)\quad\text{in}\quad \mathbf{R}^N\times(0,1). \tag{5.10}$$

On the other hand, it follows from assertion (b) of Theorem 4.1 and (1.12) that

$$\|U(t)\|_\infty\le Ct^{-\frac{N}{2m}}\sup_{x\in\mathbf{R}^N}\mu(B(x,t^{\frac{1}{2m}}))\le Ct^{-\frac{N}{2m}}\sup_{x\in\mathbf{R}^N}\mu(B(x,1))\le C\gamma t^{-\frac{N}{2m}} \tag{5.11}$$

for all $0<t<1$. Since $1<p<1+2m/N$, by (5.11) we find $c_*>0$ such that

$$\int_0^1\|U(s)\|_\infty^{p-1}\,ds\le(C\gamma)^{p-1}\int_0^1 s^{-\frac{N}{2m}(p-1)}\,ds\le c_*\gamma^{p-1}. \tag{5.12}$$

We apply Theorem 5.1 with

$$\Psi(s)=s,\qquad V=U,\qquad T=1,\qquad \delta=\frac{1}{2}\qquad\text{and}\qquad M=1.$$

Then, by (5.10) we have

$$|||u_0|||\le\frac{1}{2}. \tag{5.13}$$

Furthermore, by (5.2) and (5.12) we see that

$$D_* \equiv \sup_{0<t\le 1} \int_0^t \|U(s)\|_\infty^{p-1}\,ds = \int_0^1 \|U(s)\|_\infty^{p-1}\,ds \le c_*\gamma^{p-1}. \tag{5.14}$$

Then, by (5.13) and (5.14), taking a sufficiently small $\gamma > 0$ satisfying

$$\max\{2, p\}c_*d_0d_*\gamma^{p-1} < 1,$$

we find a function $u \in B_M \subset X_V$ such that

$$u(t) = S_m(t)\mu + \int_0^t S_m(t-s)|u(s)|^p\,ds$$

for all $t \in (0, 1)$. Furthermore, by (5.8) and (5.11) we see that inequality (2.13) also holds with $T = 1$. Therefore u is a solution of integral equation (I). Thus Theorem 1.3 follows. □

Remark 5.1. The argument in the proof of Theorem 1.3 is applicable to the case when μ is a signed Radon measure in $\mathbf{R}^N$. Indeed, the same conclusion as in Theorem 1.3 holds if μ is a signed Radon measure satisfying

$$\sup_{x\in\mathbf{R}^N} |\mu|(B(x, T^{\frac{1}{2m}})) \le \gamma_2 T^{\frac{N}{2m}-\frac{1}{p-1}}$$

for some $T > 0$, instead of (1.12). Here $|\mu|$ is the total variation of μ.

Similarly to Remark 5.1, we consider problem (1.1) without the nonnegativity of the initial data and obtain sufficient conditions for the existence of solutions of problem (1.1).

Theorem 5.2. *Let $N \ge 1$, $m = 2, 3, \dots$ and $1 < \alpha < p$. Then there exists $\gamma = \gamma(N, m, p, \alpha) > 0$ such that, if μ is a measurable function in $\mathbf{R}^N$ satisfying*

$$\sup_{x\in\mathbf{R}^N} \left[⨍_{B(x,\sigma)} |\mu(y)|^\alpha\,dy \right]^{\frac{1}{\alpha}} \le \gamma\sigma^{-\frac{2m}{p-1}}, \qquad 0 < \sigma \le T^{\frac{1}{2m}}, \tag{5.15}$$

for some $T > 0$, then problem (1.1) possesses a solution in $\mathbf{R}^N \times [0, T)$.

Proof. Similarly to the proof of Theorem 1.3, it suffices to show the existence of solution of integral equation (I) in $\mathbf{R}^N \times [0, 1)$. We apply Theorem 5.1 with

$$\begin{aligned} &T = 1, \quad u_0(x,t) := S_m(t)\mu, \quad U(x,t) := (2d_0)^\alpha S_K(t)|\mu|^\alpha, \quad \Psi(s) := s^{\frac{1}{\alpha}}, \\ &V(x,t) := 2d_0\left(S_K(t)|\mu|^\alpha\right)^{\frac{1}{\alpha}}. \end{aligned} \tag{5.16}$$

Similarly to (5.9), by (5.16) we see that U satisfies inequality (5.1). Furthermore, by (5.15) we have

$$\sup_{x\in\mathbf{R}^N} \int_{B(x,\sigma)} |\mu(x)|^\alpha\,dx \le C\gamma^\alpha\sigma^{N-\frac{2\alpha m}{p-1}} \quad \text{for all } \sigma \in (0, 1).$$

This together with assertion (b) of Theorem 4.1 implies that

$$\begin{aligned} \|U(t)\|_\infty &\le Ct^{-\frac{N}{2m}} \sup_{x\in\mathbf{R}^N} |\mu|^\alpha(B(x, t^{\frac{1}{2m}})) \\ &\le C\gamma^\alpha t^{-\frac{N}{2m}}\left(t^{\frac{1}{2m}}\right)^{N-\frac{2\alpha m}{p-1}} = C\gamma^\alpha t^{-\frac{\alpha}{p-1}} \end{aligned} \tag{5.17}$$

for all $0 < t < 1$. By (5.17) we obtain

$$\int_0^t \left\| \frac{\Psi(U(s))^p}{U(s)} \right\|_\infty ds = \int_0^t \|U(s)\|_\infty^{\frac{p-\alpha}{\alpha}} ds \le C\gamma^{p-\alpha} \int_0^t s^{-\frac{p-\alpha}{p-1}} ds \le C\gamma^{p-\alpha} t^{\frac{\alpha-1}{p-1}},$$
$$\left\| \frac{U(t)}{\Psi(U(t))} \right\|_\infty = \|U(t)\|_\infty^{\frac{\alpha-1}{\alpha}} \le C\gamma^{\alpha-1} t^{-\frac{\alpha-1}{p-1}},$$

for all $0 < t < 1$. This implies that

$$D_* \equiv \sup_{0<t\le 1} \left\| \frac{U(t)}{\Psi(U(t))} \right\|_\infty \int_0^t \left\| \frac{\Psi(U(s))^p}{U(s)} \right\|_\infty ds \le C\gamma^{p-1}. \tag{5.18}$$

On the other hand, it follows from assertion (a) of Theorem 4.1 with $j = 0$ and the Jensen inequality that

$$|S_m(t)\mu| \le d_0 S_K(t)|\mu| \le d_0 \left(S_K(t)|\mu|^\alpha\right)^{\frac{1}{\alpha}} = \frac{1}{2} V(t) \tag{5.19}$$

for all $0 < t < 1$. Similarly to the proof of Theorem 1.3, by (5.18) and (5.19), taking a sufficiently small $\gamma > 0$ and applying Theorem 5.1 with (5.16), $\delta = 1/2$ and $M = 1$, we see that integral equation (I) possesses a solution in $\mathbf{R}^N \times [0, 1)$. Thus Theorem 5.2 follows. □

Theorem 5.3. *Let $N \ge 1$, $m = 2, 3, \dots$, $p = p_m$ and $\beta > 0$. For $s > 0$, set*

$$\Phi(s) := s[\log(e+s)]^\beta, \qquad \rho(s) := s^{-N}\left[\log\left(e + \frac{1}{s}\right)\right]^{-\frac{N}{2m}}.$$

Then there exists $\gamma = \gamma(N, m, \beta) > 0$ such that, if μ is a measurable function in $\mathbf{R}^N$ satisfying

$$\sup_{x\in\mathbf{R}^N} \Phi^{-1}\left[\ \fint_{B(x,\sigma)} \Phi(T^{\frac{1}{p-1}}|\mu(y)|)\, dy\right] \le \gamma\rho(\sigma T^{-\frac{1}{2m}}), \qquad 0 < \sigma \le T^{\frac{1}{\theta}}, \tag{5.20}$$

for some $T > 0$, then problem (1.1) possesses a solution in $\mathbf{R}^N \times [0, T)$.

Proof. Similarly to the proof of Theorem 1.3, it suffices to show the existence of solutions of integral equation (I) in $\mathbf{R}^N \times [0, 1)$. Let $0 < \gamma < 1$ and assume relation (5.20). Let $L \ge e$ and set $\Phi_L(s) := s[\log(L+s)]^\beta$ for $s > 0$. Then, taking a sufficiently large $L \ge e$ if necessary, we have:

(a) Φ_L is convex in $(0, \infty)$;
(b) the function $(0, \infty) \ni s \mapsto s^{\frac{p-1}{2}}[\log(L+s)]^{-\beta p}$ is monotone increasing.

Define a positive function $\Psi_L = \Psi_L(s)$ in $(0, \infty)$ by $\Psi_L(s) := \Phi_L^{-1}(s)$. Then

$$\begin{aligned} C^{-1}\Phi_L(s) \le \Phi(s) \le C\Phi_L(s), \\ C^{-1}s[\log(L+s)]^{-\beta} \le \Psi_L(s) \le Cs[\log(L+s)]^{-\beta}, \end{aligned} \tag{5.21}$$

for all $s > 0$. We apply Theorem 5.1 with

$$\begin{aligned} &T = 1, \quad u_0(x,t) := S_m(t)\mu, \quad U(x,t) := S_K(t)\Phi_L(|\mu|), \quad \Psi(s) := \Phi_L^{-1}(s), \\ &V(x,t) := \Phi_L^{-1}(S_K(t)\Phi_L(|\mu|)). \end{aligned} \tag{5.22}$$

It follows from (5.20) and (5.21) that

$$\sup_{x\in\mathbf{R}^N} \Phi_L^{-1}\left[\ \fint_{B(x,\sigma)} \Phi_L(|\mu(y)|)\, dy\right] \le C\gamma\rho(\sigma) \quad \text{for all } 0 < \sigma < 1. \tag{5.23}$$

Applying assertion (b) of Theorem 4.1 with (5.23), we see that

$$\begin{aligned}\|U(t)\|_\infty &= \|S_K(t)\Phi_L(|\mu|)\|_\infty \le Ct^{-\frac{N}{2m}} \sup_{x\in\mathbf{R}^N} \int_{B(x,t^{1/2m})} \Phi_L(|\mu(y)|)\,dy \\ &\le Ct^{-\frac{N}{2m}} \left(t^{\frac{1}{2m}}\right)^N \Phi_L(C\gamma\rho(t^{\frac{1}{2m}})) \\ &\le C\gamma\rho(t^{\frac{1}{2m}})[\log(L+C\gamma\rho(t^{\frac{1}{2m}}))]^\beta \\ &\le C\gamma\rho(t^{\frac{1}{2m}})[\log(L+C\rho(t^{\frac{1}{2m}}))]^\beta \le C\gamma t^{-\frac{N}{2m}} \left|\log\frac{t}{2}\right|^{-\frac{N}{2m}+\beta} =: \gamma\xi(t)\end{aligned} \tag{5.24}$$

for all $0<t<1$. Here the last inequality in (5.24) follows from

$$\rho(t^{\frac{1}{2m}})[\log(L+C\rho(t^{\frac{1}{2m}}))]^\beta = O\left(t^{-\frac{N}{2m}}|\log t|^{-\frac{N}{2m}}|\log t|^\beta\right) = O\left(t^{-\frac{N}{2m}}|\log t|^{-\frac{N}{2m}+\beta}\right)$$

as $t\to +0$. Since $p=p_m=1+2m/N$, it follows from property (b), (5.21) and (5.24) that

$$\begin{aligned}0 &\le \frac{\Psi_L(U(x,t))^p}{U(x,t)} \le CU(x,t)^{p-1}[\log(L+U(x,t))]^{-\beta p} \\ &= CU(x,t)^{\frac{p-1}{2}}U(x,t)^{\frac{p-1}{2}}[\log(L+U(x,t))]^{-\beta p} \\ &\le C(\gamma\xi(t))^{\frac{p-1}{2}}(\gamma\xi(t))^{\frac{p-1}{2}}[\log(L+\gamma\xi(t))]^{-\beta p} \\ &\le C\gamma^{\frac{p-1}{2}}\xi(t)^{p-1}[\log(L+\xi(t))]^{-\beta p} \\ &\le C\gamma^{\frac{p-1}{2}}t^{-\frac{N}{2m}(p-1)}\left|\log\frac{t}{2}\right|^{-\frac{N}{2m}(p-1)+\beta(p-1)-\beta p} = C\gamma^{\frac{p-1}{2}}t^{-1}\left|\log\frac{t}{2}\right|^{-1-\beta}\end{aligned} \tag{5.25}$$

for all $(x,t)\in\mathbf{R}^N\times(0,1)$. Similarly, we have

$$\begin{aligned}0 &\le \frac{U(x,t)}{\Psi_L(U(x,t))} \le C[\log(L+U(x,t))]^\beta \le C[\log(L+\gamma\xi(t))]^\beta \\ &\le C[\log(L+\xi(t))]^\beta \le C\left|\log\frac{t}{2}\right|^\beta\end{aligned} \tag{5.26}$$

for all $(x,t)\in\mathbf{R}^N\times(0,1)$. By (5.25) and (5.26) we obtain

$$\begin{aligned}D_* &\equiv \sup_{0<t<1}\left\|\frac{U(t)}{\Psi_L(U(t))}\right\|_\infty \int_0^t \left\|\frac{\Psi_L(U(s))^p}{U(s)}\right\|_\infty ds \\ &\le C\gamma^{\frac{p-1}{2}} \sup_{0<t<1}\left\{\left|\log\frac{t}{2}\right|^\beta \int_0^t s^{-1}\left|\log\frac{s}{2}\right|^{-1-\beta} ds\right\} \le C\gamma^{\frac{p-1}{2}}.\end{aligned} \tag{5.27}$$

On the other hand, it follows from Theorem 4.1 and the Jensen inequality that

$$|u_0(t)| = |S_m(t)\mu| \le d_0 S_K(t)|\mu| \le d_0\Phi_L^{-1}(S_K(t)\Phi_L(|\mu|)) = d_0 V(t) \tag{5.28}$$

for all $0<t<1$. Similarly to the proof of Theorem 1.3, by (5.27) and (5.28), taking a sufficiently small $\gamma>0$ and applying Theorem 5.1 with (5.22), $\delta=d_0$ and $M=2d_0$, we see that integral equation (I) possesses a solution in $\mathbf{R}^N\times[0,1)$. Thus Theorem 5.3 follows. □

Proof of Theorem 1.4. The proof is similar to those of [16] and [17]. Let $p>p_m$ and assume relation (1.14). Then we find $\alpha\in(1,p)$ such that $2m\alpha<N(p-1)$ and obtain

$$\sup_{x\in\mathbf{R}^N}\left[\fint_{B(x,\sigma)}|\mu(y)|^\alpha\,dy\right]^{\frac{1}{\alpha}}\le\gamma_3\left[\fint_{B(0,\sigma)}|y|^{-\frac{2m\alpha}{p-1}}\,dy\right]^{\frac{1}{\alpha}}+\gamma_3\le C\gamma_3\sigma^{-\frac{2m}{p-1}}+\gamma_3$$

for all $\sigma>0$. Taking a sufficiently small $\gamma_3>0$ if necessary, we apply Theorem 5.2 to obtain Theorem 1.4 with $p>p_m$. Similarly, under (1.15), problem (1.1) possesses a global-in-times solution.

Next we let $p=p_m$ and assume relation (1.13). Let $0<\beta<N/2m$. Then

$$\begin{aligned}&\sup_{x\in\mathbf{R}^N}\fint_{B(x,\sigma)}\Phi(|\mu(y)|)\,dy=\fint_{B(0,\sigma)}\Phi(|\mu(y)|)\,dy\\&\le C\gamma_3\fint_{B(0,\sigma)}|y|^{-N}|\log|y||^{-\frac{N}{2m}-1+\beta}\,dy\le C\gamma_3\sigma^{-N-\frac{N}{2m}+\beta}\end{aligned}$$

for all sufficiently small $\sigma>0$. This implies that

$$\sup_{x\in\mathbf{R}^N}\Phi^{-1}\left[\fint_{B(x,\sigma)}\Phi(|\mu(y)|)\,dy\right]\le C\gamma_3\sigma^{-N-\frac{N}{2m}}$$

for all sufficiently small $\sigma>0$. Therefore, taking a sufficiently small $\gamma_3>0$ if necessary, we see that inequality (5.20) holds with $T=1$. Therefore we apply Theorem 5.3 to obtain Theorem 1.4 with $p=p_m$. Thus Theorem 1.4 follows. □

Remark 5.2. Let $p>1$. Consider the Cauchy problem

$$\begin{cases}\partial_t u+(-\Delta)^m u=F_p(u), & x\in\mathbf{R}^N,\ t>0,\\ u(x,0)=\mu(x)\ge 0, & x\in\mathbf{R}^N,\end{cases}\tag{P}$$

where $m=2,3,\dots$. Here F_p is a continuous function in $\mathbf{R}$ satisfying

$$|F_p(u)|\le|u|^p,\qquad |F_p(u)-F_p(v)|\le C_F(|u|^{p-1}+|v|^{p-1})|u-v|,$$

for all $u,v\in\mathbf{R}$, where C_F is a positive constant. Then, applying the same arguments in Section 5, we can show that the same conclusions as in Theorems 1.3, 5.2 and 5.3 and the same statement as in Remark 5.1 hold for problem (P), in particular, in the case of $F_p(u)=|u|^{p-1}u$.

Declaration of competing interest

None declared.

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