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Exact controllability of semilinear heat equations in spaces of analytic functions

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Abstract

It is by now well known that the use of Carleman estimates allows to establish the controllability to trajectories of nonlinear parabolic equations. However, by this approach, it is not clear how to decide whether a given function is indeed reachable. In this paper, we pursue the study of the reachable states of parabolic equations based on a direct approach using control inputs in Gevrey spaces by considering a semilinear heat equation in dimension one. The nonlinear part is assumed to be an analytic function of the spatial variable x, the unknown y, and its derivative $\partial_x y$. By investigating carefully a nonlinear Cauchy problem in the spatial variable and the relationship between the jet of space derivatives and the jet of time derivatives, we derive an exact controllability result for small initial and final data that can be extended as analytic functions on some ball of the complex plane.

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1. Introduction

The null controllability of nonlinear parabolic equations is well understood since the nineties. It was derived in [6] in dimension one by solving some "ill-posed problem" with Cauchy data in some Gevrey spaces (see also [13]), and in [4,5] in any dimension and for any control region by using some "parabolic Carleman estimates".

The null controllability was actually extended to the controllability to trajectories in [5]. However, it is a quite hard task to decide whether a given state is the value at some time of a trajectory of the system without control (free evolution). In practice, the only known examples of such states are the steady states.

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As noticed in [17], in the linear case, the steady states are Gevrey functions of order $1/2$ in x (and thus analytic over $\mathbb{C}$) for which infinitely many traces vanish at the boundary, a fact which is also a very conservative condition leading to exclude e.g. all the nontrivial polynomial functions.

The recent paper [17] used the flatness approach and a Borel theorem to provide an explicit set of reachable states composed of states that can be extended as analytic functions on a ball B. It was also noticed in [17] that any reachable state could be extended as an analytic function on a square included in the ball B. We refer the reader to [1,7] for new sets of reachable states for the linear 1D heat equation, with control inputs chosen in $L^2(0,T)$. We notice that the flatness approach applied to the control of PDEs, first developed in [12,3,19,24], was revisited recently to recover the null controllability of (i) the heat equation in cylinders [15]; (ii) a family of parabolic equations with unsmooth coefficients [16]; (iii) the Schrödinger equation [18]; (iv) the Korteweg-de Vries equation with a control at the left endpoint [14]. One of the main features of the flatness approach is that it provides control inputs developed as explicit series, which leads to very efficient numerical schemes.

The aim of the present paper is to extend the results of [17] to semilinear heat equations. Roughly, we shall prove that a reachable state for the linear heat equation is also reachable for the semilinear one, provided that its magnitude is not too large and its poles and those of the nonlinear term are sufficiently far from the origin. The method of proof is inspired by [6] where a Cauchy problem in the variable x is investigated. The main novelty is that we prove an *exact* controllability result (and not only a *null* controllability result as in [6]), and we need to investigate the influence of the nonlinear terms on the jets of the time derivatives of two traces at $x = 0$. Here, we do not use some series expansions of the control inputs as in the flatness approach, but we still use some Borel theorem as in [22,17]. It is unclear whether the same results could be obtained by the classical approach using the exact controllability of the linearized system and a fixed-point argument.

To be more precise, we are concerned with the exact controllability of the following semilinear heat equation

$$\partial_t y = \partial_x^2 y + f(x, y, \partial_x y), \qquad x \in [-1,1],\ t \in [0,T], \tag{1.1}$$

$$y(-1,t) = h_{-1}(t), \qquad t \in [0,T], \tag{1.2}$$

$$y(1,t) = h_1(t), \qquad t \in [0,T], \tag{1.3}$$

$$y(x,0) = y^0(x), \qquad x \in [-1,1], \tag{1.4}$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ is analytic with respect to all its arguments in a neighborhood of $(0,0,0)$. More precisely, we assume that

$$f(x,0,0) = 0 \quad \forall x \in (-4,4), \tag{1.5}$$

and that

$$f(x, y_0, y_1) = \sum_{(p,q,r)\in\mathbb{N}^3} a_{p,q,r} (y_0)^p (y_1)^q x^r \quad \forall (x, y_0, y_1) \in (-4,4)^3, \tag{1.6}$$

with

$$|a_{p,q,r}| \le \frac{M}{b_0^p b_1^q b_2^r} \qquad \forall p,q,r \in \mathbb{N} \tag{1.7}$$

for some constants

$$M > 0, \quad b_0 > 4,\ b_1 > 4,\ \text{and } b_2 > 4. \tag{1.8}$$

Note that $a_{0,0,r} = 0$ for all $r \in \mathbb{N}$ by (1.5). For $p,q \in \mathbb{N}$ let

$$A_{p,q}(x) = \sum_{r\in\mathbb{N}} a_{p,q,r} x^r, \quad |x| < b_2.$$

We infer from (1.6) and (1.7) that

$$f(x, y_0, y_1) = \sum_{\substack{p,q\in\mathbb{N},\\ p+q>0}} A_{p,q}(x) (y_0)^p (y_1)^q,$$

$$|A_{p,q}(x)| \le \frac{M}{b_0^p b_1^q} \frac{1}{1 - \frac{|x|}{b_2}}, \quad |x| < b_2.$$

Among the many physically relevant instances of (1.1) satisfying (1.5)-(1.8), we quote:

(1) the *heat equation with an analytic potential*:

$$\partial_t y = \partial_x^2 y + \varphi(x) y$$

where $\varphi(x) = \sum_{r\geq 0} a_r x^r$, with $|a_r| \leq M/b_2^r$ for all $r \in \mathbb{N}$ and some constants $M > 0$, $|b_2| > 4$.

(2) the *Allen-Cahn equation*

$$\partial_t y = \partial_x^2 y + y - y^3$$

(3) the *viscous Burgers' equation*

$$\partial_t y = \partial_x^2 y - y\partial_x y. \tag{1.9}$$

Note that our controllability result is still valid when the nonlinear term $-y\partial_x y$ in (1.9) is replaced by a term like $\varphi(x)y^p(\partial_x y)^q$ with φ as in (1), and $p, q \in \mathbb{N}$.

Because of the smoothing effect, the exact controllability result has to be stated in a space of analytic functions (see [17] for the linear heat equation). For given $R > 1$ and $C > 0$, we denote by $\mathcal{R}_{R,C}$ the set

$$\mathcal{R}_{R,C} := \{y : [-1,1] \to \mathbb{R};\ \exists (\alpha_n)_{n\geq 0} \in \mathbb{R}^{\mathbb{N}},\ |\alpha_n| \leq C\frac{n!}{R^n}\ \ \forall n \geq 0 \text{ and } y(x) = \sum_{n=0}^{\infty} \alpha_n \frac{x^n}{n!}\ \ \forall x \in [-1,1]\}.$$

We say that a function $h \in C^\infty([t_1, t_2])$ is *Gevrey of order* $s \geq 0$ *on* $[t_1, t_2]$, and we write $h \in G^s([t_1, t_2])$, if there exist some positive constants M, R such that

$$|\partial_t^p h(t)| \leq M\frac{(p!)^s}{R^p} \quad \forall t \in [t_1, t_2],\ \forall p \geq 0.$$

Similarly, we say that a function $y \in C^\infty([x_1, x_2] \times [t_1, t_2])$ is *Gevrey of order* s_1 *in* x *and* s_2 *in* t, with $s_1, s_2 \geq 0$, and we write $y \in G^{s_1,s_2}([x_1, x_2] \times [t_1, t_2])$, if there exist some positive constants M, R_1, R_2 such that

$$|\partial_x^{p_1}\partial_t^{p_2} y(x,t)| \leq M\frac{(p_1!)^{s_1}(p_2!)^{s_2}}{R_1^{p_1}R_2^{p_2}} \quad \forall (x,t) \in [x_1, x_2] \times [t_1, t_2],\ \forall (p_1, p_2) \in \mathbb{N}^2.$$

The main result in this paper is the following *exact controllability* result.

Theorem 1.1. *Let $f = f(x, y_0, y_1)$ be as in (1.5)-(1.8) with $b_2 > \hat{R} := 4e^{(2e)^{-1}} \approx 4.81$. Let $R > \hat{R}$ and $T > 0$. Then there exists some number $\hat{C} > 0$ such that for all $y^0, y^1 \in \mathcal{R}_{R,\hat{C}}$, there exists $h_{-1}, h_1 \in G^2([0,T])$ such that the solution y of (1.1)-(1.4) is defined for all $(x,t) \in [-1,1] \times [0,T]$ and satisfies $y(x,T) = y^1(x)$ for all $x \in [-1,1]$. Furthermore, we have that $y \in G^{1,2}([-1,1] \times [0,T])$.*

It is likely that using the smoothing effect, a similar result could be obtained with a less regular initial data (e.g. in $L^2(-1,1)$) as for the linear Korteweg-de Vries equation in [14, Corollary 1.1]. This would however require to estimate carefully the domain of analyticity in x of the solution.

A similar result with only *one control* can be derived assuming that f is odd w.r.t. (x, y_0). Consider the control system

$$\partial_t y = \partial_x^2 y + f(x, y, \partial_x y), \qquad x \in [0,1],\ t \in [0,T], \tag{1.10}$$

$$y(0,t) = 0, \qquad t \in [0,T], \tag{1.11}$$

$$y(1,t) = h(t), \qquad t \in [0,T], \tag{1.12}$$

$$y(x,0) = y^0(x), \qquad x \in [0,1]. \tag{1.13}$$

Corollary 1.2. *Let $f = f(x, y_0, y_1)$ be as in (1.5)-(1.8) with $b_2 > \hat{R} := 4e^{(2e)^{-1}} \approx 4.81$, and assume that*

$$f(-x, -y_0, y_1) = -f(x, y_0, y_1) \quad \forall x \in [-1, 1],\ \forall y_0, y_1 \in (-4, 4). \tag{1.14}$$

Let $R > \hat{R}$ and $T > 0$. Then there exists some number $\hat{C} > 0$ such that for all $y^0, y^1 \in \mathcal{R}_{R,\hat{C}}$ with $(y^0(-x), y^1(-x)) = (-y^0(x), -y^1(x))$ for all $x \in [-1, 1]$, there exists $h \in G^2([0, T])$ such that the solution y of (1.10)-(1.13) is defined for all $(x, t) \in [-1, 1]\times \in [0, T]$ and satisfies $y(x, T) = y^1(x)$ for all $x \in [0, 1]$. Furthermore, we have that $y \in G^{1,2}([0, 1] \times [0, T])$.

Corollary 1.2 can be applied e.g. to (i) the heat equation with an *even* analytic potential; (ii) the Allen-Cahn equation; (iii) the viscous Burgers' equation.

The constant $\hat{R} := 4e^{(2e)^{-1}}$ is probably not optimal, but our main aim was to provide an explicit (reasonable) constant. It is expected that the optimal constant is $\hat{R} := 1$, with a diamond-shaped domain of analyticity as in [1] and [7] for the linear heat equation.

Our method of proof combines two steps.

- The first one is the analysis of a Cauchy problem in the spatial variable. We prove the existence of *global* solutions of the semilinear heat equation defined for x in the full interval $[-1, 1]$ associated with two initial data $y(0, t) = g_0(t)$, $y_x(0, t) = g_1(t)$ for $t \in [0, T]$. To do that, we refine the method developed in [20,21] which gives solely *local* solutions in x. The solution is completely defined in terms of the initial data g_0 and g_1 (flatness property) but, in contrast to [17], there is no representation of the solution as an explicit series.
- The second one is a "jets analysis" which investigate the relationship between the jet $(\partial_x^n y(0, 0))_{n\geq 0}$ and the jets $(\partial_t^n y(0, 0))_{n\geq 0}$ and $(\partial_x \partial_t^n y(0, 0))_{n\geq 0}$. This step is needed to reach a given state in the reachable space.

We notice that the method of proof in the linear case (see [17]) was also based on the same two steps, the computations being however easier and explicit. We note also that our approach does not follow the classical "linearization + fixed point-argument" approach which is widely used to deal with the controllability of nonlinear PDEs. Among the advantages of our method, we could mention (i) its *robustness*, in the sense that it can be adapted to many other PDEs (see [11] for an extension to PDEs of the form $\partial_t^N y = \partial_x^M y + f(x, y, ..., \partial_x^{M-1} y)$ with $N < M$) (ii) its possible use to elaborate efficient numerical schemes (see [15] in the linear case). For the restrictions of the method, we should say that (i) the constants are not optiomal and (ii) it is (to date) only applicable in dimension one.

The paper is organized as follows. Section 2 is concerned with the wellposedness of the Cauchy problem in the x-variable (Theorem 2.1). The relationship between the jet of space derivatives and the jet of time derivatives at some point (jet analysis) for a solution of (1.1) is studied in Section 3. In particular, we show that the semilinear heat equation (1.1) can be (locally) solved forward and backward if the initial data y_0 can be extended as an analytic function in some ball of $\mathbb{C}$ (Proposition 3.6). Finally, the proofs of Theorem 1.1 and Corollary 1.2 are displayed in Section 4.

2. Cauchy problem in the space variable

2.1. Statement of the global wellposedness result

Let $f = f(x, y_0, y_1)$ be as in (1.5)-(1.8). We are concerned with the wellposedness of the Cauchy problem in the variable x:

$$\partial_x^2 y = \partial_t y - f(x, y, \partial_x y), \qquad x \in [-1, 1],\ t \in [t_1, t_2], \tag{2.1}$$

$$y(0, t) = g_0(t), \qquad t \in [t_1, t_2], \tag{2.2}$$

$$\partial_x y(0, t) = g_1(t), \qquad t \in [t_1, t_2], \tag{2.3}$$

for some given functions $g_0, g_1 \in G^2([t_1, t_2])$. The aim of this section is to prove the following result.

Theorem 2.1. *Let $f = f(x, y_0, y_1)$ be as in (1.5)-(1.8). Let $-\infty < t_1 < t_2 < +\infty$ and $R > 4$. Then there exists some number $C > 0$ such that for all $g_0, g_1 \in G^2([t_1, t_2])$ with*

$$|g_i^{(n)}(t)| \leq C\frac{(n!)^2}{R^n}, \quad i=0,1,\ n\geq 0,\ t\in[t_1,t_2], \tag{2.4}$$

there exist some numbers R_1, R_2 with $4/e < R_1 < R_2$ and a solution $y \in G^{1,2}([-1,1]\times[t_1,t_2])$ of (2.1)-(2.3) satisfying for some constant $M>0$

$$|\partial_x^{p_1}\partial_t^{p_2} y(x,t)| \leq M\frac{(p_1+2p_2)!}{R_1^{p_1}R_2^{2p_2}} \quad \forall (x,t)\in[-1,1]\times[t_1,t_2],\ \forall (p_1,p_2)\in\mathbb{N}^2. \tag{2.5}$$

Assuming that f is defined for $x\in(-4,4)$ instead of $x\in(-1,1)$ is likely not optimal, and only technical.

2.2. Abstract existence theorem

We consider a family of Banach spaces $(X_s)_{s\in[0,1]}$ satisfying for $0\leq s'\leq s\leq 1$

$$X_s \subset X_{s'}, \tag{2.6}$$

$$\|f\|_{X_{s'}} \leq \|f\|_{X_s}; \tag{2.7}$$

that is, the natural embedding $X_s\subset X_{s'}$ for $s'\leq s$ is of norm less than 1.

We are concerned with an abstract Cauchy problem

$$\partial_x U(x) = G(x)U(x), \quad -1\leq x\leq 1,$$
$$U(0)=U^0,$$

where $U^0\in X_1$ and $\big(G(x)\big)_{x\in[-1,1]}$ is a familly of possibly *nonlinear* operators.

Our first result is a global wellposedness result. It extends the abstract result in [2,20,21] which gives solely *local* solutions.

Theorem 2.2. *For any $\varepsilon\in(0,1/4)$, there exists a constant $D>0$ such that for any family $(G(x))_{x\in[-1,1]}$ of nonlinear maps from X_s to $X_{s'}$ for $0\leq s'<s\leq 1$ satisfying*

$$\|G(x)U\|_{X_{s'}} \leq \frac{\varepsilon}{s-s'}\|U\|_{X_s} \tag{2.8}$$

$$\|G(x)U - G(x)V\|_{X_{s'}} \leq \frac{\varepsilon}{s-s'}\|U-V\|_{X_s} \tag{2.9}$$

for $0\leq s'<s\leq 1$, $x\in[-1,1]$ and $U,\ V\in X_s$ with $\|U\|_{X_s}\leq D$, $\|V\|_{X_s}\leq D$, there exists $\eta>0$ so that for any $U^0\in X_1$ with $\left\|U^0\right\|_{X_1}\leq\eta$, there exists a solution $U\in C([-1,1],X_{s_0})$ for some $s_0\in(0,1)$ to the integral equation

$$U(x) = U^0 + \int_0^x G(\tau)U(\tau)d\tau. \tag{2.10}$$

Moreover, we have the estimate

$$\|U(x)\|_{X_s} \leq C_1\left(1-\frac{\lambda|x|}{a_\infty(1-s)}\right)^{-1}\left\|U^0\right\|_{X_1}, \quad \text{for } 0\leq s<1,\ |x|<\frac{a_\infty}{\lambda}(1-s),$$

where $\lambda\in(0,1)$, $a_\infty\in(\lambda,1)$ and $C_1>0$ are some constants. In particular, we have

$$\|U(x)\|_{X_s} \leq C_1\left(1-\frac{2}{\frac{a_\infty}{\lambda}+1}\right)^{-1}\left\|U^0\right\|_{X_1}, \quad \text{for } 0\leq s\leq s_0=\frac{1}{2}(1-\frac{\lambda}{a_\infty}),\ |x|\leq 1.$$

If, in addition, we assume that

$$\textit{for all } U_0\in X_s \textit{ with } \|U_0\|_{X_s}\leq D,\ \textit{the map } \tau\in[-1,1]\to G(\tau)U_0\in X_{s'} \quad \textit{is continuous}, \tag{2.11}$$

then U is solution in the classical sense of

$$\begin{cases} \partial_x U(x) &= G(x)U(x), \quad -1\leq x\leq 1,\\ U(0) &= U^0. \end{cases} \tag{2.12}$$

We prove first the existence of a solution of (2.10) on an interval $[-(1-\delta), 1-\delta]$, where $\delta \in (0,1)$. Next, we use a scaling argument to obtain a solution of (2.10) for $x \in [-1,1]$.

Consider a sequence of numbers $(a_k)_{k\geq 0}$ satisfying the following properties (the existence of such a sequence is proved in Lemma 2.4, see below):

(i) $a_0 = 1$;
(ii) $(a_k)_{k\geq 0}$ is a decreasing sequence converging to $a_\infty > 1-\delta$;
(iii) $\sum_{i=0}^{\infty} \frac{(4\varepsilon)^i}{1-\frac{a_{i+1}}{a_i}} < +\infty$.

Next, we pick η small enough so that $\eta \sum_{i=0}^{\infty} \frac{(4\varepsilon)^i}{1-\frac{a_{i+1}}{a_i}} < D$.

We define, for $k \in \mathbb{N} \cup \{\infty\}$, the space $Y_k = \{U \in \bigcap_{0\leq s<1} C(-a_k(1-s), a_k(1-s), X_s);\ \|U\|_{Y_k} < \infty\}$ with the norm

$$\|U\|_{Y_k} := \sup_{\substack{|x|<a_k(1-s)\\ 0\leq s<1}} \|U(x)\|_{X_s} \left(1 - \frac{|x|}{a_k(1-s)}\right) \qquad \text{if } k \in \mathbb{N}, \tag{2.13}$$

$$\|U\|_{Y_\infty} := \sup_{\substack{|x|<a_\infty(1-s)\\ 0\leq s<1}} \|U(x)\|_{X_s} \left(1 - \frac{|x|}{a_\infty(1-s)}\right) \qquad \text{if } k = \infty. \tag{2.14}$$

Clearly, Y_k for $k < \infty$ and Y_∞ are Banach spaces (see [20,21]). Note that for $|x| < a_k(1-s)$, $0 \leq s < 1$, we have that $1 - \frac{|x|}{a_k(1-s)} \in (0,1]$. Note also that we have $Y_k \subset Y_{k+1}$ and $\|U\|_{Y_{k+1}} \leq \|U\|_{Y_k}$, for the sequence $(a_k)_{k\in\mathbb{N}}$ is decreasing.

Proposition 2.3. *For any $\varepsilon \in (0,1/4)$, any $\delta \in (0,1)$ and any G and D as in (2.9) and (2.11), there exists some numbers $a_\infty > 1-\delta$ and $\eta > 0$ such that for any $U^0 \in X_1$ with $\left\|U^0\right\|_{X_1} \leq \eta$, there exists a unique solution for $x \in (-a_\infty, a_\infty)$ to (2.10) in the space Y_∞. Moreover, we have the estimate*

$$\|U(x)\|_{X_s} \leq C_1 \left(1 - \frac{|x|}{a_\infty(1-s)}\right)^{-1} \left\|U^0\right\|_{X_1}, \quad \text{for } 0 \leq s < 1,\ |x| < a_\infty(1-s),$$

where $C_1 > 0$ is a constant.

Proof of Proposition 2.3. We follow closely the proof of [8], taking care of the choice of the constants and of the time of existence.

We want to define a sequence $(U_k)_{k\geq 0}$ by the relations

$$U_0 = 0, \quad U_{k+1} = TU_k \quad \text{for } k \in \mathbb{N}$$

where

$$(TU)(x) = U^0 + \int_0^x G(\tau)U(\tau)d\tau.$$

Note that $U_1(x) = (TU_0)(x) = U^0$ for $|x| < 1$. Introduce

$$V_k := U_{k+1} - U_k, \quad k \in \mathbb{N}.$$

We prove by induction on $k \in \mathbb{N}$ the following statements (that contain the fact that the sequence $(U_k)_{k\in\mathbb{N}}$ is indeed well defined):

$$\lambda_k := \|V_k\|_{Y_k} \leq (4\varepsilon)^k \eta, \tag{2.15}$$

$$\|U_{k+1}(x)\|_{X_s} \leq \sum_{i=0}^{k} \frac{\lambda_i}{1 - \frac{a_{i+1}}{a_i}} \leq D \ \text{ for } |x| < a_{k+1}(1-s), \tag{2.16}$$

so that $G(x)U_{k+1}(x)$ is well defined in $X_{s'}$ for $|x| \le a_{k+1}(1-s)$.

Let us first check that (2.15)-(2.16) are valid for $k=0$. For (2.15), we have that

$$\lambda_0 = \|U_1 - U_0\|_{Y_0} \le \|U^0\|_{X_1} \le \eta.$$

For (2.16), we notice that

$$\|U_1\|_{X_s} = \|U^0\|_{X_s} \le \|U^0\|_{X_1} \le \eta \le \frac{\eta}{1-a_1} \le D.$$

Assume that (2.15)-(2.16) are true up to the rank k. Let us check that they are also true at the rank $k+1$.

Take s and x (for simplicity, we assume $x \ge 0$) so that $0 \le x < a_{k+1}(1-s)$. For any $0 \le \tau \le x$, (2.16) gives $\max\left(\|U_{k+1}(\tau)\|_s, \|U_k(\tau)\|_s\right) \le D$ (recall $a_{k+1} \le a_k$). In particular, we can apply (2.9) replacing s' by s and s by $s(\tau) = \frac{1}{2}\left(1 + s - \frac{\tau}{a_{k+1}}\right)$, obtaining

$$\|G(\tau)U_{k+1}(\tau) - G(\tau)U_k(\tau)\|_{X_s} \le \frac{\varepsilon}{s(\tau)-s} \|U_{k+1}(\tau) - U_k(\tau)\|_{X_{s(\tau)}}$$

Note that we have indeed $0 \le s < s(\tau) < 1$. Next

$$\begin{aligned}
\|V_{k+1}(x)\|_{X_s} &= \|(TU_{k+1})(x) - (TU_k)(x)\|_{X_s} \\
&\le \int_0^x \|G(\tau)U_{k+1}(\tau) - G(\tau)U_k(\tau)\|_{X_s}\, d\tau \\
&\le \int_0^x \frac{\varepsilon}{s(\tau)-s} \|U_{k+1}(\tau) - U_k(\tau)\|_{X_{s(\tau)}}\, d\tau \\
&\le \varepsilon \|V_k\|_{Y_k} \int_0^x \frac{a_{k+1}}{s(\tau)-s} \left(\frac{1-s(\tau)}{a_{k+1}(1-s(\tau)) - \tau}\right) d\tau
\end{aligned}$$

where we have used the fact that $s(\tau)$ satisfies $\tau < a_{k+1}(1-s(\tau))$ (for $a_{k+1}(1-s(\tau)) - \tau = \frac{1}{2}(a_{k+1}(1-s) - \tau) > 0$) and $0 < s(\tau) < 1$, so that with (2.13)

$$\|V_k(\tau)\|_{X_{s(\tau)}} \le \left(1 - \frac{|\tau|}{a_{k+1}(1-s(\tau))}\right)^{-1} \|V_k\|_{Y_{k+1}} \le \left(1 - \frac{|\tau|}{a_{k+1}(1-s(\tau))}\right)^{-1} \|V_k\|_{Y_k}. \tag{2.17}$$

Let us go back to the estimate of the integral. To simplify the notations, we denote $A = a_{k+1}(1-s)$ and recall $0 \le \tau \le x < A$. We have

$$\begin{aligned}
\int_0^x \frac{a_{k+1}(1-s(\tau))}{(s(\tau)-s)(a_{k+1}(1-s(\tau)) - \tau)} d\tau &= 2a_{k+1} \int_0^x \frac{A+\tau}{(A-\tau)^2} d\tau \\
&\le \int_0^x \frac{4a_{k+1}A}{(A-\tau)^2} d\tau = a_{k+1} \left.\frac{4A}{A-\tau}\right]_0^x \le a_{k+1} \frac{4A}{A-x}.
\end{aligned}$$

So, recalling $\frac{A}{A-x} = \left(1 - \frac{x}{A}\right)^{-1} = \left(1 - \frac{|x|}{a_{k+1}(1-s)}\right)^{-1}$, we have obtained

$$\|V_{k+1}(x)\|_s \left(1 - \frac{|x|}{a_{k+1}(1-s)}\right) \le 4a_{k+1}\varepsilon \|V_k\|_{Y_k}.$$

So, we have proved that

$$\|V_{k+1}\|_{Y_{k+1}} \le 4a_0\varepsilon \|V_k\|_{Y_k}, \tag{2.18}$$

and hence $\lambda_{k+1} \le 4a_0\varepsilon\lambda_k$. This yields (2.15) at rank $k+1$.

Let us proceed with the proof of (2.16) at rank $k+1$.

Since $U_{k+2} = U_{k+1} + V_{k+1}$, we only need to prove $\|V_{k+1}(x)\|_s \leq \frac{\lambda_{k+1}}{1-\frac{a_{k+2}}{a_{k+1}}}$ for $|x| < a_{k+2}(1-s)$. This is obtained by noticing that

$$\|V_{k+1}(x)\|_s \leq \left(1 - \frac{|x|}{a_{k+1}(1-s)}\right)^{-1} \|V_{k+1}\|_{Y_{k+1}} \leq \left(1 - \frac{a_{k+2}}{a_{k+1}}\right)^{-1} \lambda_{k+1}$$

since $|x| < a_{k+2}(1-s)$. The proof by induction of (2.15)-(2.16) is complete.

We are now in a position to prove the existence of a solution to (2.12). Let us introduce the function $U_\infty := \lim_{k\to+\infty} U_k = \sum_{k=0}^{+\infty} V_k$. Note that the convergence of the series is normal in Y_∞. Indeed,

$$\sum_{k=0}^{+\infty} \|V_k\|_{Y_\infty} \leq \sum_{k=0}^{+\infty} \|V_k\|_{Y_k} \leq \sum_{k=0}^{+\infty} \lambda_k < +\infty.$$

Note also that (2.16) remains true for U_∞ for $|x| < a_\infty(1-s)$, so that $G(\tau)U_\infty$ is well defined.

Let us prove that U_∞ is indeed a solution of (2.10). Using the fact that $U_{k+1} - TU_k = 0$, we have

$$U_\infty - (TU_\infty)(x) = U_\infty - U_{k+1} + \int_0^x [G(\tau)U_k(\tau) - G(\tau)U_\infty(\tau)]\,d\tau$$

where all the terms in the equation are in Y_∞. The same estimates as before yield for $0 \leq s < 1$ and $|x| < a_\infty(1-s)$

$$\| \int_0^x [G(\tau)U_k(\tau) - G(\tau)U_\infty(\tau)]\,d\tau \|_{X_s} \leq 4a_\infty \varepsilon \|U_k - U_\infty\|_{Y_\infty} \left(1 - \frac{|x|}{a_\infty(1-s)}\right)^{-1},$$

and hence

$$\|U_\infty - TU_\infty\|_{Y_\infty} \leq \|U_\infty - U_{k+1}\|_{Y_\infty} + 4a_\infty\varepsilon \|U_k - U_\infty\|_{Y_\infty} \to 0$$

as $k \to +\infty$. Thus U_∞ is a solution of (2.10).

Let us prove the uniqueness of the solution of (2.10) in the same space Y_∞. Assume that U and $\widetilde{U}$ are two solutions of (2.10) in Y_∞. Pick $\lambda \in (0,1)$, and let $a'_0 = 1$, $a'_k = \lambda a_k$ for $k \in \mathbb{N}^* \cup \{\infty\}$. Note that (i), (ii) and (iii) are still valid for the a'_k. Denote by Y'_k, $k \in \mathbb{N} \cup \{\infty\}$, the space associated with a'_k. Note that $a'_k < a_\infty$ and hence $Y_\infty \subset Y'_k$ for $k \gg 1$. Then we have by the same computations as above that

$$\|U - \widetilde{U}\|_{Y'_k} \leq (4\varepsilon)^k \|U - \widetilde{U}\|_{Y'_0}$$

which yields

$$\|U - \widetilde{U}\|_{Y'_\infty} \leq \lim_{k\to+\infty} \|U - \widetilde{U}\|_{Y'_k} = 0.$$

Thus $U(x) = \widetilde{U}(x)$ for $|x| < a'_\infty = \lambda a_\infty$, with λ as close to 1 as desired. □

It remains to prove the existence of the sequence $(a_k)_{k\geq 0}$. This is done in the next lemma.

Lemma 2.4. *There exists a sequence $(a_k)_{k\in\mathbb{N}}$ satisfying (i), (ii) and (iii).*

Proof. We denote $C_0 = 4\varepsilon < 1$ and we require $\sum_{i=0}^\infty \frac{C_0^i}{1-\frac{a_{i+1}}{a_i}} < +\infty$. Picking $a_0 = 1$ and $\gamma > 0$ small enough, we define the sequence $(a_k)_{k\in\mathbb{N}}$ by induction by setting

$$a_{k+1} = a_k \left(1 - \frac{\gamma}{(1+k)^2}\right), \quad k \in \mathbb{N}.$$

The sequence $(a_k)_{k\in\mathbb{N}}$ is clearly decreasing, $\frac{C_0^i}{1-\frac{a_{i+1}}{a_i}} = \gamma^{-1}(1+i)^2 C_0^i$, and hence $\sum_{i=0}^{\infty} \frac{C_0^i}{1-\frac{a_{i+1}}{a_i}} < +\infty$, for $C_0 < 1$. Finally, $b_k = \ln(a_k)$ converges to $\sum_{k=0}^{\infty} \ln\left(1 - \frac{\gamma}{(1+k)^2}\right) \geq -2\gamma \sum_{k=0}^{\infty} \frac{1}{(1+k)^2}$ for γ small enough. In particular, $a_\infty \geq e^{-2\gamma\zeta(2)}$ which can be made greater than $1-\delta$ for γ small. $\square$

Let us complete the proof of Theorem 2.2 by using a scaling argument. Pick any number $\lambda \in (0,1)$ with

$$\frac{\varepsilon}{\lambda} < \frac{1}{4}.$$

Let $\delta = 1-\lambda \in (0,1)$ and pick $a_\infty \in (\lambda, 1)$. Introduce the new variables $\tilde{x} := \lambda x \in [-\lambda, \lambda] = [-(1-\delta), 1-\delta]$ for $x \in [-1,1]$, and the new unknown

$$\tilde{U}(\tilde{x}) := U(x) = U(\lambda^{-1}\tilde{x}).$$

Then $\tilde{U}$ should solve

$$\tilde{U}(\tilde{x}) = U^0 + \int_0^{\tilde{x}} \tilde{G}(\tau)\tilde{U}(\tau)d\tau, \tag{2.19}$$

where

$$\tilde{G}(\tilde{x}) := \begin{cases} \lambda^{-1}G(\lambda^{-1}\tilde{x}) & \text{if } \tilde{x} \in [-\lambda, \lambda], \\ \lambda^{-1}G(1) & \text{if } \tilde{x} \in [\lambda, 1], \\ \lambda^{-1}G(-1) & \text{if } \tilde{x} \in [-1, -\lambda]. \end{cases}$$

Then $\left(\tilde{G}(\tilde{x})\right)_{\tilde{x}\in[-1,1]}$ is a family of nonlinear maps from X_s to $X_{s'}$ for $0 \leq s' < s \leq 1$ satisfying for $0 \leq s' < s \leq 1$, $\tilde{x} \in [-1,1]$ and $U, V \in X_s$ with $\|U\|_{X_s} \leq D$, $\|V\|_{X_s} \leq D$

$$\|\tilde{G}(\tilde{x})U\|_{X_{s'}} \leq \frac{\tilde{\varepsilon}}{s-s'}\|U\|_{X_s}$$
$$\|\tilde{G}(\tilde{x})U - \tilde{G}(\tilde{x})V\|_{X_{s'}} \leq \frac{\tilde{\varepsilon}}{s-s'}\|U-V\|_{X_s}$$

where $\tilde{\varepsilon} = \varepsilon/\lambda \in (0, 1/4)$. We infer from Proposition 2.3 the existence of a solution $\tilde{U}$ of (2.19). A simple change of variables shows that the function U defined for $x \in [-1,1]$ by $U(x) = \tilde{U}(\tilde{x})$ solves (2.10).

Finally, assume that the continuity conditions (2.9) and (2.11) hold. We notice that $\tilde{x} \mapsto \tilde{G}(\tilde{x})\tilde{U}(\tilde{x})$ is continuous from $(-a_\infty(1-s), a_\infty(1-s))$ to $X_{s'}$ for all $0 \leq s' < s < 1$. We infer that $\tilde{U}$ is a classical solution of the Cauchy problem

$$\partial_{\tilde{x}}\tilde{U} = \tilde{G}(\tilde{x})\tilde{U}(\tilde{x}), \quad \tilde{U}(0) = U^0$$

for $x \in [-(1-\delta), 1-\delta] = [-\lambda, \lambda]$, and that is satisfies

$$\tilde{U} \in \bigcap_{0\leq s<1} C(-a_\infty(1-s), a_\infty(1-s), X_s).$$

Then the function U defined for $x \in [-1,1]$ by $U(x) = \tilde{U}(\tilde{x})$ solves

$$\partial_x U = G(x)U(x), \quad U(0) = U^0,$$

for $x \in [-1,1]$, and it satisfies $U \in \bigcap_{0\leq s<1} C(-\frac{a_\infty}{\lambda}(1-s), \frac{a_\infty}{\lambda}(1-s), X_s)$ and

$$\|U(x)\|_{X_s} \leq C_1 \left(1 - \frac{\lambda|x|}{a_\infty(1-s)}\right)^{-1} \left\|U^0\right\|_{X_1}, \quad \text{for } 0 \leq s < 1, \ |x| < \frac{a_\infty}{\lambda}(1-s).$$

The proof of Theorem 2.2 is complete. $\square$

2.3. Gevrey type functional spaces

We follow closely [9,25].

2.4. Definitions

We define several spaces of Gevrey λ functions for $\lambda > 1$. For our application to the heat equation, we shall take $\lambda = 2$, but for the moment we stay in the generality. Introduce

$$\Gamma_\lambda(k) = 2^{-5}(k!)^\lambda(1+k)^{-2},$$

and let Γ denote the Gamma function of Euler. It is increasing on $[2, +\infty)$.

We also introduce a variant of those functions with a parameter $a \in \mathbb{R}$ (a is not necessarily an integer):

$$\Gamma_{\lambda,a}(k) = 2^{-5}(\Gamma(k+1-a))^\lambda(1+k)^{-2}, \quad \text{for } k \in \mathbb{N} \text{ s.t. } k > |a|+1, \tag{2.20}$$

$$\Gamma_{\lambda,a}(k) = \Gamma_\lambda(k), \quad \text{for } k \in \mathbb{N} \text{ s.t. } 0 \le k \le |a|+1. \tag{2.21}$$

Clearly, $\Gamma_{\lambda,0} = \Gamma_\lambda$. Note that for $k > |a|+1$, we have $k+1-a \ge 2$, so we are in an interval where Γ is increasing. Thus we have for all $k \in \mathbb{N}$

$$\Gamma_{\lambda,a}(k) \le \Gamma_\lambda(k), \quad \text{if } a \ge 0 \tag{2.22}$$

$$\Gamma_\lambda(k) \le \Gamma_{\lambda,a}(k), \quad \text{if } a \le 0. \tag{2.23}$$

For any $L > 0$, we consider the space of functions $u \in C^\infty(K)$ (where $K = [t_1, t_2]$ with $-\infty < t_1 < t_2 < \infty$) such that

$$|u|_{L,a} := \sup\left\{ \frac{\left|u^{(k)}(t)\right|}{L^{|k-a|}\Gamma_{\lambda,a}(k)}, t \in K, k \in \mathbb{N} \right\} < \infty.$$

Note that for $a = 0$, we recover the spaces defined earlier in [25], and $|u|_{L,0} = |u|_L$.

Definition 1. Yamanaka [25] defined the norms

$$\|u\|_L := \max\left\{ 2^6 \|u\|_{L^\infty(K)}, 2^3 L^{-1} \left|u'\right|_L \right\},$$

and similarly we define for $a \in \mathbb{R}$

$$\|u\|_{L,a} := \max\left\{ 2^6 \|u\|_{L^\infty(K)}, 2^3 L^{-1} \left|u'\right|_{L,a} \right\}.$$

We denote by G^λ_L (resp. $G^\lambda_{L,a}$) the (Banach) space of functions $u \in C^\infty(K)$ such that $\|u\|_L < \infty$ (resp. $\|u\|_{L,a} < \infty$).

The space $G^\lambda_{L,a}$ is supposed to "represent" the space of functions Gevrey λ with radius L^{-1} with a derivatives. Roughly, we may think that $u \in G^\lambda_{L,a}$ if $D^a u \in G^\lambda_L$, even if it is not completely true if $a \notin \mathbb{N}$.

Note that, as a direct consequence of (2.22)-(2.23), we have the embeddings $G^\lambda_{L,a} \subset G^\lambda_L$ if $a \ge 0$, $G^\lambda_L \subset G^\lambda_{L,a}$ if $a \le 0$, together with the inequalities

$$\|u\|_L \le \max(L^a, L^{-a}) \|u\|_{L,a} \quad \text{if } a \ge 0, \tag{2.24}$$

$$\|u\|_{L,a} \le \max(L^a, L^{-a}) \|u\|_L \quad \text{if } a \le 0. \tag{2.25}$$

Furthermore, for any $a \in \mathbb{R}$ and $0 < L < L'$, we have the embedding $G^\lambda_{L,a} \subset G^\lambda_{L',a}$ with

$$\|u\|_{L',a} \le \|u\|_{L,a}. \tag{2.26}$$

The following result [25, Theorem 5.4] will be used several times in the sequel.

Lemma 2.5. *(Algebra property) [25, Theorem 5.4]*

$$\|uv\|_L \le \|u\|_L \|v\|_L \quad \forall u, v \in G^\lambda_L. \tag{2.27}$$

2.5. Cost of derivation

The following result is a variant of Proposition 2.3 of Kawagishi-Yamanaka [9], where the spaces we consider contain some non-integer "derivatives".

Lemma 2.6 *(Cost of derivatives for Gevrey spaces containing derivatives). Let $\lambda > 0$ and $\delta > 0$. Let $q \in \mathbb{N}$ and $a, b \in \mathbb{R}$ with $d = q - a + b > 0$. Then there exists some number $C = C(\lambda, \delta, a, b, q) > 0$ such that for all $L > 0$, all $\alpha > 1$, and all $u \in G^{\lambda}_{L,a}$, we have*

$$\left|u^{(q)}\right|_{\alpha L,b} \leq \left(C(L^{-d} + L^d) + (1+\delta)\alpha^b L^d \left(\frac{\lambda d}{e \ln \alpha}\right)^{\lambda d}\right) |u|_{L,a} \tag{2.28}$$

and hence

$$\left\|u^{(q)}\right\|_{\alpha L,b} \leq \left(C(L^{-d} + \langle L\rangle^C) + (1+\delta)\alpha^b L^d \left(\frac{\lambda d}{e \ln \alpha}\right)^{\lambda d}\right) \|u\|_{L,a}\,. \tag{2.29}$$

Proof. The main tool will be the asymptotic of the Gamma function $\frac{\Gamma(x+d)}{\Gamma(x)} \sim x^d$ as $x \to +\infty$, which follows at once from Stirling's formula (see [23])

$$\lim_{x \to +\infty} \frac{\Gamma(x+1)}{(x/e)^x \sqrt{2\pi x}} = 1.$$

In particular, for any $\delta > 0$, there exists a number $N = N(\lambda, \delta, a, b, q)$ such that for all $k \in \mathbb{N}$ with $k \geq N$,

$$\left(\frac{\Gamma(k+1+q-a)}{\Gamma(k+1-b)}\right)^{\lambda} \leq (1+\delta)k^{\lambda d}.$$

We can also assume that $k \geq N$ implies $k + q > |a| + 1$, and $k > |b| + 1$, so that $\Gamma_{\lambda,a}(k+q)$ and $\Gamma_{\lambda,b}(k)$ are given by (2.20). Note that we always have $\frac{(1+k)^2}{(1+k+q)^2} \leq 1$ if $k \in \mathbb{N}$, for $q \geq 0$.

Let $k \in \mathbb{N}$. If $k \geq N$, we have

$$\begin{aligned}
\frac{|u^{(k+q)}(t)|}{(\alpha L)^{k-b}\Gamma_{\lambda,b}(k)} &\leq \frac{|u|_{L,a}\, L^{k+q-a}\Gamma_{\lambda,a}(k+q)}{(\alpha L)^{k-b}\Gamma_{\lambda,b}(k)} \\
&\leq \frac{|u|_{L,a}\, L^d}{\alpha^{k-b}} \left(\frac{\Gamma(k+1+q-a)}{\Gamma(k+1-b)}\right)^{\lambda} \\
&\leq (1+\delta)\frac{|u|_{L,a}\, L^d}{\alpha^{k-b}} k^{\lambda d} \\
&\leq (1+\delta)\alpha^b\, |u|_{L,a}\, L^d \sup_{t \geq 0}(\alpha^{-t} t^{\lambda d}) \\
&\leq (1+\delta)\alpha^b\, |u|_{L,a}\, L^d \left(\frac{\lambda d}{e \ln(\alpha)}\right)^{\lambda d},
\end{aligned}$$

where we used the fact that $\sup_{t \geq 0}(\alpha^{-t} t^{\lambda d}) = \left(\frac{\lambda d}{e \ln(\alpha)}\right)^{\lambda d}$, where $\alpha > 1$ and $\lambda d > 0$.

If $k \leq N$, we still have

$$\frac{|u^{(k+q)}(t)|}{(\alpha L)^{|k-b|}\Gamma_{\lambda,b}(k)} \leq \frac{|u|_{L,a}\, L^{|k+q-a|-|k-b|}}{\alpha^{|k-b|}} \frac{\Gamma_{\lambda,a}(k+q)}{\Gamma_{\lambda,b}(k)}.$$

Noticing that $|k+q-a| - |k-b| = |k-b+d| - |k-b| \in [-d, d]$, $\alpha^{-|k-b|} \leq 1$, $\frac{\Gamma_{\lambda,a}(k+q)}{\Gamma_{\lambda,b}(k)} \leq C(\lambda, \delta, a, b, q)$ for $0 \leq k \leq N(\lambda, \delta, a, b, q)$, we infer that

$$\frac{|u^{(k+q)}(t)|}{(\alpha L)^{k-b}\Gamma_{\lambda,b}(k)} \leq C(L^{-d} + L^d)|u|_{L,a}$$

and (2.28) follows.

Let us now prove (2.29). The estimate

$$\left\|u^{(q)}\right\|_{L^\infty(K)} \le L^{|q-1-a|}\Gamma_{\lambda,a}(q-1)\left|u'\right|_{L,a} \le C(\lambda,q,b,a)L^{|q-1-a|+1}\|u\|_{L,a} \le C\langle L\rangle^C\|u\|_{L,a}$$

is valid for any $q \ge 1$, and it is obvious for $q = 0$. On the other hand, we infer from (2.28) that

$$\left|u^{(q+1)}\right|_{\alpha L,b} \le \left(C(L^{-d}+L^d) + (1+\delta)\alpha^b L^d \left(\frac{\lambda d}{e\ln\alpha}\right)^{\lambda d}\right)\left|u'\right|_{L,a}$$

Using the definition of $\|\cdot\|_{\alpha L,b}$, we obtain at once (2.29). □

2.6. Application to the semilinear heat equation

We aim to solve the system:

$$\partial_x^2 u = \partial_t u - f(x,u,\partial_x u), \qquad x \in [-1,1],\ t \in [0,T], \tag{2.30}$$

$$u(0,t) = \bar{u_0}(t), \qquad t \in [0,T], \tag{2.31}$$

$$\partial_x u(0,t) = \bar{u_1}(t), \qquad t \in [0,T]. \tag{2.32}$$

This is equivalent to solve the first order system

$$\partial_x U = AU + F(x,U), \tag{2.33}$$

$$U(0) = U_0, \tag{2.34}$$

with $U = (u, \partial_x u)$, $U_0 = (\bar{u_0}, \bar{u_1})$, $A = \begin{pmatrix} 0 & 1 \\ \partial_t & 0 \end{pmatrix}$, and $F(x,(u_0,u_1)) = \begin{pmatrix} 0 \\ -f(x,u_0,u_1) \end{pmatrix}$.

Let $L > 0$. We define the space $\mathcal{X}_L := \{U = (u_0,u_1) \in G^2_{L,\frac{1}{2}} \times G^2_L\}$, with

$$\|U\|_{\mathcal{X}_L} = \|(u_0,u_1)\|_{\mathcal{X}_L} = \|u_0\|_{L,1/2} + \|u_1\|_L\,,$$

where the norms are those defined in Definition 1 with $\lambda = 2$. (Note that u_0 is more regular than u_1 of one half derivative.) In particular, we have that

$$\|AU\|_{\mathcal{X}_L} = \|u_1\|_{L,1/2} + \|\partial_t u_0\|_L\,.$$

In the following result, L_1 stands for the inverse of the radius of the initial datum.

Theorem 2.7. *Pick any $L_1 < 1/4$. Then there exists a number $\eta > 0$ such that for any $U_0 \in \mathcal{X}_{L_1}$ with $\|U_0\|_{\mathcal{X}_{L_1}} \le \eta$, there exists a solution to (2.33)-(2.34) for $x \in [-1,1]$ in $C([-1,1],\mathcal{X}_{L_0})$ for some $L_0 > 0$.*

Proof of Theorem 2.7. In order to apply Theorem 2.2, we introduce a scale of Banach spaces $(X_s)_{s\in[0,1]}$ as follows: for $s \in [0,1]$ and $U = (u_0,u_1)$, we set $X_s = \mathcal{X}_{L(s)}$ and

$$\|U\|_{X_s} = e^{-\tau(1-s)}\|(u_0,u_1)\|_{\mathcal{X}_{L(s)}} = e^{-\tau(1-s)}\left(\|u_0\|_{L(s),1/2} + \|u_1\|_{L(s)}\right), \tag{2.35}$$

where

$$L(s) = e^{r(1-s)}L_1 \tag{2.36}$$

and the parameters $r > 0$, $\tau > 0$ will be chosen thereafter. Note that (2.7) is satisfied because of (2.26) and the fact that $L(s') > L(s)$ for $s' < s$. Actually, we have even that

$$\|U\|_{X_{s'}} \le e^{-\tau(s-s')}\|U\|_{X_s}\,. \tag{2.37}$$

Lemma 2.8 and Lemma 2.9 (see below) will allow us to select the parameters so that $G = A + F$ satisfies the assumptions of Theorem 2.2.

Lemma 2.8. *Let $L_1 < 1/4$. There exist $\tau_0 > 0$ large enough and $\varepsilon_0 < 1/4$ such that we have the estimates*

$$\|AU\|_{X_{s'}} \le \frac{\varepsilon_0}{s-s'}\|U\|_{X_s} \quad \forall U \in X_s,$$

for all $\tau \ge \tau_0$ and all s, s' with $0 \le s' < s \le 1$.

Proof. By assumption, we have $L_1^{1/2}/2 < 1/4$ and we can pick $\delta > 0$ so that

$$(1+\delta)\frac{L_1^{1/2}}{2} < \frac{1}{4}. \tag{2.38}$$

Applying Lemma 2.6 with $\lambda = 2$ and $\delta > 0$ as in (2.38), and with $q = 0$, $b = 1/2$, $a = 0$ (respectively $q = 1$, $b = 0$, $a = 1/2$), so that $\lambda d = 1$ in both cases, we obtain the existence of some number $C = C_\delta > 0$ such that

$$\begin{aligned}\|AU\|_{\mathcal{X}_{\alpha L}} &= \|u_1\|_{\alpha L,1/2} + \|\partial_t u_0\|_{\alpha L}\\ &\le \left(C(L^{-d} + \langle L\rangle^C) + \frac{1+\delta}{e\ln\alpha}(\alpha L)^{1/2}\right)\left(\|u_1\|_L + \|u_0\|_{L,1/2}\right)\\ &\le \left(C(L^{-d} + \langle L\rangle^C) + \frac{1+\delta}{e\ln\alpha}(\alpha L)^{1/2}\right)\|U\|_{\mathcal{X}_L}\end{aligned} \tag{2.39}$$

uniformly for $\alpha > 1$ and $L > 0$. So, (2.39) applied with $L = L(s)$, $\alpha = \frac{L(s')}{L(s)} = e^{r(s-s')} > 1$ becomes for $s' < s$ (we also use $L_1 \le C$, for $0 < L_1 < 1/4$)

$$\begin{aligned}\|AU\|_{X_{s'}} &\le e^{-\tau(s-s')}\left(C(L_1^{-1} + e^{rC}) + (1+\delta)\frac{e^{r\frac{1-s'}{2}}L_1^{1/2}}{er(s-s')}\right)\|U\|_{X_s}\\ &\le \left(Ce^{-\tau(s-s')}(L_1^{-1} + e^{rC}) + (1+\delta)\frac{e^{\frac{r}{2}}L_1^{1/2}}{er(s-s')}\right)\|U\|_{X_s}\\ &\le \left(\frac{e^{-1}}{\tau(s-s')}C(L_1^{-1} + e^{rC}) + (1+\delta)\frac{e^{\frac{r}{2}}L_1^{1/2}}{er(s-s')}\right)\|U\|_{X_s}\end{aligned}$$

where we have used the fact that

$$e^{-\tau(s-s')} = \frac{\tau(s-s')e^{-\tau(s-s')}}{\tau(s-s')} \le \frac{e^{-1}}{\tau(s-s')} \tag{2.40}$$

for $te^{-t} \le e^{-1}$ for $t \ge 0$. Minimizing the constant in the r.h.s. leads to the choice $r = 2$. (Note that the initial space $X_1 = \mathcal{X}_{L_1}$ is independent on the choice of r.) We arrive to the estimate

$$\|AU\|_{X_{s'}} \le \left(\frac{Ce^{-1}(L_1^{-1} + e^{2C})}{\tau} + (1+\delta)\frac{L_1^{1/2}}{2}\right)\frac{1}{s-s'}\|U\|_{X_s}\cdot$$

By (2.38), we can then pick τ_0 large enough so that

$$\varepsilon_0 := \frac{Ce^{-1}(L_1^{-1} + e^{2C})}{\tau_0} + (1+\delta)\frac{L_1^{1/2}}{2} < \frac{1}{4}\cdot$$

The proof of Lemma 2.8 is complete. □

Lemma 2.9. *Let f be as in (1.5)-(1.8), and let $F(x,U) = \begin{pmatrix} 0 \\ -f(x,u_0,u_1)\end{pmatrix}$ for $x \in [-1,1]$ and $U = (u_0,u_1) \in L^\infty(K)^2$ with $\sup(\|u_0\|_{L^\infty(K)}, \|u_1\|_{L^\infty(K)}) < 4$. Let $r = 2$, $L_1 > 0$, and $\varepsilon > 0$. Then there exists $\tau_0 > 0$ (large enough) such that for $\tau \ge \tau_0$, there exists $D > 0$ (small enough) such that we have the estimates*

$$\|F(x,U)\|_{X_{s'}} \le \frac{\varepsilon}{s-s'}\,\|U\|_{X_s} \tag{2.41}$$

$$\|F(x,U)-F(x,V)\|_{X_{s'}} \le \frac{\varepsilon}{s-s'}\,\|U-V\|_{X_s} \tag{2.42}$$

for $0 \le s' < s \le 1$, *and* $U=(u_0,u_1)\in X_s$, $V=(v_0,v_1)\in X_s$ *with*

$$\|U\|_{X_s} \le D,\ \ \|V\|_{X_s} \le D. \tag{2.43}$$

Finally, for $0\le s\le 1$ *and* $U=(u_0,u_1)\in X_s$ *with* $\|U\|_{X_s}\le D$, *the map* $x\in[-1,1]\to F(x,U)\in X_s$ *is continuous.*

Proof. Since (2.41) follows from (2.42), for $F(x,0)=0$, it is sufficient to prove (2.42). Pick $0\le s'<s\le 1$, $D>0$ and $U,V\in X_s$ satisfying (2.43). Then

$$\begin{aligned}\|F(x,U)-F(x,V)\|_{X_{s'}} &= \|-\begin{pmatrix}0\\ f(x,U)-f(x,V)\end{pmatrix}\|_{X_{s'}}\\ &= e^{-\tau(1-s')}\|f(x,u_0,u_1)-f(x,v_0,v_1)\|_{L(s')}\\ &\le e^{-\tau(1-s')}\sum_{p+q>0}\|A_{p,q}(x)[u_0^pu_1^q-v_0^pv_1^q]\|_{L(s')}\\ &\le e^{-\tau(1-s')}\sum_{p+q>0}|A_{p,q}(x)|\big(\|u_0^p-v_0^p\|_{L(s')}\|u_1^q\|_{L(s')}\\ &\qquad +\|v_0^p\|_{L(s')}\|u_1^q-v_1^q\|_{L(s')}\big)\end{aligned}$$

where we used the triangle inequality and Lemma 2.5. Note that, by (2.24), we have for a constant $C=C(L_1)\ge 1$ and any $0\le s'<1$

$$\|u_0\|_{L(s')}+\|u_1\|_{L(s')}\le C\,\|u_0\|_{L(s'),1/2}+\|u_1\|_{L(s')}\le Ce^{\tau(1-s')}\,\|U\|_{X_{s'}}\le CDe^{\tau}, \tag{2.44}$$

and similarly

$$\|v_0\|_{L(s')}+\|v_1\|_{L(s')}\le CDe^{\tau}.$$

Since, by Lemma 2.5,

$$\begin{aligned}\|u_0^p-v_0^p\|_{L(s')} &= \|(u_0-v_0)(u_0^{p-1}+u_0^{p-2}v_0+\cdots+v_0^{p-1})\|_{L(s')}\\ &\le \|u_0-v_0\|_{L(s')}\left(\|u_0\|_{L(s')}^{p-1}+\|u_0\|_{L(s')}^{p-2}\|v_0\|_{L(s')}+\cdots+\|v_0\|_{L(s')}^{p-1}\right)\\ &\le p(CDe^\tau)^{p-1}\|u_0-v_0\|_{L(s')},\end{aligned}$$

we infer that

$$\begin{aligned}\|F(x,U)-F(x,V)\|_{X_{s'}} \ &\le e^{-\tau(1-s')}\left(\sum_{p+q>0}|A_{p,q}(x)|p(CDe^\tau)^{p+q-1}\|u_0-v_0\|_{L(s')}\right.\\ &\qquad\left.+\sum_{p+q>0}|A_{p,q}(x)|q(CDe^\tau)^{p+q-1}\|u_1-v_1\|_{L(s')}\right)\\ &=: e^{-\tau(1-s')}(S_1+S_2).\end{aligned} \tag{2.45}$$

Let us estimate S_1. Set $M':=M/(1-b_2^{-1})$. Since

$$|A_{p,q}(x)|\le \frac{M}{b_0^pb_1^q}(1-\frac{|x|}{b_2})^{-1}\le \frac{M'}{b_0^pb_1^q}\qquad \text{for } |x|\le 1,$$

we have that

$$
\begin{aligned}
S_1 &\le \sum_{p>0} \frac{M'}{b_0} p \left(\frac{CDe^\tau}{b_0}\right)^{p-1} \sum_{q\ge 0}\left(\frac{CDe^\tau}{b_1}\right)^q \|u_0 - v_0\|_{L(s')} \\
&= \frac{M'}{b_0}\sum_{p\ge 0}(p+1)\left(\frac{CDe^\tau}{b_0}\right)^p (1-\frac{CDe^\tau}{b_1})^{-1}\|u_0-v_0\|_{L(s')} \\
&= \frac{M'}{b_0}(1-\frac{CDe^\tau}{b_0})^{-2}(1-\frac{CDe^\tau}{b_1})^{-1}\|u_0-v_0\|_{L(s')} \\
&\le \frac{8M'}{b_0}\|u_0-v_0\|_{L(s')} \\
&\le 2M'\|u_0-v_0\|_{L(s')}
\end{aligned}
$$

provided that

$$
D \le \min(\frac{b_0 e^{-\tau}}{2C}, \frac{b_1 e^{-\tau}}{2C}). \tag{2.46}
$$

Similarly, we can prove that

$$
S_2 \le 2M'\|u_1 - v_1\|_{L(s')}.
$$

Therefore, using (2.37), (2.40) and (2.44), we infer that

$$
\begin{aligned}
\|F(x,U)-F(x,V)\|_{X_{s'}} &\le 2M'e^{-\tau(1-s')}\left(\|u_0-v_0\|_{L(s')} + \|u_1-v_1\|_{L(s')}\right) \\
&\le 2M'e^{-\tau(1-s')}\left(C\|u_0-v_0\|_{L(s'),\frac{1}{2}} + \|u_1-v_1\|_{L(s')}\right) \\
&\le 2CM'\|U-V\|_{X_{s'}} \\
&\le 2CM'e^{-\tau(s-s')}\|U-V\|_{X_s} \\
&\le \frac{2CM'}{e}\frac{1}{\tau(s-s')}\|U-V\|_{X_s}.
\end{aligned}
$$

To complete the proof of (2.42), it is sufficient to pick $\tau \ge \tau_0$ with τ_0 such that $2CM'/(e\tau_0) \le \epsilon$, and D as in (2.46).

For given $0 \le s \le 1$ and $U = (u_0, u_1) \in X_s$ with $\|U\|_{X_s} \le D$, let us prove that the map $x \in [-1, 1] \to F(x, U) \in X_s$ is continuous. Pick any $x, x' \in [-1, 1]$. From the mean value theorem, we have for $r \in \mathbb{N}$ that $|x^r - x'^r| \le r|x - x'|$ for $r \in \mathbb{N}$, and hence

$$
|A_{p,q}(x) - A_{p,q}(x')| \le |x-x'|\sum_{r\in\mathbb{N}} \frac{rM}{b_0^p b_1^q b_2^r} = \frac{M}{b_0^p b_1^q b_2}(1-\frac{1}{b_2})^{-2}|x-x'|.
$$

We infer that

$$
\begin{aligned}
\|F(x,U)-F(x',U)\|_{X_s} &= e^{-\tau(1-s)}\|f(x,u_0,u_1)-f(x',u_0,u_1)\|_{L(s)} \\
&\le e^{-\tau(1-s)}\sum_{p+q>0}|A_{p,q}(x)-A_{p,q}(x')|\|u_0^p u_1^q\|_{L(s)} \\
&\le e^{-\tau(1-s)}Mb_2^{-1}(1-\frac{1}{b_2})^{-2}|x-x'|\sum_{p+q>0}\left(\frac{\|u_0\|_{L(s)}}{b_0}\right)^p\left(\frac{\|u_1\|_{L(s)}}{b_1}\right)^q,
\end{aligned}
$$

the last series being convergent for $\|U\|_{X_s} \le D$. □

We are in a position to prove Theorem 2.1.

Proof of Theorem 2.1. Let $f = f(x, y_0, y_1)$ be as in (1.5)-(1.8), $-\infty < t_1 < t_2 < +\infty$ and $R > 4$. Pick $g_0, g_1 \in G^2([t_1, t_2])$ such that (2.4) holds. We will show that Theorem 2.7 can be applied provided that the constant C in (2.4) is small enough. Pick $L_1 \in (1/R, 1/4)$. Let $\eta = \eta(L_1) > 0$ be as in Theorem 2.7. Let $U_0 = (u_0, u_1) = (g_0, g_1)$. We have to show that

$$\|U_0\|_{\mathcal{X}_{L_1}} = \|g_0\|_{L_1,\frac{1}{2}} + \|g_1\|_{L_1} \le \eta$$

for C small enough. It is sufficient to have

$$\|g_0\|_{L_1,\frac{1}{2}} \le \frac{\eta}{2}, \tag{2.47}$$

$$\|g_1\|_{L_1} \le \frac{\eta}{2}. \tag{2.48}$$

Recall that

$$\|g_0\|_{L_1\frac{1}{2}} = \max\left(2^6\|g_0\|_{L^\infty([t_1,t_2])}, 2^3 L_1^{-1} \sup_{t\in[t_1,t_2],k\in\mathbb{N}} \frac{|g_0^{(k+1)}(t)|}{L_1^{|k-\frac{1}{2}|}\Gamma_{2,\frac{1}{2}}(k)}\right), \tag{2.49}$$

$$\|g_1\|_{L_1} = \max\left(2^6\|g_1\|_{L^\infty([t_1,t_2])}, 2^3 L_1^{-1} \sup_{t\in[t_1,t_2],k\in\mathbb{N}} \frac{|g_1^{(k+1)}(t)|}{L_1^k 2^{-5}(k!)^2(1+k)^{-2}}\right), \tag{2.50}$$

where

$$\Gamma_{2,\frac{1}{2}}(k) = \begin{cases} 2^{-5}\big(\Gamma(k+\frac{1}{2})\big)^2(1+k)^{-2}, & \text{if } k > 3/2, \\ 2^{-5}(k!)^2(1+k)^{-2} & \text{if } 0 \le k \le 3/2. \end{cases}$$

Then, if follows that (2.47) is satisfied provided that

$$\|g_0\|_{L^\infty([t_1,t_2])} \le 2^{-7}\eta, \tag{2.51}$$

$$\|g_0^{(k+1)}\|_{L^\infty([t_1,t_2])} \le 2^{-4}\eta L_1^{1+|k-\frac{1}{2}|}\Gamma_{2,\frac{1}{2}}(k) \quad \forall k \in \mathbb{N}. \tag{2.52}$$

Since $\Gamma(k+\frac{1}{2}) \sim \Gamma(k)k^{\frac{1}{2}} \sim (k-1)!k^{\frac{1}{2}}$ as $k \to +\infty$, we have that $\big(\Gamma(k+\frac{1}{2})\big)^2 \sim (k!)^2/k$. Thus, the r.h.s. of (2.52) is equivalent to $2^{-9}\eta L_1^{k+\frac{1}{2}}(k!)^2k^{-3}$ as $k \to +\infty$. Using (2.4) and the fact that $L_1 > 1/R$, we have that (2.52) holds if C is small enough. The same is true for (2.51). Similarly, we see that (2.48) is satisfied provided that

$$\|g_1\|_{L^\infty([t_1,t_2])} \le 2^{-7}\eta, \tag{2.53}$$

$$\|g_1^{(k+1)}\|_{L^\infty([t_1,t_2])} \le 2^{-9}\eta L_1^{k+1}(k!)^2(k+1)^{-2} \quad \forall k \in \mathbb{N}. \tag{2.54}$$

Again, (2.53) and (2.54) are satisfied if the constant C in (2.4) is small enough.

We infer from Theorem 2.7 the existence of a solution $U = (y, \partial_x y) \in C([-1,1], X_{s_0})$ for some $s_0 \in (0,1)$ of (2.1)-(2.3). Let us check that $y \in C^\infty([-1,1] \times [t_1,t_2])$. To this end, we prove by induction on $n \in \mathbb{N}$ the following statement

$$U \in C^n([-1,1], C^k([t_1,t_2])^2) \quad \forall k \in \mathbb{N}. \tag{2.55}$$

The assertion (2.55) is true for $n = 0$, for $X_{s_0} \subset C^k([t_1,t_2])^2$ for all $k \in \mathbb{N}$. Assume (2.55) true for some $n \in \mathbb{N}$. Since A is a continuous linear map from X_s to $X_{s'}$ for $0 \le s' < s \le 1$, we have that

$$AU \in C^n([-1,1], X_s) \subset C^n([-1,1], C^k([t_1,t_2])^2) \quad \forall s \in (0,s_0),\ \forall k \in \mathbb{N}.$$

On the other hand, as f is analytic and hence of class C^∞, we infer from (2.55) that $F(x,U) \in C^n([-1,1], C^k([t_1,t_2])^2)$ for all $k \in \mathbb{N}$. Since $\partial_x U = AU + F(x,U)$, we obtain that (2.55) is true with n replaced by $n+1$. The proof of $y \in C^\infty([-1,1] \times [t_1,t_2])$ is complete. Finally, the proof of $y \in G^{1,2}([-1,1] \times [t_1,t_2])$, which uses some estimates of the next section, is given in appendix, with eventually a stronger smallness assumption on the initial data. □

3. Correspondence between the space derivatives and the time derivatives

We are concerned with the relationship between the time derivatives and the space derivatives of any solution of a general semilinear heat equation

$$\partial_t y = \partial_x^2 y + f(x, y, \partial_x y), \tag{3.1}$$

where $f = f(x, y_0, y_1)$ is of class C^∞ on $\mathbb{R}^3$.

When $f = 0$, then the jet $(\partial_x^n y(0,0))_{n\geq 0}$ is nothing but the reunion of the jets $(\partial_t^n y(0,0))_{n\geq 0}$ and $(\partial_t^n \partial_x y(0,0))_{n\geq 0}$, for

$$\partial_t^n y = \partial_x^{2n} y, \quad \forall n \in \mathbb{N}, \tag{3.2}$$
$$\partial_t^n \partial_x y = \partial_x^{2n+1} y, \quad \forall n \in \mathbb{N}. \tag{3.3}$$

When f is no longer assumed to be 0, then the relations (3.2)-(3.3) do not hold anymore. Nevertheless, there is still a one-to-one correspondence between the jet $(\partial_x^n y(0,0))_{n\geq 0}$ and the jets $(\partial_t^n y(0,0))_{n\geq 0}$ and $(\partial_t^n \partial_x y(0,0))_{n\geq 0}$.

Proposition 3.1. *Let $-\infty < t_1 \leq \tau \leq t_2 < +\infty$. Assume that $f \in C^\infty(\mathbb{R}^3)$ and that $y \in C^\infty([-1,1]\times[t_1,t_2])$ satisfies (3.1) on $[-1,1]\times[t_1,t_2]$. Then the determination of the jet $(\partial_x^n y(0,\tau))_{n\geq 0}$ is equivalent to the determination of the jets $(\partial_t^n y(0,\tau))_{n\geq 0}$ and $(\partial_t^n \partial_x y(0,\tau))_{n\geq 0}$.*

Proof. The proof of Proposition 3.1 is a direct consequence of the following

Lemma 3.2. *Let $f \in C^\infty(\mathbb{R}^3)$ and $n \in \mathbb{N}^*$. Then there exist two smooth functions $H_n = H_n(x, y_0, y_1, ..., y_{2n-1})$ and $\tilde{H}_n = \tilde{H}_n(x, y_0, y_1, ..., y_{2n})$ such that any solution $y \in C^\infty([-1,1]\times[t_1,t_2])$ of (3.1) satisfies*

$$\partial_t^n y = \partial_x^{2n} y + H_n(x, y, \partial_x y, ..., \partial_x^{2n-1} y) \quad \text{for } (x,t) \in [-1,1]\times[t_1,t_2], \tag{3.4}$$
$$\partial_t^n \partial_x y = \partial_x^{2n+1} y + \tilde{H}_n(x, y, \partial_x y, ..., \partial_x^{2n} y) \quad \text{for } (x,t) \in [-1,1]\times[t_1,t_2]. \tag{3.5}$$

Proof of Lemma 3.2. Assume first that $n = 1$. Then (3.4) holds with $H_1(x, y_0, y_1) = f(x, y_0, y_1)$. Taking the derivative with respect to x in (3.1) yields

$$\partial_x \partial_t y = \partial_x^3 y + \frac{\partial f}{\partial x}(x, y, \partial_x y) + \frac{\partial f}{\partial y_0}(x, y, \partial_x y)\partial_x y + \frac{\partial f}{\partial y_1}(x, y, \partial_x y)\partial_x^2 y,$$

and hence (3.5) holds with $\tilde{H}_1(x, y_0, y_1, y_2) = \frac{\partial f}{\partial x}(x, y_0, y_1) + \frac{\partial f}{\partial y_0}(x, y_0, y_1) y_1 + \frac{\partial f}{\partial y_1}(x, y_0, y_1) y_2$.

Assume now that (3.4) and (3.5) are satisfied at rank $n-1$, and let us prove that they are satisfied at rank n. For (3.4), we notice that

$$\begin{aligned}
\partial_t^n y &= \partial_t(\partial_t^{n-1} y) \\
&= \partial_t(\partial_x^{2n-2} y + H_{n-1}(x, y, \partial_x y, ..., \partial_x^{2n-3} y)) \\
&= \partial_x^{2n-2}\partial_t y + \sum_{k=0}^{2n-3} \frac{\partial H_{n-1}}{\partial y_k}(x, y, \partial_x y, ..., \partial_x^{2n-3} y)\partial_t \partial_x^k y \\
&= \partial_x^{2n-2}(\partial_x^2 y + f(x, y, \partial_x y)) + \sum_{k=0}^{2n-3} \frac{\partial H_{n-1}}{\partial y_k}(x, y, \partial_x y, ..., \partial_x^{2n-3} y)\partial_x^k(\partial_x^2 y + f(x, y, \partial_x y)) \qquad (3.6) \\
&=: \partial_x^{2n} y + H_n(x, y, \partial_x y, ..., \partial_x^{2n-1} y)
\end{aligned}$$

for some smooth function $H_n = H_n(x, y_0, ..., y_{2n-1})$. For (3.5), we notice that

$$\begin{aligned}
\partial_t^n \partial_x y &= \partial_t(\partial_t^{n-1}\partial_x y) \\
&= \partial_t(\partial_x^{2n-1} y + \tilde{H}_{n-1}(x, y, \partial_x y, ..., \partial_x^{2n-2} y))
\end{aligned}$$

$$= \partial_x^{2n-1}\partial_t y + \sum_{k=0}^{2n-2} \frac{\partial \tilde{H}_{n-1}}{\partial y_k}(x, y, \partial_x y, ..., \partial_x^{2n-2} y)\partial_t \partial_x^k y$$
$$= \partial_x^{2n-1}(\partial_x^2 y + f(x, y, \partial_x y)) + \sum_{k=0}^{2n-2} \frac{\partial \tilde{H}_{n-1}}{\partial y_k}(x, y, \partial_x y, ..., \partial_x^{2n-2} y)\partial_x^k(\partial_x^2 y + f(x, y, \partial_x y))$$
$$=: \partial_x^{2n+1} y + \tilde{H}_n(x, y, \partial_x y, ..., \partial_x^{2n} y)$$

for some smooth function $\tilde{H}_n = \tilde{H}_n(x, y_0, y_1, ..., y_{2n})$. □

Next, we relate the behavior as $n \to +\infty$ of the jets $(\partial_t^n y(0, \tau))_{n\geq 0}$ and $(\partial_t^n \partial_x y(0, \tau))_{n\geq 0}$ to those of the jet $(\partial_x^n y(0, \tau))_{n\geq 0}$. To do that, we assume that in (3.1) the nonlinear term reads

$$f(x, y_0, y_1) = \sum_{(p,q,r)\in\mathbb{N}^3} a_{p,q,r}(y_0)^p (y_1)^q x^r \quad \forall (x, y_0, y_1) \in (-4, 4)^3, \tag{3.7}$$

where the coefficients $a_{p,q,r}$, $(p, q, r) \in \mathbb{N}^3$, satisfy (1.7)-(1.8).

The following result is a refined and quantified version of Proposition 3.1, in the sense that it gives a *domain* and a *codomain* for the map which associates (by an algorithm) a jet $(\partial_x^n y(0, \tau))_{n\in\mathbb{N}}$ to a pair of jets $(\partial_t^n y(0, \tau))_{n\in\mathbb{N}}$ and $(\partial_x \partial_t^n y(0, \tau))_{n\in\mathbb{N}}$. A C^∞ smooth solution y of the semilinear heat equation (3.1) is assumed to exist in order to perform the computations, but its existence will be fully justified thereafter (see below Proposition 3.6).

Proposition 3.3. *Let $-\infty < t_1 \leq \tau \leq t_2 < +\infty$ and $f = f(x, y_0, y_1)$ be as in (1.5)-(1.6) with the coefficients $a_{p,q,r}$, $(p, q, r) \in \mathbb{N}^3$, satisfying (1.7)-(1.8). Pick any $\widetilde{R} > 4$ and any numbers R, R' with $4 < R' < R < \min(\widetilde{R}, b_2)$. Then there exists some number $\widetilde{C} > 0$ such that for any $C \in (0, \widetilde{C}]$, we can find a number $C' = C'(C, R, R') > 0$ with $\lim_{C\to 0^+} C'(C, R, R') = 0$ such that for any function $y \in C^\infty([-1, 1] \times [t_1, t_2])$ satisfying (3.1) on $[-1, 1] \times [t_0, t_1]$ and*

$$y(x, \tau) = y^0(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}, \quad \forall x \in [-1, 1] \tag{3.8}$$

for some $y^0 \in \mathcal{R}_{\tilde{R},C}$, is such that

$$|\partial_x^k \partial_t^n y(0, \tau)| \leq C' \frac{(2n + k)!}{R^k R'^{2n}}, \quad \forall k, n \in \mathbb{N}. \tag{3.9}$$

In particular, we have

$$|\partial_t^n y(0, \tau)| \leq C' \frac{(2n)!}{R'^{2n}}, \quad \forall n \in \mathbb{N}, \tag{3.10}$$
$$|\partial_x \partial_t^n y(0, \tau)| \leq C' \frac{(2n + 1)!}{R R'^{2n}}, \quad \forall n \in \mathbb{N}. \tag{3.11}$$

Proof. We know from Proposition 3.1 that the jets $(\partial_t^n y(0, \tau))_{n\geq 0}$ and $(\partial_t^n \partial_x y(0, \tau))_{n\geq 0}$ are completely determined by the jet $(\partial_x^n y(0, \tau))_{n\geq 0}$, that is by y^0. A *direct* proof of estimates (3.10) and (3.11) (which follow at once from (3.9)) seems hard to be derived, whereas a proof of (3.9) can be obtained by induction on n. We shall need several lemmas.

Lemma 3.4. *(see [10, Lemma A.1]) For all $k, q \in \mathbb{N}$ and $a \in \{0, ..., k + q\}$, we have*

$$\sum_{\substack{j+p=a\\0\leq j\leq k\\0\leq p\leq q}} \binom{k}{j}\binom{q}{p} = \binom{k+q}{a}.$$

The following Lemma gives the algebra property for the mixed Gevrey spaces $G^{1,2}([-1, 1] \times [t_1, t_2])$. A slight modification of its proof actually yields Lemma 2.5, making the paper almost self-contained.

Lemma 3.5. *Let* $(x_0,t_0)\in[-1,1]\times[t_1,t_2]$, $R,R'\in(0,+\infty)$, $q\in\mathbb{N}$, $\mu\in(q+2,+\infty)$, $k_0,n_0\in\mathbb{N}$, $C_1,C_2\in(0,+\infty)$, *and* $y_1,y_2\in C^\infty([-1,1]\times[t_1,t_2])$ *be such that*

$$|\partial_x^k\partial_t^n y_i(x_0,t_0)|\le C_i\frac{(2n+k+q)!}{R^kR'^{2n}(2n+k+1)^\mu}\quad \forall i=1,2,\quad \forall k\in\{0,...,k_0\},\ \forall n\in\{0,...,n_0\}. \tag{3.12}$$

Then we have

$$|\partial_x^k\partial_t^n(y_1y_2)(x_0,t_0)|\le K_{q,\mu}C_1C_2\frac{(2n+k+q)!}{R^kR'^{2n}(2n+k+1)^\mu}\quad \forall k\in\{0,...,k_0\},\ \forall n\in\{0,...,n_0\}, \tag{3.13}$$

where

$$K_{q,\mu}:=2^{\mu-q}(1+q)^{2q}\sum_{j\ge0}\sum_{i\ge0}\frac{1}{(2i+j+1)^{\mu-q}}<\infty.$$

Proof of Lemma 3.5. Using $(2n+k+q)^q\le(1+q)^q\,(1+2n+k)^q$, we obtain

$$(2n+k+q)!\le(2n+k)!(2n+k+q)^q\le(1+q)^q(2n+k)!\,(1+2n+k)^q\,.$$

So, denoting $\widetilde{\mu}:=\mu-q>2$ and $\widetilde{C}_i:=(1+q)^qC_i$, we have

$$|\partial_x^k\partial_t^n y_i(x_0,t_0)|\le\widetilde{C}_i\frac{(2n+k)!}{R^kR'^{2n}(2n+k+1)^{\widetilde{\mu}}}\quad \forall i=1,2,\quad \forall k\in\{0,...,k_0\},\ \forall n\in\{0,...,n_0\}. \tag{3.14}$$

From Leibniz' rule, we have that

$$\begin{aligned}
&|\partial_x^k\partial_t^n(y_1y_2)(x_0,t_0)|\\
&\quad=\left|\sum_{0\le j\le k}\sum_{0\le i\le n}\binom{k}{j}\binom{n}{i}(\partial_x^j\partial_t^i y_1)(x_0,t_0)(\partial_x^{k-j}\partial_t^{n-i}y_2)(x_0,t_0)\right|\\
&\quad\le\sum_{0\le j\le k}\sum_{0\le i\le n}\binom{k}{j}\binom{n}{i}\widetilde{C}_1\frac{(2i+j)!}{R^jR'^{2i}(2i+j+1)^{\widetilde{\mu}}}\widetilde{C}_2\frac{(2(n-i)+k-j)!}{R^{k-j}R'^{2(n-i)}(2(n-i)+k-j+1)^{\widetilde{\mu}}}\\
&\quad=\frac{\widetilde{C}_1\widetilde{C}_2}{R^kR'^{2n}}(2n+k)!\underbrace{\sum_{0\le j\le k}\sum_{0\le i\le n}\frac{\binom{k}{j}\binom{n}{i}\binom{2n+k}{2i+j}^{-1}}{(2i+j+1)^{\widetilde{\mu}}(2(n-i)+k-j+1)^{\widetilde{\mu}}}}_{I}.
\end{aligned}$$

We infer from Lemma 3.4 that

$$\binom{k}{j}\binom{q}{p}\le\binom{k+q}{j+p},\qquad \text{for } 0\le j\le k,\ 0\le p\le q. \tag{3.15}$$

This yields

$$\binom{n}{i}\le\binom{n}{i}^2\le\binom{2n}{2i},$$

and hence (using again (3.15))

$$\binom{k}{j}\binom{n}{i}\le\binom{k}{j}\binom{2n}{2i}\le\binom{2n+k}{2i+j}.$$

Finally, by convexity of $x\to x^{\widetilde{\mu}}$, we have that

$$\begin{aligned}
&\sum_{0\le j\le k}\sum_{0\le i\le n}\frac{(2n+k+1)^{\widetilde{\mu}}}{(2i+j+1)^{\widetilde{\mu}}(2(n-i)+k-j+1)^{\widetilde{\mu}}}\le\sum_{0\le j\le k}\sum_{0\le i\le n}\left(\frac{1}{2i+j+1}+\frac{1}{2(n-i)+k-j+1}\right)^{\widetilde{\mu}}\\
&\le 2^{\widetilde{\mu}-1}\sum_{0\le j\le k}\sum_{0\le i\le n}\left(\frac{1}{(2i+j+1)^{\widetilde{\mu}}}+\frac{1}{(2(n-i)+k-j+1)^{\widetilde{\mu}}}\right)\le2^{\widetilde{\mu}}\sum_{j\ge0}\sum_{i\ge0}\frac{1}{(2i+j+1)^{\widetilde{\mu}}}<\infty,
\end{aligned}$$

where we used the fact that $\widetilde{\mu} = \mu - q > 2$.

It follows that

$$I \le 2^{\widetilde{\mu}} \left(\sum_{j\ge 0} \sum_{i\ge 0} \frac{1}{(2i+j+1)^{\widetilde{\mu}}} \right) \frac{1}{(2n+k+1)^{\widetilde{\mu}}} = 2^{\mu-q} \left(\sum_{j\ge 0} \sum_{i\ge 0} \frac{1}{(2i+j+1)^{\mu-q}} \right) \frac{(2n+k+1)^q}{(2n+k+1)^{\mu}},$$

and hence the proof of Lemma 3.5 is complete once we have noticed that $(2n+k)!(2n+k+1)^q \le (2n+k+q)!$. □

Let us go back to the proof of Proposition 3.3. Pick any number $\mu > 3$. We shall prove by induction on $n \in \mathbb{N}$ that

$$|\partial_x^k \partial_t^n y(0,\tau)| \le C_n \frac{(2n+k)!}{R^k R'^{2n}(2n+k+1)^{\mu}}, \quad \forall k \in \mathbb{N}, \tag{3.16}$$

where $0 < C_n \le C' < +\infty$. For $n = 0$, using the fact that $R < \widetilde{R}$, we have that

$$|\partial_x^k y(0,\tau)| = |a_k| \le C \frac{k!}{\widetilde{R}^k} \le C \left(\sup_{p\in\mathbb{N}} (\frac{R}{\widetilde{R}})^p (p+1)^{\mu} \right) \frac{k!}{R^k (k+1)^{\mu}} \le C_0 \frac{k!}{R^k (k+1)^{\mu}}$$

provided that

$$C \le \widetilde{C} = \left(\sup_{p\in\mathbb{N}} (\frac{R}{\widetilde{R}})^p (p+1)^{\mu} \right)^{-1} C_0. \tag{3.17}$$

Assume that (3.16) is satisfied at the rank $n \in \mathbb{N}$ for some constant $C_n > 0$. Then, by (1.1), (1.6), we have that

$$\begin{aligned} |\partial_x^k \partial_t^{n+1} y(0,\tau)| &= |\partial_x^k \partial_t^n \big(\partial_x^2 y + \sum_{p,q,r\in\mathbb{N}} a_{p,q,r} y^p (\partial_x y)^q x^r\big)(0,\tau)| \\ &= |\partial_x^k \partial_t^n \big(\partial_x^2 y + \sum_{p,q\in\mathbb{N}} A_{p,q}(x) y^p (\partial_x y)^q\big)(0,\tau)| \\ &\le |\partial_t^n \partial_x^{k+2} y(0,\tau)| + \sum_{p\ge 1} |\partial_x^k \partial_t^n \big(A_{p,0}(x) y^p\big)(0,\tau)| \\ &\quad + \sum_{q\ge 1} \sum_{p\ge 0} |\partial_x^k \partial_t^n \big(A_{p,q}(x) y^p (\partial_x y)^q\big)(0,\tau)| \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{3.18}$$

(Note that the sum for I_2 is over $p \ge 1$, for $A_{0,0}(x) = 0$.)

Since $R' < R$, we can pick some $\varepsilon \in (0,1)$ such that

$$R'^2 \le (1-\varepsilon) R^2.$$

For I_1, we have that

$$I_1 \le C_n \frac{(2n+k+2)!}{R^{k+2} R'^{2n} (2n+k+3)^{\mu}} \le (1-\varepsilon) C_n \frac{(2n+k+2)!}{R^k R'^{2n+2} (2n+k+3)^{\mu}}. \tag{3.19}$$

Since $A_{p,q}$ does not depend on t, we have that $\partial_x^k \partial_t^n A_{p,k} = 0$ for $n \ge 1$ and $k \ge 0$. Next, for $k \ge 0$, we have that

$$|\partial_x^k A_{p,q}(0)| = k!\,|a_{p,q,k}| \le \frac{k!}{b_2^k} \frac{M}{b_0^p b_1^q} \le \frac{\overline{C}\,k!}{(k+1)^{\mu} R^k b_0^p b_1^q}$$

for some constant $\overline{C} > 0$ depending on R, b_2, μ, for $R < b_2$.

Note that, still by (3.16), the function $\partial_x y$ satisfies the estimate

$$|\partial_x^k \partial_t^n (\partial_x y)(0,\tau)| \le \frac{C_n}{R} \frac{(2n+k+1)!}{R^k R'^{2n} (2n+k+2)^{\mu}}, \quad \forall k \in \mathbb{N}.$$

Using $\mu - 1 > 2$, we infer from iterated applications of Lemma 3.5 that

$$\left|\partial_x^k \partial_t^n \left(A_{p,0} y^p\right)(0,\tau)\right| \le \frac{\overline{C} C_n^p K^p (2n+k)!}{R^k R'^{2n} (2n+k+1)^\mu b_0^p}, \tag{3.20}$$

$$\left|\partial_x^k \partial_t^n \left(A_{p,q} y^p (\partial_x y)^q\right)(0,\tau)\right| \le \frac{\overline{C} C_n^{p+q} K^{p+q} (2n+k+1)!}{R^k R'^{2n} (2n+k+1)^\mu b_0^p b_1^q R^q} \quad \forall q \ge 1, \tag{3.21}$$

where we denote $K := \max(K_{0,\mu}, K_{1,\mu})$. We infer from (3.20)-(3.21) that

$$I_2 \le \sum_{p\ge 1} \frac{\overline{C} C_n^p K^p (2n+k)!}{R^k R'^{2n} (2n+k+1)^\mu b_0^p}, \tag{3.22}$$

$$I_3 \le \sum_{q\ge 1}\sum_{p\ge 0} \frac{\overline{C} C_n^{p+q} K^{p+q} (2n+k+1)!}{R^k R'^{2n} (2n+k+1)^\mu b_0^p b_1^q R^q}. \tag{3.23}$$

Using (3.18)-(3.19) and (3.22)-(3.23), we see that the condition

$$|\partial_x^k \partial_t^{n+1} y(0,\tau)| \le C_{n+1} \frac{(2n+k+2)!}{R^k R'^{2n+2} (2n+k+3)^\mu}, \quad \forall k \in \mathbb{N},$$

is satisfied provided that

$$(1-\varepsilon) C_n + \sum_{p\ge 1} \frac{\overline{C} R'^2}{(2n+k+1)(2n+k+2)} \left(\frac{2n+k+3}{2n+k+1}\right)^\mu \left(\frac{C_n K}{b_0}\right)^p + \sum_{q\ge 1}\sum_{p\ge 0} \frac{\overline{C} R'^2}{(2n+k+2)} \left(\frac{2n+k+3}{2n+k+1}\right)^\mu \left(\frac{C_n K}{b_0}\right)^p \left(\frac{C_n K}{b_1 R}\right)^q \le C_{n+1}. \tag{3.24}$$

Pick a number $\delta \in (0,1)$. Assume that

$$C_n \le \delta \cdot \min\left(\frac{b_0}{K}, \frac{b_1 R}{K}\right), \tag{3.25}$$

so that $C_n K / b_0 \le \delta$ and $C_n K/(b_1 R) \le \delta$. Set

$$C_{n+1} = \lambda_n C_n := \left[(1-\varepsilon) + \frac{K}{b_0} \frac{\overline{C} R'^2}{(2n+1)(2n+2)} \frac{3^\mu}{1-\delta} + \frac{K}{b_1 R} \frac{\overline{C} R'^2}{(2n+2)} \frac{3^\mu}{(1-\delta)^2}\right] C_n.$$

Then, with this choice of C_{n+1}, (3.24) holds provided that (3.25) is satisfied. Next, one can pick some $n_0 \in \mathbb{N}$ such that for $n \ge n_0$, we have

$$\lambda_n \le 1.$$

This yields $C_{n+1} \le C_n$ for $n \ge n_0$, provided that (3.25) holds for $n = n_0$. To ensure (3.25) for $n = 0, 1, ..., n_0$, it is sufficient to choose C_0 small enough (or, equivalently, $\widetilde{C}$ small enough) so that

$$\max\left(C_0, \lambda_0 C_0, \lambda_1 \lambda_0 C_0, ..., \lambda_{n_0-1} \cdots \lambda_0 C_0\right) \le \delta \cdot \min\left(\frac{b_0}{K}, \frac{b_1 R}{K}\right).$$

The proof by induction of (3.16) is achieved.

We can pick

$$C'(C, R, R') := \max\left(C_0, \lambda_0 C_0, \lambda_1 \lambda_0 C_0, ..., \lambda_{n_0-1} \cdots \lambda_0\, C_0\right)$$

with $C_0 = C \sup_{p\in\mathbb{N}} (\frac{R}{\hat{R}})^p (p+1)^\mu$, so that $C'(C,R,R') \to 0$ as $C \to 0$. The proof of Proposition 3.3 is complete. □

Proposition 3.6. *Let $-\infty < t_1 \le \tau \le t_2 < +\infty$ and $f = f(x, y_0, y_1)$ be as in (1.5)-(1.6) with the coefficients $a_{p,q,r}$, $(p,q,r) \in \mathbb{N}^3$, satisfying (1.7)-(1.8). Assume in addition that $b_2 > \hat{R} := 4e^{(2e)^{-1}} \approx 4.81$. Let $\tilde{R} > \hat{R}$. Then there exists some number $\widetilde{C} > 0$ such that for any $C \in (0, \widetilde{C}]$ and any numbers R, R', L with $\hat{R} < R' < R < \min(\tilde{R}, b_2)$ and*

$4e^{e^{-1}}/R'^2 < L < 1/4$, *there exists a number* $C'' = C''(C, R, R', L) > 0$ *with* $\lim_{C\to 0^+} C''(C, R, R', L) = 0$ *such that for any* $y^0 \in \mathcal{R}_{\tilde{R},C}$, *we can pick a function* $y \in G^{1,2}([-1, 1] \times [t_1, t_2])$ *satisfying* (3.1) *for* $(x, t) \in [-1, 1] \times [t_1, t_2]$ *and*

$$y(x,\tau) = y^0(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}, \quad \forall x \in [-1, 1], \tag{3.26}$$

and such that for all $t \in [t_1, t_2]$

$$\left|\partial_t^n y(0,t)\right| \le C'' L^n (n!)^2, \tag{3.27}$$

$$\left|\partial_x \partial_t^n y(0,t)\right| \le C'' L^n (n!)^2. \tag{3.28}$$

Proof. Let $\hat{R} := 4e^{(2e)^{-1}}$, $\tilde{R} > \hat{R}$ and R, R' with $\hat{R} < R' < R < \min(\tilde{R}, b_2)$. Pick $\widetilde{C}, C$ as in Proposition 3.3, and pick any $y^0 \in \mathcal{R}_{\tilde{R},C}$. If a function y as in Proposition 3.6 does exists, then both sequences of numbers

$$d_n := \partial_t^n y(0,\tau), \quad n \in \mathbb{N},$$

$$\tilde{d}_n := \partial_x \partial_t^n y(0,\tau), \quad n \in \mathbb{N}$$

can be computed inductively in terms of the coefficients $a_n = \partial_x^n y^0(0)$, $n \in \mathbb{N}$, according to Proposition 3.1. Furthermore, it follows from Proposition 3.3 (see (3.10)-(3.11)) that we have for some $C' = C'(C, R, R') > 0$ and all $n \in \mathbb{N}$

$$|d_n| \le C' \frac{(2n)!}{R'^{2n}},$$

$$|\tilde{d}_n| \le C' \frac{(2n+1)!}{RR'^{2n}}.$$

Note that both sequences $(d_n)_{n\in\mathbb{N}}$ and $(\tilde{d}_n)_{n\in\mathbb{N}}$ (as above) can be defined in terms of the coefficients a_n's, even if the existence of the function y is not yet established.

Let $L \in (\frac{4e^{e^{-1}}}{R'^2}, \frac{1}{4})$, $R'' \in (\sqrt{\frac{4e^{e^{-1}}}{L}}, R')$ and $M = M(R, R', R'') > 0$ such that

$$|\tilde{d}_n| \le MC' \frac{(2n)!}{R''^{2n}}, \quad \forall n \in \mathbb{N}.$$

The following lemma is a particular case of [17, Proposition 3.6] (with $a_p = [2p(2p-1)]^{-1}$ for $p \ge 1$).

Lemma 3.7. *Let* $(d_q)_{q\ge 0}$ *be a sequence of real numbers such that*

$$|d_q| \le CH^q (2q)! \quad \forall q \ge 0$$

for some $H > 0$ *and* $C > 0$. *Then for all* $\tilde{H} > e^{e^{-1}} H$ *there exists a function* $f \in C^\infty(\mathbb{R})$ *such that*

$$f^{(q)}(0) = d_q \quad \forall q \ge 0, \tag{3.29}$$

$$|f^{(q)}(t)| \le C\tilde{H}^q (2q)! \quad \forall q \ge 0,\ \forall t \in \mathbb{R}. \tag{3.30}$$

Pick $H = 1/R''^2$ and $\tilde{H} = L/4 > e^{e^{-1}} H$. Then by Lemma 3.7, there exist two functions $g_0, g_1 \in G^2([-1, 1])$ such that

$$g_0^{(n)}(0) = d_n, \quad n \ge 0, \tag{3.31}$$

$$g_1^{(n)}(0) = \tilde{d}_n, \quad n \ge 0, \tag{3.32}$$

$$|g_0^{(n)}(t)| \le C'\tilde{H}^n (2n)!, \quad n \ge 0,\ t \in [-1, 1], \tag{3.33}$$

$$|g_1^{(n)}(t)| \le MC'\tilde{H}^n (2n)!, \quad n \ge 0,\ t \in [-1, 1]. \tag{3.34}$$

It follows at once from Stirling' formula that $(2n)! \le C_s 4^n (n!)^2$ for some universal constant $C_s > 0$, so that (with $4\tilde{H} = L$)

$$|g_0^{(n)}(t)| \le C' C_s (4\tilde{H})^n (n!)^2, \quad n \ge 0,\ t \in [-1,1], \tag{3.35}$$

$$|g_1^{(n)}(t)| \le M C' C_s (4\tilde{H})^n (n!)^2, \quad n \ge 0,\ t \in [-1,1]. \tag{3.36}$$

Note that $4\tilde{H} = L < 1/4$. If C is sufficiently small, then C' is as small as desired, and it follows then from Theorem 2.1 that we can pick a function $y \in G^{1,2}([-1,1] \times [t_1, t_2])$ satisfying (2.1)-(2.3). Using again Proposition 3.1, we infer that $\partial_x^k y(0,\tau) = a_k = \partial_x^k y^0(0)$ for all $k \ge 0$, and hence (3.26) holds. The estimate (3.27)-(3.28) follow from (2.2)-(2.3) and (3.35)-(3.36) with $L = 4\tilde{H}$ and $C'' = \max(C' C_s, M C' C_s)$. The proof of Proposition 3.6 is complete. □

4. Proofs of the main results

Let us start with the proof of Theorem 1.1. Let $R > \hat{R} = 4e^{(2e)^{-1}}$ and let $\hat{C}$ be (for the moment) the constant $\tilde{C}$ given by Proposition 3.6. Pick any $y^0, y^1 \in \mathcal{R}_{R,\hat{C}}$. We infer from Proposition 3.6 applied with $[t_1, t_2] = [0, T]$ and $\tau = 0$ (resp. $\tau = T$) the existence of two functions $\hat{y}, \tilde{y} \in G^{1,2}([-1,1] \times [0,T])$ satisfying (2.1) and such that

$$\hat{y}(x,0) = y^0(x) \ \text{ and } \ \tilde{y}(x,T) = y^1(x), \quad \forall x \in [-1,1].$$

Let $\rho \in C^\infty(\mathbb{R})$ be such that

$$\rho(t) = \begin{cases} 1 & \text{if } t \le \frac{T}{4}, \\ 0 & \text{if } t \ge \frac{3T}{4}, \end{cases}$$

and $\rho_{|[0,T]} \in G^{\frac{3}{2}}([0,T])$. Let

$$g_0(t) = \rho(t)\hat{y}(0,t) + (1-\rho(t))\tilde{y}(0,t), \quad t \in [0,T],$$
$$g_1(t) = \rho(t)\partial_x \hat{y}(0,t) + (1-\rho(t))\partial_x \tilde{y}(0,t), \quad t \in [0,T].$$

Then by [17, Lemma 3.7] $g_0, g_1 \in G^2([0,T])$ and using (3.27)-(3.28) and picking a smaller value of $\hat{C}$ if necessary, we can assume that (2.4) is satisfied with $R = 1/L$. It follows then from Theorem 2.1 that there exists a solution $y \in G^{1,2}([-1,1] \times [0,T])$ of (2.1)-(2.3). The control inputs h_{-1} and h_1 are defined by using (1.2)-(1.3). Then y satisfies (1.1)-(1.4) together with $y(x,T) = y_1(x)$ for $x \in [-1,1]$. Indeed, since $\rho(t) = 0$ for $t > 3T/4$, we have

$$\partial_t^n y(0,T) = g_0^{(n)}(T) = \partial_t^n \tilde{y}(0,T), \quad \forall n \in \mathbb{N},$$
$$\partial_x \partial_t^n y(0,T) = g_1^{(n)}(T) = \partial_x \partial_t^n \tilde{y}(0,T), \quad \forall n \in \mathbb{N}.$$

It follows then from Proposition 3.1 that $\partial_x^n y(0,T) = \partial_x^n \tilde{y}(0,T) = \partial_x^n y^1(0)$ for all $n \in \mathbb{N}$, and hence $y(.,T) = y^1$. The proof of (1.4) is similar. The proof of Theorem 1.1 is achieved. □

Let us now proceed to the proof of Corollary 1.2. Pick any solution $y = y(x,t)$ for $x \in [-1,1]$ and $t \in [t_1, t_2]$ of (3.1), and set $y^0(x) = y(x,\tau)$ where $\tau \in [t_1, t_2]$. Assume that $y^0(-x) = -y^0(x)$ for $x \in [-1,1]$. The following claims are needed.

Claim 1. For all $n \ge 0$, there exists a smooth function $\widehat{H}_n$ such that we have $\partial_x^n[\partial_x^2 y + f(x, y, \partial_x y)] = \widehat{H}_n(x, y, \partial_x y, ..., \partial_x^{n+2} y)$, where

$$\widehat{H}_n(-x, -y_0, y_1, -y_2, ..., (-1)^{n+1} y_{n+2}) = (-1)^{n+1} \widehat{H}_n(x, y_0, y_1, ..., y_{n+2}).$$

The proof is by induction on $n \ge 0$. Claim 1 is obvious for $n = 0$ (take $\widehat{H}_0(x, y_0, y_1, y_2) = y_2 + f(x, y_0, y_1)$), and if it is true for some $n \in \mathbb{N}$, then

$$\begin{aligned}\partial_x^{n+1}[\partial_x^2 y + f(x,y,\partial_x y)] &= \partial_x \partial_x^n[\partial_x^2 y + f(x,y,\partial_x y)]\\ &= \partial_x[\widehat{H}_n(x,y,\partial_x y,\dots,\partial_x^{n+2}y)]\\ &= \partial_x \widehat{H}_n(x,y,\partial_x y,\dots,\partial_x^{n+2}y) + \sum_{k=0}^{n+2}\partial_{y_k}\widehat{H}_n(x,y,\partial_x y,\dots,\partial_x^{n+2}y)\partial_x^{k+1}y\\ &=: \widehat{H}_{n+1}(x,y,\partial_x y,\dots,\partial_x^{n+2}y,\partial_x^{n+3}y).\end{aligned}$$

Then it can be seen that

$$\widehat{H}_{n+1}(-x,-y_0,y_1,-y_2,\dots,(-1)^{n+1}y_{n+2},(-1)^{n+2}y_{n+3}) = (-1)^{n+2}\widehat{H}_{n+1}(x,y_0,y_1,\dots,y_{n+3}).$$

Our second claim is concerned with the function H_n in Lemma 3.2.

Claim 2. For all $n \ge 1$ we have $H_n(-x,-y_0,y_1,\dots.,(-1)^{2n-1}y_{2n-1}) = -H_n(x,y_0,y_1,\dots,y_{2n-1})$.

We prove Claim 2 by induction on $n \ge 1$. For $n = 1$, the result is obvious, for $H_1(x,y_0,y_1) = f(x,y_0,y_1)$. Assume the result true at rank $n-1 \ge 1$. Then we infer from (3.4) and (3.6) that

$$\begin{aligned}H_n(x,y,\partial_x y,\dots,\partial_x^{2n-1}y) &= \partial_x^{2n-2}[\partial_x^2 y + f(x,y,\partial_x y)] - \partial_x^{2n}y\\ &\quad + \sum_{k=0}^{2n-3}\frac{\partial H_{n-1}}{\partial y_k}(x,y,\partial_x y,\dots,\partial_x^{2n-3}y)\partial_x^k(\partial_x^2 y + f(x,y,\partial_x y))\\ &= \widehat{H}_{2n-2}(x,y,\partial_x y,\dots,\partial_x^{2n}y) - \partial_x^{2n}y\\ &\quad + \sum_{k=0}^{2n-3}\frac{\partial H_{n-1}}{\partial y_k}(x,y,\partial_x y,\dots,\partial_x^{2n-3}y)\widehat{H}_k(x,y,\partial_x y,\dots,\partial_x^{k+2}y).\end{aligned}$$

Using Claim 1 and the induction hypothesis, one readily sees that

$$H_n(-x,-y_0,y_1,\dots.,(-1)^{2n-1}y_{2n-1}) = -H_n(x,y_0,y_1,\dots,y_{2n-1}).$$

Claim 2 is proved.

Claim 3. $\partial_t^n y(0,\tau) = 0 \ \ \forall n \in \mathbb{N}$.

Note that the result is true for $n = 0$, for $y(0,\tau) = y^0(0) = 0$. By Claim 2, we have

$$\begin{aligned}\partial_t^n y(0,\tau) &= \partial_x^{2n}y(0,\tau) + H_n(0,y(0,\tau),\partial_x y(0,\tau),\dots,\partial_x^{2n-1}y(0,\tau))\\ &= \partial_x^{2n}y^0(0) + H_n(0,y^0(0),\partial_x y^0(0),\dots,\partial_x^{2n-1}y^0(0)).\end{aligned}$$

It is clear that the function $\partial_x^{2n}y^0$ is odd, and it follows from Claim 2 that the function

$$x \to H_n(x,y^0(x),\partial_x y^0(x),\dots,\partial_x^{2n-1}y^0(x))$$

is odd as well. It follows that $\partial_t^n y(0,\tau) = 0$. The proof of Claim 3 is achieved.

Let us go back to the proof of Corollary 1.2. Let us show that $\hat{y}(0,t) = \tilde{y}(0,t) = 0$ for all $t \in [0,T]$. Let us consider $\hat{y}(0,t)$ only, the property for $\tilde{y}(0,t)$ being similar. The function $\hat{y}$ is given by Proposition 3.6. But in the proof of Proposition 3.6, as $d_n = \partial_t y^n(0,\tau) = 0$ for all $n \in \mathbb{N}$, it is sufficient to pick $g_0(t) = 0$ for all $t \in [0,T]$, so that $\hat{y}(0,t) = 0$ for $t \in [0,T]$. Finally, the function $y = y(x,t)$ for $(x,t) \in [-1,1] \times [0,T]$ given by Theorem 1.1 yields by restriction to $[0,1] \times [0,T]$ the solution of the control problem (1.10)-(1.13). □

Declaration of competing interest

No competing interest.

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Appendix A. Gevrey regularity of the solution of (2.1)-(2.3) provided in Theorem 2.1

Assume that f satisfies (1.5)-(1.8). Let us show that $y \in G^{1,2}([-1,1]\times[t_1,t_2])$. Pick any numbers R_1, R_2 such that $4/e < R_1 < R_2$, and let us prove that there exists some constant $M>0$ such that (2.5) holds. To this end, picking any $\mu > 3$, we prove by induction on $l \in \mathbb{N}$ that

$$|\partial_x^k \partial_t^n y(x,t)| \le C_l \frac{(2n+k)!}{R_1^k R_2^{2n}(2n+k+1)^\mu} \quad \forall (x,t)\in[-1,1]\times[t_1,t_2],\ \forall n\in\mathbb{N}, \tag{A.1}$$

for $l\in\mathbb{N}$ and $k\in\{2l,2l+1\}$, with $\sup_{l\in\mathbb{N}} C_l < \infty$. Let us start with $l=0$. Then (A.1) reads

$$|\partial_t^n y(x,t)| \le C_0 \frac{(2n)!}{R_2^{2n}(2n+1)^\mu}, \tag{A.2}$$

$$|\partial_x\partial_t^n y(x,t)| \le C_0 \frac{(2n+1)!}{R_1 R_2^{2n}(2n+2)^\mu} \tag{A.3}$$

for $(x,t)\in[-1,1]\times[t_1,t_2]$ and $n\in\mathbb{N}$.

We already know that $y\in C^\infty([-1,1]\times[t_1,t_2])$ and that $(y,\partial_x y)\in C([-1,1],X_{s_0})$ for some $s_0\in(0,1)$, i.e. $(y,\partial_x y)\in C([-1,1],\mathcal{X}_{L_0})$ with $L_0 = L(s_0) = e^{2(1-s_0)}L_1 \le e^2 L_1 < (e/2)^2$. Thus we have for some constant $C>0$ and for all $n\in\mathbb{N}$ and all $(x,t)\in[-1,1]\times[t_1,t_2]$,

$$|\partial_t^{n+1} y(x,t)| \le C\, L_0^{|n-\frac{1}{2}|+1}\Gamma(n+\frac{1}{2})^2(1+n)^{-2},$$

$$|\partial_x\partial_t^{n+1} y(x,t)| \le C\, L_0^{n+1}(n!)^2(1+n)^{-2}.$$

Using the estimate $\Gamma(n+\frac{1}{2}) \sim \Gamma(n+1)(n+\frac{1}{2})^{-\frac{1}{2}}$ and the estimate $(n!)^2 \sim \sqrt{\pi n}(2n)!/2^{2n}$ that follows from Stirling formula, we infer the existence of a universal constant $C_0>0$ such that (A.2)-(A.3) hold for some constants R_1, R_2 with $4/e < R_1 < R_2 < \sqrt{4/L_0}$.

Assume now that (A.1) is true for all $k\in\{0,1,...,2l+1\}$ for some $l\in\mathbb{N}$. Let us pick $k\in\{2l,2l+1\}$, and let us check that (A.1) is true for $k+2\in\{2l+2,2l+3\}$. Then

$$\begin{aligned}
|\partial_x^{k+2}\partial_t^n y(x,t)| &= |\partial_x^k\partial_t^n\partial_x^2 y| \\
&= |\partial_x^k\partial_t^n(\partial_t y - f(x,y,\partial_x y))| \\
&\le |\partial_x^k\partial_t^{n+1}y| + \sum_{p\ge1}|\partial_x^k\partial_t^n(A_{p,0}(x)y^p)| + \sum_{q\ge1}\sum_{p\ge0}|\partial_x^k\partial_t^n(A_{p,q}(x)y^p(\partial_x y)^q)| \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

Then

$$I_1 \le C_l \frac{(2n+2+k)!}{R_1^k R_2^{2n+2}(2n+k+3)^\mu} = C_l\left(\frac{R_1}{R_2}\right)^2 \frac{(2n+2+k)!}{R_1^{k+2}R_2^{2n}(2n+k+3)^\mu}.$$

On the other hand, we have as in the proof of Proposition 3.3 that for some positive constant $K=K(\mu)$

$$|\partial_x^k\partial_t^n(A_{p,0}(x)y^p)| \le \frac{\overline{C}C_l^p K^p(2n+k)!}{R_1^k R_2^{2n}(2n+k+1)^\mu b_0^p},$$

$$|\partial_x^k\partial_t^n(A_{p,q}(x)y^p(\partial_x y)^q)| \le \frac{\overline{C}C_l^{p+q}K^{p+q}(2n+k+1)!}{R_1^k R_2^{2n}(2n+k+1)^\mu b_0^p b_1^q R_1^q} \quad \forall q\ge1.$$

This yields

$$I_2 + I_3 \leq \sum_{p\geq 1} \frac{\overline{C} C_l^p K^p (2n+k)!}{R_1^k R_2^{2n} (2n+k+1)^\mu b_0^p} + \sum_{q\geq 1}\sum_{p\geq 0} \frac{\overline{C} C_l^{p+q} K^{p+q} (2n+k+1)!}{R_1^k R_2^{2n} (2n+k+1)^\mu b_0^p b_1^q R_1^q}.$$

The desired estimate

$$|\partial_x^{k+2}\partial_t^n y(x,t)| \leq C_{l+1} \frac{(2n+k+2)!}{R_1^{k+2} R_2^{2n} (2n+k+3)^\mu} \tag{A.4}$$

is satisfied provided that

$$\begin{aligned}&\left(\frac{R_1}{R_2}\right)^2 C_l + \overline{C} R_1^2 \sum_{p\geq 1} \frac{1}{(2n+k+1)(2n+k+2)} \left(\frac{2n+k+3}{2n+k+1}\right)^\mu \left(\frac{C_l K}{b_0}\right)^p \\ &+ \overline{C} R_1^2 \sum_{q\geq 1}\sum_{p\geq 0} \frac{1}{2n+k+2} \left(\frac{2n+k+3}{2n+k+1}\right)^\mu \left(\frac{C_l K}{b_0}\right)^p \left(\frac{C_l K}{b_1 R_1}\right)^q \leq C_{l+1}.\end{aligned} \tag{A.5}$$

We assume that for some number $\delta \in (0,1)$,

$$C_l \leq \delta \cdot \min(\frac{b_0}{K}, \frac{b_1 R_1}{K}). \tag{A.6}$$

We set

$$C_{l+1} := \lambda_l C_l := [\left(\frac{R_1}{R_2}\right)^2 + \frac{K}{b_0} \frac{\overline{C} R_1^2}{(2l+1)(2l+2)} \frac{3^\mu}{1-\delta} + \frac{K}{b_1} \frac{\overline{C} R_1}{2l+2} \frac{3^\mu}{(1-\delta)^2}] C_l.$$

With this choice, (A.5) and (A.4) are satisfied. Since $R_1 < R_2$, there exist some number $l_0 \in \mathbb{N}$ such that $\lambda_l \leq 1$ (and hence $C_{l+1} \leq C_l$) for $l \geq l_0$. For (A.6) to be satisfied for all $l \geq 0$, it remains then to choose C_0 sufficiently small so that

$$\max(C_0, \lambda_0 C_0, \lambda_1 \lambda_0 C_0, ..., \lambda_{l_0-1} \cdots \lambda_0 C_0) \leq \delta \cdot \min(\frac{b_0}{K}, \frac{b_1 R_1}{K}).$$

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