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Ann. I. H. Poincaré – AN 37 (2020) 785–815

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

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Unique continuation principles in cones under nonzero Neumann boundary conditions ☆

Serena Dipierro [a,*], Veronica Felli [b], Enrico Valdinoci [a]

[a] *Department of Mathematics and Statistics, University of Western Australia, 35 Stirling Hwy, Crawley, WA 6009, Australia*
[b] *Dipartimento di Scienza dei Materiali, Università di Milano-Bicocca, Via Cozzi 55, 20125 Milano, Italy*

Received 27 March 2019; accepted 28 January 2020
Available online 4 February 2020

Abstract

We consider an elliptic equation in a cone, endowed with (possibly inhomogeneous) Neumann conditions. The operator and the forcing terms can also allow non-Lipschitz singularities at the vertex of the cone.

In this setting, we provide unique continuation results, both in terms of interior and boundary points.

The proof relies on a suitable Almgren-type frequency formula with remainders. As a byproduct, we obtain classification results for blow-up limits.
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MSC: 35J15; 35J25; 35J75

Keywords: Unique continuation; Singular weights; Conical geometry; Blow-up limits; Almgren's frequency formula

1. Introduction

In this article we consider an elliptic equation with Neumann boundary condition. The domain taken into consideration is a cone, and the equation and the boundary condition can be inhomogeneous and be singular at the origin.

The main results that we provide are of unique continuation type. Roughly speaking, we will show that *if a solution vanishes at any order at the vertex of the cone, then the solution must necessarily vanish in a neighborhood of the vertex* (and then everywhere, up to suitable assumptions).

☆ The authors are member of INdAM/GNAMPA. S. Dipierro is supported by the Australian Research Council DECRA DE180100957 "PDEs, free boundaries and applications" and the Fulbright Foundation. S. Dipierro and E. Valdinoci are supported by the Australian Research Council Discovery Project DP170104880 NEW "Nonlocal Equations at Work". V. Felli is partially supported by the PRIN-2015 grant "Variational methods, with applications to problems in mathematical physics and geometry". This work was started on the occasion of a very fruitful visit of V. Felli to the University of Melbourne.
* Corresponding author.
E-mail addresses: serena.dipierro@uwa.edu.au (S. Dipierro), veronica.felli@unimib.it (V. Felli), enrico.valdinoci@uwa.edu.au (E. Valdinoci).

https://doi.org/10.1016/j.anihpc.2020.01.005
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The notion of vanishing can be framed both with respect to the convergence of points coming from the interior of the domain and, under the appropriate assumptions, with respect to the convergence of points coming from the boundary.

From these results, we also obtain classification results for the blow-up limits. The method of proof will rely on the special geometric structure of the cone, which is a set invariant under dilations and in which the normal on the side of the cone is perpendicular to the radial direction. The main analytic tool in use will be an appropriate type of frequency function. Differently from the classical case in [5], the choice of the frequency function in our case has to comprise additional quantities and reminders to deal with the forcing terms and possibly compensate for the singular behaviors near the vertex.

The mathematical setting in which we work is the following. We let $\Omega \subseteq \mathbb{R}^n$, with $n \geqslant 2$, be a cone with vertex at the origin (namely, we assume that $x \in \Omega$ if and only if $tx \in \Omega$ for all $t > 0$). We consider the spherical cap

$$\Sigma = \left\{ \frac{x}{|x|} : x \in \Omega \right\} \subset \mathbb{S}^{n-1} \tag{1.1}$$

and we assume that Σ has C^2 boundary in $\mathbb{S}^{n-1}$.

We also take into account a positive function $A \in W^{1,1}(\Omega)$ such that

$$c \leqslant A(x) \leqslant \frac{1}{c} \quad \text{for some } c > 0 \text{ and a.e. } x \in \Omega. \tag{1.2}$$

For every $r > 0$ we denote $B_r = \{x \in \mathbb{R}^n : |x| < r\}$. We deal with weak solutions of the following partial differential equation in a neighbourhood of the vertex of the cone (to fix the notations we consider $\Omega \cap B_1$) with possibly inhomogeneous Neumann datum:

$$\begin{cases} \operatorname{div}\big(A(x)\,\nabla u(x)\big) = g(x, u(x)), & \text{for every } x \in \Omega \cap B_1, \\ A(x)\nabla u(x) \cdot \nu(x) = f(x, u(x)), & \text{for every } x \in B_1 \cap \partial\Omega, \end{cases} \tag{1.3}$$

where $\nu(x)$ denotes the exterior unit normal of Ω at $x \in \partial\Omega$, $f \in C^1((\overline{\Omega} \setminus \{0\}) \times \mathbb{R})$, and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

We say that a function $u \in H^1(B_1 \cap \Omega)$ is a weak solution to (1.3) if, for all $\varphi \in C^\infty_c(B_1 \cap \overline{\Omega})$,

$$\int_{B_1 \cap \Omega} A(x)\,\nabla u(x) \cdot \nabla\varphi(x)\,dx = -\int_{B_1 \cap \Omega} g(x, u(x))\varphi(x)\,dx + \int_{B_1 \cap \partial\Omega} f(x, u(x))\varphi(x)\,d\mathscr{H}^{n-1}_x. \tag{1.4}$$

As a technical observation, we point out that the integrals at the right hand side of the above identity are finite under the assumptions of Theorem 1.1 below in view of the Poincaré-type Inequality and the Trace Inequality proved in Corollary 2.3 and Lemma 2.5 respectively.

The use of Almgren-type frequency functions to study unique continuation properties of elliptic partial differential equations dates back to the pioneering contribution of Garofalo and Lin [9] and relies essentially on the possibility of deducing from the boundedness of the frequency quotient a doubling-type condition. Unique continuation from boundary points was investigated via Almgren-type monotonicity arguments in [1,2,8,11,16]. As far as elliptic equations with Neumann-type boundary conditions are concerned, we mention that in [15] boundary unique continuation theorems and doubling properties near the boundary were established under zero Neumann boundary conditions. The main novelty of the present paper is a strong unique continuation result for solutions whose restriction to the boundary vanishes at any order at the vertex under non-homogeneous Neumann boundary conditions, while in [15, Theorem 1.7] unique continuation from the boundary was proved for solutions vanishing on positive surface measure subsets of the boundary and satisfying a zero Neumann condition on such set. The achievement of such a result requires a combination of the monotonicity argument with a blow-up analysis for scaled solutions, in the spirit of [7,6].

We now introduce the notation needed to define the frequency function for our setting. For $r > 0$, we define

$$\begin{aligned} D(r) &:= r^{2-n} \int_{B_r\cap\Omega} A(x)\,|\nabla u(x)|^2\,dx - r^{2-n} \int_{B_r\cap\partial\Omega} f(x,u(x))\,u(x)\,d\mathscr{H}^{n-1}_x \\ &\quad + r^{2-n} \int_{B_r\cap\Omega} g(x,u(x))\,u(x)\,dx \\ \text{and} \qquad H(r) &:= r^{1-n} \int_{\partial B_r\cap\Omega} A(x)\,u^2(x)\,d\mathscr{H}^{n-1}_x \\ &= \int_{\Sigma} A(ry)\,u^2(ry)\,d\mathscr{H}^{n-1}_y. \end{aligned} \tag{1.5}$$

We also introduce the "Almgren frequency function" in our framework, given by

$$\mathscr{N}(r) := \frac{D(r)}{H(r)}. \tag{1.6}$$

With this setting, the pivotal result that we obtain is an appropriate monotonicity formula with reminders, which we state as follows:

Theorem 1.1. *Suppose that* (1.2) *holds and*

$$|\nabla A(x)\cdot x| \leqslant \varepsilon_r\,A(x), \qquad \text{for a.e. } x\in B_r\cap\Omega, \text{ with } \lim_{r\searrow 0}\varepsilon_r = 0, \tag{1.7}$$

$$|\nabla A(x)| \leqslant \frac{C\,A(x)}{|x|}, \qquad \text{for a.e. } x\in B_1\cap\Omega, \tag{1.8}$$

$$|f(x,t)| \leqslant C\,A(x)\,|x|^{\delta-1}\,|t|, \qquad \text{for a.e. } x\in \Omega\cap B_1 \text{ and any } t\in\mathbb{R}, \tag{1.9}$$

$$|\nabla_x f(x,t)| \leqslant C\,A(x)\,|x|^{\delta-2}\,|t|, \qquad \text{for a.e. } x\in \Omega\cap B_1 \text{ and any } t\in\mathbb{R}, \tag{1.10}$$

$$\text{and} \quad |g(x,t)| \leqslant C\,A(x)\,|x|^{\delta-2}\,|t|, \qquad \text{for a.e. } x\in B_1\cap\Omega \text{ and any } t\in\mathbb{R}, \tag{1.11}$$

for some $C > 0$ and $\delta > 0$.

Let also

$$F(x,t) := \int_0^t f(x,\tau)\,d\tau. \tag{1.12}$$

Let

$$u \in H^1(\Omega\cap B_1)\cap L^\infty(\Omega\cap B_1) \tag{1.13}$$

be a solution of (1.3) *in the sense of* (1.4)*, such that*

$$u \not\equiv 0 \text{ in } \Omega\cap B_r, \tag{1.14}$$

for all $r\in(0,1)$.

Then the following holds true.

(i) *There exists $r_0 > 0$ such that*

$$H(r) > 0 \quad \text{and} \quad \mathscr{N}(r) + 1 > 0 \quad \text{for all } r\in(0,r_0); \tag{1.15}$$

in particular the function $\mathscr{N}$ defined in (1.6) *is well defined on $(0,r_0)$.*

(ii) *There exist $r_1\in(0,r_0)$ and $C_1 > 0$ such that*

$$\mathscr{N}'(r) \geqslant -C_1 \max\{r^\delta, \varepsilon_r\} r^{-1}(2+\mathscr{N}(r)) \quad \text{for all } r\in(0,r_1). \tag{1.16}$$

(iii) *If also*

$$r \mapsto \frac{\varepsilon_r}{r} \in L^1(0, r_1), \tag{1.17}$$

then the limit

$$\gamma := \lim_{r \searrow 0} \mathcal{N}(r) \tag{1.18}$$

exists, is finite and $\gamma \geqslant 0$.

We observe that the assumptions of Theorem 1.1 are very general and do not necessarily require the weight A to be Lipschitz continuous or the source terms f and g to be bounded. In particular, estimate (1.16) requires assumptions (1.7) and (1.8) which could be satisfied even by unbounded potentials, as for example $A(x) = \log|x|(\cos(x_n/|x|) - 2)$. On the other hand, to prove that $\mathcal{N}$ is bounded and has finite limit as $r \to 0^+$ assumption (1.17) is also needed; we observe that (1.17) forces the boundedness of A but could be satisfied by non-Lipschitz continuous weights, like $A(x) = 1 + |x|^\delta$ with δ positive and small, for example.

The functions f and g can be singular as well, in accordance with (1.9) and (1.11). To allow all these possible singularities, it is crucial that the "frequency function" also takes into account the special behaviors of A, f and g, as in (1.5). Moreover, the special geometry of the cone Ω will turn out to be the cornerstone for our main estimates to hold, thus providing an interesting interplay between analytic and geometric properties of the problem.

We also observe that condition (1.14) is quite natural, since it requires that the solution is nontrivial in any neighborhood of the vertex of the cone. Furthermore, under the additional assumption that A is locally Lipschitz continuous, assumption (1.14) is satisfied by all nontrivial solutions, in light of the classical unique continuation principle in [10], see also [12] (similarly, if A satisfies a Muckenhoupt-type assumption, then (1.14) is a consequence of the unique continuation principle in [16], see also [9]).

From Theorem 1.1 and a "doubling property" method one obtains a number of results of unique continuation type. In this spirit, we first provide a unique continuation result from the vertex of the cone with respect to interior points:

Theorem 1.2. *Let u be a solution of* (1.3)*, under assumptions* (1.2)*,* (1.7)*,* (1.8)*,* (1.9)*,* (1.10)*,* (1.11)*,* (1.13) *and* (1.17)*.*

Assume also that u vanishes at the origin at any order with respect to interior points, namely that for every $k \in \mathbb{N}$

$$\lim_{\Omega \ni x \to 0} \frac{u(x)}{|x|^k} = 0. \tag{1.19}$$

Then there exists $r > 0$ such that

$$u \equiv 0 \text{ in } \Omega \cap B_r. \tag{1.20}$$

If, in addition, A is locally Lipschitz continuous, then

$$u \equiv 0 \text{ in } \Omega \cap B_1. \tag{1.21}$$

An interesting consequence of our Theorem 1.1 deals with blow-up limits. Namely, for each $\lambda > 0$, we define

$$u_\lambda(x) := \frac{u(\lambda x)}{\sqrt{H(\lambda)}}. \tag{1.22}$$

We consider the Laplace-Beltrami operator $\mathcal{L}_\Sigma := -\Delta_{\mathbb{S}^{n-1}}$ on the spherical cap Σ under null Neumann boundary conditions. By classical spectral theory, the spectrum of the operator $\mathcal{L}_\Sigma$ is discrete and consists in a nondecreasing diverging sequence of eigenvalues $0 = \lambda_1(\Sigma) < \lambda_2(\Sigma) \leqslant \cdots \leqslant \lambda_k(\Sigma) \leqslant \cdots$ with finite multiplicity.

In the following theorem we describe the limit profiles of the blowed-up family (1.22) in terms of the eigenvalues and the eigenfunctions of $\mathcal{L}_\Sigma$.

Theorem 1.3. *Let u be a solution of* (1.3)*, under assumptions* (1.2)*,* (1.7)*,* (1.8)*,* (1.9)*,* (1.10)*,* (1.11)*,* (1.13) *and* (1.17)*.*

Assume that (1.14) *holds true,*

$$|f_t(x,t)| \leqslant C\,|x|^{\delta-1}, \quad \text{for a.e. } x \in \Omega \cap B_1 \text{ and any } t \in \mathbb{R}, \tag{1.23}$$

and that

$$\lim_{x\to 0} A(x) = 1. \tag{1.24}$$

Then, up to a subsequence, as $\lambda \searrow 0$*, we have that* u_λ *converges strongly in* $H^1(\Omega\cap B_1)$ *to a function* $\tilde u$ *which is positively homogeneous and can be written in the form*

$$\tilde u(x) = |x|^\gamma \,\psi\left(\frac{x}{|x|}\right), \tag{1.25}$$

where

$$\gamma = -\frac{n-2}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_{k_0}(\Sigma)} \geqslant 0$$

for some $k_0 \in \mathbb{N}\setminus\{0\}$ *and* ψ *is an eigenfunction of the operator* $\mathscr{L}_\Sigma$ *associated to the eigenvalue* $\lambda_{k_0}(\Sigma)$ *such that*

$$\int_\Sigma \psi^2(x)\, d\mathscr{H}^{n-1}_x = 1. \tag{1.26}$$

From Theorem 1.3, one can also obtain a unique continuation result from the vertex of the cone with respect to boundary points:

Theorem 1.4. *Let* u *be a solution of* (1.3)*, under assumptions* (1.2)*,* (1.7)*,* (1.8)*,* (1.9)*,* (1.10)*,* (1.11)*,* (1.13)*,* (1.17)*,* (1.23) *and* (1.24)*.*

Assume also that u *vanishes at the origin at any order with respect to boundary points, namely that for every* $k\in\mathbb{N}$

$$\lim_{\partial\Omega\ni x\to 0} \frac{u(x)}{|x|^k} = 0. \tag{1.27}$$

Then there exists $r>0$ *such that*

$$u \equiv 0 \text{ in } \Omega\cap B_r. \tag{1.28}$$

If, in addition, A *is locally Lipschitz continuous, then*

$$u \equiv 0 \text{ in } \Omega. \tag{1.29}$$

We stress that while (1.19) is assumed for interior points, we have that hypothesis (1.27) focuses on boundary points.

The rest of the article is organized as follows. Section 2 presents a number of ancillary results, to be exploited in the proofs of the main theorems. In particular, we will collect there some observations on the geometry of the cone and suitable functional inequalities.

The proof of Theorem 1.1 is presented in Section 3 and will serve as a pivotal result for the main theorems of this paper. Namely, Theorem 1.2 will be proved in Section 4, Theorem 1.3 will be proved in Section 5, and Theorem 1.4 will be proved in Section 6.

2. Toolbox

This section collects ancillary results used in the main proofs.

2.1. Cone structure

We recall here an elementary property of the cones:

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a cone with respect to the origin. Then*

$$\nu(x) \cdot x = 0 \qquad \text{for any } x \in \partial\Omega \setminus \{0\}. \tag{2.1}$$

Proof. Fixed $x_0 \in \partial\Omega \setminus \{0\}$, we have that there exists $r_0 > 0$ such that $\Omega \cap B_r(x_0)$ coincides with the sublevel sets of some nondegenerate function $\Phi_0 : B_r(x_0) \to \mathbb{R}$, with $\nu(x) = \frac{\nabla\Phi_0(x)}{|\nabla\Phi_0(x)|}$. By the cone structure of Ω, we thereby see that, for any t close to 1,

$$0 = \Phi_0(x_0) = \Phi_0(tx_0),$$

and so

$$0 = \frac{d}{dt}\Phi_0(tx_0)\bigg|_{t=1} = \nabla\Phi_0(x_0) \cdot x_0 = |\nabla\Phi(x_0)|\, \nu(x_0) \cdot x_0.$$

This proves that $\nu(x_0) \cdot x_0 = 0$ and establishes (2.1). □

2.2. A Poincaré-type Inequality

In this subsection, we provide some results concerning suitable weighted Poincaré-type Inequalities which will play an important role in some of the technical estimates needed to prove the main results.

Lemma 2.2. *Let $\mu \in (-\infty, n)$. Let $\Omega \subset \mathbb{R}^n$ be a cone with respect to the origin such that the spherical cap Σ defined in (1.1) is smooth. Let $A \in Ł^\infty(\Omega)$ satisfy (1.8). For every $r > 0$ and $u \in C^\infty(\overline{\Omega \cap B_r})$*

$$\int\limits_{\Omega\cap B_r} \left(\frac{n-\mu}{2}A(x) + \nabla A(x)\cdot x\right)\frac{u^2(x)}{|x|^\mu}\,dx \leqslant \frac{1}{r^{\mu-1}}\int\limits_{\partial B_r\cap\Omega} A(x)u^2(x)\,d\mathscr{H}^{n-1}_x + \frac{2}{n-\mu}\int\limits_{\Omega\cap B_r}\frac{A(x)|\nabla u(x)|^2}{|x|^{\mu-2}}\,dx.$$

Proof. Let $u \in C^\infty(\overline{\Omega \cap B_r})$. Since

$$\operatorname{div}\left(Au^2\frac{x}{|x|^\mu}\right) = \frac{n-\mu}{|x|^\mu}Au^2 + u^2\nabla A\cdot\frac{x}{|x|^\mu} + 2Au\nabla u\cdot\frac{x}{|x|^\mu},$$

by the Divergence Theorem and (2.1) we deduce that

$$\begin{aligned}
&(n-\mu)\int\limits_{\Omega\cap B_r}\frac{A(x)u^2(x)}{|x|^\mu}\,dx\\
&= \int\limits_{\Omega\cap B_r}\left[\operatorname{div}\left(A(x)u^2(x)\frac{x}{|x|^\mu}\right) - u^2\nabla A(x)\cdot\frac{x}{|x|^\mu} - 2A(x)\,u(x)\nabla u(x)\cdot\frac{x}{|x|^\mu},\right]\\
&= \frac{1}{r^{\mu-1}}\int\limits_{\partial B_r\cap\Omega} A(x)\,u^2(x)\,d\mathscr{H}^{n-1} - \int\limits_{\Omega\cap B_r}\frac{(\nabla A(x)\cdot x)u^2(x)}{|x|^\mu}\,dx - 2\int\limits_{\Omega\cap B_r}\frac{A(x)u\nabla u(x)\cdot x}{|x|^\mu}\,dx\\
&\leqslant \frac{1}{r^{\mu-1}}\int\limits_{\partial B_r\cap\Omega} A(x)u^2(x)\,d\mathscr{H}^{n-1} - \int\limits_{\Omega\cap B_r}\frac{(\nabla A(x)\cdot x)u^2(x)}{|x|^\mu}\,dx + \frac{n-\mu}{2}\int\limits_{\Omega\cap B_r}\frac{A(x)u^2(x)}{|x|^\mu}\,dx\\
&\quad + \frac{2}{n-\mu}\int\limits_{\Omega\cap B_r}\frac{A(x)|\nabla u(x)|^2}{|x|^{\mu-2}}\,dx,
\end{aligned}$$

and hence the conclusion follows. □

Corollary 2.3. *Let $\mu \in (-\infty, n)$. Let $\Omega \subset \mathbb{R}^n$ be a cone with respect to the origin such that the spherical cap Σ defined in (1.1) is smooth. Let $c \in \left(0, \frac{n-\mu}{2}\right)$ and $A \in Ł^\infty(\Omega)$ satisfy (1.8) and (1.7). Then there exists $r_\mu > 0$ such that for every $r \in (0, r_\mu)$ and $u \in C^\infty(\overline{\Omega \cap B_r})$*

$$c \int\limits_{\Omega\cap B_r} \frac{A(x)u^2(x)}{|x|^\mu}\,dx \leqslant \frac{1}{r^{\mu-1}} \int\limits_{\partial B_r\cap\Omega} A(x)u^2(x)\,d\mathscr{H}^{n-1}_x + \frac{2}{n-\mu} \int\limits_{\Omega\cap B_r} \frac{A(x)|\nabla u(x)|^2}{|x|^{\mu-2}}\,dx.$$

Proof. Exploiting (1.7), we observe that

$$\frac{n-\mu}{2}A(x) + \nabla A(x)\cdot x \geqslant \frac{n-\mu}{2}A(x) - \varepsilon_r\, A(x) \geqslant cA(x),$$

as long as r is small enough, and hence the desired result follows by Lemma 2.2. □

For $\mu < 2$ the previous corollary yields the following result.

Corollary 2.4. *Let $\mu < 2$. Let $\Omega \subset \mathbb{R}^n$ be a cone with respect to the origin such that the spherical cap Σ defined in (1.1) is smooth. Let $c \in \big(0, \frac{n-\mu}{2}\big)$ and $A \in Ł^\infty(\Omega)$ satisfy (1.8) and (1.7). Then there exists $r_\mu > 0$ such that, for every $r \in (0, r_\mu)$ and $u \in H^1(\Omega\cap B_r)$, $u|x|^{-\mu/2} \in L^2(\Omega\cap B_r)$ and*

$$c \int\limits_{\Omega\cap B_r} \frac{A(x)u^2(x)}{|x|^\mu}\,dx \leqslant \frac{1}{r^{\mu-1}} \int\limits_{\partial B_r\cap\Omega} A(x)u^2(x)\,d\mathscr{H}^{n-1}_x + \frac{2\,r^{2-\mu}}{n-\mu} \int\limits_{\Omega\cap B_r} A(x)|\nabla u(x)|^2\,dx.$$

Proof. The inequality for $u \in C^\infty(\overline{\Omega\cap B_r})$ follows easily from Corollary 2.3 and the fact that, since $2-\mu > 0$, $|x|^{2-\mu} \leqslant r^{2-\mu}$ in $\Omega\cap B_r$. The conclusion follows by density and the Fatou's Lemma. □

2.3. Trace Inequalities

Now we present a result of trace-type which will be exploited in the proofs of the main theorems.

Lemma 2.5. *Let $\gamma \in (-\infty, n-1)$. Let $\Omega \subset \mathbb{R}^n$ be a cone with respect to the origin such that the spherical cap Σ defined in (1.1) is smooth. Let $A \in Ł^\infty(\Omega)$ satisfy (1.2). For every $r > 0$ and $u \in C^\infty(\overline{\Omega\cap B_r})$ we have that*

$$\int\limits_{\partial\Omega\cap B_r} \frac{A(x)\,u^2(x)}{|x|^\gamma}\,d\mathscr{H}^{n-1} \leqslant C \int\limits_{\Omega\cap B_r} \left[\frac{A(x)\,|\nabla u(x)|^2}{|x|^{\gamma-1}} + \frac{A(x)u^2(x)}{|x|^{\gamma+1}}\right] dx,$$

for some $C > 0$ independent of r. Furthermore, if $\gamma < 1$, then every function $u \in H^1(\Omega\cap B_r)$ has a trace belonging to $L^2(\partial\Omega\cap B_r; |x|^{-\gamma/2})$ and

$$\int\limits_{\partial\Omega\cap B_r} \frac{A(x)\,u^2(x)}{|x|^\gamma}\,d\mathscr{H}^{n-1} \leqslant C \left[r^{1-\gamma} \int\limits_{\Omega\cap B_r} A(x)\,|\nabla u(x)|^2\,dx + \int\limits_{\Omega\cap B_r} \frac{A(x)u^2(x)}{|x|^{\gamma+1}} \right] dx.$$

Proof. We let $u \in C^\infty(\overline{\Omega\cap B_r})$. Also, for all $\rho \in (0, r)$ and $\theta \in \Sigma$, we define $u^{(\rho)}(\theta) := u(\rho\theta)$. By Fubini's Theorem and the Sobolev Trace Theorem on manifolds we have that

$$\begin{aligned}\int_{\partial\Omega\cap B_r} \frac{u^2(x)}{|x|^\gamma}\,d\mathscr{H}^{n-1} &= \int_0^r \rho^{-\gamma}\left(\int_{\partial\Omega\cap\partial B_\rho} u^2\,d\mathscr{H}^{n-2}\right)d\rho\\ &= \int_0^r \rho^{-\gamma+n-2}\left(\int_{\partial\Sigma} u^2(\rho\theta)\,d\theta\right)d\rho\\ &= \int_0^r \rho^{-\gamma+n-2}\left(\int_{\partial\Sigma} |u^{(\rho)}(\theta)|^2\,d\theta\right)d\rho\\ &\leqslant C\int_0^r \rho^{-\gamma+n-2}\left(\int_{\Sigma}\Big(|\nabla_\theta u^{(\rho)}(\theta)|^2+|u^{(\rho)}(\theta)|^2\Big)\,d\theta\right)d\rho,\end{aligned}$$

where ∇_θ denotes the tangential gradient along Σ, so that, if $x=\rho\theta$,

$$|\nabla_\theta u^{(\rho)}(\theta)| = \rho\left|\nabla u(x) - \big(\nabla u(x)\cdot x\big)\frac{x}{|x|^2}\right| \leqslant \rho\,|\nabla u(x)|.$$

Hence, in view of (1.2), we find that

$$\begin{aligned}\int_{\partial\Omega\cap B_r} \frac{A(x)\,u^2(x)}{|x|^\gamma}\,d\mathscr{H}^{n-1} &\leqslant \frac{C}{c}\int_0^r \rho^{-\gamma+n-2}\left(\int_{\Sigma}\Big(\rho^2|\nabla u(\rho\theta)|^2+u^2(\rho\theta)\Big)\,d\theta\right)d\rho\\ &= \frac{C}{c}\int_{\Omega\cap B_r} |x|^{-\gamma-1}\Big(|x|^2|\nabla u(x)|^2+u^2(x)\Big)\,dx\\ &\leqslant \frac{C}{c^2}\int_{\Omega\cap B_r} |x|^{-\gamma-1}\Big(|x|^2 A(x)|\nabla u(x)|^2+A(x)u^2(x)\Big)\,dx,\end{aligned}$$

which yields the inequality for functions in $C^\infty(\overline{\Omega\cap B_r})$. If $\gamma<1$, then $|x|^{1-\gamma}\leqslant r^{1-\gamma}$ in $\Omega\cap B_r$, then the conclusion follows by density and Fatou's Lemma. □

3. Proof of Theorem 1.1

We first observe that, by elliptic regularity theory (see e.g. Theorem 8.13 in [14], [3,4] or [13]) we have that, under the assumptions of Theorem 1.1,

$$u\in H^2\big(\Omega\cap(B_r\setminus B_\delta)\big),\qquad \text{for all } 0<\delta<r<1. \tag{3.1}$$

We denote by ν both the exterior normal at $\partial\Omega$ and the exterior normal at ∂B_r, since no confusion can arise. Testing the equation in (1.3) against the solution itself, we see that

$$\begin{aligned}\int_{B_r\cap\Omega} gu &= \int_{B_r\cap\Omega} \operatorname{div}(A\nabla u)\,u\\ &= \int_{B_r\cap\Omega}\Big(\operatorname{div}(Au\nabla u)-A|\nabla u|^2\Big)\\ &= \int_{\partial B_r\cap\Omega} Au\nabla u\cdot\nu + \int_{B_r\cap\partial\Omega} Au\nabla u\cdot\nu - \int_{B_r\cap\Omega} A|\nabla u|^2\\ &= \int_{\partial B_r\cap\Omega} Au\nabla u\cdot\nu + \int_{B_r\cap\partial\Omega} fu - \int_{B_r\cap\Omega} A|\nabla u|^2.\end{aligned}$$

Hence, recalling (1.5),

$$\int_{\partial B_r\cap\Omega} Au\nabla u\cdot\nu = r^{n-2}D(r). \tag{3.2}$$

Using again (1.3), we also observe that

$$\begin{aligned}
&\operatorname{div}\left(A(\nabla u\cdot x)\nabla u-\frac{A}{2}|\nabla u|^2x\right)-(\nabla u\cdot x)g\\
&=(\nabla u\cdot x)\operatorname{div}(A\nabla u)-(\nabla u\cdot x)g+A\nabla u\cdot\nabla(\nabla u\cdot x)-\frac{1}{2}\operatorname{div}(A|\nabla u|^2x)\\
&=\sum_{i,j=1}^{n}\Big(A\partial_i u\,\partial_i(\partial_j u\,x_j)-\frac12\partial_i(A(\partial_j u)^2x_i)\Big)\\
&=\sum_{i,j=1}^{n}\Big(A\partial_i u\,\partial^2_{ij}u\,x_j+A(\partial_i u)^2\delta_{ij}-\frac12\partial_i A(\partial_j u)^2x_i-A\partial_j u\,\partial^2_{ij}u\,x_i-\frac12A(\partial_j u)^2\Big)\\
&=\sum_{i,j=1}^{n}\Big(A(\partial_i u)^2\delta_{ij}-\frac12\partial_i A(\partial_j u)^2x_i-\frac12A(\partial_j u)^2\Big)\\
&=\frac{(2-n)\,A}{2}|\nabla u|^2-\frac12|\nabla u|^2\nabla A\cdot x.
\end{aligned} \tag{3.3}$$

On the other hand, from (1.5) we know that

$$\begin{aligned}
D'(r)=(2-n)r^{1-n}&\int_{B_r\cap\Omega}A|\nabla u|^2+r^{2-n}\int_{\partial B_r\cap\Omega}A|\nabla u|^2\\
&-(2-n)r^{1-n}\int_{B_r\cap\partial\Omega}fu-r^{2-n}\int_{\partial B_r\cap\partial\Omega}fu\\
&+(2-n)r^{1-n}\int_{B_r\cap\Omega}gu+r^{2-n}\int_{\partial B_r\cap\Omega}gu,
\end{aligned} \tag{3.4}$$

and (recalling that Ω is a cone, hence $\Omega/r=\Omega$ for each $r>0$)

$$\begin{aligned}
H'(r)&=\int_\Sigma \nabla A(ry)\cdot y\,u^2(ry)\,d\mathscr{H}^{n-1}_y+2\int_\Sigma A(ry)\,u(ry)\nabla u(ry)\cdot y\,d\mathscr{H}^{n-1}_y\\
&=r^{1-n}\int_{\partial B_r\cap\Omega}\nabla A\cdot\nu\,u^2+2r^{1-n}\int_{\partial B_r\cap\Omega}Au\nabla u\cdot\nu.
\end{aligned} \tag{3.5}$$

Thus, comparing (3.2) with (3.5) we conclude that

$$H'(r)-r^{1-n}\int_{\partial B_r\cap\Omega}\nabla A\cdot\nu\,u^2=2r^{1-n}\int_{\partial B_r\cap\Omega}Au\nabla u\cdot\nu=\frac{2D(r)}{r},$$

and therefore

$$D(r)=\frac{rH'(r)}{2}-\frac{r^{2-n}}{2}\int_{\partial B_r\cap\Omega}\nabla A\cdot\nu\,u^2. \tag{3.6}$$

From (3.1) it follows that, for all $0<\delta<r<1$, $A(\nabla u\cdot x)\nabla u-\frac{A}{2}|\nabla u|^2x\in W^{1,1}(\Omega\cap(B_r\setminus B_\delta)$ so that

$$\int_{\Omega\cap(B_r\setminus B_\delta)} \operatorname{div}\left(A(\nabla u\cdot x)\nabla u - \frac{A}{2}|\nabla u|^2 x\right) = \int_{\partial B_r\cap\Omega} r\left(A(\nabla u\cdot\nu)^2 - \frac{A}{2}|\nabla u|^2\right)$$
$$- \int_{\partial B_\delta\cap\Omega} \delta\left(A(\nabla u\cdot\nu)^2 - \frac{A}{2}|\nabla u|^2\right) + \int_{(B_r\setminus B_\delta)\cap\partial\Omega} \left(f\,\nabla u\cdot x - \tfrac{A}{2}|\nabla u|^2 x\cdot\nu\right). \quad (3.7)$$

Since

$$\int_0^1 \left[\int_{\Omega\cap\partial B_r} |\nabla u|^2\right] dr = \int_{\Omega\cap B_1} |\nabla u|^2 < +\infty,$$

there exists a decreasing sequence $\{\delta_n\}\subset(0,1)$ such that $\lim_{n\to+\infty}\delta_n = 0$ and

$$\delta_n \int_{\Omega\cap\partial B_{\delta_n}} |\nabla u|^2 \longrightarrow 0 \quad \text{as } n\to+\infty.$$

Choosing $\delta=\delta_n$ in (3.7) and letting $n\to\infty$ we then obtain

$$\int_{\Omega\cap B_r} \operatorname{div}\left(A(\nabla u\cdot x)\nabla u - \frac{A}{2}|\nabla u|^2 x\right)$$
$$= \int_{\partial B_r\cap\Omega} r\left(A(\nabla u\cdot\nu)^2 - \frac{A}{2}|\nabla u|^2\right) + \int_{B_r\cap\partial\Omega}\left(f\,\nabla u\cdot x - \tfrac{A}{2}|\nabla u|^2 x\cdot\nu\right).$$

Therefore, taking into account (3.3),

$$(2-n)r^{1-n}\int_{B_r\cap\Omega} A|\nabla u|^2$$
$$= 2r^{1-n}\int_{B_r\cap\Omega}\left[\frac{1}{2}|\nabla u|^2\nabla A\cdot x + \operatorname{div}\left(A(\nabla u\cdot x)\nabla u - \frac{A}{2}|\nabla u|^2 x\right) - (\nabla u\cdot x)g\right]$$
$$= r^{1-n}\int_{B_r\cap\Omega}\left(|\nabla u|^2\nabla A\cdot x - 2(\nabla u\cdot x)g\right) + \int_{\partial B_r\cap\Omega}\left(2r^{-n}A(\nabla u\cdot x)^2 - r^{2-n}A|\nabla u|^2\right)$$
$$+ r^{1-n}\int_{B_r\cap\partial\Omega}\left(2f\,\nabla u\cdot x - A|\nabla u|^2 x\cdot\nu\right).$$

We thereby substitute this identity into (3.4) and we conclude that

$$D'(r) = r^{1-n}\int_{B_r\cap\Omega}\left(|\nabla u|^2\nabla A\cdot x - 2(\nabla u\cdot x)g\right) + 2r^{-n}\int_{\partial B_r\cap\Omega} A(\nabla u\cdot x)^2$$
$$+ r^{1-n}\int_{B_r\cap\partial\Omega}\left(2f\,\nabla u\cdot x - A|\nabla u|^2 x\cdot\nu\right)$$
$$- (2-n)r^{1-n}\int_{B_r\cap\partial\Omega} fu - r^{2-n}\int_{\partial B_r\cap\partial\Omega} fu$$
$$+ (2-n)r^{1-n}\int_{B_r\cap\Omega} gu + r^{2-n}\int_{\partial B_r\cap\Omega} gu.$$

From this and (3.6), we find that

$$
\begin{aligned}
&D'(r)H(r) - H'(r)D(r) \\
&= H(r)\Bigg[r^{1-n} \int_{B_r\cap\Omega} \Big(|\nabla u|^2 \nabla A\cdot x - 2(\nabla u\cdot x) g \Big) + 2r^{-n} \int_{\partial B_r\cap\Omega} A(\nabla u\cdot x)^2 \\
&\quad + r^{1-n} \int_{B_r\cap\partial\Omega} \Big(2f\,\nabla u\cdot x - A|\nabla u|^2 x\cdot\nu \Big) \\
&\quad - (2-n) r^{1-n} \int_{B_r\cap\partial\Omega} fu - r^{2-n} \int_{\partial B_r\cap\partial\Omega} fu \\
&\quad + (2-n) r^{1-n} \int_{B_r\cap\Omega} gu + r^{2-n} \int_{\partial B_r\cap\Omega} gu \Bigg] \\
&\quad - \frac{r(H'(r))^2}{2} + \frac{r^{2-n}\,H'(r)}{2} \int_{\partial B_r\cap\Omega} \nabla A\cdot\nu\, u^2.
\end{aligned} \tag{3.8}
$$

On the other hand, recalling (3.5), we see that

$$
\begin{aligned}
&- \frac{r(H'(r))^2}{2} + \frac{r^{2-n}\,H'(r)}{2} \int_{\partial B_r\cap\Omega} \nabla A\cdot\nu\, u^2 \\
&= -\frac{r}{2}\left(r^{1-n} \int_{\partial B_r\cap\Omega} \nabla A\cdot\nu\, u^2 + 2r^{1-n} \int_{\partial B_r\cap\Omega} Au\nabla u\cdot\nu \right)^2 \\
&\quad + \frac{r^{2-n}}{2} \left(\int_{\partial B_r\cap\Omega} \nabla A\cdot\nu\, u^2 \right) \left(r^{1-n} \int_{\partial B_r\cap\Omega} \nabla A\cdot\nu\, u^2 + 2r^{1-n} \int_{\partial B_r\cap\Omega} Au\nabla u\cdot\nu \right) \\
&= -\frac{r^{3-2n}}{2} \left(\int_{\partial B_r\cap\Omega} \nabla A\cdot\nu\, u^2 \right)^2 - 2r^{3-2n} \left(\int_{\partial B_r\cap\Omega} \nabla A\cdot\nu\, u^2 \right) \left(\int_{\partial B_r\cap\Omega} Au\nabla u\cdot\nu \right) \\
&\quad - 2r^{3-2n} \left(\int_{\partial B_r\cap\Omega} Au\nabla u\cdot\nu \right)^2 \\
&\quad + \frac{r^{3-2n}}{2} \left(\int_{\partial B_r\cap\Omega} \nabla A\cdot\nu\, u^2 \right)^2 + r^{3-2n} \left(\int_{\partial B_r\cap\Omega} \nabla A\cdot\nu\, u^2 \right) \left(\int_{\partial B_r\cap\Omega} Au\nabla u\cdot\nu \right) \\
&= -r^{3-2n} \left(\int_{\partial B_r\cap\Omega} \nabla A\cdot\nu\, u^2 \right) \left(\int_{\partial B_r\cap\Omega} Au\nabla u\cdot\nu \right) - 2r^{3-2n} \left(\int_{\partial B_r\cap\Omega} Au\nabla u\cdot\nu \right)^2.
\end{aligned}
$$

Hence, substituting this identity into (3.8), we conclude that

$$
\begin{aligned}
&D'(r)H(r) - H'(r)D(r) \\
&= H(r)\Bigg[r^{1-n} \int_{B_r\cap\Omega} \Big(|\nabla u|^2 \nabla A\cdot x - 2(\nabla u\cdot x)g\Big) + 2r^{-n} \int_{\partial B_r\cap\Omega} A(\nabla u\cdot x)^2 \\
&\quad + r^{1-n} \int_{B_r\cap\partial\Omega} \Big(2f\,\nabla u\cdot x - A|\nabla u|^2 x\cdot\nu\Big) \\
&\quad - (2-n)r^{1-n} \int_{B_r\cap\partial\Omega} fu - r^{2-n} \int_{\partial B_r\cap\partial\Omega} fu \\
&\quad +(2-n)r^{1-n} \int_{B_r\cap\Omega} gu + r^{2-n} \int_{\partial B_r\cap\Omega} gu \Bigg] \\
&\quad - r^{3-2n} \left(\int_{\partial B_r\cap\Omega} \nabla A\cdot\nu\, u^2 \right) \left(\int_{\partial B_r\cap\Omega} Au\nabla u\cdot\nu \right) - 2r^{3-2n} \left(\int_{\partial B_r\cap\Omega} Au\nabla u\cdot\nu \right)^2 .
\end{aligned}
\tag{3.9}
$$

Moreover, from the Cauchy-Schwarz Inequality, we know that

$$
\int_{\partial B_r\cap\Omega} Au\nabla u\cdot x \leqslant \sqrt{\int_{\partial B_r\cap\Omega} Au^2 \int_{\partial B_r\cap\Omega} A(\nabla u\cdot x)^2}\,.
$$

Consequently, using again (1.5), we also observe that

$$
\begin{aligned}
&2r^{-n}H(r) \int_{\partial B_r\cap\Omega} A(\nabla u\cdot x)^2 - 2r^{3-2n} \left(\int_{\partial B_r\cap\Omega} Au\nabla u\cdot\nu \right)^2 \\
&= 2r^{1-2n} \left[\left(\int_{\partial B_r\cap\Omega} A\,u^2 \right) \left(\int_{\partial B_r\cap\Omega} A(\nabla u\cdot x)^2 \right) - \left(\int_{\partial B_r\cap\Omega} Au\nabla u\cdot x \right)^2 \right] \\
&\geqslant 0.
\end{aligned}
$$

Plugging this information into (3.9), we thus obtain that

$$
\begin{aligned}
&D'(r)H(r) - H'(r)D(r) \\
&\geqslant H(r)\Bigg[r^{1-n} \int_{B_r\cap\Omega} \Big(|\nabla u|^2 \nabla A\cdot x - 2(\nabla u\cdot x)g\Big) \\
&\quad + r^{1-n} \int_{B_r\cap\partial\Omega} \Big(2f\,\nabla u\cdot x - A|\nabla u|^2 x\cdot\nu\Big) \\
&\quad - (2-n)r^{1-n} \int_{B_r\cap\partial\Omega} fu - r^{2-n} \int_{\partial B_r\cap\partial\Omega} fu \\
&\quad +(2-n)r^{1-n} \int_{B_r\cap\Omega} gu + r^{2-n} \int_{\partial B_r\cap\Omega} gu \Bigg]
\end{aligned}
\tag{3.10}
$$

$$- r^{3-2n} \left(\int_{\partial B_r \cap \Omega} \nabla A \cdot \nu \, u^2 \right) \left(\int_{\partial B_r \cap \Omega} A u \nabla u \cdot \nu \right).$$

Then, from (3.10) and (2.1), we obtain that

$$\begin{aligned} & D'(r)H(r) - H'(r)D(r) \\ & \geqslant H(r) \Bigg[r^{1-n} \int_{B_r \cap \Omega} \left(|\nabla u|^2 \nabla A \cdot x - 2(\nabla u \cdot x) g \right) \\ & \quad + 2r^{1-n} \int_{B_r \cap \partial\Omega} f \, \nabla u \cdot x - (2-n) r^{1-n} \int_{B_r \cap \partial\Omega} f u - r^{2-n} \int_{\partial B_r \cap \partial\Omega} f u \\ & \quad + (2-n) r^{1-n} \int_{B_r \cap \Omega} g u + r^{2-n} \int_{\partial B_r \cap \Omega} g u \Bigg] \\ & \quad - r^{3-2n} \left(\int_{\partial B_r \cap \Omega} \nabla A \cdot \nu \, u^2 \right) \left(\int_{\partial B_r \cap \Omega} A u \nabla u \cdot \nu \right). \end{aligned} \tag{3.11}$$

Now, we define

$$E(r) := r^{2-n} \int_{B_r \cap \Omega} A |\nabla u|^2. \tag{3.12}$$

By (1.9), we have that

$$\left| \int_{B_r \cap \partial\Omega} f u \right| \leqslant C \int_{B_r \cap \partial\Omega} A \, |x|^{\delta - 1} \, |u|^2. \tag{3.13}$$

On the other hand, by Lemma 2.5 (used here with $\gamma := 1 - \delta$), we see that

$$\int_{B_r \cap \partial\Omega} A \, |x|^{\delta-1} \, |u|^2 \leqslant C \int_{\Omega \cap B_r} \left[r^\delta \, A \, |\nabla u|^2 + \frac{A \, u^2}{|x|^{2-\delta}} \right].$$

Hence, in view of Corollary 2.4 (used here with $\mu := 2 - \delta$), (1.5) and (3.12)

$$\begin{aligned} \int_{B_r \cap \partial\Omega} A \, |x|^{\delta-1} \, |u|^2 & \leqslant C \, r^\delta \int_{\Omega \cap B_r} A \, |\nabla u|^2 + \frac{C}{r^{1-\delta}} \int_{\partial B_r \cap \Omega} A u^2 \\ & \leqslant C \, r^{n-2+\delta} \big(H(r) + E(r) \big). \end{aligned} \tag{3.14}$$

Therefore, in light of (3.13)

$$\left| \int_{B_r \cap \partial\Omega} f u \right| \leqslant C \, r^{n-2+\delta} \big(H(r) + E(r) \big). \tag{3.15}$$

Also, by (1.11) and Corollary 2.4 (used here with $\mu := 2 - \delta$),

$$\begin{aligned} \int_{B_r \cap \Omega} |g| \, |u| & \leqslant C \int_{B_r \cap \Omega} A |x|^{\delta - 2} \, |u|^2 \\ & \leqslant C \, r^\delta \int_{\Omega \cap B_r} A \, |\nabla u|^2 + \frac{C}{r^{1-\delta}} \int_{\partial B_r \cap \Omega} A u^2 \leqslant C \, r^{n-2+\delta} \big(H(r) + E(r) \big). \end{aligned} \tag{3.16}$$

Consequently, by (3.15) and (3.16)

$$E(r)-D(r)\leqslant |D(r)-E(r)|\leqslant r^{2-n}\left|\int_{B_r\cap\partial\Omega} fu-\int_{B_r\cap\Omega} gu\right|\leqslant C\,r^{\delta}\big(H(r)+E(r)\big),$$

and therefore, for any $r\in(0,1)$ sufficiently small,

$$\frac{E(r)}{2}\leqslant(1-Cr^{\delta})\,E(r)\leqslant Cr^{\delta}H(r)+D(r). \tag{3.17}$$

Estimate (3.17) implies statement (i) with $r_0>0$ so small as to satisfy condition (3.17) and $Cr_0^{\delta}<1$. Indeed, let us argue by contradiction and assume that there exists $\bar r\in(0,r_0)$ such that $H(\bar r)=0$. By (1.5) this would imply that $u\equiv 0$ on $\Omega\cap\partial B_{\bar r}$ and hence, in view of (3.2), $D(\bar r)=0$. Then (3.17) yields that $E(\bar r)=0$ and hence u is constant in $\Omega\cap B_{\bar r}$. Therefore $u\equiv 0$ in $\Omega\cap B_{\bar r}$, which is in contradiction with (1.14).

Furthermore, for all $r\in(0,r_0)$, (3.17) implies that

$$0\leqslant\frac{E(r)}{2}<H(r)+D(r), \tag{3.18}$$

and hence $\mathcal{N}(r)+1>0$.

Moreover, from the Sobolev Trace Theorem on manifolds applied on the spherical cap $\partial B_r\cap\partial\Omega=r\partial\Sigma$, we have that, recalling (1.2),

$$\begin{aligned}\int_{\partial B_r\cap\partial\Omega} A\,|u|^2&\leqslant\frac{1}{c}\int_{\partial B_r\cap\partial\Omega}|u|^2=\frac{r^{n-2}}{c}\int_{\partial\Sigma}|u(r\theta)|^2\\&\leqslant Cr^{n-2}\int_{\Sigma}\Big(u^2(r\theta)+|\nabla(u^2(r\theta))|\Big)\\&\leqslant Cr^{n-2}\int_{\Sigma}\Big(u^2(r\theta)+2r|u(r\theta)||\nabla u(r\theta)|\Big)\\&\leqslant Cr^{n-2}H(r)+2Cr^{n-1}\sqrt{\int_{\Sigma}u^2(r\theta)}\sqrt{\int_{\Sigma}|\nabla u(r\theta)|^2}\\&\leqslant Cr^{n-2}H(r)+Cr^{n-\frac{3}{2}}\sqrt{H(r)}\sqrt{r^{2-n}\int_{\Omega\cap\partial B_r}A|\nabla u|^2}\\&\leqslant C\left(r\int_{\partial B_r\cap\Omega}A\,|\nabla u|^2+r^{n-2}H(r)\right)\end{aligned} \tag{3.19}$$

for some $C>0$ independent of r (varying from line to line). Now, we recall (1.9) and we observe that

$$\left|\int_{\partial B_r\cap\partial\Omega} fu\right|\leqslant C\int_{\partial B_r\cap\partial\Omega}A\,|x|^{\delta-1}\,|u|^2=Cr^{\delta-1}\int_{\partial B_r\cap\partial\Omega}A\,|u|^2. \tag{3.20}$$

In addition, from (1.11),

$$\left|\int_{\partial B_r\cap\Omega} gu\right|\leqslant C\int_{\partial B_r\cap\Omega}A\,|x|^{\delta-2}|u|^2=Cr^{\delta-2}\int_{\partial B_r\cap\Omega}A\,|u|^2=Cr^{n+\delta-3}H(r). \tag{3.21}$$

From (3.4), (3.20) and (3.21), we obtain that

$$r^{2-n}\int_{\partial B_r\cap\Omega} A|\nabla u|^2 = D'(r)+\frac{n-2}{r}D(r)+r^{2-n}\int_{\partial B_r\cap\partial\Omega} fu - r^{2-n}\int_{\partial B_r\cap\Omega} gu$$
$$\leqslant D'(r)+\frac{n-2}{r}D(r)+C\left(r^{\delta+1-n}\int_{\partial B_r\cap\partial\Omega} A\,|u|^2+r^{\delta-1}H(r)\right).$$

Then from (3.19) it follows that

$$r^{2-n}\int_{\partial B_r\cap\Omega} A|\nabla u|^2 \leqslant D'(r)+\frac{n-2}{r}D(r)+Cr^{\delta-1}H(r)+Cr^{\delta+2-n}\int_{\partial B_r\cap\Omega} A\,|\nabla u|^2,$$

from which it follows that

$$r^{2-n}\int_{\partial B_r\cap\Omega} A|\nabla u|^2 \leqslant C\left(D'(r)+\frac{n-2}{r}D(r)+Cr^{\delta-1}H(r)\right), \tag{3.22}$$

for some $C>0$ and for all $r>0$ sufficiently small.

Plugging (3.22) into (3.19) we conclude that

$$\begin{aligned}\int_{\partial B_r\cap\partial\Omega} A\,|u|^2 &\leqslant Cr^{n-2}H(r)+Cr^{n-\frac32}\sqrt{H(r)}\sqrt{r^{2-n}\int_{\Omega\cap\partial B_r} A|\nabla u|^2}\\ &\leqslant Cr^{n-2}H(r)+Cr^{n-\frac32}\sqrt{H(r)}\sqrt{D'(r)+\tfrac{n-2}{r}D(r)+Cr^{\delta-1}H(r)}\end{aligned} \tag{3.23}$$

as long as r is sufficiently small. It is now our goal to use the previously obtained information in order to estimate the right hand side of (3.11). To this end, we first observe that, from (1.7),

$$\begin{aligned}& r^{3-2n}\left|\left(\int_{\partial B_r\cap\Omega}\nabla A\cdot\nu\,u^2\right)\left(\int_{\partial B_r\cap\Omega} Au\nabla u\cdot\nu\right)\right|\\ &\leqslant \varepsilon_r\, r^{2-2n}\left|\left(\int_{\partial B_r\cap\Omega} A\,u^2\right)\left(\int_{\partial B_r\cap\Omega} Au\nabla u\cdot\nu\right)\right|\\ &=\varepsilon_r\, r^{1-n}\,H(r)\left|\int_{\partial B_r\cap\Omega} Au\nabla u\cdot\nu\right|\\ &\leqslant \varepsilon_r\, r^{1-n}\,H(r)\sqrt{\int_{\partial B_r\cap\Omega} A\,|u|^2}\sqrt{\int_{\partial B_r\cap\Omega} A\,|\nabla u|^2}\\ &=\varepsilon_r\, r^{\frac{1-n}{2}}\left(H(r)\right)^{\frac32}\sqrt{\int_{\partial B_r\cap\Omega} A\,|\nabla u|^2}.\end{aligned}$$

This and (3.22) lead to

$$\begin{aligned}& r^{3-2n}\left|\left(\int_{\partial B_r\cap\Omega}\nabla A\cdot\nu\,u^2\right)\left(\int_{\partial B_r\cap\Omega} Au\nabla u\cdot\nu\right)\right|\\ &\quad\leqslant C\varepsilon_r\, r^{-\frac12}\left(H(r)\right)^{\frac32}\sqrt{D'(r)+\tfrac{n-2}{r}D(r)+Cr^{\delta-1}H(r)}.\end{aligned} \tag{3.24}$$

Furthermore, by (1.7) and (1.11),

$$\begin{aligned}
& r^{1-n}\left|\int_{B_r\cap\Omega}\left(|\nabla u|^2\nabla A\cdot x-2(\nabla u\cdot x)g\right)\right| \\
& \leqslant Cr^{1-n}\int_{B_r\cap\Omega}\left(\varepsilon_r\, A|\nabla u|^2+A\,|x|^{\delta-1}|\nabla u|\,|u|\right) \\
& \leqslant Cr^{1-n}\left(\varepsilon_r\int_{B_r\cap\Omega}A\,|\nabla u|^2+\sqrt{\int_{B_r\cap\Omega}A\,|\nabla u|^2}\sqrt{\int_{B_r\cap\Omega}A\,|x|^{2(\delta-1)}|u|^2}\right) \\
& = Cr^{1-n}\left(\varepsilon_r r^{n-2}E(r)+r^{\frac{n-2}{2}}\sqrt{E(r)}\sqrt{\int_{B_r\cap\Omega}A\,|x|^{2(\delta-1)}|u|^2}\right).
\end{aligned}$$

Consequently, exploiting Corollary 2.4 with $\mu:=2(1-\delta)$,

$$\begin{aligned}
& r^{1-n}\left|\int_{B_r\cap\Omega}\left(|\nabla u|^2\nabla A\cdot x-2(\nabla u\cdot x)g\right)\right| \\
& \leqslant Cr^{1-n}\left(\varepsilon_r r^{n-2}E(r)+r^{\frac{n-2}{2}}\sqrt{E(r)}\sqrt{r^{2\delta-1}\int_{\partial B_r\cap\Omega}A\,|u|^2+r^{2\delta}\int_{B_r\cap\Omega}A\,|\nabla u|^2}\right) \\
& \leqslant C\left(\frac{\varepsilon_r}{r}E(r)+r^{\delta-1}\sqrt{E(r)}\sqrt{H(r)+E(r)}\right).
\end{aligned}$$

Now, plugging the latter inequality, (3.15), (3.16), (3.21) and (3.24) into (3.11), we conclude that

$$\begin{aligned}
& D'(r)H(r)-H'(r)D(r) \\
& \geqslant H(r)\Bigg[-C\left(\frac{\varepsilon_r}{r}E(r)+r^{\delta-1}\sqrt{E(r)}\sqrt{H(r)+E(r)}\right) \\
& \quad +2r^{1-n}\int_{B_r\cap\partial\Omega}f\,\nabla u\cdot x-C\,r^{\delta-1}\big(H(r)+E(r)\big)-r^{2-n}\int_{\partial B_r\cap\partial\Omega}fu \\
& \quad -C\,r^{\delta-1}\big(H(r)+E(r)\big)-Cr^{\delta-1}H(r)\Bigg] \\
& \quad -C\varepsilon_r\,r^{-\frac{1}{2}}\left(H(r)\right)^{\frac{3}{2}}\sqrt{D'(r)+\tfrac{n-2}{r}D(r)+Cr^{\delta-1}H(r)} \\
& \geqslant H(r)\Bigg[-C\left(\frac{\varepsilon_r}{r}E(r)+r^{\delta-1}\sqrt{E(r)}\sqrt{H(r)+E(r)}\right)-C\,r^{\delta-1}\big(H(r)+E(r)\big) \\
& \quad +2r^{1-n}\int_{B_r\cap\partial\Omega}f\,\nabla u\cdot x-r^{2-n}\int_{\partial B_r\cap\partial\Omega}fu\Bigg] \\
& \quad -C\varepsilon_r\,r^{-\frac{1}{2}}\left(H(r)\right)^{\frac{3}{2}}\sqrt{D'(r)+\tfrac{n-2}{r}D(r)+Cr^{\delta-1}H(r)},
\end{aligned} \tag{3.25}$$

for some $C>0$.

Now, recalling (3.20) and (3.23), we notice that

$$\left|\int_{\partial B_r\cap\partial\Omega} fu\right| \leqslant C\left(r^{n+\delta-3}H(r)+r^{n+\delta-\frac{5}{2}}\sqrt{H(r)}\sqrt{D'(r)+\tfrac{n-2}{r}D(r)+Cr^{\delta-1}H(r)}\right).$$

This and (3.25) give that

$$\begin{aligned}
&D'(r)H(r)-H'(r)D(r)\\
&\geqslant H(r)\Bigg[-C\left(\frac{\varepsilon_r}{r}E(r)+r^{\delta-1}\sqrt{E(r)}\sqrt{H(r)+E(r)}\right)-C\,r^{\delta-1}\big(H(r)+E(r)\big)\\
&\qquad+2r^{1-n}\int_{B_r\cap\partial\Omega} f\,\nabla u\cdot x - C\,r^{\delta-1}H(r)-Cr^{\delta-\frac{1}{2}}\sqrt{H(r)}\sqrt{D'(r)+\tfrac{n-2}{r}D(r)+Cr^{\delta-1}H(r)}\Bigg]\\
&\qquad-C\varepsilon_r\,r^{-\frac{1}{2}}\big(H(r)\big)^{\frac{3}{2}}\sqrt{D'(r)+\tfrac{n-2}{r}D(r)+Cr^{\delta-1}H(r)}\\
&\geqslant H(r)\Bigg[-C\left(\frac{\varepsilon_r}{r}E(r)+r^{\delta-1}\sqrt{E(r)}\sqrt{H(r)+E(r)}\right)-C\,r^{\delta-1}\big(H(r)+E(r)\big)\\
&\qquad+2r^{1-n}\int_{B_r\cap\partial\Omega} f\,\nabla u\cdot x\Bigg]\\
&\qquad-C\max\{r^{\delta},\varepsilon_r\}\,r^{-\frac{1}{2}}\big(H(r)\big)^{\frac{3}{2}}\sqrt{D'(r)+\tfrac{n-2}{r}D(r)+Cr^{\delta-1}H(r)}.
\end{aligned}\tag{3.26}$$

Now, we denote by $\partial_\star := \frac{x}{|x|}\cdot\nabla$ and we observe that $\partial_\star$ is the "radial" component of the tangential gradient along $\partial\Omega$, since Ω is a cone. Hence, since, by (1.12),

$$\nabla\big(F(x,u(x))\big)=\nabla\left(\int_0^{u(x)} f(x,\tau)\,d\tau\right)=\int_0^{u(x)}\nabla_x f(x,\tau)\,d\tau+f(x,u(x))\nabla u(x),$$

we obtain that

$$|x|\,\partial_\star\big(F(x,u(x))\big)=\int_0^{u(x)}\nabla_x f(x,\tau)\cdot x\,d\tau+f(x,u(x))\nabla u(x)\cdot x.$$

As a consequence, by (1.10),

$$\begin{aligned}
f(x,u(x))\nabla u(x)\cdot x &\leqslant |x|\,\partial_\star\big(F(x,u(x))\big)+C\int_0^{|u(x)|}A(x)\,|x|^{\delta-1}\,\tau\,d\tau\\
&\leqslant |x|\,\partial_\star\big(F(x,u(x))\big)+C\,A(x)\,|x|^{\delta-1}\,|u(x)|^2.
\end{aligned}\tag{3.27}$$

Moreover, integrating by parts along $\partial\Omega$,

$$\left|\int_{B_r\cap\partial\Omega}|x|\,\partial_\star\big(F(x,u(x))\big)\right|\leqslant C\int_{B_r\cap\partial\Omega}\big|F(x,u(x))\big|+C\int_{\partial(B_r\cap\partial\Omega)}|x|\,\big|F(x,u(x)\big|.\tag{3.28}$$

In addition, by (1.9) and (1.12), we know that

$$|F(x,t)|\leqslant C\,A(x)\,|x|^{\delta-1}\int_0^{|t|}\tau\,d\tau\leqslant C\,A(x)\,|x|^{\delta-1}\,|t|^2.$$

This and (3.28) lead to

$$\left|\int_{B_r\cap\partial\Omega}|x|\,\partial_\star\big(F(x,u(x))\big)\right|\leqslant C\int_{B_r\cap\partial\Omega}A(x)\,|x|^{\delta-1}\,|u(x)|^2+C\int_{\partial B_r\cap\partial\Omega}A(x)\,|x|^{\delta}\,|u(x)|^2.$$

Hence, recalling (3.27),

$$\begin{aligned}\left|\int_{B_r\cap\partial\Omega}f(x,u(x))\nabla u(x)\cdot x\right| &\leqslant \left|\int_{B_r\cap\partial\Omega}|x|\,\partial_\star\big(F(x,u(x))\big)\right|+C\int_{B_r\cap\partial\Omega}A(x)\,|x|^{\delta-1}\,|u(x)|^2\\ &\leqslant C\int_{B_r\cap\partial\Omega}A(x)\,|x|^{\delta-1}\,|u(x)|^2+C\int_{\partial B_r\cap\partial\Omega}A(x)\,|x|^{\delta}\,|u(x)|^2,\end{aligned}$$

up to renaming $C>0$.

Therefore, recalling (3.14) and (3.23),

$$\begin{aligned}&\left|\int_{B_r\cap\partial\Omega}f(x,u(x))\nabla u(x)\cdot x\right|\\ &\qquad\leqslant C\,r^{n-2+\delta}\Big(H(r)+E(r)\Big)+Cr^{n+\delta-\frac{3}{2}}\sqrt{H(r)}\sqrt{D'(r)+\tfrac{n-2}{r}D(r)+Cr^{\delta-1}H(r)}.\end{aligned}$$

Then, we insert this information into (3.26) and we conclude that

$$\begin{aligned}&D'(r)H(r)-H'(r)D(r)\\ &\quad\geqslant H(r)\Bigg[-C\left(\frac{\varepsilon_r}{r}\,E(r)+r^{\delta-1}\sqrt{E(r)}\,\sqrt{H(r)+E(r)}\right)-C\,r^{\delta-1}\big(H(r)+E(r)\big)\\ &\qquad\quad-C\,r^{\delta-1}\big(H(r)+E(r)\big)-Cr^{\delta-\frac{1}{2}}\sqrt{H(r)}\sqrt{D'(r)+\tfrac{n-2}{r}D(r)+Cr^{\delta-1}H(r)}\Bigg]\\ &\qquad-C\max\{r^{\delta},\varepsilon_r\}\,r^{-\frac{1}{2}}\,\big(H(r)\big)^{\frac{3}{2}}\,\sqrt{D'(r)+\tfrac{n-2}{r}D(r)+Cr^{\delta-1}H(r)}\\ &\quad\geqslant H(r)\Bigg[-C\left(\frac{\varepsilon_r}{r}\,E(r)+r^{\delta-1}\sqrt{E(r)}\,\sqrt{H(r)+E(r)}\right)-C\,r^{\delta-1}\big(H(r)+E(r)\big)\Bigg]\\ &\qquad-C\max\{r^{\delta},\varepsilon_r\}\,r^{-\frac{1}{2}}\,\big(H(r)\big)^{\frac{3}{2}}\,\sqrt{D'(r)+\tfrac{n-2}{r}D(r)+Cr^{\delta-1}H(r)}.\end{aligned}$$

Accordingly, by (1.6),

$$\begin{aligned}\mathscr{N}'(r)&=\frac{d}{dr}\left(\frac{D(r)}{H(r)}\right)=\frac{D'(r)H(r)-H'(r)D(r)}{H^2(r)}\\ &\geqslant -C\left(\frac{\varepsilon_r}{r}\,\frac{E(r)}{H(r)}+r^{\delta-1}\sqrt{\frac{E(r)}{H(r)}}\sqrt{1+\frac{E(r)}{H(r)}}\right)-C\,r^{\delta-1}\left(1+\frac{E(r)}{H(r)}\right)\\ &\qquad-C\max\{r^{\delta},\varepsilon_r\}\,r^{-\frac{1}{2}}\sqrt{\frac{D'(r)}{H(r)}+\frac{n-2}{r}\,\frac{D(r)}{H(r)}+Cr^{\delta-1}}.\end{aligned}$$

From this inequality and (3.17) we find that

$$\begin{aligned}\mathcal{N}'(r) &\geqslant -C\left(\frac{\varepsilon_r}{r}(1+\mathcal{N}(r)) + r^{\delta-1}\sqrt{1+\mathcal{N}(r)}\sqrt{2+\mathcal{N}(r)}\right) - C\,r^{\delta-1}(2+\mathcal{N}(r)) \\ &\quad - C\max\{r^\delta,\varepsilon_r\}\,r^{-\frac{1}{2}}\sqrt{\frac{D'(r)}{H(r)} + \frac{n-2}{r}\frac{D(r)}{H(r)} + Cr^{\delta-1}} \\ &\geqslant -C\,\frac{\varepsilon_r}{r}(1+\mathcal{N}(r)) - C\,r^{\delta-1}(1+\mathcal{N}(r)) - C\,r^{\delta-1} \\ &\quad - C\,\max\{r^\delta,\varepsilon_r\}r^{-\frac{1}{2}}\sqrt{\frac{D'(r)}{H(r)} + \frac{n-2}{r}\frac{D(r)}{H(r)} + Cr^{\delta-1}} \\ &\geqslant -C\,\max\{r^\delta,\varepsilon_r\}r^{-1}\left[(2+\mathcal{N}(r)) + \sqrt{r\frac{D'(r)}{H(r)} + (n-2)\frac{D(r)}{H(r)} + Cr^\delta}\right].\end{aligned} \tag{3.29}$$

Let

$$\Lambda = \{r \in (0, r_0) : D'(r)H(r) \leqslant D(r)H'(r)\}.$$

In view of (3.6), (1.15), and (1.7), for $r \in \Lambda$ we can estimate $D'(r)$ as follows:

$$\begin{aligned}D'(r) &\leqslant \frac{D(r)H'(r)}{H(r)} = \frac{2}{r}\frac{D^2(r)}{H(r)} + r^{1-n}(\mathcal{N}(r)+1)\int_{\partial B_r\cap\Omega} \nabla A\cdot\nu\, u^2 - r^{1-n}\int_{\partial B_r\cap\Omega}\nabla A\cdot\nu\, u^2 \\ &\leqslant \frac{2}{r}\frac{D^2(r)}{H(r)} + (\mathcal{N}(r)+1)\frac{\varepsilon_r}{r}H(r) + \frac{\varepsilon_r}{r}H(r) = \frac{2}{r}\frac{D^2(r)}{H(r)} + (\mathcal{N}(r)+2)\frac{\varepsilon_r}{r}H(r).\end{aligned}$$

It follows that, for all $r \in \Lambda$,

$$\begin{aligned}\sqrt{r\frac{D'(r)}{H(r)} + (n-2)\frac{D(r)}{H(r)} + Cr^\delta} &\leqslant \sqrt{2\mathcal{N}^2(r) + \varepsilon_r(\mathcal{N}(r)+2) + (n-2)\mathcal{N}(r) + Cr^\delta} \\ &\leqslant C(\mathcal{N}(r)+2).\end{aligned}$$

Combining the previous estimate with (3.29) we obtain that, for all $r \in \Lambda$ sufficiently small

$$\mathcal{N}'(r) \geqslant -C\,\max\{r^\delta,\varepsilon_r\}r^{-1}(2+\mathcal{N}(r)). \tag{3.30}$$

For $r \notin \Lambda$ estimate (3.30) is trivial, since the left hand side of (3.30) is nonnegative outside Λ whereas the right hand side is nonpositive because of (1.15). Estimate (1.16) and statement (ii) are thereby proved.

To prove statement (iii), let $h(r) := \max\{r^\delta,\varepsilon_r\}r^{-1}$. By assumption (1.17), we have that $h \in L^1(0, r_1)$. Then, from (1.16) it follows that

$$\left((2+\mathcal{N}(r))e^{-C_1\int_r^1 h(s)\,ds}\right)' = e^{-C_1\int_r^1 h(s)\,ds}\left(\mathcal{N}'(r) + C_1 h(r)(2+\mathcal{N}(r))\right) \geqslant 0$$

hence the function $w(r) := (2+\mathcal{N}(r))e^{-C_1\int_r^1 h(s)\,ds}$ is nondecreasing in $(0, r_1)$.

Moreover $w \geqslant 0$ in view of (1.15). Therefore w admits a finite limit as $r \to 0^+$ and then also $\mathcal{N}$ has a finite limit γ as $r \to 0^+$. Since estimate (3.17) implies that $\mathcal{N}(r) \geqslant -Cr^\delta$ in $(0, r_0)$, we conclude that $\gamma \geqslant 0$.

4. Proof of Theorem 1.2

We start by proving (1.20). To this end, we argue for a contradiction and we suppose that (1.20) is violated. Then, we have that (1.14) is satisfied and hence all the hypotheses of Theorem 1.1 are fulfilled. In particular, by the fact that the limit in (1.18) is finite and $\mathcal{N}$ is continuous in $(0, r_0)$, we find that $\mathcal{N}$ is bounded, i.e. for all $r \in (0, r_0)$,

$$\mathcal{N}(r) \leqslant C, \tag{4.1}$$

for some $C > 0$.

Moreover, by (3.6),

$$\frac{2D(r)}{rH(r)} = \frac{H'(r)}{H(r)} - \frac{r^{1-n}}{H(r)} \int\limits_{\partial B_r \cap \Omega} \nabla A \cdot \nu \, u^2.$$

As a consequence, recalling (1.8),

$$\begin{aligned} \frac{H'(r)}{H(r)} &= \frac{2D(r)}{rH(r)} + \frac{r^{1-n}}{H(r)} \int\limits_{\partial B_r \cap \Omega} \nabla A \cdot \nu \, u^2 \\ &= \frac{2\mathscr{N}(r)}{r} + \frac{r^{1-n}}{H(r)} \int\limits_{\partial B_r \cap \Omega} \nabla A \cdot \nu \, u^2 \\ &\leqslant \frac{2\mathscr{N}(r)}{r} + C\,\frac{r^{1-n}}{rH(r)} \int\limits_{\partial B_r \cap \Omega} A\, u^2 \\ &= \frac{2\mathscr{N}(r)}{r} + \frac{C}{r} \\ &\leqslant \frac{C}{r}\big(\mathscr{N}(r)+1\big), \end{aligned}$$

for some $C > 0$ independent of r (varying from line to line). This and (4.1) yield that

$$\frac{H'(r)}{H(r)} \leqslant \frac{C}{r}, \tag{4.2}$$

up to renaming $C > 0$ and therefore, if $r \in (0, r_0/2)$,

$$\begin{aligned} \frac{H(2r)}{H(r)} &= \exp\left(\log H(2r) - \log H(r)\right) \\ &= \exp\left(\int\limits_r^{2r} \frac{H'(\rho)}{H(\rho)}\, d\rho\right) \\ &\leqslant \exp\left(C \int\limits_r^{2r} \frac{d\rho}{\rho}\right) \\ &= C, \end{aligned} \tag{4.3}$$

up to renaming C line after line. More in general, integration of (4.2) over the interval (r, rR) yields that for every $R > 1$ there exists $C_R > 0$ (depending on R but independent of r) such that

$$H(Rr) \leqslant C_R H(r) \quad \text{for all } r \in (0, r_0/R). \tag{4.4}$$

The inequality in (4.3) provides a pivotal "doubling property" in our setting. From this, we obtain that

$$\int\limits_{\partial B_{2r} \cap \Omega} A(x)\, u^2(x)\, d\mathscr{H}^{n-1}_x \leqslant C \int\limits_{\partial B_r \cap \Omega} A(x)\, u^2(x)\, d\mathscr{H}^{n-1}_x,$$

up to renaming C.

Integrating the latter inequality in r, we find that

$$\int\limits_{B_{2r} \cap \Omega} A(x)\, u^2(x)\, dx \leqslant C_0 \int\limits_{B_r \cap \Omega} A(x)\, u^2(x)\, dx,$$

for some $C_0 > 0$ independent of r, which gives that

$$\int_{B_r\cap\Omega} A(x)\,u^2(x)\,dx \leqslant C_0^m \int_{B_{r/2^m}\cap\Omega} A(x)\,u^2(x)\,dx, \tag{4.5}$$

for all $m\in\mathbb{N}$ and $r\in(0,r_0)$.

Now we fix $k\in\mathbb{N}$ such that $2^{2k}\geqslant 2C_0$. In light of (1.19) we can write that

$$|u(x)|\leqslant |x|^k,$$

as long as $x\in\Omega$ and $|x|$ is sufficiently small. Hence, we can exploit (4.5) for m sufficiently large and conclude that

$$\begin{aligned}\int_{B_{r_0}\cap\Omega} A(x)\,u^2(x)\,dx &\leqslant C_0^m \int_{B_{r_0/2^m}\cap\Omega} A(x)\,|x|^{2k}\,dx\\ &\leqslant C_0^m\left(\frac{r_0}{2^m}\right)^{2k}\int_{B_{r_0/2^m}\cap\Omega} A(x)\,dx \leqslant \frac{r_0^{2k}}{2^m}\,\|A\|_{L^1(\Omega)}.\end{aligned}$$

Then, sending $m\to+\infty$, we conclude that

$$\int_{B_{r_0}\cap\Omega} A(x)\,u^2(x)\,dx = 0,$$

and therefore, by (1.2), it follows that u must vanish necessarily in $B_{r_0}\cap\Omega$. This proves (1.20), against our initial contradictory assumption.

Having established (1.20), we can now complete the proof of Theorem 1.2, since, if A is Lipschitz, we can use (1.20) and the classical unique continuation principle in [10] and obtain (1.21), as desired.

5. Proof of Theorem 1.3

By (1.3) and (1.22), we see that, if $x\in\Omega$ and λ is sufficiently small,

$$\begin{aligned}0 &= \operatorname{div}\big(A(\lambda x)\,\nabla u_\lambda(x)\big) - \frac{\lambda^2}{\sqrt{H(\lambda)}}\,g(\lambda x,\sqrt{H(\lambda)}\,u_\lambda(x))\\ &= \operatorname{div}\big(A_\lambda(x)\,\nabla u_\lambda(x)\big) - g_\lambda(x,u_\lambda(x)),\end{aligned} \tag{5.1}$$

where

$$A_\lambda(x) := A(\lambda x)$$

and $$g_\lambda(x,t) := \frac{\lambda^2}{\sqrt{H(\lambda)}}\,g(\lambda x,\sqrt{H(\lambda)}\,t).$$

Similarly, we see that, if $x\in\partial\Omega$,

$$\begin{aligned}0 &= \frac{\lambda}{\sqrt{H(\lambda)}}\Big(A(\lambda x)\nabla u(\lambda x)\cdot\nu(\lambda x) - f(\lambda x,u(\lambda x))\Big)\\ &= A_\lambda(x)\nabla u_\lambda(x)\cdot\nu(x) - \frac{\lambda}{\sqrt{H(\lambda)}}\,f(\lambda x,\sqrt{H(\lambda)}\,u_\lambda(x))\\ &= A_\lambda(x)\nabla u_\lambda(x)\cdot\nu(x) - f_\lambda(x,u_\lambda(x)),\end{aligned} \tag{5.2}$$

where

$$f_\lambda(x,t) := \frac{\lambda}{\sqrt{H(\lambda)}}\,f(\lambda x,\sqrt{H(\lambda)}\,t).$$

Now, in the notation of (1.5), we write $D_{u,A,f,g}$ and $H_{u,A}$ to emphasize their dependences. In the same way, in the notation of (1.6), we write $\mathscr{N}_{u,A,f,g}$. For short, we drop the indexes when they refer to the original configuration in (1.3) and we write

$$D_\lambda := D_{u_\lambda, A_\lambda, f_\lambda, g_\lambda}, \qquad H_\lambda := H_{u_\lambda, A_\lambda} \qquad \text{and} \qquad \mathscr{N}_\lambda := \mathscr{N}_{u_\lambda, A_\lambda, f_\lambda, g_\lambda}. \tag{5.3}$$

We remark that

$$\begin{aligned} H_\lambda(r) &= r^{1-n} \int_{\partial B_r \cap \Omega} A_\lambda(x)\, u_\lambda^2(x)\, d\mathscr{H}_x^{n-1} \\ &= \frac{r^{1-n}}{H(\lambda)} \int_{\partial B_r \cap \Omega} A(\lambda x)\, u^2(\lambda x)\, d\mathscr{H}_x^{n-1} \\ &= \frac{(\lambda r)^{1-n}}{H(\lambda)} \int_{\partial B_{\lambda r} \cap \Omega} A(y)\, u^2(y)\, d\mathscr{H}_y^{n-1} \\ &= \frac{H(\lambda r)}{H(\lambda)}. \end{aligned}$$

In addition,

$$\begin{aligned} D_\lambda(r) &= r^{2-n} \int_{B_r \cap \Omega} A_\lambda(x)\, |\nabla u_\lambda(x)|^2\, dx - r^{2-n} \int_{B_r \cap \partial\Omega} f_\lambda(x, u_\lambda(x))\, u_\lambda(x)\, d\mathscr{H}_x^{n-1} \\ &\qquad + r^{2-n} \int_{B_r \cap \Omega} g_\lambda(x, u_\lambda(x))\, u_\lambda(x)\, dx \\ &= \frac{\lambda^2 r^{2-n}}{H(\lambda)} \int_{B_r \cap \Omega} A(\lambda x)\, |\nabla u(\lambda x)|^2\, dx - \frac{\lambda r^{2-n}}{H(\lambda)} \int_{B_r \cap \partial\Omega} f(\lambda x, u(\lambda x))\, u(\lambda x)\, d\mathscr{H}_x^{n-1} \\ &\qquad + \frac{\lambda^2 r^{2-n}}{H(\lambda)} \int_{B_r \cap \Omega} g(\lambda x, u(\lambda x))\, u(\lambda x)\, dx \\ &= \frac{(\lambda r)^{2-n}}{H(\lambda)} \int_{B_{\lambda r} \cap \Omega} A(y)\, |\nabla u(y)|^2\, dy - \frac{(\lambda r)^{2-n}}{H(\lambda)} \int_{B_{\lambda r} \cap \partial\Omega} f(y, u(y))\, u(y)\, d\mathscr{H}_y^{n-1} \\ &\qquad + \frac{(\lambda r)^{2-n}}{H(\lambda)} \int_{B_{\lambda r} \cap \Omega} g(y, u(y))\, u(y)\, dy \\ &= \frac{D(\lambda r)}{H(\lambda)}, \end{aligned}$$

and therefore

$$\mathscr{N}_\lambda(r) = \frac{D(\lambda r)}{H(\lambda r)} = \mathscr{N}(\lambda r). \tag{5.4}$$

This and (1.18) give that, for all $r > 0$,

$$\lim_{\lambda \searrow 0} \mathscr{N}_\lambda(r) = \gamma,$$

for some finite $\gamma \geqslant 0$.

Now we claim that, for all $R > 0$ and $\lambda \in (0, r_0/R)$,

$$\|u_\lambda\|_{H^1(\Omega \cap B_R)} \leqslant C_R, \tag{5.5}$$

for some $C_R > 0$ (eventually depending on R). To this end, we exploit (3.12), (3.18), (4.4), and (4.1) to see that, for all $\lambda \in (0, r_0/R)$,

$$\begin{aligned}\int_{\Omega\cap B_R} A_\lambda(x)\,|\nabla u_\lambda(x)|^2\,dx &= \frac{\lambda^2}{H(\lambda)}\int_{\Omega\cap B_R} A(\lambda x)\,|\nabla u(\lambda x)|^2\,dx\\ &= \frac{\lambda^{2-n}}{H(\lambda)}\int_{\Omega\cap B_{R\lambda}} A(y)\,|\nabla u(y)|^2\,dy = R^{n-2}\frac{E(\lambda R)}{H(\lambda)} \leqslant 2R^{n-2}\,\frac{H(\lambda R)+D(\lambda R)}{H(\lambda)}\\ &= 2R^{n-2}\,\frac{H(\lambda R)}{H(\lambda)}(1+\mathcal{N}(\lambda R)) \leqslant C_R,\end{aligned} \tag{5.6}$$

for some $C_R > 0$ depending on R.

Moreover, using again (4.4), we observe that

$$\begin{aligned}\int_{\partial B_R\cap\Omega} A_\lambda(x)\,u_\lambda^2(x)\,d\mathcal{H}_x^{n-1} &= \frac{1}{H(\lambda)}\int_{\partial B_R\cap\Omega} A(\lambda x)\,u^2(\lambda x)\,d\mathcal{H}_x^{n-1}\\ &= \frac{\lambda^{1-n}}{H(\lambda)}\int_{\partial B_{R\lambda}\cap\Omega} A(y)\,u^2(y)\,d\mathcal{H}_y^{n-1} = R^{n-1}\frac{H(R\lambda)}{H(\lambda)} \leqslant C_R,\end{aligned} \tag{5.7}$$

up to renaming C_R. Hence, recalling Corollary 2.4 (used here with $\mu := 0$, $r := R$, and on the function u_λ and with weight A_λ) and (5.6),

$$\begin{aligned}\int_{\Omega\cap B_R} A_\lambda(x)\,u_\lambda^2(x)\,dx &\leqslant C_R\left(\int_{\partial B_R\cap\Omega} A_\lambda(x)\,u_\lambda^2(x)\,d\mathcal{H}_x^{n-1} + \int_{\Omega\cap B_R} A_\lambda(x)|\nabla u_\lambda(x)|^2\,dx\right)\\ &\leqslant C_R,\end{aligned} \tag{5.8}$$

up to renaming C_R. This inequality and (5.6), combined with (1.2), give (5.5), as desired.

Now, from (5.5) and a diagonal process, we deduce that, along a subsequence, u_λ converges a.e. in Ω, strongly in $L^2(\Omega\cap B_R)$ and weakly in $H^1(\Omega\cap B_R)$ for all $R>0$, as $\lambda\searrow 0$. Consistently with the notation in Theorem 1.3, we denote by $\tilde u$ this limit; we observe that $\tilde u \in \bigcap_{R>0} H^1(\Omega\cap B_R)$.

As a particular case of (5.7) with $R=1$ we have that

$$\int_{\partial B_1\cap\Omega} A_\lambda(x)\,u_\lambda^2(x)\,d\mathcal{H}_x^{n-1} = 1$$

which, in view of the compactness of the trace embedding $H^1(\Omega\cap B_1)\hookrightarrow\hookrightarrow L^2(\Omega\cap\partial B_1)$, implies that

$$\int_{\partial B_1\cap\Omega} \tilde u^2(x)\,d\mathcal{H}_x^{n-1} = \lim_{\lambda\searrow 0}\int_{\partial B_1\cap\Omega} A_\lambda(x)\,u_\lambda^2(x)\,d\mathcal{H}_x^{n-1} = 1. \tag{5.9}$$

Hence $\tilde u \not\equiv 0$.

We observe that, by (1.11), for every $x\in B_1$,

$$\begin{aligned}|g_\lambda(x,u_\lambda(x))| &= \frac{\lambda^2}{\sqrt{H(\lambda)}}\left|g(\lambda x,\sqrt{H(\lambda)}\,u_\lambda(x))\right|\\ &\leqslant C\,\lambda^\delta\,A(\lambda x)\,|x|^{\delta-2}\,|u_\lambda(x)|\\ &\leqslant C\,\lambda^\delta\,|x|^{\delta-2}\,|u_\lambda(x)|,\end{aligned} \tag{5.10}$$

up to renaming C line after line.

Moreover, by (1.9),

$$\begin{aligned}|f_\lambda(x,u_\lambda(x))| &= \frac{\lambda}{\sqrt{H(\lambda)}}\left|f(\lambda x,\sqrt{H(\lambda)}\,u_\lambda(x))\right|\\ &\leqslant C\,\lambda^\delta\,A(\lambda x)\,|x|^{\delta-1}\,|u_\lambda(x)|\\ &\leqslant C\,\lambda^\delta\,|x|^{\delta-1}\,|u_\lambda(x)|.\end{aligned} \tag{5.11}$$

Now we claim that, for all $R > 0$,

$$\begin{aligned} &\lim_{\lambda\searrow 0} \int_{\Omega\cap B_R} g_\lambda(x,u_\lambda(x))\,u_\lambda(x)\,dx = 0 \\ \text{and}\quad &\lim_{\lambda\searrow 0} \int_{\partial\Omega\cap B_R} f_\lambda(x,u_\lambda(x))\,u_\lambda(x)\,d\mathscr{H}^{n-1}_x = 0. \end{aligned} \tag{5.12}$$

Indeed, using (5.10), Corollary 2.4 (used here with $A := 1$, $r := R$, and $\mu := 2 - \delta$), and (1.2), we see that

$$\begin{aligned} \left| \int_{\Omega\cap B_R} g_\lambda(x,u_\lambda(x))\,u_\lambda(x)\,dx \right| &\leqslant C\,\lambda^\delta \int_{\Omega\cap B_R} |x|^{\delta-2}\,u_\lambda^2(x)\,dx \\ &\leqslant C_R\,\lambda^\delta \left(\int_{\partial B_R\cap\Omega} u_\lambda^2(x)\,d\mathscr{H}^{n-1}_x + \int_{\Omega\cap B_R} |\nabla u_\lambda(x)|^2\,dx \right) \\ &\leqslant C_R\,\lambda^\delta \left(\int_{\partial B_R\cap\Omega} A_\lambda(x)\,u_\lambda^2(x)\,d\mathscr{H}^{n-1}_x + \int_{\Omega\cap B_R} A_\lambda(x)\,|\nabla u_\lambda(x)|^2\,dx \right). \end{aligned}$$

From this, (5.7) and (5.6), we deduce that

$$\left| \int_{\Omega\cap B_1} g_\lambda(x,u_\lambda(x))\,u_\lambda(x)\,dx \right| \leqslant C_R\lambda^\delta,$$

up to renaming $C_R > 0$.

This proves the first claim in (5.12), and we now prove the second. For this, using (5.11), and then Lemma 2.5 (with $A := 1$, $r := R$ and $\gamma := 1 - \delta$) we find that

$$\begin{aligned} \left| \int_{\partial\Omega\cap B_R} f_\lambda(x,u_\lambda(x))\,u_\lambda(x)\,d\mathscr{H}^{n-1}_x \right| &\leqslant C\,\lambda^\delta \int_{\partial\Omega\cap B_R} |x|^{\delta-1}\,u_\lambda^2(x)\,d\mathscr{H}^{n-1}_x \\ &\leqslant C_R\,\lambda^\delta \int_{\Omega\cap B_R} \left(|\nabla u_\lambda(x)|^2 + \frac{u_\lambda^2(x)}{|x|^{2-\delta}} \right) dx. \end{aligned}$$

Hence, using Corollary 2.4 as before, we obtain that

$$\left| \int_{\partial\Omega\cap B_R} f_\lambda(x,u_\lambda(x))\,u_\lambda(x)\,d\mathscr{H}^{n-1}_x \right| \leqslant C_R\,\lambda^\delta,$$

which implies the second claim in (5.12). This completes the proof of (5.12).

Now we claim that

$$\begin{cases} \Delta\tilde{u} = 0 & \text{in } \Omega, \\ \dfrac{\partial\tilde{u}}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.13}$$

To this end, we exploit (5.1) and (5.2) and, given $\varphi \in C_0^\infty(\mathbb{R}^n)$, we write that

$$\begin{aligned}
0 &= \int_{\Omega} \operatorname{div}\Big(A_\lambda(x)\,\nabla u_\lambda(x)\Big)\varphi(x)\,dx - \int_{\Omega} g_\lambda(x,u_\lambda(x))\,\varphi(x)\,dx \\
&= \int_{\partial\Omega} A_\lambda(x)\,\varphi(x)\nabla u_\lambda(x)\cdot\nu(x)\,d\mathscr{H}^{n-1}_x \\
&\qquad - \int_{\Omega} A_\lambda(x)\,\nabla u_\lambda(x)\cdot\nabla\varphi(x)\,dx - \int_{\Omega} g_\lambda(x,u_\lambda(x))\,\varphi(x)\,dx \\
&= \int_{\partial\Omega} f_\lambda(x,u_\lambda(x))\,\varphi(x)\,d\mathscr{H}^{n-1}_x \\
&\qquad - \int_{\Omega} A_\lambda(x)\,\nabla u_\lambda(x)\cdot\nabla\varphi(x)\,dx - \int_{\Omega} g_\lambda(x,u_\lambda(x))\,\varphi(x)\,dx.
\end{aligned}$$

Hence, in light of (1.24), (5.10) and (5.11),

$$\begin{aligned}
&\left|\int_{\Omega} \nabla\tilde u(x)\cdot\nabla\varphi(x)\,dx\right| = \lim_{\lambda\searrow 0}\left|\int_{\Omega} A_\lambda(x)\,\nabla u_\lambda(x)\cdot\nabla\varphi(x)\,dx\right| \\
&\quad = \lim_{\lambda\searrow 0}\left|\int_{\partial\Omega} f_\lambda(x,u_\lambda(x))\,\varphi(x)\,d\mathscr{H}^{n-1}_x - \int_{\Omega} g_\lambda(x,u_\lambda(x))\,\varphi(x)\,dx\right| \\
&\quad \leqslant C\,\lim_{\lambda\searrow 0}\lambda^\delta\left(\int_{\partial\Omega}|x|^{\delta-1}\,|u_\lambda(x)|\,|\varphi(x)|\,d\mathscr{H}^{n-1}_x + \int_{\Omega}|x|^{\delta-2}\,|u_\lambda(x)|\,|\varphi(x)|\,dx\right) \\
&\quad \leqslant C\,\lim_{\lambda\searrow 0}\lambda^\delta\left(\int_{\partial\Omega\cap B_R}|x|^{\delta-1}\,|u_\lambda(x)|^2\,d\mathscr{H}^{n-1}_x + \int_{\partial\Omega\cap B_R}|x|^{\delta-1}\,|\varphi(x)|^2\,d\mathscr{H}^{n-1}_x\right. \\
&\qquad\qquad \left. + \int_{\Omega\cap B_R}|x|^{\delta-2}\,|u_\lambda(x)|^2\,dx + \int_{\Omega\cap B_R}|x|^{\delta-2}\,|\varphi(x)|^2\,dx\right) \\
&\quad \leqslant C'\,\lim_{\lambda\searrow 0}\lambda^\delta\left(1 + \int_{\partial\Omega\cap B_R}|x|^{\delta-1}\,|u_\lambda(x)|^2\,d\mathscr{H}^{n-1}_x + \int_{\Omega\cap B_R}|x|^{\delta-2}\,|u_\lambda(x)|^2\,dx\right),
\end{aligned}$$

where $C', R > 0$ may also depend on φ. Consequently, using Corollary 2.4 and Lemma 2.5 as before, we obtain

$$\left|\int_{\Omega} \nabla\tilde u(x)\cdot\nabla\varphi(x)\,dx\right| \leqslant C'\,\lim_{\lambda\searrow 0}\lambda^\delta,$$

up to renaming $C' > 0$, that is

$$\int_{\Omega} \nabla\tilde u(x)\cdot\nabla\varphi(x)\,dx = 0.$$

Since this identity holds true for all $\varphi \in C^\infty_0(\mathbb{R}^n)$, we have completed the proof of (5.13).

We now show that

$$u_\lambda \text{ converges strongly to } \tilde u \text{ in } H^1(\Omega\cap B_1), \text{ as } \lambda\searrow 0. \tag{5.14}$$

for some $C > 0$. To accomplish this, we will exploit elliptic regularity theory, see e.g. Theorem 8.13 in [14] (with the notation in Example 6.2 on page 314 in [14] for the definition of the norms) or [3,4] and Theorem 5.1 in [13], considering a set Ω_1 with smooth boundary and such that $\Sigma \subset \Omega_1 \subset \Omega \cap (B_2 \setminus B_{1/2})$. In this way, by (5.1) and (5.2),

$$\|u_\lambda\|_{H^2(\Omega_1)} \leqslant C\Big(1 + \|u_\lambda\|_{L^2(\Omega_1)} + \|g_\lambda(\cdot, u_\lambda)\|_{L^2(\Omega\cap(B_2\setminus B_{1/2}))} + \|f_\lambda(\cdot, u_\lambda)\|_{H^{1/2}((\partial\Omega)\cap(B_2\setminus B_{1/2}))}\Big). \tag{5.15}$$

Moreover, in light of (5.8) and (5.10),

$$\begin{aligned} \|g_\lambda(\cdot, u_\lambda)\|^2_{L^2(\Omega\cap(B_2\setminus B_{1/2}))} &= \int_{\Omega\cap(B_2\setminus B_{1/2})} |g_\lambda(x, u_\lambda(x))|^2\,dx \\ &\leqslant C\,\lambda^{2\delta} \int_{\Omega\cap(B_2\setminus B_{1/2})} |x|^{2(\delta-2)}\,|u_\lambda(x)|^2\,dx \\ &\leqslant C\,\lambda^{2\delta} \int_{\Omega\cap B_2} |u_\lambda(x)|^2\,dx \\ &\leqslant C\,\lambda^{2\delta}. \end{aligned} \tag{5.16}$$

Similarly, recalling (5.11) and (5.8),

$$\begin{aligned} \|f_\lambda(\cdot, u_\lambda)\|^2_{L^2(\Omega\cap(B_2\setminus B_{1/2}))} &= \int_{\Omega\cap(B_2\setminus B_{1/2})} |f_\lambda(x, u_\lambda(x))|^2\,dx \\ &\leqslant C\,\lambda^{2\delta} \int_{\Omega\cap(B_2\setminus B_{1/2})} |x|^{2(\delta-1)}|u_\lambda(x)|^2\,dx \\ &\leqslant C\,\lambda^{2\delta} \int_{\Omega\cap B_2} |u_\lambda(x)|^2\,dx \\ &\leqslant C\,\lambda^{2\delta}. \end{aligned}$$

Furthermore, from (1.10), (1.23), and (5.5) it follows that

$$\begin{aligned} &\|\nabla(f_\lambda(\cdot, u_\lambda))\|^2_{L^2(\Omega\cap(B_2\setminus B_{1/2}))} \\ &= \int_{\Omega\cap(B_2\setminus B_{1/2})} \left|\frac{\lambda}{\sqrt{H(\lambda)}}\Big(\lambda\nabla_x f\Big(\lambda x, \sqrt{H(\lambda)}u_\lambda(x)\Big) + f_t\Big(\lambda x, \sqrt{H(\lambda)}u_\lambda(x)\Big)\sqrt{H(\lambda)}\nabla u_\lambda(x)\Big)\right|^2 dx \\ &\leqslant C\lambda^{2\delta} \int_{\Omega\cap B_2} (|u_\lambda(x)|^2 + |\nabla u_\lambda(x)|^2)\,dx \\ &\leqslant C\lambda^{2\delta} \end{aligned}$$

Therefore

$$\|f_\lambda(\cdot, u_\lambda)\|_{H^1(\Omega\cap(B_2\setminus B_{1/2}))} \leqslant C\lambda^\delta$$

which, in view of the continuous trace embedding $H^1(\Omega \cap (B_2 \setminus B_{1/2})) \hookrightarrow H^{1/2}(\partial\Omega \cap (B_2 \setminus B_{1/2}))$, yields

$$\|f_\lambda(\cdot, u_\lambda)\|_{H^{1/2}(\partial\Omega\cap(B_2\setminus B_{1/2}))} \leqslant C\lambda^\delta$$

up to renaming C. From this, (5.8), (5.16) and (5.15), we conclude that

$$\|u_\lambda\|_{H^2(\Omega_1)} \leqslant C,$$

again up to renaming $C > 0$. Thus, using the trace embedding,

$$\|u_\lambda\|_{H^{3/2}(\Sigma)} \leqslant C,$$

up to renaming $C > 0$, and consequently, up to a subsequence, we obtain that

$$u_\lambda \text{ converges to } \tilde{u} \text{ in } H^1(\Sigma). \tag{5.17}$$

Now we notice that, exploiting (5.1) and (5.2),

$$\begin{aligned}
0 &= \int_{\Omega\cap B_1} \operatorname{div}\Big(A_\lambda(x)\,\nabla u_\lambda(x)\Big)u_\lambda(x)\,dx - \int_{\Omega\cap B_1} g_\lambda(x,u_\lambda(x))\,u_\lambda(x)\,dx \\
&= \int_{\partial(\Omega\cap B_1)} A_\lambda(x)\,u_\lambda(x)\nabla u_\lambda(x)\cdot\nu(x)\,d\mathscr{H}^{n-1}_x \\
&\qquad - \int_{\Omega\cap B_1} A_\lambda(x)\,|\nabla u_\lambda(x)|^2\,dx - \int_{\Omega\cap B_1} g_\lambda(x,u_\lambda(x))\,u_\lambda(x)\,dx \\
&= \int_{\Sigma} A_\lambda(x)\,u_\lambda(x)\nabla u_\lambda(x)\cdot\nu(x)\,d\mathscr{H}^{n-1}_x + \int_{\partial\Omega\cap B_1} f_\lambda(x,u_\lambda(x))\,u_\lambda(x)\,d\mathscr{H}^{n-1}_x \\
&\qquad - \int_{\Omega\cap B_1} A_\lambda(x)\,|\nabla u_\lambda(x)|^2\,dx - \int_{\Omega\cap B_1} g_\lambda(x,u_\lambda(x))\,u_\lambda(x)\,dx.
\end{aligned}$$

Using this, (1.24), (5.12) and (5.17), we conclude that

$$\begin{aligned}
&\lim_{\lambda\searrow 0} \int_{\Omega\cap B_1} |\nabla u_\lambda(x)|^2\,dx \\
&= \lim_{\lambda\searrow 0} \int_{\Omega\cap B_1} A_\lambda(x)\,|\nabla u_\lambda(x)|^2\,dx \\
&= \lim_{\lambda\searrow 0} \int_{\Sigma} A_\lambda(x)\,u_\lambda(x)\nabla u_\lambda(x)\cdot\nu(x)\,d\mathscr{H}^{n-1}_x + \int_{\partial\Omega\cap B_1} f_\lambda(x,u_\lambda(x))\,u_\lambda(x)\,d\mathscr{H}^{n-1}_x \\
&\qquad - \int_{\Omega\cap B_1} g_\lambda(x,u_\lambda(x))\,u_\lambda(x)\,dx \\
&= \lim_{\lambda\searrow 0} \int_{\Sigma} A_\lambda(x)\,u_\lambda(x)\nabla u_\lambda(x)\cdot\nu(x)\,d\mathscr{H}^{n-1}_x \\
&= \int_{\Sigma} \tilde{u}(x)\nabla\tilde{u}(x)\cdot\nu(x)\,d\mathscr{H}^{n-1}_x.
\end{aligned}$$

Hence, recalling (5.13),

$$\begin{aligned}
\lim_{\lambda\searrow 0} \int_{\Omega\cap B_1} |\nabla u_\lambda(x)|^2\,dx &= \int_{\partial(\Omega\cap B_1)} \tilde{u}(x)\nabla\tilde{u}(x)\cdot\nu(x)\,d\mathscr{H}^{n-1}_x \\
&= \int_{\Omega\cap B_1} \operatorname{div}\Big(\tilde{u}(x)\nabla\tilde{u}(x)\Big)\,dx \\
&= \int_{\Omega\cap B_1} |\nabla\tilde{u}(x)|^2\,dx.
\end{aligned}$$

Since the weak convergence and the convergence of the norm imply the strong convergence in $L^2(\Omega\cap B_1)$, we thereby conclude that ∇u_λ converges to $\nabla\tilde u$ strongly in $L^2(\Omega\cap B_1,\mathbb{R}^n)$, and this gives (5.14), as desired.

From (5.14) and (5.12), recalling (5.13) and the notation in (5.3), we conclude that

$$\lim_{\lambda\searrow 0}\mathscr{N}_\lambda(r)=\mathscr{N}_{\tilde u,1,0,0}(r).$$

As a consequence, exploiting (5.4),

$$\mathscr{N}_{\tilde u,1,0,0}(r)=\gamma. \tag{5.18}$$

From this, we conclude that

$$\tilde u \text{ is positively homogeneous of degree } \gamma, \tag{5.19}$$

and hence we can write $\tilde u$ as in (1.25).

For completeness, we give a self-contained proof of (5.19) by arguing as follows. By (5.18), we know that $\mathscr{N}'_{\tilde u,1,0,0}(r)=0$, and therefore, by (1.6),

$$D'_{\tilde u,1,0,0}(r)\,H_{\tilde u,1}(r)-H'_{\tilde u,1}(r)D_{\tilde u,1,0,0}(r)=0\quad\text{for all } r>0.$$

Hence, exploiting (3.9) in this setting, and recalling (2.1), we see that

$$\begin{aligned}0&=r^{-n}H_{\tilde u,1}(r)\int_{\partial B_r\cap\Omega}(\nabla\tilde u\cdot x)^2-r^{3-2n}\left(\int_{\partial B_r\cap\Omega}\tilde u\nabla\tilde u\cdot\nu\right)^2\\&=r^{3-2n}\left[\int_{\partial B_r\cap\Omega}\tilde u^2\int_{\partial B_r\cap\Omega}(\nabla\tilde u\cdot\nu)^2-\left(\int_{\partial B_r\cap\Omega}\tilde u\nabla\tilde u\cdot\nu\right)^2\right]\quad\text{for all } r>0.\end{aligned}$$

By the Cauchy-Schwarz Inequality, the latter term is nonnegative, and consequently we find that $\tilde u$ is proportional to $\nabla\tilde u\cdot\nu$. Accordingly, we have that $\tilde u$ is a positively homogeneous function, of some degree γ'.

Then, using (5.18) once again

$$\begin{aligned}\gamma\int_{\partial B_1\cap\Omega}\tilde u(x)\frac{\partial\tilde u}{\partial\nu}(x)\,d\mathscr{H}^{n-1}_x&=\gamma\,\gamma'\int_{\partial B_1\cap\Omega}\tilde u^2(x)\,d\mathscr{H}^{n-1}_x=\gamma\,\gamma'\,H_{\tilde u,1}(1)\\&=\gamma' D_{\tilde u,1,0,0}(1)=\gamma'\int_{B_1\cap\Omega}|\nabla\tilde u(x)|^2\,dx.\end{aligned}\tag{5.20}$$

On the other hand, by (5.13),

$$\int_{\partial B_1\cap\Omega}\tilde u(x)\frac{\partial\tilde u}{\partial\nu}(x)\,d\mathscr{H}^{n-1}_x=\int_{\partial(B_1\cap\Omega)}\tilde u(x)\frac{\partial\tilde u}{\partial\nu}(x)\,d\mathscr{H}^{n-1}_x=\int_{B_1\cap\Omega}|\nabla\tilde u(x)|^2\,dx.$$

Plugging this information into (5.20), we thereby conclude that

$$\gamma\int_{B_1\cap\Omega}|\nabla\tilde u(x)|^2\,dx=\gamma'\int_{B_1\cap\Omega}|\nabla\tilde u(x)|^2\,dx,$$

and then $\gamma'=\gamma$. This completes the proof of (5.19) (and thus of (1.25)).

We also remark that, by (1.25) and (5.13), using the notation $\rho:=|x|$ and $\vartheta:=x/|x|$,

$$0=\Delta\tilde u(x)=\gamma(\gamma-1)\rho^{\gamma-2}\psi(\vartheta)+(n-1)\gamma\rho^{\gamma-2}\psi(\vartheta)+\rho^{\gamma-2}\Delta_{S^{n-1}}\psi(\vartheta),$$

and therefore ψ is an eigenfunction of te operator $\mathscr{L}_\Sigma$; the Neumann boundary condition of ψ also follows from the one of $\tilde u$ in (5.13).

Furthermore, by (5.9) and (1.25)

$$\begin{aligned}1 &= \int_{\partial B_1\cap\Omega} \tilde{u}^2(x)\, d\mathscr{H}_x^{n-1} \\ &= \int_{\partial B_1\cap\Omega} |x|^{2\gamma}\psi^2\left(\frac{x}{|x|}\right) d\mathscr{H}_x^{n-1} \\ &= \int_{\partial B_1\cap\Omega} \psi^2(x)\, d\mathscr{H}_x^{n-1},\end{aligned}$$

which gives (1.26). The proof of Theorem 1.3 is thereby complete.

6. Proof of Theorem 1.4

First, we prove (1.28). We argue by contradiction, supposing that (1.28) does not hold, and therefore (1.14) is satisfied. Hence, we are in the position of using Theorem 1.3, and we let $\tilde{u}$ and ψ as in (1.25). We note that, by (5.13) and elliptic regularity theory, we have that $\tilde{u}$ is smooth on $\overline{\Omega}\setminus\{0\}$.

We observe that the trace of $\tilde{u}$ on $B_1\cap\partial\Omega$ (which belongs to $L^2(B_1\cap\partial\Omega)$ by trace embeddings) cannot vanish identically, i.e.

$$\tilde{u}\not\equiv 0 \quad \text{on } B_1\cap\partial\Omega, \tag{6.1}$$

otherwise $\tilde{u}$ would be a harmonic function with homogeneous Dirichlet and Neumann conditions on $B_1\cap\partial\Omega$, and then necessarily $\tilde{u}$ would vanish identically in $B_1\cap\Omega$ (otherwise its trivial extension would violate classical unique continuation principles), in contradiction with (1.26).

From assumption (1.27) it follows that, for all $k\in\mathbb{N}$

$$\lambda^{-k}u(\lambda\cdot)\to 0 \quad \text{in } L^2(B_1\cap\partial\Omega). \tag{6.2}$$

Since, in view of (1.22),

$$\frac{\sqrt{H(\lambda)}}{\lambda^k} = \frac{\|\lambda^{-k}u(\lambda\cdot)\|_{L^2(B_1\cap\partial\Omega)}}{\|u_\lambda\|_{L^2(B_1\cap\partial\Omega)}}$$

and, by Theorem 1.3, $u_\lambda\to\tilde{u}$ in $L^2(B_1\cap\partial\Omega)$ along a subsequence, from (6.1) and (6.2) we conclude that

$$\lim_{\lambda\searrow 0}\frac{\sqrt{H(\lambda)}}{\lambda^k}=0,$$

for all $k\in\mathbb{N}$. Consequently, for all $k\in\mathbb{N}$, there exists $\lambda_0(k)\in(0,r_0/2)$ such that, for all $\lambda\in(0,\lambda_0(k)]$,

$$\frac{\sqrt{H(\lambda)}}{\lambda^k}\leqslant 1. \tag{6.3}$$

On the other hand, by (4.3),

$$H(2^m\lambda)\leqslant C^m H(\lambda)$$

for all $m\in\mathbb{N}$ and $\lambda\in(0,2^{-m}r_0)$, for a suitable $C>0$ independent of λ and m. This and (6.3) give that, for all k, $m\in\mathbb{N}$ and for all $\lambda\in(0,\min\{\lambda_0(k),2^{-m}r_0\})$,

$$H(2^m\lambda)\leqslant C^m\lambda^{2k}.$$

As a consequence, recalling (1.5) and integrating,

$$\begin{aligned}\frac{1}{2^m}\int\limits_{B_{2^m\lambda}\cap\Omega} A(x)\,u^2(x)\,dx &= \frac{1}{2^m}\int\limits_0^{2^m\lambda}\left[\,\int\limits_{\partial B_\rho\cap\Omega} A(x)\,u^2(x)\,d\mathscr{H}^{n-1}_x\right]d\rho\\ &=\int\limits_0^{\lambda}\left[\,\int\limits_{\partial B_{2^m r}\cap\Omega} A(x)\,u^2(x)\,d\mathscr{H}^{n-1}_x\right]dr=\int\limits_0^{\lambda}(2^m r)^{n-1}H(2^m r)\,dr\\ &\leqslant 2^{m(n-1)}C^m\int\limits_0^{\lambda} r^{n-1+2k}\,dr=\frac{2^{m(n-1)}C^m\,\lambda^{n+2k}}{n+2k},\end{aligned}$$

for all $k, m \in \mathbb{N}$ and for all $\lambda \in (0, \min\{\lambda_0(k), 2^{-m}r_0\})$.

We choose $m_\lambda \in \mathbb{N}$ such that

$$\left|\log_2\left(\frac{2\lambda}{r_0}\right)\right| \leqslant m_\lambda < 1+\left|\log_2\left(\frac{2\lambda}{r_0}\right)\right|, \tag{6.4}$$

so that $\lambda < 2^{-m_\lambda}r_0$ for all $\lambda < \frac{r_0}{2}$. Then we find that

$$\int\limits_{B_{2^{m_\lambda}\lambda}\cap\Omega} A(x)\,u^2(x)\,dx \leqslant \frac{2^{m_\lambda n}\,C^{m_\lambda}\,\lambda^{n+2k}}{n+2k},$$

for all $k \in \mathbb{N}$ and for all $\lambda \in (0, \lambda_0(k)]$.

Hence, since, by (6.4), we know that $2^{m_\lambda}\lambda \in \left[\frac{r_0}{2}, r_0\right]$,

$$\int\limits_{B_{\frac{r_0}{2}}\cap\Omega} A(x)\,u^2(x)\,dx \leqslant \frac{(2^n\,C)^{1+|\log_2\frac{2\lambda}{r_0}|}\,\lambda^{n+2k}}{n+2k} \leqslant \frac{(2^n\,C)^{-2\log_2\frac{2\lambda}{r_0}}\,\lambda^{n+2k}}{n+2k} = \kappa\,\frac{\lambda^{n+2k-\theta}}{n+2k},$$

for some suitable $\theta, \kappa > 0$ depending only on n, C, r_0 (but independent of k), for all $k \in \mathbb{N}$ and for all $\lambda \in (0, \min\{\lambda_0(k), r_0/4\})$.

Accordingly, choosing $k \in \mathbb{N}$ sufficiently large such that $n + 2k - \theta > 0$ and sending $\lambda \searrow 0$, we conclude that

$$\int\limits_{B_{\frac{r_0}{2}}\cap\Omega} A(x)\,u^2(x)\,dx = 0.$$

This gives that (1.28) holds true, in contradiction with our initial hypothesis.

This completes the proof of (1.28). Finally, the proof of (1.29) is identical to the proof of (1.21), hence the proof of Theorem 1.4 is complete.

Declaration of competing interest

The authors declare that there is no competing interest.

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