# Stability analysis and Hopf bifurcation at high Lewis number in a combustion model with free interface

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## Abstract

In this paper we analyze the stability of the traveling wave solution for an ignition-temperature first-order reaction model of diffusional-thermal combustion in the case of high Lewis numbers (Le > 1). In contrast to conventional Arrhenius kinetics where the reaction zone is infinitely thin, the reaction zone for stepwise temperature kinetics is of order unity. The system of two parabolic PDEs is characterized by a free interface at which ignition temperature $\Theta_i$ is reached. We turn the model to a fully nonlinear problem in a fixed domain. When the Lewis number is large, we define a bifurcation parameter $m = \Theta_i/(1-\Theta_i)$ and a perturbation parameter $\varepsilon = 1/\text{Le}$. The main result is the existence of a critical value $m^c(\varepsilon)$ close to $m^c = 6$ at which Hopf bifurcation holds for $\varepsilon$ small enough. Proofs combine spectral analysis and non-standard application of Hurwitz's Theorem with asymptotics as $\varepsilon \to 0$.
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## 1. Introduction

This paper is devoted to the stability analysis of the unique (up to translation) traveling wave solution to a diffusional-thermal model of flame propagation with stepwise ignition-temperature kinetics and a first-order reaction. In contrast to conventional Arrhenius kinetics where the reaction zone is infinitely thin (see, e.g. [13]), the reaction zone for stepwise temperature kinetics is of order unity (*thick flame*), see [3] for details and [1,4] for mathematical work on zero-order stepwise kinetics.

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We recall that the Lewis number Le is defined as the ratio of thermal diffusivity and mass diffusivity. Our concern in this paper is with high Lewis numbers, namely Le > 1. It is known (see [3, Section 3.2]) that large enough Lewis numbers give rise to *pulsating instabilities*, i.e., oscillatory behavior of the flame. This is very unlike *cellular instabilities* for relatively small Lewis number (Le < 1), that is pattern formation; in the latter case, a paradigm for the evolution of the disturbed flame front is the Kuramoto-Sivashinsky equation (see [22,25], and also [5–8,11]).

The problem reads in one spatial dimension:

$$\begin{cases} \dfrac{\partial \Theta}{\partial t} = \dfrac{\partial^2 \Theta}{\partial x^2} + W(\Theta, \Phi), \\[2mm] \dfrac{\partial \Phi}{\partial t} = \mathrm{Le}^{-1} \dfrac{\partial^2 \Phi}{\partial x^2} - W(\Theta, \Phi). \end{cases} \tag{1.1}$$

Here, $\Theta$ and $\Phi$ are appropriately normalized temperature and concentration of deficient reactant, $x \in \mathbb{R}$ denotes the spatial coordinate, $t > 0$ the time. The nonlinear term $W(\Theta, \Phi)$ is a scaled reaction rate given by (see [3, Section 2, formula (3)])

$$W(\Theta, \Phi) = \begin{cases} A\Phi, & \text{if } \Theta \geq \Theta_i, \\ 0, & \text{if } \Theta < \Theta_i, \end{cases} \tag{1.2}$$

or, equivalently, $W(\Theta, \Phi) = A\Phi H(\Theta - \Theta_i)$ where $H$ is the Heaviside function. In (1.2), $0 < \Theta_i < 1$ is the reduced ignition temperature, $A > 0$ is a normalized factor depending on $\Theta_i$ and Le, to be determined hereafter for the purpose of ensuring that the speed of traveling wave is set at unity. Moreover, the following boundary conditions hold at $\pm\infty$:

$$\begin{aligned} \Theta(t, -\infty) &= 1, & \Theta(t, \infty) &= 0, \\ \Phi(t, -\infty) &= 0, & \Phi(t, \infty) &= 1. \end{aligned} \tag{1.3}$$

At least prima facie, $\Phi$ does not vanish except as $x$ tends to $-\infty$. Thus, problem (1.1)–(1.3) belongs to the class of parabolic Partial Differential Equations with discontinuous nonlinearities. Models in combustion theory and other fields (see, e.g. [2, Section 1]) involving discontinuous reaction terms have been used by physicists and engineers for long because of their manageability; as a result, elliptic and parabolic PDEs with discontinuous nonlinearities have received a close attention from the mathematical community, see [1, Section 1] and references therein.

In this paper, we adopt the physical perspective of a Free Boundary Problem (see [1, Eq. (7)]) and consider the case of a free *ignition interface* $g(t)$ defined by

$$\Theta(t, g(t)) = \Theta_i, \tag{1.4}$$

such that $\Theta(t, x) > \Theta_i$ for $x < g(t)$ and $\Theta(t, x) < \Theta_i$ for $x > g(t)$. Formula (1.4) means that the ignition temperature $\Theta_i$ is reached at the ignition interface which defines the flame front.

According to (1.4), the system for $\boldsymbol{X} = (\Theta, \Phi)$ reads as follows, for $t > 0$ and $x \in \mathbb{R}, x \neq g(t)$:

$$\begin{cases} \dfrac{\partial \Theta}{\partial t} = \dfrac{\partial^2 \Theta}{\partial x^2} + A\Phi, & x < g(t), \\[2mm] \dfrac{\partial \Phi}{\partial t} = \mathrm{Le}^{-1} \dfrac{\partial^2 \Phi}{\partial x^2} - A\Phi, & x < g(t), \end{cases} \tag{1.5}$$

$$\begin{cases} \dfrac{\partial \Theta}{\partial t} = \dfrac{\partial^2 \Theta}{\partial x^2}, & x > g(t), \\[2mm] \dfrac{\partial \Phi}{\partial t} = \mathrm{Le}^{-1} \dfrac{\partial^2 \Phi}{\partial x^2}, & x > g(t). \end{cases} \tag{1.6}$$

At the free interface $x = g(t)$, in addition to (1.4) the following continuity conditions hold:

$$[\Theta] = [\Phi] = 0, \qquad \left[\dfrac{\partial \Theta}{\partial x}\right] = \left[\dfrac{\partial \Phi}{\partial x}\right] = 0, \tag{1.7}$$

where we denote by $[f]$ the jump of a function $f$ at a point $x_0$, i.e., the difference $f(x_0^+) - f(x_0^-)$.

The system above admits a unique (up to translation) traveling wave solution $\boldsymbol{U} = (\Theta^0, \Phi^0)$ which propagates with constant positive velocity $V$. In the moving frame coordinate $z = x - Vt$, by choosing

$$A = \frac{\Theta_i}{1 - \Theta_i}\left(1 + \frac{\Theta_i}{\text{Le}(1 - \Theta_i)}\right), \tag{1.8}$$

to have $V = 1$ and, hence, $z = x - t$, the traveling wave solution is explicitly given by the following formulae:

$$\Theta^0(z) = \begin{cases} 1 - (1 - \Theta_i)e^{\frac{\Theta_i}{1 - \Theta_i}z}, & z < 0, \\ \Theta_i e^{-z}, & z > 0, \end{cases}$$

$$\Phi^0(z) = \begin{cases} \dfrac{\Theta_i}{A(1 - \Theta_i)}e^{\frac{\Theta_i}{1 - \Theta_i}z}, & z < 0, \\ 1 + \left(\dfrac{\Theta_i}{A(1 - \Theta_i)} - 1\right)e^{-\text{Le}z}, & z > 0. \end{cases}$$

The paper is organized as follows: In Section 2, we first transform the free interface problem to a system of parabolic equations on a fixed domain. Then, in the spirit of [9,18,19], the perturbation $\boldsymbol{u}$ of the traveling wave $\boldsymbol{U}$ is split as $\boldsymbol{u} = s\boldsymbol{U}_\xi + \boldsymbol{v}$ ("ansatz 1"), in which $s$ is the perturbation of the front $g$. The largest part of the section is devoted to a thorough study of the linearization at 0 of the elliptic part of the parabolic system in a weighted space $\mathcal{W}$ where its realization $L$ is sectorial (see Subsection 2.3 for further details about the use of a weighted space). Furthermore, we determine the spectrum of $L$ which contains $(-\infty, -\frac{1}{4}]$, a parabola and its interior, the roots of the so-called *dispersion relation*, and the eigenvalue 0. Thereafter, an important point is getting rid of the eigenvalue 0 which is generated by the translation invariance. To this end, in Section 3 we use a spectral projection $P$ as well as "ansatz 2" and then derive the fully nonlinear problem (see, e.g. [21]) for $\boldsymbol{w}$:

$$\frac{\partial \boldsymbol{w}}{\partial \tau} = (I - P)L\boldsymbol{w} + F(\boldsymbol{w}).$$

Next, in Sections 4 and 5 we use the bifurcation parameter $m$ defined by

$$m := \frac{\Theta_i}{1 - \Theta_i}$$

to investigate the stability of the traveling wave. Because of the translation invariance, stability refers to orbital stability with asymptotic phase. Simultaneously, as one already noted that pulsating instability is likely to occur at large Lewis number, it is natural to introduce a small perturbation parameter $\varepsilon > 0$ (dimensionless diffusion coefficient) defined by $\varepsilon := \text{Le}^{-1}$, so that (1.8) reads $A = m + \varepsilon m^2$. The simplest situation arises in the asymptotic case of gasless combustion when $\text{Le} = \infty$, as in [16]. As it is readily seen, as $\varepsilon \to 0$, problem (1.5)–(1.7) converges formally to:

$$\begin{cases} \dfrac{\partial \Theta}{\partial t} = \dfrac{\partial^2 \Theta}{\partial x^2} + A\Phi, & x < g(t), \\ \dfrac{\partial \Phi}{\partial t} = -A\Phi, & x < g(t), \end{cases} \tag{1.9}$$

$$\begin{cases} \dfrac{\partial \Theta}{\partial t} = \dfrac{\partial^2 \Theta}{\partial x^2}, & x > g(t), \\ \Phi \equiv 1, & x > g(t), \end{cases} \tag{1.10}$$

with conditions (1.4), $[\Theta] = [\Phi] = 0$ and $[\partial \Theta/\partial x] = 0$ at the free interface $x = g(t)$. However, the limit free interface system (1.9)–(1.10) is only partly parabolic. At the outset, we fix $m$ in Section 4 and let $\varepsilon$ tend to 0, which allows to apply the classical Hurwitz's Theorem in complex analysis (see Appendix A) to the dispersion relation $D_\varepsilon(\lambda, m)$. Our first main result, Theorem 4.2, states that, for $2 < m < m^c = 6$ and $0 < \varepsilon < \varepsilon_0(m)$, the traveling wave $\boldsymbol{U}$ is orbitally stable with asymptotic phase and, for $m > m^c = 6$, it is unstable. To give a broad picture, we take advantage of the regular convergence of the point spectrum as $\varepsilon \to 0$.

Section 5 is devoted to the proof of Hopf bifurcation in a neighborhood of the critical value $m^c = 6$. The difficulty is twofold: first, the framework is that of a fully nonlinear problem; second, $m$ is not fixed in the sequence of

parameterized analytic functions $D_\varepsilon(\lambda, m)$ which prevents us from using Hurwitz's Theorem directly. The trick is to find a proper approach to combining $m$ with $\varepsilon$: to this end we construct a sequence of critical values $m^c(\varepsilon)$ such that $m^c(0) = m^c$ and apply Hurwitz's Theorem to $D_\varepsilon(\lambda, m^c(\varepsilon))$. Proposition 5.1 and Theorem 5.3 are crucial to prove Hopf bifurcation at $m^c(\varepsilon)$ for $\varepsilon$ small enough. Finally, in four appendices, we collect some formulae and results that we use to prove our main results.

## 2. The linearized operator

In this section, we first derive the governing equations for the perturbations of the traveling wave solution. As usual, it is convenient to transform the free interface problem to a system on a fixed domain. More specifically, we use the general method of [9] that converts free interface problems to fully nonlinear problems with transmission conditions at a fixed interface (see [1]). Then, we are going to focus on the linearized system.

### 2.1. The system with fixed interface

To begin with, we rewrite problem (1.4)–(1.7) in a new system of coordinates that fixes the position of the ignition interface at the origin:

$$\tau = t, \ \ \xi = x - g(\tau).$$

Hereafter, we are going to use, whenever it is convenient, the superdot to denote differentiation with respect to time and the prime to denote partial differentiation with respect to the space variable.

Then, the system for $\boldsymbol{X} = (\Theta, \Phi)$ and $g$ reads:

$$\begin{cases} \dfrac{\partial \Theta}{\partial \tau} - \dot{g}\dfrac{\partial \Theta}{\partial \xi} = \dfrac{\partial^2 \Theta}{\partial \xi^2} + A\Phi, & \xi < 0, \\[2mm] \dfrac{\partial \Phi}{\partial \tau} - \dot{g}\dfrac{\partial \Phi}{\partial \xi} = \mathrm{Le}^{-1}\dfrac{\partial^2 \Phi}{\partial \xi^2} - A\Phi, & \xi < 0, \end{cases} \tag{2.1}$$

$$\begin{cases} \dfrac{\partial \Theta}{\partial \tau} - \dot{g}\dfrac{\partial \Theta}{\partial \xi} = \dfrac{\partial^2 \Theta}{\partial \xi^2}, & \xi > 0, \\[2mm] \dfrac{\partial \Phi}{\partial \tau} - \dot{g}\dfrac{\partial \Phi}{\partial \xi} = \mathrm{Le}^{-1}\dfrac{\partial^2 \Phi}{\partial \xi^2}, & \xi > 0. \end{cases} \tag{2.2}$$

Moreover, $\Theta$, $\Phi$ and their first-order space derivatives are continuous at the fixed interface $\xi = 0$, thus

$$\Theta(\cdot, 0) = \Theta_i, \qquad [\Theta] = [\Phi] = 0, \qquad \left[\dfrac{\partial \Theta}{\partial \xi}\right] = \left[\dfrac{\partial \Phi}{\partial \xi}\right] = 0. \tag{2.3}$$

In addition, at $\xi = \pm\infty$, $\Theta$ and $\Phi$ satisfy (1.3).

Next, we introduce the small perturbations $\boldsymbol{u} = (u_1, u_2)$ and $s$, respectively of the traveling wave $\boldsymbol{U}$ and of the front $g$, more precisely,

$$u_1(\tau, \xi) = \Theta(\tau, \xi) - \Theta^0(\xi),$$
$$u_2(\tau, \xi) = \Phi(\tau, \xi) - \Phi^0(\xi),$$
$$s(\tau) = g(\tau) - \tau.$$

It then follows that the perturbations $\boldsymbol{u}$ and $s$ verify the system

$$\begin{cases} \dfrac{\partial u_1}{\partial \tau} = \dfrac{\partial^2 u_1}{\partial \xi^2} + \dfrac{\partial u_1}{\partial \xi} + A u_2 + \dot{s}\dfrac{d\Theta^0}{d\xi} + \dot{s}\dfrac{\partial u_1}{\partial \xi}, & \xi < 0, \\[2mm] \dfrac{\partial u_2}{\partial \tau} = \mathrm{Le}^{-1}\dfrac{\partial^2 u_2}{\partial \xi^2} + \dfrac{\partial u_2}{\partial \xi} - A u_2 + \dot{s}\dfrac{d\Phi^0}{d\xi} + \dot{s}\dfrac{\partial u_2}{\partial \xi}, & \xi < 0, \end{cases} \tag{2.4}$$

$$\begin{cases} \dfrac{\partial u_1}{\partial \tau} = \dfrac{\partial^2 u_1}{\partial \xi^2} + \dfrac{\partial u_1}{\partial \xi} + \dot{s}\dfrac{d\Theta^0}{d\xi} + \dot{s}\dfrac{\partial u_1}{\partial \xi}, & \xi > 0, \\[3mm] \dfrac{\partial u_2}{\partial \tau} = \mathrm{Le}^{-1}\dfrac{\partial^2 u_2}{\partial \xi^2} + \dfrac{\partial u_2}{\partial \xi} + \dot{s}\dfrac{d\Phi^0}{d\xi} + \dot{s}\dfrac{\partial u_2}{\partial \xi}, & \xi > 0, \end{cases} \tag{2.5}$$

and the corresponding interface conditions obtained from (2.3) are:

$$u_1(\cdot, 0) = 0, \qquad [u_1] = [u_2] = \left[\dfrac{\partial u_1}{\partial \xi}\right] = \left[\dfrac{\partial u_2}{\partial \xi}\right] = 0. \tag{2.6}$$

### 2.2. Ansatz 1

In the spirit of [9,18], we introduce the following splitting or ansatz:

$$\begin{aligned} u_1(\tau, \xi) &= s(\tau)\dfrac{d\Theta^0}{d\xi}(\xi) + v_1(\tau, \xi), \\[2mm] u_2(\tau, \xi) &= s(\tau)\dfrac{d\Phi^0}{d\xi}(\xi) + v_2(\tau, \xi), \end{aligned} \tag{2.7}$$

in which $v_1$, $v_2$ are new unknown functions. In a more abstract setting, the ansatz reads

$$\boldsymbol{u}(\tau, \xi) = s(\tau)\dfrac{d\boldsymbol{U}}{d\xi} + \boldsymbol{v}(\tau, \xi), \qquad \boldsymbol{v} = (v_1, v_2).$$

Substituting (2.7) into (2.4)–(2.5), we get the system for $\boldsymbol{v}$ and $s$:

$$\begin{cases} \dfrac{\partial v_1}{\partial \tau} = \dfrac{\partial^2 v_1}{\partial \xi^2} + \dfrac{\partial v_1}{\partial \xi} + A v_2 + \dot{s}\left(s\dfrac{d^2\Theta^0}{d\xi^2} + \dfrac{\partial v_1}{\partial \xi}\right), & \xi < 0, \\[3mm] \dfrac{\partial v_2}{\partial \tau} = \mathrm{Le}^{-1}\dfrac{\partial^2 v_2}{\partial \xi^2} + \dfrac{\partial v_2}{\partial \xi} - A v_2 + \dot{s}\left(s\dfrac{d^2\Phi^0}{d\xi^2} + \dfrac{\partial v_2}{\partial \xi}\right), & \xi < 0, \end{cases} \tag{2.8}$$

$$\begin{cases} \dfrac{\partial v_1}{\partial \tau} = \dfrac{\partial^2 v_1}{\partial \xi^2} + \dfrac{\partial v_1}{\partial \xi} + \dot{s}\left(s\dfrac{d^2\Theta^0}{d\xi^2} + \dfrac{\partial v_1}{\partial \xi}\right), & \xi > 0, \\[3mm] \dfrac{\partial v_2}{\partial \tau} = \mathrm{Le}^{-1}\dfrac{\partial^2 v_2}{\partial \xi^2} + \dfrac{\partial v_2}{\partial \xi} + \dot{s}\left(s\dfrac{d^2\Phi^0}{d\xi^2} + \dfrac{\partial v_2}{\partial \xi}\right), & \xi > 0. \end{cases} \tag{2.9}$$

At $\xi = 0$, it is easy to see that the new interface conditions are:

$$[v_1] = [v_2] = 0, \qquad \left[\dfrac{\partial v_1}{\partial \xi}\right] = -s\left[\dfrac{d^2\Theta^0}{d\xi^2}\right], \qquad \left[\dfrac{\partial v_2}{\partial \xi}\right] = -s\left[\dfrac{d^2\Phi^0}{d\xi^2}\right], \qquad v_1(\cdot, 0) = -s\dfrac{d\Theta^0}{d\xi}(0).$$

Taking advantage of the conditions

$$\dfrac{d\Theta^0}{d\xi}(0) = -\Theta_i, \qquad \left[\dfrac{d^2\Theta^0}{d\xi^2}\right] = \dfrac{\Theta_i}{1 - \Theta_i}, \qquad \left[\dfrac{d^2\Phi^0}{d\xi^2}\right] = -\dfrac{\mathrm{Le}\,\Theta_i}{1 - \Theta_i},$$

where we used (1.8) to derive the last condition, it follows that

$$s(\tau) = \dfrac{v_1(\tau, 0)}{\Theta_i}, \qquad \left[\dfrac{\partial v_1}{\partial \xi}\right] = -\dfrac{v_1(\tau, 0)}{1 - \Theta_i}, \qquad \left[\dfrac{\partial v_2}{\partial \xi}\right] = \dfrac{v_1(\cdot, 0)\mathrm{Le}}{1 - \Theta_i}. \tag{2.10}$$

Summarizing, the free interface problem (1.4)–(1.7) has been converted to (2.4)–(2.6), which constitutes a non-linear system for $v_1$, $v_2$ and $s$, with transmission conditions (2.10) at $\xi = 0$. The next subsections are devoted to the study of the linearized problem (at zero) in an abstract setting, with notation $\boldsymbol{u} = (u, v)$ for convenience.

### 2.3. The linearized problem

Now, we consider the linearization at 0 of the system (2.8)–(2.10), which reads as follows:

$$
\begin{cases}
\dfrac{\partial u}{\partial \tau} = \dfrac{\partial^2 u}{\partial \xi^2} + \dfrac{\partial u}{\partial \xi} + A v, & \xi < 0, \\[2mm]
\dfrac{\partial v}{\partial \tau} = \mathrm{Le}^{-1} \dfrac{\partial^2 v}{\partial \xi^2} + \dfrac{\partial v}{\partial \xi} - A v, & \xi < 0,
\end{cases}
\tag{2.11}
$$

$$
\begin{cases}
\dfrac{\partial u}{\partial \tau} = \dfrac{\partial^2 u}{\partial \xi^2} + \dfrac{\partial u}{\partial \xi}, & \xi > 0, \\[2mm]
\dfrac{\partial v}{\partial \tau} = \mathrm{Le}^{-1} \dfrac{\partial^2 v}{\partial \xi^2} + \dfrac{\partial v}{\partial \xi}, & \xi > 0,
\end{cases}
\tag{2.12}
$$

with the interface conditions

$$
[u] = [v] = 0, \qquad \left[\dfrac{\partial u}{\partial \xi}\right] = -\dfrac{u(\cdot, 0)}{1 - \Theta_i}, \qquad \left[\dfrac{\partial v}{\partial \xi}\right] = \dfrac{u(\cdot, 0)\mathrm{Le}}{1 - \Theta_i}.
\tag{2.13}
$$

Problem (2.11)–(2.12) can be written in the more compact form $\dfrac{\partial \boldsymbol{u}}{\partial \tau} = \mathcal{L}\boldsymbol{u}$, where $\boldsymbol{u} = (u, v)$,

$$
\mathcal{L} = \begin{pmatrix}
\dfrac{\partial^2}{\partial \xi^2} + \dfrac{\partial}{\partial \xi} & A\chi_- \\[4mm]
0 & \mathrm{Le}^{-1}\dfrac{\partial^2}{\partial \xi^2} + \dfrac{\partial}{\partial \xi} - A\chi_-
\end{pmatrix}
$$

and $\chi_-$ denotes the characteristic function of the set $(-\infty, 0)$.

We now introduce the weighted space $\mathcal{W}$ where we analyze the system (2.11)–(2.13). As a matter of fact, the introduction of exponentially weighted spaces for proving stability of traveling waves has been a tool used in many problems since the pioneering work of Sattinger (see [24]). In our situation, due to the structure of the operator $\mathcal{L}$ and of the boundary conditions, choosing a suitable weight function, we can shift the continuous spectrum of the realization of the operator $\mathcal{L}$ in the space $\mathcal{W}$ to the left, thus, creating a gap with the imaginary axis which simplifies the analysis. Moreover, such a weight works well also with the nonlinearity term of our problem, as we will show. We stress that working in the space $\mathcal{W}$ implies that the conditions (1.3) at $-\infty$ have been relaxed; in this respect, the mathematical problem is slightly different from the original one.

**Definition 2.1.** The exponentially weighted Banach space $\mathcal{W}$ is defined by

$$
\mathcal{W} = \left\{ \boldsymbol{u} = (u, v) : e^{\frac{1}{2}\xi}u, e^{\frac{1}{2}\xi}v \in C_b((-\infty, 0); \mathbb{C}), \ e^{\frac{1}{2}\xi}u, e^{\frac{\mathrm{Le}}{2}\xi}v \in C_b((0, \infty); \mathbb{C}), \lim_{\xi \to 0^{\pm}} u(\xi), \lim_{\xi \to 0^{\pm}} v(\xi) \in \mathbb{R} \right\},
$$

equipped with the norm:

$$
\|\boldsymbol{u}\|_{\mathcal{W}} = \sup_{\xi < 0} |e^{\frac{1}{2}\xi}u(\xi)| + \sup_{\xi > 0} |e^{\frac{1}{2}\xi}u(\xi)| + \sup_{\xi < 0} |e^{\frac{1}{2}\xi}v(\xi)| + \sup_{\xi > 0} |e^{\frac{\mathrm{Le}}{2}\xi}v(\xi)|.
$$

In the above definition, $C_b(I; \mathbb{C})$ denotes the space of bounded and continuous functions from $I$ to $\mathbb{C}$, $I$ being either the interval $(-\infty, 0)$ or $(0, \infty)$. We finally introduce the realization $L$ of the operator $\mathcal{L}$ in $\mathcal{W}$ defined by

$$
D(L) = \left\{ \boldsymbol{u} \in \mathcal{W} : \dfrac{d\boldsymbol{u}}{d\xi}, \dfrac{d^2\boldsymbol{u}}{d\xi^2} \in \mathcal{W}, \ [u] = [v] = 0, \ \left[\dfrac{du}{d\xi}\right] = -\dfrac{u(0)}{1 - \Theta_i}, \ \left[\dfrac{dv}{d\xi}\right] = \dfrac{\mathrm{Le}\, u(0)}{1 - \Theta_i} \right\},
$$

$$
L\boldsymbol{u} = \mathcal{L}\boldsymbol{u}, \qquad \boldsymbol{u} \in D(L).
$$

**Remark 2.2.** We observe that, for any Lewis number, the pair $\dfrac{d\boldsymbol{U}}{d\xi} = \left( \dfrac{d\Theta^0}{d\xi}, \dfrac{d\Phi^0}{d\xi} \right)$ verifies system (2.11)–(2.13), and it belongs to the space $\mathcal{W}$. In other words, $\dfrac{d\boldsymbol{U}}{d\xi}$ is an eigenfunction of the operator $L$ associated with the eigenvalue 0.

The above remark gives a first justification for the choice of the exponential weights in the definition of $\mathcal{W}$. We also stress that, following the same strategy as in the proof of the forthcoming Theorem 2.3, it can be easily checked that the spectrum of the realization of the operator $\mathcal{L}$ in the nonweighted space of pairs $(u, v)$ such that $u$, $v$ are bounded and continuous in $(-\infty, 0) \cup (0, \infty)$, contains a parabola which is tangent at 0 to the imaginary axis.

### 2.4. Analysis of the operator L

Next theorem is devoted to a deep study of the operator $L$. For simplicity of notation, we set

$$H_{1,\lambda} = \sqrt{1 + 4\lambda}, \qquad H_{2,\lambda} = \sqrt{\mathrm{Le}^2 + 4\mathrm{Le}(A + \lambda)}, \qquad H_{3,\lambda} = \sqrt{\mathrm{Le}^2 + 4\mathrm{Le}\lambda} \tag{2.14}$$

and for $j = 1, 2$,

$$k_{j,\lambda} = \frac{-1 + (-1)^{j+1} H_{1,\lambda}}{2}, \qquad k_{2+j,\lambda} = \frac{-\mathrm{Le} + (-1)^{j+1} H_{2,\lambda}}{2}, \qquad k_{4+j,\lambda} = \frac{-\mathrm{Le} + (-1)^{j+1} H_{3,\lambda}}{2}. \tag{2.15}$$

**Theorem 2.3.** *The operator $L$ is sectorial and therefore generates an analytic semigroup. Moreover, its spectrum has components:*

(1) *$(-\infty, -1/4] \cup \mathcal{P}$, where $\mathcal{P} = \{\lambda \in \mathbb{C} : a \operatorname{Re}\lambda + b(\operatorname{Im}\lambda)^2 + c \leq 0\}$ with*

$$a = \left(1 - \frac{1}{\mathrm{Le}}\right)^2, \qquad b = \frac{1}{\mathrm{Le}}, \qquad c = \frac{2A + 1}{2} + \frac{8A - 5}{4\mathrm{Le}} + \frac{1 + A}{\mathrm{Le}^2} - \frac{1}{4\mathrm{Le}^3};$$

(2) *the simple isolated eigenvalue 0, the kernel of $L$ being spanned by $\dfrac{d\boldsymbol{U}}{d\xi}$;*

(3) *additional eigenvalues given by the roots of the dispersion relation*

$$D(\lambda; \Theta_i, \mathrm{Le}) := (k_{6,\lambda} - k_{3,\lambda})(k_{3,\lambda} - k_{2,\lambda})\left[1 - (1 - \Theta_i)\sqrt{1 + 4\lambda}\right] + A\mathrm{Le}, \tag{2.16}$$

*where $A$ is given by (1.8).*

**Proof.** Since the proof is rather lengthy, we split it into four steps. In the first two steps, we prove properties (1) and (3). Step 3 is devoted to the proof of property (2). Finally, in Step 4, we prove that the operator $L$ is sectorial in $\mathcal{W}$.

For notational convenience, throughout the proof, we set

$$\mathscr{I}_1 := \int_0^\infty f_1(s)e^{-k_1 s}ds, \qquad \mathscr{I}_2 := \int_{-\infty}^0 f_1(s)e^{-k_2 s}ds, \qquad \mathscr{I}_3 := \int_{-\infty}^0 f_2(s)e^{-k_2 s}ds,$$

$$\mathscr{I}_4 := \int_{-\infty}^0 f_2(s)e^{-k_4 s}ds, \qquad \mathscr{I}_5 := \int_0^\infty f_2(s)e^{-k_5 s}ds,$$

for any fixed $\boldsymbol{f} = (f_1, f_2) \in \mathcal{W}$, where, here and Steps 1 to 3, we simply write $k_j$ instead of $k_{j,\lambda}$ to enlighten the notation.

*Step 1.* To begin with, we prove that the interval $(-\infty, -1/4]$ belongs to the point spectrum of $L$. We first assume that $\lambda \leq -\mathrm{Le}/4$ (recall that $\mathrm{Le} > 1$). In such a case, $\operatorname{Re}(k_1) = \operatorname{Re}(k_2) = -1/2$, $\operatorname{Re}(k_5) = \operatorname{Re}(k_6) = -\mathrm{Le}/2$ and the function $\boldsymbol{u}$ defined by

$$u(\xi) = \begin{cases} c_1 e^{k_1 \xi} + c_2 e^{k_2 \xi}, & \xi < 0, \\ c_5 e^{k_1 \xi} + c_6 e^{k_2 \xi}, & \xi \geq 0, \end{cases} \qquad v(\xi) = \begin{cases} 0, & \xi < 0, \\ c_7 e^{k_5 \xi} + c_8 e^{k_6 \xi}, & \xi \geq 0, \end{cases} \tag{2.17}$$

belongs to $\mathcal{W}$ and solves the equation $\lambda \boldsymbol{u} - \mathcal{L}\boldsymbol{u} = \boldsymbol{0}$ for any choice of the complex parameters $c_1$, $c_2$, $c_5$, $c_6$, $c_7$ and $c_8$. Since there are only four boundary conditions to impose to guarantee that $\boldsymbol{u} \in D(L)$, the resolvent equation $\lambda \boldsymbol{u} - L\boldsymbol{u} = \boldsymbol{0}$ is not uniquely solvable in $\mathcal{W}$. Thus, $\lambda$ belongs to the point spectrum of $L$.

Next, we consider the case when $\lambda \in (-\text{Le}/4, -1/4]$. In this situation, $\text{Re}(k_1) = \text{Re}(k_2) = -1/2$, however, $\text{Re}(k_5) + \text{Le}/2 > 0$, $\text{Re}(k_6) + \text{Le}/2 < 0$. Thanks to the fact that $e^{\frac{\text{Le}}{2}\xi}v(\xi)$ should be bounded in $(0, \infty)$, the constant $c_7$ in (2.17) is zero, whereas the constants $c_1$, $c_2$, $c_5$, $c_6$ $c_8$ are arbitrary. As above, the resolvent equation $\lambda \boldsymbol{u} - L\boldsymbol{u} = \boldsymbol{0}$ cannot be solved uniquely. Consequently, we conclude that $(-\infty, -1/4]$ belongs to the point spectrum of the operator $L$.

From now on, we consider the case when $\lambda \notin (-\infty, -1/4]$ and the equation $\lambda \boldsymbol{u} - \mathcal{L}\boldsymbol{u} = \boldsymbol{f}$, with $\boldsymbol{f} \in \mathcal{W}$, whose solution is given by (B.2)–(B.5). Then, $\text{Re}(k_1) + 1/2 > 0$, $\text{Re}(k_2) + 1/2 < 0$, $\text{Re}(k_5) + \text{Le}/2 > 0$ and $\text{Re}(k_6) + \text{Le}/2 < 0$. Similarly to the previous procedure, using the formulae (B.4), (B.5) and (B.2) as well as the fact that the functions $\xi \mapsto e^{\frac{1}{2}\xi}u(\xi)$ and $\xi \mapsto e^{\frac{\text{Le}}{2}\xi}v(\xi)$ should be bounded in $\mathbb{R}$ and in $(0, \infty)$ respectively, the constants $c_2$, $c_5$, $c_7$ can be determined explicitly and they are given by

$$c_2 = \frac{1}{H_{1,\lambda}} \int_{-\infty}^{0} (Av(s) + f_1(s))e^{-k_2 s}ds, \qquad c_5 = \frac{1}{H_{1,\lambda}}\mathscr{I}_1, \qquad c_7 = \frac{\text{Le}}{H_{3,\lambda}}\mathscr{I}_5.$$

We now consider formula (B.3). Since $\text{Le} > 1$, it follows that $\text{Re}(k_4) + 1/2 < 0$. Moreover, we observe that the inequality $\text{Re}(k_3) + 1/2 \leq 0$ is satisfied if and only if $\lambda \in \mathcal{P}$. Indeed, fix any $\lambda \in \overset{\circ}{\mathcal{P}}$, the interior of $\mathcal{P}$, so that $\text{Re}(k_3) + 1/2 < 0$, and take

$$f_1(\xi) = \begin{cases} e^{-\frac{1}{2}\xi}, & \xi < 0, \\ 0, & \xi \geq 0, \end{cases} \qquad f_2 \equiv 0 \text{ in } \mathbb{R}.$$

In such a case, the more general solution, $\boldsymbol{u} \in \mathcal{W}$, to the equation $\lambda \boldsymbol{u} - \mathcal{L}\boldsymbol{u} = \boldsymbol{f}$ is given by $u(\xi) = c_6 e^{k_2 \xi}$ and $v(\xi) = c_8 e^{k_6 \xi}$ for $\xi \geq 0$, whereas $v \equiv 0$ in $(-\infty, 0)$ and $u(\xi) = c_1 e^{k_1 \xi} + 2H_{1,\lambda}^{-2}(2e^{-\frac{1}{2}\xi} - e^{k_1 \xi})$ for $\xi < 0$. Note that $k_1 \neq k_3$ for $\lambda \in \overset{\circ}{\mathcal{P}}$. Imposing the boundary conditions, we deduce that $c_6 = c_8 = 0$, $c_1 = -2H_{1,\lambda}^{-2}$ and $k_1 c_1 = 2H_{1,\lambda}^{-2}k_2$, which is clearly a contradiction. We conclude that the domain $\overset{\circ}{\mathcal{P}}$ and, consequently, its closure belong to the continuous spectrum of $L$. Summarizing, property (1) in the statement of the theorem is established.

*Step 2.* Here, we consider the equation $\lambda \boldsymbol{u} - \mathcal{L}\boldsymbol{u} = \boldsymbol{f}$ for $\boldsymbol{f} \in \mathcal{W}$ and values of $\lambda$ which are not in $(-\infty, -1/4] \cup \mathcal{P}$. For such $\lambda$'s and $j = 1, 2$ it holds that

$$\text{Re}(k_{2j-1}) + \frac{1}{2} > 0, \qquad \text{Re}(k_{2j}) + \frac{1}{2} < 0, \qquad \text{Re}(k_5) + \frac{\text{Le}}{2} > 0, \qquad \text{Re}(k_6) + \frac{\text{Le}}{2} < 0. \tag{2.18}$$

We first assume that $k_1 \neq k_3$. Imposing that the function $\boldsymbol{u}$ defined by (B.3)–(B.4) belongs to $\mathcal{W}$, we can uniquely determine the constants $c_2$, $c_4$, $c_5$ and $c_7$ and we get

$$u(\xi) = c_1 e^{k_1 \xi} + \frac{e^{k_1 \xi}}{H_{1,\lambda}} \int_{\xi}^{0} f_1(s)e^{-k_1 s}ds + \frac{e^{k_2 \xi}}{H_{1,\lambda}} \int_{-\infty}^{\xi} f_1(s)e^{-k_2 s}ds$$

$$+ \frac{A}{H_{1,\lambda}} \left\{ \left( \frac{e^{k_3 \xi}}{k_3 - k_2} - \frac{e^{k_3 \xi} - e^{k_1 \xi}}{k_3 - k_1} \right)c_3 + \frac{\text{Le}}{H_{2,\lambda}} \left[ \left( \frac{e^{k_1 \xi} - e^{k_3 \xi}}{k_3 - k_1} - \frac{e^{k_3 \xi}}{k_3 - k_2} \right) \int_{\xi}^{0} f_2(s)e^{-k_3 s}ds \right. \right.$$

$$+ \frac{e^{k_1 \xi}}{k_3 - k_1} \int_{\xi}^{0} f_2(s)e^{-k_1 s}ds + \left( \frac{e^{k_1 \xi} - e^{k_4 \xi}}{k_4 - k_1} + \frac{e^{k_4 \xi}}{k_4 - k_2} \right) \int_{-\infty}^{\xi} f_2(s)e^{-k_4 s}ds$$

$$\left. \left. + \frac{e^{k_1 \xi}}{k_4 - k_1} \int_{\xi}^{0} f_2(s)(e^{-k_4 s} - e^{-k_1 s})ds + \frac{(k_4 - k_3)e^{k_2 \xi}}{(k_3 - k_2)(k_4 - k_2)} \int_{-\infty}^{\xi} f_2(s)e^{-k_2 s}ds \right] \right\}, \tag{2.19}$$

$$v(\xi) = \left( c_3 + \frac{\text{Le}}{H_{2,\lambda}} \int_{\xi}^{0} f_2(s)e^{-k_3 s}ds \right)e^{k_3 \xi} + \frac{\text{Le}\, e^{k_4 \xi}}{H_{2,\lambda}} \int_{-\infty}^{\xi} f_2(s)e^{-k_4 s}ds, \tag{2.20}$$

for $\xi < 0$. Note that $k_2 - k_3 \neq 0$ (see Appendix B). For $\xi > 0$, we get

$$u(\xi) = \frac{e^{k_1\xi}}{H_{1,\lambda}} \int_\xi^\infty f_1(s)e^{-k_1 s}ds + \left(c_6 + \frac{1}{H_{1,\lambda}} \int_0^\xi f_1(s)e^{-k_2 s}ds\right)e^{k_2\xi}, \tag{2.21}$$

$$v(\xi) = \frac{\mathrm{Le}\,e^{k_5\xi}}{H_{3,\lambda}} \int_\xi^\infty f_2(s)e^{-k_5 s}ds + \left(c_8 + \frac{\mathrm{Le}}{H_{3,\lambda}} \int_0^\xi f_2(s)e^{-k_6 s}ds\right)e^{k_6\xi}. \tag{2.22}$$

Imposing the boundary conditions, we obtain the following linear system for the unknowns $c_1$, $c_3$, $c_6$ and $c_8$:

$$\begin{pmatrix} 1 & \frac{A}{(k_3-k_2)H_{1,\lambda}} & -1 & 0 \\ 0 & 1 & 0 & -1 \\ k_1 & \frac{Ak_2}{(k_3-k_2)H_{1,\lambda}} & \frac{1}{\Theta_i-1}-k_2 & 0 \\ 0 & k_3 & \frac{\mathrm{Le}}{1-\Theta_i} & -k_6 \end{pmatrix} \begin{pmatrix} c_1 \\ c_3 \\ c_6 \\ c_8 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix}, \tag{2.23}$$

where

$$F_1 = -\frac{A\mathrm{Le}}{(k_4-k_2)H_{1,\lambda}H_{2,\lambda}}\mathscr{I}_4 - \frac{1}{H_{1,\lambda}}\mathscr{I}_2 + \frac{1}{H_{1,\lambda}}\mathscr{I}_1 - \frac{A\mathrm{Le}(k_4-k_3)}{(k_3-k_2)(k_4-k_2)H_{1,\lambda}H_{2,\lambda}}\mathscr{I}_3;$$

$$F_2 = \frac{\mathrm{Le}}{H_{3,\lambda}}\mathscr{I}_5 - \frac{\mathrm{Le}}{H_{2,\lambda}}\mathscr{I}_4;$$

$$F_3 = -\frac{A\mathrm{Le}k_2}{(k_4-k_2)H_{1,\lambda}H_{2,\lambda}}\mathscr{I}_4 - \frac{k_2}{H_{1,\lambda}}\mathscr{I}_2 + \frac{1}{H_{1,\lambda}}\left(k_1 + \frac{1}{1-\Theta_i}\right)\mathscr{I}_1 + \frac{A\mathrm{Le}k_2}{(k_3-k_2)(k_4-k_2)H_{1,\lambda}}\mathscr{I}_3;$$

$$F_4 = \frac{\mathrm{Le}k_5}{H_{3,\lambda}}\mathscr{I}_5 - \frac{\mathrm{Le}k_4}{H_{2,\lambda}}\mathscr{I}_4 - \frac{\mathrm{Le}}{(1-\Theta_i)H_{4,\lambda}}\mathscr{I}_1.$$

This system is uniquely solvable if and only if $\overline{D}(\lambda; \Theta_i, \mathrm{Le}) = [\mathrm{Le}(k_2 - k_3)]^{-1}D(\lambda; \Theta_i, \mathrm{Le})$, the determinant of the matrix in left-hand side of (2.23), does not vanish, where $D(\lambda; \Theta_i, \mathrm{Le})$ is defined in (2.16). Hence, the solutions to the equation $D(\lambda; \Theta_i, \mathrm{Le}) = 0$ are elements of the point spectrum of $L$. Property (3) is proved. On the other hand, as it is easily seen, if $\lambda \notin (-\infty, -1/4] \cup \mathcal{P}$ is not a root of the dispersion relation, then it is easy to check that the function $\boldsymbol{u}$ given by (2.19)–(2.23) belongs to $D(L)$, so that $\lambda$ is an element of the resolvent set of operator $L$.

Finally, we consider the case when $k_3 = k_1$, which gives $\lambda = \lambda_\pm := -\frac{A\mathrm{Le}}{\mathrm{Le}-1} \pm \frac{i\sqrt{A\mathrm{Le}(\mathrm{Le}-1)}}{\mathrm{Le}-1}$ (see Appendices B and C). It is easy to check that this pair of conjugate complex numbers does not belong to $\mathcal{P}$. It thus follows that $u$ for $\xi \geq 0$ and $v$ for $\xi \in \mathbb{R}$ are still given by (2.20), (2.21) and (2.22). On the other hand, for $\xi < 0$, $u$ is given by

$$u(\xi) = c_1 e^{k_1\xi} - \frac{Ac_3}{H_{1,\lambda}}\xi e^{k_1\xi} + \frac{e^{k_1\xi}}{H_{1,\lambda}} \int_\xi^0 f_1(s)e^{-k_1 s}ds + \frac{e^{k_2\xi}}{H_{1,\lambda}} \int_{-\infty}^\xi f_1(s)e^{-k_2 s}ds$$

$$+ \frac{A\mathrm{Le}\,e^{k_1\xi}}{H_{1,\lambda}H_{2,\lambda}} \int_\xi^0 (s-\xi)f_2(s)ds - \frac{A\mathrm{Le}\,e^{k_1\xi}}{H_{1,\lambda}H_{2,\lambda}^2} \int_{-\infty}^0 f_2(s)e^{-k_4 s}ds$$

$$+ \frac{A\mathrm{Le}\,e^{k_1\xi}}{H_{1,\lambda}H_{2,\lambda}^2} \int_\xi^0 f_2(s)e^{-k_1 s}ds + \frac{A\mathrm{Le}\,e^{k_4\xi}}{H_{1,\lambda}H_{2,\lambda}^2} \int_{-\infty}^\xi f_2(s)e^{-k_4 s}ds$$

$$+ \frac{A}{H_{1,\lambda}}\left\{\frac{e^{k_1\xi}}{k_1-k_2}c_3 + \frac{\mathrm{Le}}{H_{2,\lambda}}\left[\frac{e^{k_4\xi}}{k_4-k_2} \int_{-\infty}^\xi f_2(s)e^{-k_4 s}ds - \frac{e^{k_1\xi}}{k_1-k_2} \int_\xi^0 f_2(s)e^{-k_1 s}ds\right.\right.$$

$$+ \frac{(k_4 - k_1)e^{k_2\xi}}{(k_1 - k_2)(k_4 - k_2)} \int\limits_{-\infty}^{\xi} f_2(s)e^{-k_2 s}\,ds \Bigg]\Bigg\}.$$

Notice that $\sup_{\xi<0} e^{\frac{1}{2}\xi}|u(\xi)| < \infty$; therefore, $\boldsymbol{u}$ belongs to $\mathcal{W}$. Imposing the boundary conditions, we get a linear system for the unknowns $(c_1, c_3, c_6, c_8)$, whose matrix is the same as in (2.23). Since the determinant is not zero when $\lambda = \lambda_\pm$ (see Appendix C) and the first- and second-order derivatives of $\boldsymbol{u}$ belong to $\mathcal{W}$, we conclude that $\lambda_\pm$ are in the resolvent set of operator $L$.

*Step 3.* Now, we proceed to show that 0 is an isolated simple eigenvalue of the operator $L$. In view of the previous steps, in a neighborhood of $\lambda = 0$ the solution $\boldsymbol{u} = R(\lambda, L)\boldsymbol{f}$ of the equation $\lambda \boldsymbol{u} - L\boldsymbol{u} = \boldsymbol{f}$ is given by (2.19)–(2.22) for any $\boldsymbol{f} \in \mathcal{W}$, where

$$c_1 = \frac{\mathrm{Le}(k_2 - k_3)}{D(\lambda; \Theta_i, \mathrm{Le})}\left\{\left[\frac{(k_6 - k_3)(1 - \Theta_i)}{\mathrm{Le}} - \frac{A}{(k_3 - k_2)H_{1,\lambda}}\right]\mathscr{I}_1 + \frac{k_6 - k_3}{\mathrm{Le}\,H_{1,\lambda}}\mathscr{I}_2 - \frac{A(k_6 - k_3)}{(k_3 - k_2)(k_4 - k_2)H_{1,\lambda}}\mathscr{I}_3\right.$$
$$\left. + \frac{A}{H_{1,\lambda}H_{2,\lambda}}\left(\frac{k_6 - k_3}{k_4 - k_2} - \frac{k_6 - k_4}{k_3 - k_2}\right)\mathscr{I}_4 - \frac{A}{(k_3 - k_2)H_{1,\lambda}}\mathscr{I}_5\right\},$$

$$c_3 = \frac{\mathrm{Le}(k_2 - k_3)}{D(\lambda; \Theta_i, \mathrm{Le})}\left\{\mathscr{I}_1 + \mathscr{I}_2 - \frac{A\mathrm{Le}}{(k_4 - k_2)(k_3 - k_2)}\mathscr{I}_3\right.$$
$$\left. + \frac{1}{H_{2,\lambda}}\left[(k_6 - k_4)\left[1 - H_{1,\lambda}(1 - \Theta_i)\right] + \frac{A\mathrm{Le}}{k_4 - k_2}\right]\mathscr{I}_4 + \left[1 - H_{1,\lambda}(1 - \Theta_i)\right]\mathscr{I}_5\right\},$$

$$c_6 = \frac{\mathrm{Le}(k_2 - k_3)}{D(\lambda; \Theta_i, \mathrm{Le})}\left\{\frac{1}{H_{1,\lambda}}\left(\frac{A}{k_3 - k_2} + \frac{k_6 - k_3}{\mathrm{Le}}\right)\mathscr{I}_1 + \frac{(k_6 - k_3)(1 - \Theta_i)}{\mathrm{Le}}\mathscr{I}_2 - \frac{A(k_6 - k_3)(1 - \Theta_i)}{(k_3 - k_2)(k_4 - k_2)}\mathscr{I}_3\right.$$
$$\left. + \frac{A(1 - \Theta_i)}{H_{2,\lambda}}\left(\frac{k_6 - k_3}{k_4 - k_2} - \frac{k_6 - k_4}{k_3 - k_2}\right)\mathscr{I}_4 - \frac{A(1 - \Theta_i)}{k_3 - k_2}\mathscr{I}_5\right\},$$

$$c_8 = \frac{\mathrm{Le}(k_2 - k_3)}{D(\lambda; \Theta_i, \mathrm{Le})}\left\{\mathscr{I}_1 + \mathscr{I}_2 - \frac{A\mathrm{Le}}{(k_3 - k_2)(k_4 - k_2)}\mathscr{I}_3 + \left[1 - H_{1,\lambda}(1 - \Theta_i) + \frac{A\mathrm{Le}}{(k_3 - k_2)(k_4 - k_2)}\right]\mathscr{I}_4\right.$$
$$\left. + \left[\frac{A\mathrm{Le}}{(k_3 - k_2)H_{3,\lambda}} + [1 - H_{1,\lambda}(1 - \Theta_i)]\left(1 + \frac{k_6 - k_3}{H_{3,\lambda}}\right)\right]\mathscr{I}_5\right\}.$$

As it is immediately seen, the function $D(\cdot; \Theta_i, \mathrm{Le})$ is analytic in a neighborhood of $\lambda = 0$, which is simple zero of such a function, and the other functions appearing in (2.19)–(2.22) are holomorphic in a neighborhood of $\lambda = 0$. Hence, we conclude that zero is a simple pole of the resolvent operator $R(\lambda, L)$. Since $\dfrac{d\boldsymbol{U}}{d\xi}$ belongs to the kernel of $L$ (see Remark 2.2) and the matrix in (2.23) has rank three at $\lambda = 0$, this function generates the kernel, so that the geometric multiplicity of the eigenvalue $\lambda = 0$ is one. This is enough to conclude that $\lambda = 0$ is a simple eigenvalue of $L$. Property (2) is established and the spectrum of $L$ is completely characterized.

*Step 4.* In order to prove that $L$ is sectorial, it is sufficient to show that there exist two positive constants $C$ and $M$ such that

$$\|R(\lambda, L)\|_{L(\mathcal{W})} \le C|\lambda|^{-1}, \qquad \mathrm{Re}\,\lambda \ge M. \tag{2.24}$$

(See e.g., [21, Proposition 2.1.11].) Without loss of generality, we can assume that $k_{1,\lambda} \ne k_{3,\lambda}$ and the conditions in (2.18) are all satisfied if $\mathrm{Re}\,\lambda \ge M$. Throughout this step, $C_j$ denotes a positive constant, independent of $\lambda$ and $\boldsymbol{f} \in \mathcal{W}$.

We begin by estimating the terms $H_{j,\lambda}$ ($j = 1, 2, 3$). As it is easily seen,

$$|H_{2,\lambda}| \ge \mathrm{Re}(H_{2,\lambda}) = \sqrt{\frac{|\mathrm{Le}^2 + 4\mathrm{Le}(A + \lambda)| + \mathrm{Le}^2 + 4\mathrm{Le}(A + \mathrm{Re}\,\lambda)}{2}} \ge \sqrt{2\mathrm{Le}|\lambda|} \tag{2.25}$$

for any $\lambda \in \mathbb{C}$ with positive real part. Since $H_{1,\lambda}$ and $H_{3,\lambda}$ can be obtained from $H_{2,\lambda}$, by taking, $(\mathrm{Le}, A) = (1, 0)$ and $(\mathrm{Le}, A) = (\mathrm{Le}, 0)$ respectively, we also deduce that

$$|H_{1,\lambda}| \geq \mathrm{Re}(H_{1,\lambda}) \geq \sqrt{2|\lambda|}, \qquad |H_{3,\lambda}| \geq \mathrm{Re}(H_{3,\lambda}) \geq \sqrt{2\mathrm{Le}|\lambda|} \tag{2.26}$$

for the same values of $\lambda$. Thanks to (2.25) and (2.26), we can easily estimate the terms $\mathscr{I}_j$ ($j = 1, \ldots, 5$). Indeed, since $\mathrm{Re}(k_1) + 1/2 > 0$, we obtain

$$|\mathscr{I}_1| = \left| \int_0^\infty f_1(s) e^{-k_1 s} ds \right| \leq \sup_{\xi > 0} e^{\frac{1}{2}\xi} |f_1(\xi)| \int_0^\infty e^{-\frac{1}{2} \mathrm{Re}(H_{1,\lambda})s} ds \leq C_1 |\lambda|^{-\frac{1}{2}} \|\boldsymbol{f}\|_{\mathcal{W}}.$$

The other terms $\mathscr{I}_j$ can be treated likewise and we get $\sum_{j=2}^5 |\mathscr{I}_j| \leq C_2 |\lambda|^{-\frac{1}{2}} \|\boldsymbol{f}\|_{\mathcal{W}}$ for every $\boldsymbol{f} \in \mathcal{W}$ and $\lambda \in \mathbb{C}$ with positive real part.

Next, we turn to the function $D(\cdot; \Theta_i, \mathrm{Le})$. We observe that

$$|D(\lambda; \Theta_i, \mathrm{Le})| \geq [(1 - \Theta_i)\sqrt{|1 + 4\lambda|} - 1]|k_{6,\lambda} - k_{3,\lambda}||k_{3,\lambda} - k_{2,\lambda}| - A\mathrm{Le}$$

for any $\lambda \in \mathbb{C}$. Taking (2.25) and (2.26) into account, we can show that

$$C_3 \sqrt{|\lambda|} \leq |k_{3,\lambda} - k_{2,\lambda}| + |k_{3,\lambda} - k_{6,\lambda}| \leq C_4 \sqrt{|\lambda|} \tag{2.27}$$

for $\lambda \in \mathbb{C}$ with sufficiently large positive real part. Hence, for such values of $\lambda$'s we can continue the previous inequality and get

$$|D(\lambda; \Theta_i, \mathrm{Le})| \geq C_5 |\lambda|^{\frac{3}{2}}. \tag{2.28}$$

Similarly, $|k_{6,\lambda} - k_{4,\lambda}| \leq C_6 \sqrt{|\lambda|}$ for any $\lambda$ with positive real part and

$$|k_{4,\lambda} - k_{2,\lambda}| \geq \frac{1}{2}|H_{2,\lambda}| - \frac{1}{2}|H_{1,\lambda}| - \frac{\mathrm{Le} - 1}{2} \geq \sqrt{\frac{\mathrm{Le}|\lambda|}{2}} - \sqrt{\frac{|\lambda|}{2}} - \frac{\mathrm{Le} - 1}{2} \geq C_7 \sqrt{|\lambda|}, \tag{2.29}$$

if $\mathrm{Re}\,\lambda$ is sufficiently large. From (2.25)–(2.29) we infer that $|c_1| + |c_3| + |c_6| + |c_8| \leq C_8 |\lambda|^{-1}$ for any $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq M$ and a suitable positive constant $M$. Further, observing that

$$|k_{3,\lambda} - k_{1,\lambda}| + |k_{4,\lambda} - k_{1,\lambda}| \geq C_9 \sqrt{|\lambda|}, \qquad |k_{4,\lambda} - k_{3,\lambda}| \leq C_{10} \sqrt{|\lambda|},$$

we are now able to estimate the functions $u$ and $v$ in (2.19)–(2.22) and show that (2.24) holds true. The proof is complete. $\square$

**Remark 2.4.** It is worth pointing out that, as $\mathrm{Le} \to \infty$, the set $\mathcal{P}$ degenerates into a halfplane $\mathrm{Re}\,\lambda \leq -\Theta_i(1 - \Theta_i)^{-1} - 1/2$. In the limit case, the system is partly parabolic and the semigroup is not analytic, see, e.g., [17, Section 1, p. 2435].

## 3. The fully nonlinear problem

Our goal in this section is to get rid of the eigenvalue 0 and then derive a new fully nonlinear problem. We recall that the eigenvalue 0 is related to the translation invariance of the traveling wave. In a first step, we use here a method similar to that of [12] or [21, p. 358].

### 3.1. Ansatz revisited: elimination of the eigenvalue 0

It is convenient to write system (2.4)–(2.6) with notation $\boldsymbol{u} = (u_1, u_2)$, $\boldsymbol{U} = (\Theta^0, \Phi^0)$, see Section 2.1, in an abstract form:

$$\dot{\boldsymbol{u}} = L\boldsymbol{u} + \dot{s}\boldsymbol{U}' + \dot{s}\boldsymbol{u}'. \tag{3.1}$$

Note that, in view of (2.6), $\boldsymbol{u}(\tau, \cdot)$ belongs to $D(L)$ for each $\tau$. Since 0 is an isolated simple eigenvalue of $L$, we can introduce the spectral projection $P$ onto the kernel of $L$, defined by $P\boldsymbol{f} = \langle \boldsymbol{f}, \boldsymbol{e}^* \rangle \boldsymbol{U}'$ for every $\boldsymbol{f} \in \mathcal{W}$ and a unique $\boldsymbol{e}^* \in \mathcal{W}^*$, the dual space of $\mathcal{W}$, such that $\langle \boldsymbol{U}', \boldsymbol{e}^* \rangle = 1$. For further use, we recall that $P$ commutes with $L$ on $D(L)$. We are going to apply the projections $P$ and $Q = I - P$ to system (3.1) to remove the eigenvalue 0.

**Ansatz 2.** We split $u$ into $u(\tau, \cdot) = Pu(\tau, \cdot) + Qu(\tau, \cdot) = p(\tau)U' + w(\tau, \cdot)$, i.e.,

$$u_1(\tau, \xi) = p(\tau)\frac{d\Theta^0}{d\xi}(\xi) + w_1(\tau, \xi), \tag{3.2}$$

$$u_2(\tau, \xi) = p(\tau)\frac{d\Phi^0}{d\xi}(\xi) + w_2(\tau, \xi),$$

where $p(\tau) = \langle u(\tau), e^* \rangle$ and $w = (w_1, w_2)$. Clearly, $w(\tau, \cdot) \in Q(D(L))$ for each $\tau$. It follows from (3.1) that

$$\dot{p} = \dot{s} + \dot{s}\langle u', e^* \rangle, \qquad \dot{w} = Lw + \dot{s}Qu', \tag{3.3}$$

a Lyapunov-Schmidt-like reduction of the problem. We point out that the above procedure generates a new ansatz slightly different from ansatz 1 (see (2.7)) that helped us determine the functional framework.

Thanks to new ansatz 2, we are going to derive an equation for $w$ in the space $\mathcal{W}$. Now, the spectrum of the part of $L$ in $Q(\mathcal{W})$ does not contain the eigenvalue 0.

### 3.2. Derivation of the fully nonlinear equation

To get a self-contained equation for $w$, we need to eliminate $\dot{s}$ from the right-hand side of the second equation in (3.3). For this purpose, we begin by evaluating the first component of (3.3) at $\xi = 0^+$ to get

$$\frac{\partial w_1}{\partial \tau}(\cdot, 0^+) = (Lw)_1(\cdot, 0^+) + \dot{s}(Qu')_1(\cdot, 0^+)$$

$$= (Lw)_1(\cdot, 0^+) + \dot{s}\frac{\partial u_1}{\partial \xi}(\cdot, 0^+) + \dot{s}\langle u', e^* \rangle \Theta_i. \tag{3.4}$$

Next, we observe that the function $w_1$ is continuous (but not differentiable) at $\xi = 0$, since both $u$ and $U'$ are continuous at $\xi = 0$. Therefore, evaluating (3.2) at $\xi = 0$ and recalling that $u_1(\tau, 0) = 0$ (see (2.6)), we infer that $w_1(\tau, 0) = \Theta_i p(\tau)$. Differentiating this formula yields

$$\frac{\partial w_1}{\partial \tau}(\cdot, 0) = \dot{p}\Theta_i = \dot{s}\Theta_i + \dot{s}\langle u', e^* \rangle \Theta_i. \tag{3.5}$$

From (3.4) and (3.5), it follows that

$$\dot{s}\Theta_i = (Lw)_1(\cdot, 0^+) + \dot{s}\frac{\partial u_1}{\partial \xi}(\cdot, 0^+). \tag{3.6}$$

To get rid of the spatial derivative of $u_1$ from the right-hand side of (3.6), we use (3.2) to write

$$\frac{\partial u_1}{\partial \xi}(\cdot, 0^+) = p\frac{d^2\Theta^0}{d\xi^2}(0^+) + w_1'(\cdot, 0^+) = w_1(\cdot, 0) + w_1'(\cdot, 0^+). \tag{3.7}$$

Plugging (3.7) into (3.6), we finally obtain the formula

$$\dot{s} = \frac{(Lw)_1(\cdot, 0^+)}{\Theta_i - w_1(\cdot, 0) - w_1'(\cdot, 0^+)}, \tag{3.8}$$

which can be regarded as a underlying or implicit *second-order Stefan condition*, see [10]. Hence, replacing it in (3.3), we get

$$\frac{\partial w}{\partial \tau} = Lw + \frac{(Lw)_1(\cdot, 0^+)}{\Theta_i - w_1(\cdot, 0) - w_1'(\cdot, 0^+)}Qu'$$

$$= Lw + \frac{(Lw)_1(\cdot, 0^+)}{\Theta_i - w_1(\cdot, 0) - w_1'(\cdot, 0^+)}Q\left(\frac{w_1(\cdot, 0)}{\Theta_i}U'' + w'\right),$$

which is a fully nonlinear parabolic equation in the space $\mathcal{W}$, written in a more abstract form:

$$\frac{\partial w}{\partial \tau} = Lw + F(w), \quad w \in Q(D(L)), \tag{3.9}$$

and is going to be the subject of our attention. Note that Equation (3.9) is fully nonlinear since the function $F$ depends on $\boldsymbol{w}$ also through the limit at $0^+$ of $L\boldsymbol{w}$ and 0 is an equilibrium for all values of $\Theta_i$. Moreover, the operator $L$ is sectorial in $Q(\mathcal{W})$. Hence, we can take advantage of the theory of analytic semigroups to solve Equation (3.9). We refer the reader to [21, Chapter 4] for further details.

## 4. Stability of the traveling wave solution

This section is devoted to the analysis of the stability of the traveling wave solution $\boldsymbol{U}$. Here, stability refers to orbital stability with asymptotic phase $s_\infty$. From now on, we focus on the asymptotic situation where the Lewis number, Le, is large and, in this respect, we use the notation $\varepsilon = 1/\text{Le}$ to stand for a small perturbation parameter. Simultaneously, we assume that $\Theta_i$ is close to the burning temperature normalized at unity, which is physically relevant (see [3, Section 3.2, Fig. 5]). More specifically, we restrict $\Theta_i$ to the domain $\frac{2}{3} < \Theta_i < 1$.

In what follows, we introduce $m := \Theta_i/(1 - \Theta_i)$ as the *bifurcation parameter* which runs in the interval $(2, \infty)$, due to the choice of $\Theta_i$. Accordingly, in some instances hereafter, we will denote $\Theta_i$ by $\Theta_i(m) = m/(1+m)$, in other words $\Theta_i$ is now parametrized by $m$. With the above notation, $A = m + \varepsilon m^2$ and the dispersion relation $D(\lambda; \Theta_i, \text{Le})$ (see (2.16)) in Section 2 reads:

$$D_\varepsilon(\lambda; m) = -\frac{1}{4}\left(\sqrt{1 + 4\varepsilon(m + \varepsilon m^2 + \lambda)} + \sqrt{1 + 4\varepsilon\lambda}\right)$$

$$\times \left(\frac{1}{\varepsilon}[\sqrt{1 + 4\varepsilon(m + \varepsilon m^2 + \lambda)} - 1] + 1 + \sqrt{1 + 4\lambda}\right)\left(1 - \frac{\sqrt{1 + 4\lambda}}{1 + m}\right) + m + \varepsilon m^2. \qquad (4.1)$$

This section is split into two parts. First, we study the stability of the null solution of the fully nonlinear equation (3.9). Second, we turn our attention to the stability of the traveling wave.

### 4.1. Stability of the null solution of (3.9)

To begin with, we recall that the spectrum of the part of $L$ in $\mathcal{W}_Q := Q(\mathcal{W})$ is the set

$$\left(-\infty, -\tfrac{1}{4}\right] \cup \mathcal{P} \cup \{\lambda \in \mathbb{C} \setminus \{0\} : D_\varepsilon(\lambda; m) = 0\}.$$

As we will show, the roots of the dispersion relation $D_\varepsilon(\cdot; m)$ are finitely many. As a consequence, there is a gap between the spectrum of this operator and the imaginary axis (at least for $\varepsilon$ small enough). In view of the principle of linearized stability, the main step in the analysis of the stability of the null solution of Equation (3.9) is a deep insight in the roots of the dispersion relation. More precisely, we need to determine when they are all contained in the left halfplane and when some of them lie in the right halfplane.

The limit critical value $m^c = 6$ will play an important role in the analysis hereafter.

**Theorem 4.1.** *The following properties are satisfied.*

(i) *Let $m \in (2, m^c)$ be fixed. Then, there exists $\varepsilon_0 = \varepsilon_0(m) > 0$ such that, for $\varepsilon \in (0, \varepsilon_0)$, the null solution of the fully nonlinear problem (3.9) is stable with respect to perturbations belonging to $Q(D(L))$.*

(ii) *Let $m > m^c$ be fixed. Then, there exists $\varepsilon_1 = \varepsilon_1(m)$ small enough such that, for $\varepsilon \in (0, \varepsilon_1)$, the null solution of (3.9) is unstable with respect to perturbations belonging to $Q(D(L))$.*

**Proof.** To begin with, we observe that the functions $D_\varepsilon(\cdot, m)$ are holomorphic in $\mathbb{C} \setminus (-\infty, -1/4]$ and therein they locally converge to the *limit dispersion relation* $D_0(\cdot, m)$ defined by

$$D_0(\lambda; m) = -\frac{1}{2}[2(m + \lambda) + 1 + \sqrt{1 + 4\lambda}]\left(1 - \frac{\sqrt{1 + 4\lambda}}{1 + m}\right) + m$$

$$= \frac{\sqrt{1 + 4\lambda} - 1}{4(1 + m)}[4\lambda - (m - 2)\sqrt{1 + 4\lambda} + m + 2],$$

as $\varepsilon \to 0^+$. The solutions of the equation $D_0(\lambda; m) = 0$ are $\lambda = 0$, for all $m$, and the roots of the second-order polynomial $4\lambda^2 + (6m - m^2)\lambda + 2m$, whose parts are not less than $-(m + 2)/4$. This polynomial admits conjugate roots $\lambda_{1,2} = a(m) \pm ib(m)$, where $a(m) = \frac{1}{8}(m^2 - 6m)$ and $b(m) = \frac{1}{8}(m - 2)\sqrt{|8m - m^2|}$ if $m \in (2, 8)$, and real roots $\lambda_{1,2} = a(m) \pm b(m)$ otherwise. The coefficient $a(m)$ is negative whenever $2 < m < 6$ and positive for $m > 6$. It can be easily checked that $\text{Re}(\lambda_{1,2}) \geq -(m + 2)/4$ for each $m \in (2, \infty)$, so that $\lambda_{1,2}$ solve the equation $D_0(\lambda; m) = 0$. In particular, there are two conjugate purely imaginary roots $\lambda_{1,2} = \pm\sqrt{3}i$ at $m^c = 6$.

We can now prove properties (i) and (ii).

(i) Fix $\rho > 0$ such that the closure of the disks of center $\lambda_{1,2}$ and radius $\rho$ is contained in $\{\text{Re } z < 0\} \setminus (-\infty, -\frac{1}{4}]$. Hurwitz's Theorem (see Appendix A) and the above results show that there exists $\varepsilon_0 > 0$ such that, for $\varepsilon \in (0, \varepsilon_0)$, $D_\varepsilon(\lambda; m)$ admits exactly two conjugate complex roots $\lambda_{1,2}(\varepsilon)$ in the disk $|\lambda - \lambda_i| < \rho$ and $\lambda_i(\varepsilon)$ converges to $\lambda_i$, as $\varepsilon \to 0$, for $i = 1, 2$. Therefore, all the elements of the spectrum of the part of operator $L$ in $\mathcal{W}_Q$ have negative real parts, which implies that the operator norm of the restriction to $\mathcal{W}_Q$ of the analytic semigroup $e^{\tau L}$ generated by $L$ decays to zero with exponential rate as $\tau \to \infty$. Now, the nonlinear stability follows from applying a standard machinery: the solution of Equation (3.9), with initial datum $\boldsymbol{w}_0$ in a small (enough) ball of $Q(D(L))$ centered at zero, is given by the variation-of-constants-formula

$$\boldsymbol{w}(\tau, \cdot) = e^{\tau L}\boldsymbol{w}_0 + \int_0^\tau e^{(\tau - s)L} F(\boldsymbol{w}(s, \cdot))ds, \qquad \tau > 0.$$

Applying the Banach fixed point theorem in the space

$$\mathcal{X}_\omega^\alpha = \left\{ \boldsymbol{w} \in C([0, \infty); \mathcal{W}_Q) : \sup_{\sigma \in (0,1)} \sigma^\alpha \|\boldsymbol{w}\|_{C^\alpha([\sigma,1]; D(L))} < \infty : \tau \mapsto e^{\omega\tau}\boldsymbol{w}(\tau, \cdot) \in C^\alpha([1, \infty); D(L)) \right\},$$

endowed with the natural norm, where $\alpha$ is fixed in $(0, 1)$ and $\omega$ is any positive number less than the real part of $\lambda_1(\varepsilon)$, allows us to prove the existence and uniqueness of a solution $\boldsymbol{w}$ of (3.9), defined in $(0, \infty)$ such that $\|\boldsymbol{w}(\tau, \cdot)\|_{\mathcal{W}} + \|L\boldsymbol{w}(\tau, \cdot)\|_{\mathcal{W}} \leq Ce^{-\omega\tau}\|\boldsymbol{w}_0\|_{D(L)}$ for $\tau \in (0, \infty)$ and some positive constant $C$, which yields the claim. For further details see [21, Chapter 9].

(ii) For $m > m^c$, we use again Hurwitz's Theorem to show that there exists $\varepsilon_1 = \varepsilon_1(m) > 0$ such that the equation $D_\varepsilon(\lambda, m) = 0$ admits a solution with positive real part if $\varepsilon \in (0, \varepsilon_1)$. Here, we take advantage of the fact that $D_0(\lambda; m)$ admits a couple of conjugate complex roots with positive real parts, if $m < 8$, a positive root, if $m = 8$, and two real solutions if $m > 8$. For these values of $\varepsilon$, the restriction of the semigroup $e^{\tau L}$ to $\mathcal{W}_Q$ exhibits an exponential dichotomy, i.e., there exists a spectral projection $P_+$ which allows to split $\mathcal{W}_Q = P_+(\mathcal{W}_Q) \oplus (I - P_+)(\mathcal{W}_Q)$. The semigroup $e^{\tau L}$ decays to zero with exponential rate when restricted to $(I - P)(\mathcal{W}_Q)$, whereas the restriction of $e^{\tau L}$ to $P_+(\mathcal{W}_Q)$ extends to a group which decays to zero with exponential rate as $\tau \to -\infty$. Again with a fixed point technique, we can prove the existence of a nontrivial backward solution $\boldsymbol{z}$ of the nonlinear equation (3.9), defined in $(-\infty, 0)$ such that $\|\boldsymbol{z}(\tau, \cdot)\|_{\mathcal{W}} + \|L\boldsymbol{z}(\tau, \cdot)\|_{\mathcal{W}} \leq C_\omega e^{\omega\tau}$ for $\tau \in (-\infty, 0)$ and any $\omega$ positive and smaller than the minimum of the positive real parts of the roots of the dispersion relation. The sequence $(\boldsymbol{z}_n)$ defined by $\boldsymbol{z}_n = \boldsymbol{z}(-n, \cdot)$ for any $n \in \mathbb{N}$ vanishes in $D(L)$ as $n \to +\infty$ and the solution $\boldsymbol{w}_n$ to (3.9) subject to the initial condition $\boldsymbol{w}_n(0, \cdot) = \boldsymbol{z}_n$ exists at least in the time domain $[0, n]$, where it coincides with the function $\boldsymbol{z}(\cdot - n, \cdot)$. Thus, the norm of $\|\boldsymbol{w}_n\|_{C([0,n]; \mathcal{W}_Q)}$ is positive and far way from zero, uniformly with respect to $n \in \mathbb{N}$, whence the instability of the trivial solution of (3.9) follows. Again, we refer the reader to [21, Chapter 9] for further results.   $\square$

### 4.2. Stability of the traveling wave

We can now rewrite the results in Theorem 4.1 in terms of problem (2.1)–(2.3).

**Theorem 4.2.** *The following properties are satisfied.*

(i) *For $m \in (2, m^c)$ fixed, there exists $\varepsilon_0 = \varepsilon_0(m) > 0$ such that, for $\varepsilon \in (0, \varepsilon_0)$, the traveling wave solution $\boldsymbol{U}$ is orbitally stable with asymptotic phase $s_\infty$ (see (4.2)), with respect to perturbations belonging to the weighted space $D(L)$.*

(ii) *For $m > m^c$ fixed, there exists $\varepsilon_1 = \varepsilon_1(m)$ small enough such that, for $\varepsilon \in (0, \varepsilon_1)$, the traveling wave $\boldsymbol{U}$ is unstable, with respect to perturbations belonging to the weighted space $D(L)$.*

**Proof.** (i) Let us fix $\boldsymbol{w}_0 \in Q(D(L))$ with $\|\boldsymbol{w}_0\|_{D(L)}$ small enough, so that Theorem 4.1(i) can be applied. Denote by $\boldsymbol{w}$ the classical solution to Equation (3.9) which satisfies the initial condition $\boldsymbol{w}(0, \cdot) = \boldsymbol{w}_0 = (w_{0,1}, w_{0,2})$. Observe that, since $p = \Theta_i^{-1} w_1(\cdot, 0)$ (see Subsection 3.1) it follows that the problem (3.1), subject to the initial condition $\boldsymbol{u}(0, \cdot) = \Theta_i^{-1} w_{0,1} \boldsymbol{U}' + \boldsymbol{w}_0$, admits a unique classical solution $(\boldsymbol{u}, s)$, where $\boldsymbol{u}$ decreases to zero as $\tau \to \infty$, with exponential rate. Moreover, using (3.8) it is immediate to check that $s(\tau)$ converges to

$$s_\infty = \int_0^\infty \frac{(L\boldsymbol{w}(r, \cdot))(0^+)}{\Theta_i(m) - w_1(r, 0) - w_1'(r, 0^+)} dr, \tag{4.2}$$

as $\tau \to \infty$ (assuming for simplicity that $g$ vanishes at $\tau = 0$). We point out that $s_\infty$ depends on the initial condition.

Coming back to problem (2.1)–(2.3) with initial condition $\boldsymbol{X}(0) = \boldsymbol{u}_0 + \boldsymbol{U}$ and $g(0) = 0$, we easily see that the solution $\boldsymbol{X} = (\Theta, \Phi)$ is defined by

$$\boldsymbol{X} = p\boldsymbol{U}' + \boldsymbol{w} + \boldsymbol{U} = \Theta_i^{-1} w_1(\cdot, 0)\boldsymbol{U}' + \boldsymbol{w} + \boldsymbol{U},$$

$$g(\tau) = \tau + \int_0^\tau \frac{(L\boldsymbol{w}(r, \cdot))(0^+)}{\Theta_i(m) - w_1(r, 0) - w_1'(r, 0^+)} dr, \qquad \tau \geq 0.$$

From this formula and the above result, the claim follows at once.

(ii) The proof is similar to that of property (i) and, hence, it is left to the reader. □

## 5. Hopf bifurcation

This section is devoted to investigating the dynamics of the perturbation of the traveling wave in a neighborhood, say $(6 - \delta, 6 + \delta)$, of the limit critical value $m^c = 6$ (see Section 4). As regards parameter $m$, the situation is more complicated than in Section 4 where it was fixed. Now, the dispersion relation $D_\varepsilon(\lambda; m)$ can be seen as a sequence of analytic functions parameterized by $m$. The main difficulty here is that Hurwitz's Theorem does not a priori apply, particularly because of the lack of uniformity of $D_\varepsilon(\lambda; m)$ with respect to $\varepsilon$ and $m$. We especially find a proper approach to combining $m$ with $\varepsilon$: we construct in Proposition 5.1 a sequence of critical values $m^c(\varepsilon)$ such that $m^c(0) = m^c$ and apply Hurwitz's Theorem to the sequence $D_\varepsilon(\lambda, m^c(\varepsilon))$. This proposition will be crucial for proving the existence of a Hopf bifurcation (see Theorem 5.3).

### 5.1. Local analysis of the dispersion relation

We look for the roots of the dispersion relation, see (4.1), in a neighborhood of $m^c = 6$ and of $\lambda = \pm i\sqrt{3}$, for $\varepsilon > 0$ small enough. A natural idea is to turn the dispersion relation into a polynomial by squaring, however the price to pay is double: the polynomial will be of high order without algebraic roots, and spurious roots therefore appear.

For convenience, we rewrite the equation $D_\varepsilon(\lambda; m) = 0$ into a much more useful form. Replacing $\sqrt{1 + 4\varepsilon(m + \varepsilon m^2 + \lambda)} + \sqrt{1 + 4\varepsilon\lambda}$ by $4\varepsilon(m + \varepsilon m^2)\left(\sqrt{1 + 4\varepsilon(m + \varepsilon m^2 + \lambda)} - \sqrt{1 + 4\varepsilon\lambda}\right)^{-1}$, with some straightforward algebra we obtain the equivalent equation

$$\sqrt{1 + 4\varepsilon\lambda} - \frac{1}{1+m}\sqrt{1 + 4\varepsilon(m + \varepsilon m^2 + \lambda)}\sqrt{1 + 4\lambda} + \frac{1 + \varepsilon m}{1 + m}\sqrt{1 + 4\lambda} = \varepsilon\frac{1 + 4\lambda}{1 + m} + 1 - \varepsilon. \tag{5.1}$$

We denote by $\zeta$ the right-hand side of (5.1) and set

$$\Sigma_1 = 1 + 4\varepsilon\lambda + \frac{2 + 6\varepsilon m + 5\varepsilon^2 m^2 + 4\varepsilon\lambda}{(1+m)^2}(1 + 4\lambda),$$

$$\Sigma_2 = \frac{1 + 4\lambda}{(1+m)^2}\left[(2 + 6\varepsilon m + 5\varepsilon^2 m^2 + 4\varepsilon\lambda)(1 + 4\varepsilon\lambda) + \frac{[1 + 4\varepsilon(m + \varepsilon m^2 + \lambda)](1 + \varepsilon m)^2}{(1+m)^2}(1 + 4\lambda)\right],$$

$$\Sigma_3 = \frac{[1 + 4\varepsilon(m + \varepsilon m^2 + \lambda)](1 + \varepsilon m)^2}{(1 + m)^4}(1 + 4\varepsilon\lambda)(1 + 4\lambda)^2.$$

Squaring both sides of (5.1) and rearranging terms, we get the equation

$$\zeta^2 - \Sigma_1 = \frac{2\sqrt{1 + 4\lambda}}{1 + m}\left\{\sqrt{1 + 4\varepsilon\lambda}[1 + \varepsilon m - \sqrt{1 + 4\varepsilon(m + \varepsilon m^2 + \lambda)}]\right.$$
$$\left. - \frac{1 + \varepsilon m}{1 + m}\sqrt{1 + 4\lambda}\sqrt{1 + 4\varepsilon(m + \varepsilon m^2 + \lambda)}\right\}. \tag{5.2}$$

Again, squaring both sides of (5.2) and rearranging terms gives

$$(\zeta^2 - \Sigma_1)^2 - 4\Sigma_2 = \frac{8\sqrt{1 + 4\varepsilon\lambda}(1 + 4\lambda)}{(1 + m)^2}\left[\frac{[1 + 4\varepsilon(m + \varepsilon m^2 + \lambda)](1 + \varepsilon m)}{1 + m}\sqrt{1 + 4\lambda}\right.$$
$$- \frac{(1 + \varepsilon m)^2}{1 + m}\sqrt{1 + 4\varepsilon(m + \varepsilon m^2 + \lambda)}\sqrt{1 + 4\lambda}$$
$$\left. - (1 + \varepsilon m)\sqrt{1 + 4\varepsilon\lambda}\sqrt{1 + 4\varepsilon(m + \varepsilon m^2 + \lambda)}\right]. \tag{5.3}$$

Finally, squaring both sides of (5.3) and using (5.2), we conclude that $[(\zeta^2 - \Sigma_1)^2 - 4\Sigma_2]^2 - 64\Sigma_3\zeta^2 = 0$ or, equivalently, $P_7(\lambda; m, \varepsilon) = 0$, where $P_7(\cdot; m, \varepsilon)$ is a seventh-order polynomial (see Appendix D for the expression of the coefficients of the polynomial).

Finding the roots of $P_7(\cdot; m, \varepsilon)$ is quite challenging. The Routh-Hurwitz criterion (see, e.g., [15, Chapter XV]) gives relevant information on the roots without computing them explicitly: it tells us whether the roots lie in the left halfplane $\mathrm{Re}\lambda < 0$, by computing the Hurwitz determinants $\Delta_j$ ($j = 1, \ldots, 6$) associated with $P_7(\lambda; m, \varepsilon)$. Unfortunately, our double-squaring method produces spurious roots which render Routh-Hurwitz criterion inefficient. However, Orlando's formula (see [15, Chapter XV, 7]), a generalization of the well-known property for the sum of the roots of a quadratic equation, establishes a relation between the leading Hurwitz determinant $\Delta_6$ and the sums of all different pairs of roots of $P_7(\lambda; m, \varepsilon)$. In particular, $\Delta_6 = 0$ in the case when either 0 is a double root (i.e., 0 is a root with algebraic multiplicity two) or two roots are purely imaginary and conjugate.

The following one is the main result of this subsection.

**Proposition 5.1.** *There exist $\varepsilon_0 > 0$ and $\delta > 0$, and a unique function $m^c : (0, \varepsilon_0) \to (6 - \delta, 6 + \delta)$ with $m^c(0) = 6$, such that the polynomial $\widetilde{P}_7(\lambda; \varepsilon) := P_7(\lambda; m^c(\varepsilon), \varepsilon)$ has exactly one pair of purely imaginary roots $\pm i\omega(\varepsilon)$, with $\omega(\varepsilon) > 0$. Moreover, $\omega(\varepsilon)$ converges to $\sqrt{3}$ as $\varepsilon$ tends to 0.*

We first need a preliminary technical lemma:

**Lemma 5.2.** *There exist $\upsilon_0 > 0$ and $\varepsilon_* > 0$ such that, for all m in the interval $[3, 7]$ (to fix ideas), $\varepsilon \in (0, \varepsilon_*)$ and any purely imaginary root $i\upsilon$ of $P_7(\cdot; m, \varepsilon)$, with $\upsilon > 0$, it holds that $0 < \upsilon < \upsilon_0$.*

**Proof.** We observe that, if $i\upsilon$ is a root of $P_7(\cdot; m, \varepsilon)$, then, in particular, the imaginary part of $P_7(i\upsilon; m, \varepsilon)$, i.e., the term $-a_0\upsilon^7 + a_2\upsilon^5 - a_4\upsilon^3 + a_6\upsilon$ vanishes.

A straightforward computation (see Appendix D) reveals that

$$\mathrm{Im}\, P_7(i\zeta; m, \varepsilon) = -2048(\varepsilon - 1)^4\varepsilon^2\zeta^7 - 8\varepsilon(m^2 + 3m + 2)\zeta^5 + O(\varepsilon^2)\zeta^5$$
$$- 128(2m^4 - 7m^2 - 3m - 1)\zeta^3 + O(\varepsilon)\zeta^3 + a_6\zeta,$$

for every $\zeta > 0$, where we denote by $O(\varepsilon^k)$ terms depending only on $\varepsilon$ such that the ratio $O(\varepsilon^k)/\varepsilon^k$ stays bounded and far away from zero for $\varepsilon$ in a neighborhood of zero. Since $m^2 + 3m + 2$ and $2m^4 - 7m^2 - 3m - 1$ are both positive for $m \in [3, \infty)$, we can estimate

$$|\mathrm{Im}\, P_7(i\zeta; m, \varepsilon)| \geq [8(m^2 + 3m + 2) - O(\varepsilon)]\varepsilon\zeta^5 + [128(2m^4 - 7m^2 - 3m - 1) - O(\varepsilon)]\zeta^3 - K|\zeta|,$$

where $K := \max\{|a_6(m, \varepsilon)| : m \in [3, 7], \varepsilon \in (0, 1]\}$. Hence, we can determine $\varepsilon_* > 0$ such that

$$|\operatorname{Im} P_7(i\zeta; m, \varepsilon)| \geq 64(2m^4 - 7m^2 - 3m - 1)\zeta^3 - K|\zeta|, \qquad m \in [3, 7], \ \varepsilon \in (0, \varepsilon_*). \tag{5.4}$$

The right-hand side of (5.4) diverges to $\infty$ as $\zeta \to +\infty$. From this it follows that there exists $\upsilon_0 > 0$ such that $|\operatorname{Im} P_7(i\zeta; m, \varepsilon)| > 0$ for every $\zeta > \upsilon_0$ and this clearly implies that $\upsilon \leq \upsilon_0$. $\quad\square$

**Proof of Proposition 5.1.** We split the proof into two steps.

*Step 1.* First, we prove the existence of a function $m^c$ with the properties listed in the statement of the proposition. For this purpose, we consider the sixth-order Hurwitz determinant $\Delta_6(m, \varepsilon)$ associated with the polynomial $P_7(\lambda; m, \varepsilon)$. It turns out that $\Delta_6(m, \varepsilon) = \varepsilon^2 m^2 C \widetilde{\Delta}_6(m, \varepsilon)$ for some positive constant $C$. As $\varepsilon \to 0$, $\widetilde{\Delta}_6(\cdot, \varepsilon)$ converges to the function $\Delta_0$, which is defined by

$$\begin{aligned}
\Delta_0(m) = &- m^{18} + 8m^{17} + 97m^{16} + 42m^{15} - 2129m^{14} - 9376m^{13} - 16811m^{12} \\
&- 7866m^{11} + 19913m^{10} + 31292m^9 - 4309m^8 - 55466m^7 - 66363m^6 \\
&- 35480m^5 - 4729m^4 + 4666m^3 + 2628m^2 + 500m + 24.
\end{aligned}$$

Noticing that $\Delta_0(6) = 0$ and $\frac{d}{dm}\Delta_0(6) > 0$, it then follows from the Implicit Function Theorem that there exist $\varepsilon_0 \in (0, \varepsilon_*)$, with $\varepsilon_*$ given by Lemma 5.2, $\delta > 0$ and a unique mapping $m^c : (0, \varepsilon_0) \to (6 - \delta, 6 + \delta)$ with $m^c(0) = 6$, such that $\widetilde{\Delta}_6(m^c(\varepsilon), \varepsilon) = 0$ and $\frac{\partial}{\partial m}\widetilde{\Delta}_6(m^c(\varepsilon), \varepsilon) > 0$ for $\varepsilon \in (0, \varepsilon_0)$. Then, upon an application of Orlando's formula, it follows that either 0 is a double root of $\widetilde{P}_7(\lambda; \varepsilon)$ or there exists at least one pair $\pm\omega(\varepsilon)i$ (with $\omega(\varepsilon) > 0$) of purely imaginary roots of $\widetilde{P}_7(\lambda; \varepsilon)$ for every $\varepsilon \in (0, \varepsilon_0)$. The first case is ruled out, since 0 is not a root of $\widetilde{P}_7(\lambda; \varepsilon)$. Indeed, $a_7(m, \varepsilon)$ converges to a positive limit as $\varepsilon$ tends to 0.

*Step 2.* Next, we prove that $\pm\omega(\varepsilon)i$ is the unique pair of purely imaginary roots of the polynomial $\widetilde{P}_7(\lambda; \varepsilon)$ for every $\varepsilon \in (0, \varepsilon_0)$. For this purpose, we begin by observing that $\widetilde{P}_7(\cdot; \varepsilon)$ converges, locally uniformly in $\mathbb{C}$ as $\varepsilon \to 0$, to the fourth-order polynomial $\widetilde{P}_4$, defined by $\widetilde{P}_4(\lambda) = -6272(4\lambda + 1)(\lambda - 12)(\lambda^2 + 3)$ for every $\lambda \in \mathbb{C}$. By Hurwitz's Theorem, four roots of $\widetilde{P}_7(\lambda; \varepsilon)$, say $\lambda_1(\varepsilon)$, $\lambda_2(\varepsilon)$, $\lambda_3(\varepsilon)$ and $\lambda_4(\varepsilon)$ converge respectively to $\lambda_1(0) = -\frac{1}{4}$, $\lambda_2(0) = 12$, $\lambda_3(0) = \sqrt{3}i$ and $\lambda_4(0) = -\sqrt{3}i$. More precisely, for $r_1 > 0$ small enough, $\lambda_i(\varepsilon)$ ($i = 1, \ldots, 4$) is simple in the ball $B(\lambda_i(0), r_1)$ for $\varepsilon \in (0, \varepsilon_0)$ (up to replacing $\varepsilon_0$ with a smaller value if needed). Assume by contradiction that there exists a positive infinitesimal sequence $\{\varepsilon_n\}$ such that, for any $n \in \mathbb{N}$, $(\lambda_5(\varepsilon_n), \lambda_6(\varepsilon_n))$ is another pair of purely imaginary and conjugate roots of $\widetilde{P}_7(\lambda; \varepsilon_n)$, different from $\pm\omega(\varepsilon_n)i$. By Lemma 5.2, $\nu(\varepsilon_n) = |\lambda_5(\varepsilon_n)| \leq \upsilon_0$ for every $n \in \mathbb{N}$. Take a subsequence $(\varepsilon_{n_k})$ such that $\nu(\varepsilon_{n_k})$ converges as $k \to \infty$. The local uniform convergence in $\mathbb{C}$ of $\widetilde{P}_7(\cdot; \varepsilon_n)$ to $\widetilde{P}_4$ implies that $\nu(\varepsilon_{n_k})$ tends to $\sqrt{3}$ as $k \to \infty$. Since the limit is independent of the choice of subsequence $(\varepsilon_{n_k})$, we conclude that $\nu(\varepsilon_n)$ converges to $\sqrt{3}$ as $n \to \infty$. Next, thanks to Hurwitz's Theorem and the fact that $\lambda_3(\varepsilon)$, $\lambda_4(\varepsilon)$ converge to $\sqrt{3}i, -\sqrt{3}i$ respectively, the pair $(\lambda_5(\varepsilon_{n_k}), \lambda_6(\varepsilon_{n_k}))$ coincides with $(\lambda_3(\varepsilon_{n_k}), \lambda_4(\varepsilon_{n_k}))$ in $B(\sqrt{3}i, r_1) \times B(-\sqrt{3}i, r_1)$. This contradicts the fact that $\lambda_3(\varepsilon_{n_k}), \lambda_4(\varepsilon_{n_k})$ are both simple. Up to replacing $\varepsilon_0$ with a smaller value if needed, we have proved that $(\omega(\varepsilon)i, -\omega(\varepsilon)i)$ is the unique pair of conjugate roots of $\widetilde{P}_7(\cdot; \varepsilon)$ and $\lambda_3(\varepsilon) = \omega(\varepsilon)i$ for every $\varepsilon \in (0, \varepsilon_0)$. The proof is now complete. $\quad\square$

### 5.2. Hopf bifurcation theorem

For fixed $0 < \varepsilon < \varepsilon_0$, $\varepsilon_0$ and $\delta$ given by Proposition 5.1, let us consider the fully nonlinear problem (3.9), where now we find it convenient to write $F(\mathbf{w}; m)$ instead of $F(\mathbf{w})$ to make much more explicit the dependence of the nonlinear term $F$ on the bifurcation parameter $m$. We recall that, for all $m$, the null solution (or trivial branch) is an equilibrium of (3.9). According to Proposition 5.1, the bifurcation parameter $m$ has a critical value $m^c(\varepsilon) \in (6 - \delta, 6 + \delta)$. We intend to prove that a Hopf bifurcation occurs from the trivial branch at $m = m^c(\varepsilon)$ if $\varepsilon$ is small enough. For $m$ close to $m^c(\varepsilon)$, we are going to locally parameterize $m$ and $\mathbf{w}$ by a parameter $\sigma \in (-\sigma_0, \sigma_0)$. To emphasize this dependence, we will write $\widetilde{m}(\sigma)$ and $\widetilde{\mathbf{w}}(\cdot, \cdot; \sigma)$.

**Theorem 5.3.** *For any fixed $\alpha \in (0, 1)$, there exists $\tilde{\varepsilon}_0 \in (0, \varepsilon_0)$, such that whenever $\varepsilon \in (0, \tilde{\varepsilon}_0)$ is fixed, the following properties are satisfied.*

(i) *There exist $\sigma_0 > 0$ and smooth functions $\widetilde{m}, \rho : (-\sigma_0, \sigma_0) \to \mathbb{R}$, $\widetilde{\mathbf{w}} : (-\sigma_0, \sigma_0) \to C^{1+\alpha}(\mathbb{R}; \mathcal{W}) \cap C^\alpha(\mathbb{R}; Q(D(L)))$, satisfying the conditions $\widetilde{m}(0) = m^c$, $\rho(0) = 1$ and $\widetilde{\mathbf{w}}(\cdot, \cdot; 0) = 0$. In addition, $\widetilde{\mathbf{w}}(\cdot, \cdot; \sigma)$ is not a constant if $\sigma \neq 0$, and $\widetilde{\mathbf{w}}(\cdot, \cdot; \sigma)$ is a $T(\sigma)$-periodic solution of the equation*

$$\widetilde{\boldsymbol{w}}_\tau(\cdot, \cdot; \sigma) = QL\widetilde{\boldsymbol{w}}(\cdot, \cdot; \sigma) + F(\widetilde{\boldsymbol{w}}(\cdot, \cdot; \sigma); \widetilde{m}(\sigma)), \qquad \tau \in \mathbb{R},$$

*where $T(\sigma) = 2\pi\rho(\sigma)\omega^{-1}$ and $\omega = \omega(\varepsilon)$ is defined in Proposition 5.1.*

(ii) *There exists $\eta_0$ such that, if $\overline{m} \in (6 - \delta_0, 6 + \delta_0)$, $\bar{\rho} \in \mathbb{R}$ and $\boldsymbol{w} \in C^{1+\alpha}(\mathbb{R}; \boldsymbol{\mathcal{W}}) \cap C^\alpha(\mathbb{R}; Q(D(L)))$ is a $2\pi\bar{\rho}\omega^{-1}$-periodic solution of the equation $\overline{\boldsymbol{w}}_\tau = QL\overline{\boldsymbol{w}} + F(\overline{\boldsymbol{w}}; \overline{m})$ such that*

$$\|\overline{\boldsymbol{w}}\|_{C^{1+\alpha}(\mathbb{R}; \boldsymbol{\mathcal{W}})} + \|\overline{\boldsymbol{w}}\|_{C^\alpha(\mathbb{R}; Q(D(L)))} + |\bar{m}| + |1 - \bar{\rho}| \leq \eta_0,$$

*then there exist $\sigma \in (-\sigma_0, \sigma_0)$ and $\tau_0 \in \mathbb{R}$ such that $\overline{m} = \widetilde{m}(\sigma)$, $\bar{\rho} = \rho(\sigma)$ and $\overline{\boldsymbol{w}} = \widetilde{\boldsymbol{w}}(\cdot + \tau_0, \cdot; \sigma)$.*

**Proof.** We split the proof into two steps.

*Step 1.* Here, we prove that there exists $\varepsilon_1 > 0$ such that $\pm\omega(\varepsilon)i$ are simple eigenvalues of $L$ (and, hence, of the part of $L$ in $\mathcal{W}_Q = Q(\mathcal{W})$) for every $\varepsilon \in (0, \varepsilon_1]$ and there are no other eigenvalues on the imaginary axis, i.e., we prove that this operator satisfies the so-called resonance condition.

To begin with, let us prove that $\pm\omega(\varepsilon)i$ are eigenvalues of $L$. In view of Theorem 2.3, we need to show that they are roots of the dispersion relation (4.1). For this purpose, we observe that the function $\widetilde{D}_\varepsilon := D_\varepsilon(\cdot; m^c(\varepsilon))$ converges to $\widetilde{D}_0$ locally uniformly in the strip $\{\lambda \in \mathbb{C} : |\operatorname{Re}\lambda| \leq \ell\}$ (for $\ell$ small enough), where

$$\widetilde{D}_0(\lambda) = -\lambda - \frac{1 + \sqrt{1 + 4\lambda}}{2} + \frac{1}{14}[(13 + 2\lambda)\sqrt{1 + 4\lambda} + 1 + 4\lambda], \qquad \lambda \in \mathbb{C}.$$

The function $\widetilde{D}_0$ has just one pair of purely imaginary conjugate roots $\pm\sqrt{3}i$. Hurwitz's theorem shows that there exists $r > 0$ such that the ball $B(\sqrt{3}i, r)$ contains exactly one root $\lambda(\varepsilon)$ of $\widetilde{D}_\varepsilon$ for each $\varepsilon$ small enough. By the proof of Proposition 5.1, we know that there exists $r_1 > 0$ such that $\omega(\varepsilon)i$ is the unique root of $\widetilde{P}_7$ in the ball $B(\sqrt{3}i, r_1)$. Clearly, $\lambda(\varepsilon)$ is a root of the polynomial $\widetilde{P}_7$ and Hurwitz's theorem also shows that $\lambda(\varepsilon)$ converges to $\sqrt{3}i$ as $\varepsilon \to 0^+$. Therefore, for $\varepsilon$ small enough, both $\lambda(\varepsilon)$ and $\omega(\varepsilon)i$ belong to $B(\sqrt{3}i, r_1)$ and, hence, they do coincide. The same argument implies that $-\omega(\varepsilon)i$ is also a root of $\widetilde{D}_\varepsilon$. We have proved that there exists $\varepsilon_1 \leq \varepsilon_0$ such that $\omega(\varepsilon)i$ and $-\omega(\varepsilon)i$ are both eigenvalues of $L$ of every $\varepsilon \in (0, \varepsilon_1]$. In particular, $\pm\omega(\varepsilon)i$ are simple roots of the function $\widetilde{D}_\varepsilon$ and there are no other eigenvalues of $L$ on the imaginary axis.

To conclude that $\pm\omega(\varepsilon)i$ are simple eigenvalues of $L$ for each $\varepsilon \in (0, \varepsilon_1]$, we just need to check that their geometric multiplicity is one. For this purpose, we observe that the proof of Theorem 2.3 shows that the eigenfunctions associated with the eigenvalues $\pm\omega(\varepsilon)i$ are given by

$$u(\xi) = c_1 e^{k_1\xi} + \frac{A}{H_{1,\lambda}}\left(\frac{e^{k_3\xi}}{k_3 - k_2} - \frac{e^{k_3\xi} - e^{k_1\xi}}{k_3 - k_1}\right)c_3, \qquad v(\xi) = c_3 e^{k_3\xi}, \quad \xi < 0,$$

$$u(\xi) = c_6 e^{k_2\xi}, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad v(\xi) = c_8 e^{k_6\xi}, \quad \xi \geq 0$$

with $k_j = k_{j, \pm\omega(\varepsilon)i}$ (see (2.15)) and the constants $c_1$, $c_3$, $c_6$ and $c_8$ are determined through the equation (2.23) (with $\lambda = \pm\omega(\varepsilon)i$) where $F_1 = \ldots = F_4 = 0$. Since the rank of the matrix in (2.23) is three at $\lambda = \pm\omega(\varepsilon)i$, it follows at once that the geometric multiplicity of $\pm\omega(\varepsilon)i$ is one.

*Step 2.* Now, we check the nontransversality condition. We begin by observing that, for every $\varepsilon \in (0, \varepsilon_1]$, the function $D_\varepsilon$ is analytic with respect to $\lambda$ and continuously differentiable with respect to $m$ in $B(\sqrt{3}i, r) \times (6 - \delta, 6 + \delta)$, where $r$ is such that the ball $B(\sqrt{3}i, r)$ does not intersect the half line $(-\infty, -1/4]$. We intend to apply the Implicit Function Theorem at $(\omega(\varepsilon)i, m^c(\varepsilon))$ for $\varepsilon$ small enough. In this respect, we need to show that the $\lambda$-partial derivative of $D_\varepsilon$ does not vanish at $(\lambda_3(\varepsilon), m^c(\varepsilon))$. To this aim, we observe that

$$\lim_{\varepsilon\to 0^+} \frac{\partial D_\varepsilon}{\partial\lambda}(\omega(\varepsilon)i, m^c(\varepsilon)) = \frac{\partial D_0}{\partial\lambda}(\sqrt{3}i, 6) = \frac{5\sqrt{3}i - 3}{49}.$$

Therefore, there exists $\varepsilon_2 \leq \varepsilon_1$ such that, if $\varepsilon \in (0, \varepsilon_2]$, then the $\lambda$-partial derivative of $D_\varepsilon$ at $(\omega(\varepsilon)i, m^c(\varepsilon))$ does not vanish. Then, it follows from the Implicit Function Theorem that for each $\varepsilon \in (0, \varepsilon_2]$, there exist $\delta_\varepsilon > 0$, $r_\varepsilon < r$ and a $C^1$-mapping $\lambda_\varepsilon : (m^c(\varepsilon) - \delta_\varepsilon, m^c(\varepsilon) + \delta_\varepsilon) \to B(\sqrt{3}i, r_\varepsilon)$, such that $D_\varepsilon(\lambda_\varepsilon(m), m) = 0$ for all $m \in (m^c(\varepsilon) - \delta_\varepsilon, m^c(\varepsilon) + \delta_\varepsilon)$ and $\lambda_\varepsilon(6) = \omega(\varepsilon)i$.

As a consequence, there are two branches of conjugate isolated and simple eigenvalues, $\lambda_\varepsilon(m)$ and $\overline{\lambda}_\varepsilon(m)$, which cross the imaginary axis respectively at $\pm\omega(\varepsilon)i$ for $m = m^c(\varepsilon)$.

It remains to determine the sign of the real part of the derivative of $\lambda_\varepsilon$ at $m = m^c(\varepsilon)$. Since

$$\lim_{\varepsilon \to 0^+} \frac{d\lambda_\varepsilon}{dm}(m^c(\varepsilon)) = -\left(\frac{\partial D_0}{\partial m}(\sqrt{3}i, 6)\right)\left(\frac{\partial D_0}{\partial \lambda}(\sqrt{3}i, 6)\right)^{-1} = \frac{3}{4} + \frac{\sqrt{3}}{12}i$$

there exists $\varepsilon_3 \leq \varepsilon_2$ such that the real part of the derivative of $\lambda_\varepsilon$ is positive at $m^c(\varepsilon)$ for any $\varepsilon \in (0, \varepsilon_3]$, which completes the proof of Step 2.

Applying [21, Theorem 9.3.3], the claims follow with $\tilde{\varepsilon}_0 = \varepsilon_3$.    □

### 5.3. Bifurcation from the traveling wave

As in Subsection 4.2, we rewrite the results in Theorem 5.3 in terms of problem (2.1)–(2.3). As above, $\varepsilon$ is fixed in $(0, \tilde{\varepsilon}_0)$; therefore, the traveling wave $\boldsymbol{U}$ depends only on $m$, which itself is parameterized by $\sigma \in (-\sigma_0, \sigma_0)$. So we write $\tilde{m}(\sigma)$ and accordingly $\tilde{\Theta}_i(\sigma) = \Theta_i(\tilde{m}(\sigma))$. Likewise, the traveling wave reads $\tilde{\boldsymbol{U}}(.; \sigma)$.

The following theorem reveals that there exists a bifurcated branch bifurcating from the traveling wave at the bifurcation point $m^c(\varepsilon)$. The proof can be obtained by arguing as in the proof of Theorem 4.2. Hence, the details are skipped.

**Theorem 5.4.** *For each $\sigma \in (-\sigma_0, \sigma_0)$, the problem (2.1)–(2.3) admits a non trivial solution $(\tilde{\boldsymbol{X}}(\cdot, \cdot; \sigma), \tilde{g}(\cdot; \sigma))$ defined by:*

$$\tilde{\boldsymbol{X}}(\cdot, \cdot; \sigma) = \tilde{\Theta}_i^{-1}(\sigma)\tilde{w}_1(\cdot, 0; \sigma)\tilde{\boldsymbol{U}}'(\cdot; \sigma) + \tilde{\boldsymbol{w}}(\cdot, \cdot; \sigma) + \tilde{\boldsymbol{U}}(\cdot; \sigma),$$

$$\tilde{g}(\tau; \sigma) = \tau + \frac{\tau}{T(\sigma)}\int_0^{T(\sigma)} \frac{(L\tilde{\boldsymbol{w}}(r, \cdot; \sigma))_1(0^+)}{\tilde{\Theta}_i(\sigma) - \tilde{w}_1(r, 0; \sigma) - \tilde{w}_1'(r, 0^+; \sigma)}dr + \tilde{h}(\tau; \sigma), \quad \tau \in \mathbb{R},$$

*where $\tilde{\boldsymbol{X}}(\cdot, \cdot; 0) = \tilde{\boldsymbol{U}}(.; 0)$, $\tilde{\boldsymbol{w}}$ is defined by Theorem 5.3. The function $\tilde{h}(\cdot; \sigma)$ belongs to $C^{1+\alpha}(\mathbb{R})$. Moreover, $\tilde{\boldsymbol{X}}(\cdot, \cdot; \sigma)$ and $\tilde{h}(\cdot; \sigma)$ are periodic with period $T(\sigma) = 2\pi\rho(\sigma)\omega^{-1}$. At the bifurcation point, the "virtual period" is $T(0) = 2\pi\omega^{-1}$.*

We refer to, e.g., [20,23] for solutions which are periodic modulo a linear growth.

**Declaration of competing interest**

There is no competing interest.

## Appendix A. Hurwitz's theorem

For the convenience of the reader, here we recall the well-known Hurwitz's Theorem in complex analysis (see, e.g., [14, p. 152]). The latter associates the zeros of a sequence of holomorphic functions with that of their corresponding limit. It is a key tool for our proofs.

**Theorem A.1** *(Hurwitz's Theorem). Let $\Omega \subset \mathbb{C}$ be a connected open set and $H(\Omega)$ be the collection of holomorphic functions on $\Omega$. Suppose $(f_n)$ is a sequence in $H(\Omega)$ which converges to some function $f$. If $f \not\equiv 0$, $\overline{B(z_0; \rho)} \subset \Omega$ and $f(z) \neq 0$ for every $z \in \partial B(z_0; \rho)$, then there exists an integer $N$ (depending on $\rho$) such that, for $n \geq N$, the functions $f$ and $f_n$ have the same number of zeros in $B(z_0; \rho)$. Furthermore, such zeros converge to $z_0$ as $n \to +\infty$.*

## Appendix B.  General solution to the equation $\lambda \boldsymbol{u} - \mathcal{L}\boldsymbol{u} = \boldsymbol{f}$

Here, we collect the expression of the more general classical solution to the equation $\lambda \boldsymbol{u} - \mathcal{L}\boldsymbol{u} = \boldsymbol{f}$ when $\boldsymbol{f} = (f_1, f_2)$ is a continuous function and $\lambda \in \mathbb{C}$. We preliminarily note that, since $\mathrm{Le} > 1$, the equation $k_{1,\lambda} = k_{4,\lambda}$ has no complex solutions $\lambda$. The equation $k_{1,\lambda} = k_{3,\lambda}$ admits two complex conjugate solutions

$$\lambda_j^* = \frac{-A\mathrm{Le} + (-1)^j i \sqrt{A\mathrm{Le}(\mathrm{Le}-1)}}{\mathrm{Le}-1}, \qquad j = 1, 2, \tag{B.1}$$

whose real parts are negative. Moreover, the equation $k_{2,\lambda} = k_{4,\lambda}$ admits no complex solutions. Also the equation $k_{2,\lambda} = k_{3,\lambda}$ admits no solutions. Indeed, squaring twice the equation $H_{1,\lambda} + H_{2,\lambda} = \mathrm{Le} - 1$ we get $\lambda_1^*$ and $\lambda_2^*$ as solutions, which would imply that $k_{1,\lambda} = k_{2,\lambda}$. Obviously, this can not be the case.

Setting $\boldsymbol{u} = (u, v)$, it turns out that, for any $\boldsymbol{f} = (f_1, f_2) \in \mathcal{W}$ and $\lambda \neq \{\lambda_1^*, \lambda_2^*\}$, the general classical solution to the equation $\lambda \boldsymbol{u} - \mathcal{L}\boldsymbol{u} = \boldsymbol{f}$ is given by

$$
\begin{aligned}
u(\xi) =& \left( c_1 - \frac{1}{H_{1,\lambda}} \int_0^\xi (Av(s) + f_1(s)) e^{-k_{1,\lambda}s} ds \right) e^{k_{1,\lambda}\xi} + \left( c_2 + \frac{1}{H_{1,\lambda}} \int_0^\xi (Av(s) + f_1(s)) e^{-k_{2,\lambda}s} ds \right) e^{k_{2,\lambda}\xi} \\
=& \left\{ c_1 - \frac{A}{H_{1,\lambda}} \left[ \frac{e^{(k_{3,\lambda}-k_{1,\lambda})\xi}-1}{k_{3,\lambda}-k_{1,\lambda}} c_3 + \frac{e^{(k_{4,\lambda}-k_{1,\lambda})\xi}-1}{k_{4,\lambda}-k_{1,\lambda}} c_4 \right] \right. \\
& + \frac{A\mathrm{Le}}{H_{1,\lambda}H_{2,\lambda}} \left[ \frac{e^{(k_{3,\lambda}-k_{1,\lambda})\xi}}{k_{3,\lambda}-k_{1,\lambda}} \int_0^\xi f_2(s) e^{-k_{3,\lambda}s} ds - \frac{e^{(k_{4,\lambda}-k_{1,\lambda})\xi}}{k_{4,\lambda}-k_{1,\lambda}} \int_0^\xi f_2(s) e^{-k_{4,\lambda}s} ds \right. \\
& \left. + \frac{k_{3,\lambda}-k_{4,\lambda}}{(k_{4,\lambda}-k_{1,\lambda})(k_{3,\lambda}-k_{1,\lambda})} \int_0^\xi f_2(s) e^{-k_{1,\lambda}s} ds \right] \\
& \left. - \frac{1}{H_{1,\lambda}} \int_0^\xi f_1(s) e^{-k_{1,\lambda}s} ds \right\} e^{k_{1,\lambda}\xi} \\
& + \left\{ c_2 + \frac{A}{H_{1,\lambda}} \left[ \frac{e^{(k_{3,\lambda}-k_{2,\lambda})\xi}-1}{k_{3,\lambda}-k_{2,\lambda}} c_3 + \frac{e^{(k_{4,\lambda}-k_{2,\lambda})\xi}-1}{k_{4,\lambda}-k_{2,\lambda}} c_4 \right] \right. \\
& + \frac{A\mathrm{Le}}{H_{1,\lambda}H_{2,\lambda}} \left[ \frac{e^{(k_{4,\lambda}-k_{2,\lambda})\xi}}{k_{4,\lambda}-k_{2,\lambda}} \int_0^\xi f_2(s) e^{-k_{4,\lambda}s} ds - \frac{e^{(k_{3,\lambda}-k_{2,\lambda})\xi}}{k_{3,\lambda}-k_{2,\lambda}} \int_0^\xi f_2(s) e^{-k_{3,\lambda}s} ds \right. \\
& \left. - \frac{k_{3,\lambda}-k_{4,\lambda}}{(k_{3,\lambda}-k_{2,\lambda})(k_{4,\lambda}-k_{2,\lambda})} \int_0^\xi f_2(s) e^{-k_{2,\lambda}s} ds \right] \\
& \left. + \frac{1}{H_{1,\lambda}} \int_0^\xi f_1(s) e^{-k_{2,\lambda}s} ds \right\} e^{k_{2,\lambda}\xi},
\end{aligned}
\tag{B.2}
$$

$$v(\xi) = \left( c_3 - \frac{\mathrm{Le}}{H_{2,\lambda}} \int_0^\xi f_2(s) e^{-k_{3,\lambda}s} ds \right) e^{k_{3,\lambda}\xi} + \left( c_4 + \frac{\mathrm{Le}}{H_{2,\lambda}} \int_0^\xi f_2(s) e^{-k_{4,\lambda}s} ds \right) e^{k_{4,\lambda}\xi} \tag{B.3}$$

for $\xi < 0$ and

$$u(\xi) = \left( c_5 - \frac{1}{H_{1,\lambda}} \int_0^\xi f_1(s) e^{-k_{1,\lambda}s} ds \right) e^{k_{1,\lambda}\xi} + \left( c_6 + \frac{1}{H_{1,\lambda}} \int_0^\xi f_1(s) e^{-k_{2,\lambda}s} ds \right) e^{k_{2,\lambda}\xi}, \tag{B.4}$$

$$v(\xi) = \left(c_7 - \frac{\mathrm{Le}}{H_{3,\lambda}} \int\limits_0^\xi f_2(s)e^{-k_{5,\lambda}s}ds\right)e^{k_{5,\lambda}\xi} + \left(c_8 + \frac{\mathrm{Le}}{H_{3,\lambda}} \int\limits_0^\xi f_2(s)e^{-k_{6,\lambda}s}ds\right)e^{k_{6,\lambda}\xi}, \tag{B.5}$$

for $\xi \geq 0$. Here, $H_{i,\lambda}$ ($i = 1, 2, 3$) and $k_{j,\lambda}$ ($j = 1, \dots, 6$) are defined by (2.14)–(2.15).

If $\lambda \in \{\lambda_1^*, \lambda_2^*\}$, then $k_{1,\lambda} = k_{3,\lambda}$. Hence, in the definition of $u$ for $\xi < 0$, the term

$$-\frac{A(e^{k_{3,\lambda}-k_{1,\lambda}} - 1)}{H_{1,\lambda}(k_{3,\lambda} - k_{1,\lambda})}c_3$$

$$+ \frac{A\mathrm{Le}}{H_{1,\lambda}H_{2,\lambda}}\left[\frac{e^{(k_{3,\lambda}-k_{1,\lambda})\xi}}{k_{3,\lambda} - k_{1,\lambda}} \int\limits_0^\xi f_2(s)e^{-k_{3,\lambda}s}ds + \frac{k_{3,\lambda} - k_{4,\lambda}}{(k_{3,\lambda} - k_{1,\lambda})(k_{4,\lambda} - k_{1,\lambda})} \int\limits_0^\xi f_2(s)e^{-k_{1,\lambda}s}ds\right]$$

should be replaced by

$$-\frac{A}{H_{1,\lambda}}c_3\xi - \frac{A\mathrm{Le}}{H_{1,\lambda}H_{2,\lambda}} \int\limits_0^\xi (s - \xi)f_2(s)e^{-k_{3,\lambda}s}ds - \frac{A\mathrm{Le}\,e^{(k_{4,\lambda}-k_{1,\lambda})\xi}}{H_{1,\lambda}H_{2,\lambda}(k_{4,\lambda} - k_{1,\lambda})} \int\limits_0^\xi f_2(s)e^{-k_{4,\lambda}s}ds.$$

## Appendix C. On the equality $k_{1,\lambda} = k_{3,\lambda}$

Here, we show that the solutions of the equation $k_{1,\lambda} = k_{3,\lambda}$, i.e., the complex numbers given by (B.1), are not roots of the dispersion relation. Since $(\mathrm{Le}^2 + 4\mathrm{Le}(A + \lambda_j^*))^{1/2} = \mathrm{Le} - 1 + (1 + 4\lambda_j^*)^{1/2}$, it is easy to see that $D(\lambda_j^*, \Theta_i, \mathrm{Le}) = 0$ if and only if

$$\sqrt{\mathrm{Le} + 4\lambda\mathrm{Le}}\left[1 \pm 2i\frac{\sqrt{A\mathrm{Le}}}{\sqrt{\mathrm{Le} - 1}} + (\Theta_i - 1)\left(1 - \frac{4A\mathrm{Le}}{\mathrm{Le} - 1} \pm 4i\frac{\sqrt{A\mathrm{Le}}}{\sqrt{\mathrm{Le} - 1}}\right)\right]$$

$$= 2A\mathrm{Le} - \left(\mathrm{Le} \pm 2i\frac{\sqrt{A\mathrm{Le}}}{\sqrt{\mathrm{Le} - 1}}\right)\left[1 \pm 2i\frac{\sqrt{A\mathrm{Le}}}{\sqrt{\mathrm{Le} - 1}} + (\Theta_i - 1)\left(1 - \frac{4A\mathrm{Le}}{\mathrm{Le} - 1} \pm 4i\frac{\sqrt{A\mathrm{Le}}}{\sqrt{\mathrm{Le} - 1}}\right)\right]. \tag{C.1}$$

Squaring both sides of (C.1) and identifying real and imaginary parts of the so obtained equation, after some long but straightforward computation we get the following system for Le and $\Theta_i$:

$$\begin{cases} \Theta_i^2 + A\mathrm{Le} + 16(\Theta_i - 1)^2\dfrac{A^2\mathrm{Le}^2}{(\mathrm{Le} - 1)^2} - 8\dfrac{A\mathrm{Le}}{\mathrm{Le} - 1}\Theta_i(\Theta_i - 1)(3\Theta_i - 1) - \Theta_i\mathrm{Le} + 4(\Theta_i - 1)\dfrac{A\mathrm{Le}^2}{\mathrm{Le} - 1} = 0, \\ 4A\mathrm{Le}(\Theta_i - 1)(4\Theta_i - 3) + (\mathrm{Le} - 1)(3\Theta_i - 4\Theta_i^2 - \mathrm{Le} + 2\Theta_i\mathrm{Le}) = 0. \end{cases} \tag{C.2}$$

First, we consider the second equation in (C.2). Replacing $A$ with its value given by (1.8) and solving the so obtained equation with respect to Le, we obtain that there are no positive solutions if $\Theta_i = 1/2$ and, when $\Theta_i \in (0, 1) \setminus \{1/2\}$, then the equation has two real solutions

$$\mathrm{Le}_\pm = \frac{20\Theta_i^2 - 13\Theta_i - 1 \pm (400\Theta_i^4 - 552\Theta_i^3 + 169\Theta_i^2 + 14\Theta_i + 1)^{\frac{1}{2}}}{2\Theta_i - 1}.$$

A straightforward computation reveals that $\mathrm{Le}_- > 1$ if and only if $\Theta_i \leq 1/2$, whereas $\mathrm{Le}_+ > 1$ if and only if $\Theta_i \in (\overline{\Theta}_i, 1)$, where the value $\overline{\Theta}_i = (4 + \sqrt{22})/12 \approx 0.724$ will play a significant role hereafter.

Now, we go back to the first equation in (C.2). Replacing $A$ by its value, given by (1.8), and taking $\mathrm{Le} = \mathrm{Le}_\pm$, we get the following equation

$$p(\Theta_i) = (\mathrm{signum}(1 - 2\Theta_i))(1 - \Theta_i)q(\Theta_i)\sqrt{400\Theta_i^4 - 552\Theta_i^3 + 169\Theta_i^2 + 14\Theta_i + 1} \tag{C.3}$$

for $\Theta_i \in (0, 1/2) \cup (\overline{\Theta}_i, 1)$, where

$$p(\Theta_i) = -38400\Theta_i^9 + 296896\Theta_i^8 - 800896\Theta_i^7 + 1041468\Theta_i^6 - 698658\Theta_i^5 + 218492\Theta_i^4 - 14718\Theta_i^3$$
$$- 3894\Theta_i^2 - 298\Theta_i - 8,$$
$$q(\Theta_i) = 1920\Theta_i^6 - 11600\Theta_i^5 + 19164\Theta_i^4 - 12038\Theta_i^3 + 2174\Theta_i^2 + 251\Theta_i + 8.$$

It follows from the next lemma that (C.3) admits no solutions in the set $(0, 1/2) \cup (\overline{\Theta}_i, 1)$ and, consequently, the solutions of $k_{1,\lambda} = k_{3,\lambda}$ are not zeros of the dispersion relation.

**Lemma C.1.** *Function $q$ is positive in $(0, 1/2)$ and negative in $(\overline{\Theta}_i, 1)$. On the contrary, $p$ is negative in $(0, 1/2)$ and positive in $(\overline{\Theta}_i, 1)$.*

**Proof.** Since the proof is easy but rather technical, we sketch it. In what follows, we denote by $c$ positive constants which may vary from line to line. Similarly, by $p_k$ and $q_k$ we denote polynomials of degree $k$, which may vary from estimate to estimate.

We begin by considering the function $q$. For $\Theta_i \in (0, 1/2)$, we can estimate the sum of the first three terms in the definition of $q$ by $13364\Theta_i^4$, so that $q(\Theta_i) > \Theta_i(13364\Theta_i^3 - 12038\Theta_i^2 + 2174\Theta_i + 251) + 8$ and the right-hand side of the previous inequality is not less than $-2\Theta_i + 8$, so that $q$ is positive in $(0, 1/2)$.

For $\Theta_i \in (\overline{\Theta}_i, 1)$ things are a bit trickier. Obviously, it suffices to prove that $q$ is negative in $(7/10, 1)$. For this purpose, we observe that, since $q(7/10) < 0$, we can estimate $q < q - q(7/10) =: q_5$ in such an interval and

$$q_5(\Theta_i) < c\Theta_i(10\Theta_i - 7)(120000\Theta_i^4 - 641000\Theta_i^3 + 749050\Theta_i^2 - 228040\Theta_i - 23753)$$
$$= c\Theta_i(10\Theta_i - 7)[(10\Theta_i - 7)(24000\Theta_i^3 - 111400\Theta_i^3 + 71830\Theta_i + 4673) - 73975]. \tag{C.4}$$

Computing the maximum value of the above third-order polynomial in the interval $(7/10, 1)$, we conclude that $q(\Theta_i) < c\Theta_i(10\Theta_i - 7)[17217(10\Theta_i - 7) - 73975]$, whose right-hand side is negative if $\Theta_i \in (7/10, \widehat{\Theta}_i)$, where $\widehat{\Theta}_i = 67657/86085 \approx 0.786$. On the other hand, if $\Theta_i \in [\widehat{\Theta}_i, 1)$, we can subtract from the fourth-order polynomial on the first line of (C.4) its value at $\widehat{\Theta}_i$ (which is negative) and, thus, estimate $q(\Theta_i) \le c\Theta_i(10\Theta_i - 7)(\Theta_i - \widehat{\Theta}_i)q_3(\Theta_i)$, and $q_3$ is negative in the interval $[\widehat{\Theta}_i, 1)$, as it is easily seen. Thus, $q$ is negative in $(\overline{\Theta}_i, 1)$ as claimed.

Next, we consider function $p$, first addressing the case when $\Theta_i \in (0, 1/2)$. Note that $p(\Theta_i) < p(\Theta_i) - p(0) = \Theta_i p_8(\Theta_i) < \Theta_i(p_8(\Theta_i) - p_8(1/2)) = c\Theta_i(1 - 2\Theta_i)p_7(\Theta_i)$ for every $\Theta_i \in (0, 1/2)$. Iterating this procedure, in the end we deduce that $p(\Theta_i) < c\Theta_i^3(1 - 2\Theta_i)^3 p_4(\Theta_i)$ for each $\Theta_i \in (0, 1/2)$. Since $p_4(\Theta_i) < p_4(\Theta_i) - p_4(1/2) = c(1 - 2\Theta_i)p_3(\Theta_i)$ for every $\Theta_i \in (0, 1/2)$ and $p_3$ is negative in $(0, 1/2)$, $p(\Theta_i)$ is negative for each $\Theta_i \in (0, 1/2)$.

Let us now assume that $\Theta_i \in (\overline{\Theta}_i, 1)$. Since $\overline{\Theta}_i > 18/25 =: \widetilde{\Theta}_i$, we can limit ourselves to proving that $p$ is negative in $(\widetilde{\Theta}_i, 1)$. For this purpose, we observe that

$$p(\Theta_i) < p(\Theta_i) - p(\widetilde{\Theta}_i) = c(\widetilde{\Theta}_i - \Theta_i)p_8(\Theta_i) < c(\widetilde{\Theta}_i - \Theta_i)[p_8(\Theta_i) - p_8(\widetilde{\Theta}_i)] = -c(\widetilde{\Theta}_i - \Theta_i)^2 p_7(\Theta_i)$$
$$< -c(\widetilde{\Theta}_i - \Theta_i)^2[p_7(\Theta_i) - p_7(1)] = c(\widetilde{\Theta}_i - \Theta_i)^2(1 - \Theta_i)p_6(\Theta_i).$$

If $\Theta_i \in [0.745, 0.75]$ then we estimate $\Theta_i^k \le 75 \cdot 10^{-2k}$ for $k = 4, 6$, $\Theta_i^k \ge 745 \cdot 10^{-3k}$ for $k = 1, 2, 3, 5$, and conclude that $p_6$ and, hence, $p$ is negative in $[0.745, 0.75]$. For $\Theta_i \in (0.75, 1)$, we estimate $p_6(\Theta_i) < p_6(\Theta_i) - p_6(3/4) = c(4\Theta_i - 3)p_5(\Theta_i)$. Iterating this procedure, we conclude that $p_6(\Theta_i) < (4\Theta_i - 3)^3 p_3(\Theta_i)$ and the polynomial $p_3$ is negative in $(0.75, 1)$. Finally, if $\Theta_i \in (0.72, 0.745)$ then we set $\overline{\overline{\Theta}}_i = 0.745$, estimate

$$p_6(\Theta_i) < (p_6(\Theta_i) - p_6(\overline{\overline{\Theta}}_i)) < c(\Theta_i - \overline{\overline{\Theta}}_i)p_5(\Theta_i) < c(\Theta_i - \overline{\overline{\Theta}}_i)(p_5(\Theta_i) - p_5(\overline{\overline{\Theta}}_i))$$
$$= c(\Theta_i - \overline{\overline{\Theta}}_i)^2 p_4(\Theta_i) \le c(\Theta_i - \overline{\overline{\Theta}}_i)^2(p_4(\Theta_i) - p_4(\widetilde{\Theta}_i)) = c(\Theta_i - \widetilde{\Theta}_i)(\Theta_i - \overline{\overline{\Theta}}_i)^2 p_3(\Theta_i)$$

and observe that $p_3$ is negative in $[0.72, 0.745]$. Thus, $p$ is negative in this interval as well. Summing up, we have proved that $p$ is negative in $(\overline{\Theta}_i, 1)$ as claimed. This concludes the proof.  □

## Appendix D.  The coefficients of the polynomial $P_7(\cdot; m, \varepsilon)$

We collect here the expression of the coefficients $a_i = a_i(m, \varepsilon)$ ($i = 0, 1, \ldots, 7$) of the polynomial $P_7(\lambda; m, \varepsilon) = a_0\lambda^7 + a_1\lambda^6 + a_2\lambda^5 + a_3\lambda^4 + a_4\lambda^3 + a_5\lambda^2 + a_6\lambda + a_7$, which appears in Subsection 5.1. They are given by

$$a_0 = 2^{11}(\varepsilon - 1)^4 \varepsilon^2;$$

$$a_1 = -2^{11}(\varepsilon^2 - \varepsilon)^2[(5\varepsilon^2 - 2\varepsilon + 1)m^2 + 2(\varepsilon + 1)^2 m + 4];$$

$$a_2 = (\varepsilon^2 - \varepsilon)\big[\varepsilon(59\varepsilon^3 - 9\varepsilon^2 + 17\varepsilon - 3)m^4 + 4\varepsilon(15\varepsilon^3 + 15\varepsilon^2 + 17\varepsilon + 1)m^3$$
$$+ 4(\varepsilon + 2)(\varepsilon^3 + 9\varepsilon^2 + 5\varepsilon + 1)m^2 - 8(2\varepsilon^3 - 3\varepsilon^2 - 4\varepsilon - 3)m - 8(\varepsilon - 1)(\varepsilon + 2)\big];$$

$$a_3 = 2^7\big[-\varepsilon^2(5\varepsilon - 1)(9\varepsilon^3 + \varepsilon^2 + 7\varepsilon - 1)m^6 - 2\varepsilon^2(59\varepsilon^4 - 8\varepsilon^3 + 74\varepsilon^2 + 8\varepsilon - 5)m^5$$
$$- 4\varepsilon(4\varepsilon^5 + 27\varepsilon^4 + 24\varepsilon^3 + 37\varepsilon^2 + 6\varepsilon - 2)m^4 + 4\varepsilon(4\varepsilon^5 + 20\varepsilon^4 - 31\varepsilon^3 - 23\varepsilon^2 - 33\varepsilon - 1)m^3$$
$$+ 4(9\varepsilon^5 + 27\varepsilon^4 - 15\varepsilon^3 - 24\varepsilon^2 - 12\varepsilon - 1)m^2 + 4(\varepsilon^4 + 17\varepsilon^3 - 7\varepsilon^2 - 9\varepsilon - 2)m - 4(\varepsilon - 1)^2(2\varepsilon + 1)\big];$$

$$a_4 = 2^3\big[\varepsilon^2(9\varepsilon - 1)^2(\varepsilon - 1)^2 m^8 + 8\varepsilon^2(\varepsilon - 1)(45\varepsilon^3 + 5\varepsilon^2 - 21\varepsilon + 3)m^7$$
$$+ 8\varepsilon(21\varepsilon^5 - 58\varepsilon^4 - 84\varepsilon^3 - 32\varepsilon^2 + 27\varepsilon - 2)m^6 - 16\varepsilon(34\varepsilon^5 + 42\varepsilon^4 + 113\varepsilon^3 + 81\varepsilon^2 - 7\varepsilon - 7)m^5$$
$$- 16(7\varepsilon^6 + 96\varepsilon^5 + 75\varepsilon^4 + 176\varepsilon^3 + 42\varepsilon^2 - 10\varepsilon - 2)m^4 + 16\varepsilon(6\varepsilon^4 - 75\varepsilon^3 - 65\varepsilon^2 - 117\varepsilon - 5)m^3$$
$$+ 16(29\varepsilon^4 - 7\varepsilon^3 - 48\varepsilon^2 - 31\varepsilon - 7)m^2 + 32(\varepsilon - 1)(7\varepsilon^2 + 14\varepsilon + 3)m - 16(\varepsilon - 1)^2\big];$$

$$a_5 = 2^5 m\big[\varepsilon^2(\varepsilon - 1)^2(9\varepsilon^2 + \varepsilon - 2)m^7 + (\varepsilon^2 - \varepsilon)(38\varepsilon^4 + 46\varepsilon^3 - 39\varepsilon^2 + 3)m^6$$
$$+ (36\varepsilon^6 + 33\varepsilon^5 - 123\varepsilon^4 - 95\varepsilon^3 + 54\varepsilon^2 - 1)m^5$$
$$- (8\varepsilon^6 - 8\varepsilon^5 + 169\varepsilon^4 + 233\varepsilon^3 + 25\varepsilon^2 - 41\varepsilon - 2)m^4$$
$$- (60\varepsilon^5 + 110\varepsilon^4 + 320\varepsilon^3 + 129\varepsilon^2 - 22\varepsilon - 21)m^3$$
$$- 4(16\varepsilon^4 + 37\varepsilon^3 + 41\varepsilon^2 + 7\varepsilon - 5)m^2 + 2(4\varepsilon^3 - 37\varepsilon^2 - 10\varepsilon - 5)m + 4(5\varepsilon^2 - 2\varepsilon - 3)\big];$$

$$a_6 = 2^3 m\big[2\varepsilon^2(\varepsilon - 1)^2(\varepsilon + 1)(2\varepsilon - 1)m^7 + (\varepsilon^2 - \varepsilon)(16\varepsilon^4 + 58\varepsilon^3 - 19\varepsilon^2 - 10\varepsilon + 3)m^6$$
$$+ (16\varepsilon^6 + 72\varepsilon^5 - 39\varepsilon^4 - 171\varepsilon^3 + 42\varepsilon^2 + 17\varepsilon - 1)m^5$$
$$+ 2(20\varepsilon^5 - 12\varepsilon^4 - 113\varepsilon^3 - 69\varepsilon^2 + 41\varepsilon + 5)m^4 - (2\varepsilon + 1)(18\varepsilon^3 + 81\varepsilon^2 + 80\varepsilon - 51)m^3$$
$$- 4(28\varepsilon^3 + 31\varepsilon^2 + 20\varepsilon - 15)m^2 - 4(11\varepsilon - 3)(\varepsilon + 1)m - 8(1 - \varepsilon)\big];$$

$$a_7 = 2^4(m^2 + m^3)\big[\varepsilon^2(\varepsilon^2 - 1)(2\varepsilon - 1)m^4 + \varepsilon(2\varepsilon - 1)(3\varepsilon^2 - 3\varepsilon - 2)m^3$$
$$+ (2\varepsilon^4 - 13\varepsilon^2 + 4\varepsilon + 1)m^2 - 3(2\varepsilon - 1)(\varepsilon + 1)m + 2(1 - 2\varepsilon)\big].$$

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