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A counterexample to the Liouville property of some nonlocal problems

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Abstract

In this paper, we construct a counterexample to the Liouville property of some nonlocal reaction-diffusion equations of the form

$$\int_{\mathbb{R}^N \setminus K} J(x-y)\,(u(y)-u(x))\mathrm{d}y + f(u(x)) = 0, \quad x \in \mathbb{R}^N \setminus K,$$

where $K \subset \mathbb{R}^N$ is a bounded compact set, called an "obstacle", and f is a bistable nonlinearity. When K is convex, it is known that solutions ranging in $[0, 1]$ and satisfying $u(x) \to 1$ as $|x| \to \infty$ must be identically 1 in the whole space. We construct a nontrivial family of simply connected (non-starshaped) obstacles as well as data f and J for which this property fails.

Keywords: Nonlocal diffusion; Liouville property; Bistable nonlinearity; Counterexample; Positive stationary solution; Calculus of variations

Contents

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1. Introduction

1.1. A nonlocal problem in heterogeneous media

Let K be a compact set of $\mathbb{R}^N$ with $N \geqslant 2$, and let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^N$. We are interested in the qualitative properties of positive solutions u to the following problem

$$\begin{cases} Lu + f(u) = 0 & \text{in } \overline{\mathbb{R}^N \setminus K}, \\ 0 \leqslant u \leqslant 1 & \text{in } \overline{\mathbb{R}^N \setminus K}, \\ u(x) \to 1 & \text{as } |x| \to +\infty, \end{cases} \tag{1.1}$$

where f is a bistable nonlinearity with $f(0) = f(1) = 0$ and L is the nonlocal operator

$$Lu(x) := \int_{\mathbb{R}^N \setminus K} J(x-y)(u(y) - u(x))\mathrm{d}y, \tag{1.2}$$

with $J \in L^1(\mathbb{R}^N)$ a non-negative kernel with unit mass. The precise assumptions on f and J will be given later on.

This type of model naturally arises in the study of the behaviour of particles evolving in a heterogeneous medium. The typical kind of problem we have in mind comes from population dynamics. In this setting, the movement of the individuals is modelled by a stochastic process that is defined in a domain that possesses several inaccessible regions (reflecting the heterogeneity of the environment). At the macroscopic level, the corresponding density of population $u(t,x)$ satisfies a reaction-diffusion equation that is defined outside a set K, which acts as an obstacle. When the individuals follow isotropic Poisson jump processes, this reaction-diffusion equation is given by

$$\frac{\partial u}{\partial t} = Lu + f(u) \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^N \setminus K, \tag{1.3}$$

and the solutions to (1.1) are particular stationary solutions to (1.3).

In recent years, much attention has been paid to the case where the movement of the individuals is modelled by a Brownian diffusion. In this situation, the reaction-diffusion equation takes the form

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(u) & \text{in } \mathbb{R}^+ \times \mathbb{R}^N \setminus K, \\ \nabla u \cdot \nu = 0 & \text{on } \mathbb{R} \times \partial K. \end{cases} \tag{1.4}$$

This problem was first studied by Berestycki, Hamel and Matano in [4]. There, in order to describe the invasion phenomenon modelled by this equation, it is shown that for any unit vector $e \in \mathbb{S}^{N-1}$ (where $\mathbb{S}^{N-1}$ denotes the unit sphere of $\mathbb{R}^N$), there exists a generalised transition wave in the direction e solution to (1.4), i.e. for any $e \in \mathbb{S}^{N-1}$, there exists an entire solution, $u(t,x)$, to (1.4) defined for all $t \in \mathbb{R}$ and all $x \in \mathbb{R}^N \setminus K$ that satisfies $0 < u(t,x) < 1$ for all $(t,x) \in \mathbb{R} \times \overline{\mathbb{R}^N \setminus K}$ and such that $\lim_{t\to-\infty} \sup_{x\in\mathbb{R}^N\setminus K} |u(t,x) - \varphi(x\cdot e + ct)| = 0$, where (φ, c) is a planar front of speed c; that is (φ, c) is the unique (up to shift) solution to

$$\begin{cases} \varphi''(z) - c\varphi'(z) + f(\varphi(z)) = 0 & \text{for } z \in \mathbb{R}, \\ \lim_{z\to+\infty} \varphi(z) = 1, \quad \lim_{z\to-\infty} \varphi(z) = 0. \end{cases}$$

Moreover, they prove that there exists a classical solution, u_∞, to

$$\begin{cases} \Delta u_\infty + f(u_\infty) = 0 & \text{in } \overline{\mathbb{R}^N \setminus K}, \\ \nabla u_\infty \cdot \nu = 0 & \text{on } \partial K, \\ 0 \leqslant u_\infty \leqslant 1 & \text{in } \overline{\mathbb{R}^N \setminus K}, \\ u_\infty(x) \to 1 & \text{as } |x| \to +\infty. \end{cases} \tag{1.5}$$

This latter solution is actually obtained as the large time limit of $u(t,x)$; more precisely:

$$u(t,x) \to u_\infty(x) \text{ as } t \to \infty, \text{ locally uniformly in } x \in \overline{\mathbb{R}^N \setminus K}.$$

In addition, they were able to classify the solutions u_∞ to (1.5) under some geometric assumptions on K. When the obstacle K is either starshaped or directionally convex (see [4, Definition 1.2]), they prove that the solutions to (1.5) are actually identically equal to 1 in the whole set $\overline{\mathbb{R}^N \setminus K}$. This was further extended to more complex obstacles by Bouhours who showed a sort of "stability" of this Liouville type property with respect to small regular perturbations of the obstacle, see [5]. From the biological standpoint, this means that, after some large time, *the population tends to occupy the whole space*.

Yet, when the domain is no longer starshaped nor directionally convex but merely simply connected, it is shown in [4] that *this Liouville type property does not hold in general*. In other words, the geometry of the domain may force the population to diffuse heterogeneously in space, even after some large time.

It is expected that (1.1) and (1.5) share some common properties. In particular, some of the results obtained for (1.5) should, to some extent, hold true as well for (1.1).

Recently, Brasseur et al. [7] have shown that (1.1) enjoys a similar Liouville type property when K is convex (or close to being convex) and when the data f and J satisfy some rather mild assumptions. That is, any solution u to (1.1) is identically equal to 1 in the whole set $\overline{\mathbb{R}^N \setminus K}$. They also point out that this cannot be expected for general obstacles since one can easily find counterexamples when $\mathbb{R}^N \setminus K$ is no longer connected. Indeed, take for instance $K = \overline{B_2} \setminus B_1$ and suppose that J is supported in $B_{1/2}$. Then, the function u defined by

$$u(x) = \begin{cases} 1 & \text{if } x \in \mathbb{R}^N \setminus B_2, \\ 0 & \text{if } x \in \overline{B_1}, \end{cases}$$

is a continuous solution to (1.1); yet, u is not identically 1 in the whole set $\overline{\mathbb{R}^N \setminus K}$. In view of this, it is natural to ask:

> *what are the optimal geometric assumptions on K ensuring*
> *that* (1.1) *enjoys such a Liouville property?*

So far, this question remains open.

In this paper, our main concern is to find out whether it is possible to construct a *nontrivial simply connected* obstacle K, as well as data f and J, for which (1.1) has a continuous solution u which is not identically equal to 1.

Note that this is actually a quite reasonable question. Indeed, since the Liouville property does not hold true on annuli it is quite natural to expect counterexamples on simply connected obstacles which are "ε-close" to an annulus. We will see that this is indeed the case. Precisely, we will construct a family of simply connected compact sets K_ε and data f_ε and J_ε for which the solution to (1.1) is not identically equal to 1.

Remarkably, we will see that not only our arguments provide an alternative proof for the classical problem (1.5) but they also apply to broader classes of nonlocal operators with anisotropic dispersal.

1.2. Main results

Before we state our main results, let us first specify the assumptions made all along this paper. We will assume that J is such that

$$\begin{cases} J \in L^1(\mathbb{R}^N) \text{ is a non-negative, radially symmetric kernel with unit mass,} \\ \text{there are } 0 \leqslant r_1 < r_2 \text{ such that } J(x) > 0 \text{ for a.e. } x \text{ with } r_1 < |x| < r_2, \\ M_1(J) := \displaystyle\int_{\mathbb{R}^N} J(x)|x|\mathrm{d}x < +\infty \text{ and } J \in W^{1,1}(\mathbb{R}^N), \end{cases} \tag{1.6}$$

and that $f \in C^1([0,1])$ is a "bistable" nonlinearity, namely

$$\begin{cases} \exists\theta \in (0,1), \quad f(0) = f(\theta) = f(1) = 0, \quad f < 0 \text{ in } (0,\theta), \quad f > 0 \text{ in } (\theta,1), \\ \int_0^1 f(s)\mathrm{d}s > 0, \quad f'(0) < 0, \quad f'(\theta) > 0, \quad f'(1) < 0, \quad f' < 1 \text{ in } [0,1]. \end{cases} \tag{1.7}$$

Our first result reads as follows

Theorem 1.1. *Let $N \geqslant 2$. Then, there are smooth (non-starshaped) simply connected compact obstacles K and data f and J satisfying (1.6) and (1.7) for which problem (1.1) has a positive nonconstant solution $u \in C(\overline{\mathbb{R}^N \setminus K}, [0,1])$.*

The obstacles constructed in Theorem 1.1 are almost of the same nature as those given in [4] for the local case. Namely, we consider an annulus $\mathcal{A}$ into which a small channel is pierced, see Fig. 1 for a visual illustration.

By contrast with the classical reaction-diffusion, the operator L does *not* enjoy strong compactness properties and has no regularising effects. So our construction is *not* a simple adaptation of the techniques of proof used for the local problem (1.5). One of the novelties of this paper is that we show how to circumvent these issues. As we shall explain in the sequel, our argument is in fact general enough to recover the local problem (see our remarks below).

Let us briefly describe our approach. Our strategy relies essentially on two ingredients. First, we take advantage of the fact that the kernel J and the nonlinearity f may be chosen at our convenience. That is, instead of considering the problem (1.1), we can consider a rescaled version of (1.1) given an appropriate choice of J. In our setting, J will be such that

$$J \in L^2(\mathbb{R}^N), \ \operatorname{supp}(J) = B_r \text{ for some } r > 0, \text{ and } J \text{ is radially } \textit{non-increasing}. \tag{1.8}$$

Then, given a small parameter ε, we look for a nonconstant positive solution u_ε to

$$\int_{\mathbb{R}^N\setminus K} J_\varepsilon(x-y)(u_\varepsilon(y) - u_\varepsilon(x))\mathrm{d}y + f_\varepsilon(u_\varepsilon(x)) = 0 \quad \text{in } \overline{\mathbb{R}^N \setminus K}, \tag{1.9}$$

that further satisfies $0 \leqslant u_\varepsilon \leqslant 1$ in $\overline{\mathbb{R}^N \setminus K}$ and $u_\varepsilon(x) \to 1$ as $|x| \to +\infty$, where

$$f_\varepsilon(s) := \varepsilon^2 f(s) \ \text{ and } \ J_\varepsilon(z) = \frac{1}{\varepsilon^N} J\left(\frac{z}{\varepsilon}\right).$$

In order to prove Theorem 1.1, we only need to show that, for some $\varepsilon > 0$, there is some obstacle K_ε such that (1.9) admits a positive nonconstant solution u_ε.

Second, we consider a well-chosen family of smooth simply connected obstacles $(K_\varepsilon)_{0<\varepsilon<1}$ that look like an annulus with a tiny channel of diameter of the order of $\varepsilon^{N/(N-1)}$ pierced in it (see Fig. 1). Given such a family, we prove that, for ε small enough, (1.9) indeed admits a positive nonconstant continuous solution. More precisely, we prove the following

Theorem 1.2. *Let $N \geqslant 2$. Let J and f be such that (1.6), (1.7) and (1.8) hold. Then, there exist $\varepsilon_* > 0$ and a family of smooth simply connected obstacles $(K_\varepsilon)_{0<\varepsilon<1} \subset \mathbb{R}^N$ such that, for all $0 < \varepsilon < \varepsilon^*$, there is a positive nonconstant solution $u_\varepsilon \in C(\overline{\mathbb{R}^N \setminus K_\varepsilon}, [0,1])$ to (1.9).*

Due to the lack of a strong regularising property of (1.9), the construction of u_ε relies essentially on elementary arguments. In particular, we obtain a solution u_ε to (1.9) using an adequate monotone iterative scheme and elementary estimates. The main difficulty in our proof lies in the construction of an adequate pair of ordered continuous sub- and super-solution in a context where the equation (1.1) does not allow the use of traditional schemes based on compactness arguments. To cope with this major difficulty, we make a detailed construction of the obstacle K_ε and design it in such a way that we still can obtain standard L^2-estimates by elementary means. This requires a detailed analysis of all the parameters involved at each step of our construction, especially when we construct our super-solution. To construct our super-solution we rely on the fact that a solution u_ε to (1.9) satisfies in particular

$$\frac{1}{\varepsilon^2}\int\limits_{\mathbb{R}^N\setminus K_\varepsilon} J_\varepsilon(x-y)(u_\varepsilon(y)-u_\varepsilon(x))\mathrm{d}x + f(u_\varepsilon(x)) = 0, \tag{1.10}$$

and, from there, relying essentially on the Bourgain-Brezis-Mironescu characterisation of Sobolev spaces (see e.g. [6,18]), we can interpret the first term on the left-hand side as a *nonlocal approximation* of Δu in the sense that its energy approximates the L^2-variation of u. This, in turn, with a pertinent choice of K_ε and a well-chosen auxiliary problem, allows one to derive *a priori* bounds to construct a super-solution by means of variational methods.

A striking consequence of our construction is that it adapts almost straightforwardly to other situations. For example, it applies to the standard reaction-diffusion equation (1.5) providing so an alternative proof of the existence of a counterexample. But it also extends to broader classes of nonlocal operators where the dispersal process need not be isotropic but instead depends on the geodesic distance between points in $\overline{\mathbb{R}^N\setminus K}$. Indeed, our proof also adapts (with almost no changes) to operators of the form

$$L_{\mathrm{g}}u(x) := \int\limits_{\mathbb{R}^N\setminus K} \widetilde{J}(d_{\mathrm{g}}(x,y))(u(y)-u(x))\mathrm{d}y, \tag{1.11}$$

where $d_{\mathrm{g}}(\cdot,\cdot)$ is the geodesic distance on $\overline{\mathbb{R}^N\setminus K}$ and $\widetilde{J}\in L^1_{\mathrm{loc}}(0,\infty)$ is such that

$$\sup_{x\in\mathbb{R}^N\setminus K}\int\limits_{\mathbb{R}^N\setminus K} \widetilde{J}(d_{\mathrm{g}}(x,y))\mathrm{d}y < \infty, \tag{1.12}$$

and $z\mapsto \widetilde{J}(|z|)$ satisfies (1.6).

More precisely, we have

Theorem 1.3. *Let $N\geqslant 2$. Then, there are smooth (non-starshaped) simply connected compact obstacles K and data f and $\widetilde{J}$ satisfying* (1.6)*,* (1.7) *and* (1.12) *for which the problem*

$$\begin{cases} L_{\mathrm{g}}u + f(u) = 0 & \text{in } \overline{\mathbb{R}^N\setminus K}, \\ u(x)\to 1 & \text{as } |x|\to+\infty, \end{cases} \tag{1.13}$$

has a solution $u\in C(\overline{\mathbb{R}^N\setminus K},[0,1])$ which is not identically equal to 1 in $\overline{\mathbb{R}^N\setminus K}$.

The obstacle K and the data f and J $(=\widetilde{J}(|\cdot|))$ constructed at Theorem 1.3 are exactly the same as in Theorem 1.1. When needed we will state in side remarks the necessary changes to make to the proofs in order to handle this type of dispersal processes.

Remark 1.4. It turns out that the techniques of proof used in [7] to establish the Liouville property of (1.1) for convex domains also apply to this modified setting (at least when J is non-increasing), but we leave this to a subsequent paper.

1.3. Further comments

Before going to the proofs of our results, we would like to make some comments on their motivation and meaning, as well as on some modelling considerations for ongoing/future works.

First, we would like to emphasize that problem (1.13) is of interest in its own right. The way the dispersal process is modelled gives an alternative way to describe the evolution of particles within a perforated domain which, in some situations, may be regarded as more realistic. The point here is that particles *cannot* travel through K (as is it the case for problem (1.1)). Instead, they are compelled to "bypass" K as if it was a material obstacle. This particularity may be helpful to study the dynamics of some species living in landscapes that present some physical dispersal barriers such as many land animals in presence of large areas of water (lake, river, seas, ...) or, conversely, aquatic animals in presence of lands (island, coral reef, artificial construction, ...), see for example [13,17,20].

As we already mentioned in the introduction, in terms of population dynamics, the stationary solution of (1.3) is expected to be the outcome of an invasion process. As such, the purpose of this paper can be interpreted as follows: *can we find a connected geometrical configuration that prevents a total invasion*?

With this in mind, for a given dispersal process, it is then legitimate to investigate which type of obstacle may prevent a full invasion and whether the characteristics of this dispersal process influence the design of the obstacle K. In this regard, the connectedness of the domain $\mathbb{R}^N \setminus K$ is of great importance since we can easily construct a disconnected domain $\mathbb{R}^N \setminus K$ that can still be fully invaded when the dispersal is modelled using the Euclidean distance and for which it will never be fully invaded when the dispersal process is modelled using the geodesic distance. Our construction of obstacles is a first step towards a better understanding of the impact of the dispersal processes in the design of obstacles that prevent total invasion. A more thorough study of the differences between the different types of dispersal process and their influence on the design of the obstacle K is currently under consideration and is left to a subsequent work.

In this spirit, the assumption on the nonlinearity f may also play an important role on the design of K. Indeed, as shown in [1], the evolution equation (1.3) with no obstacle K may exhibit a peculiar behaviour. That is, for an unbalanced f with $\int_0^1 f(s)\,\mathrm{d}s > 0$, there exists fronts to the equation (1.3) with zero speed and particular nontrivial patterns are stable, see [21]. As a consequence, there is no need to introduce an obstacle to prevent total invasion. This type of behaviour is inherent to the lack of regularising property of the generator of the dispersal process considered. As a consequence, in order to get some continuity for the solution to (1.3), some constraints on the nonlinearity f have to be imposed. In particular, this behaviour can only appear when the function $s - f(s)$ is not invertible in $[0, 1]$ and never appears when the function $s - f(s)$ is invertible in $[0, 1]$. From an applied point of view, this peculiar behaviour can serve as a tool to help identify which process is more suitable to describe the dispersal of the individuals of a population.

From a modelling standpoint, we may also wonder which type of dispersal modelling is the most relevant to describe the evolution of a population. Unfortunately or fortunately this question has no simple straight answer since, as observed in [11], the perception of some geographical components of the landscape as an obstacle to dispersal is strongly dependent of the species studied and, as a consequence, for the same domain $\mathbb{R}^N \setminus K$ different representations of the dispersal may prove to be pertinent.

On a broader applicative point of view, to some extend, our results fall within the study of the impact of the landscape connectivity on the evolution of a population and its genetic consequences. We point to [15] for a more detailed introduction and description of this vast subject and to [19] for a general review of the main current challenges in the description of dispersal processes and its consequences.

Lastly, we would like to comment further on our construction. Although the design of the obstacle and the construction of the counterexample are rather similar to the one used in [4], the proof in [4] crucially relies on the Sobolev embedding theorem in order to obtain the right energy estimate that enables the construction of a nontrivial solution. In this paper, we show that this estimate can be obtained by using *only* a Poincaré-Wirtinger type inequality provided the obstacle is well chosen, thus avoiding some difficult regularity issues. Our estimate is then more robust to a change of dispersal process as Poincaré-Wirtinger type inequalities are nothing less than spectral quantities that can be computed and estimated for a broad class of dispersal operators. In particular, with very few modifications, our construction should hold as well for a dispersal process whose infinitesimal generator is a truncated regional fractional Laplacian, Δ_R^α, defined as follows:

$$\Delta_R^\alpha[\varphi](x) := C_{N,s} \lim_{\varepsilon \to 0^+} \int\limits_{|x-y|>\varepsilon,\, y\in B_R\setminus K} \frac{\varphi(y)-\varphi(x)}{|x-y|^{N+2\alpha}}\,\mathrm{d}y.$$

Note that, in our framework, this situation corresponds to take a kernel J which is no more integrable (due to the singularity at the origin). Nevertheless, this singularity does not play any role in our construction, except that it gives more regularity to the solution we construct.

The paper is organized as follows. After describing our notations, we recall some results from the literature in Section 2. In Section 3, given a pair (J, f) we construct an adequate family of obstacles. Then, in Section 4, we construct some particular super-solutions to the problem (1.9). Finally, in Section 5, we use the super-solution constructed at Section 4 to prove Theorem 1.2.

1.4. Notations

Let us list a few notations that will be used throughout the paper.

As usual, $\mathbb{S}^{N-1}$ denotes the unit sphere of $\mathbb{R}^N$ and $B_R(x)$ the open Euclidean ball of radius $R > 0$ centred at $x \in \mathbb{R}^N$ (when $x = 0$, we simply write B_R). We denote by $\mathcal{A}(R_1, R_2)$ the open annulus $B_{R_2} \setminus \overline{B_{R_1}}$.

For a compact set $\Omega \subset \mathbb{R}^N$, we denote by $\mathrm{diam}(\Omega)$ its diameter, given by

$$\mathrm{diam}(\Omega) := \sup_{x,y\in\Omega} |x - y|.$$

The N-dimensional Hausdorff measure will be denoted by $\mathscr{H}^N$. For a measurable set $E \subset \mathbb{R}^N$, we denote by $|E|$ its Lebesgue measure and by $\mathbb{1}_E$ its characteristic function. If $0 < |E| < \infty$ and if $g : \mathbb{R}^N \to \mathbb{R}$ is locally integrable, we denote by

$$⨍_E g(x)\mathrm{d}x = \frac{1}{|E|} \int_E g(x)\mathrm{d}x,$$

the average of g in the set E. Also, we denote by $L^p(E)$, $1 \leqslant p \leqslant \infty$, the Lebesgue space of (equivalence classes of) measurable functions g for which the p-th power of the absolute value is Lebesgue integrable when $p < \infty$ (resp. essentially bounded when $p = \infty$).

2. Preliminaries

In this section, we recall some known results that will be used throughout the paper. In most cases, we will omit their proofs and point the interested reader to the relevant references.

We first state a general existence result.

Lemma 2.1. *Assume that f and J satisfy (1.6) and (1.7). Let $K \subset \mathbb{R}^N$ be a compact set and let $\underline{u}, \overline{u} \in C(\mathbb{R}^N \setminus K)$ be such that*

$$\begin{cases} L\overline{u} + f(\overline{u}) \leqslant 0 & \text{in } \mathbb{R}^N \setminus K, \\ L\underline{u} + f(\underline{u}) \geqslant 0 & \text{in } \mathbb{R}^N \setminus K. \end{cases}$$

Assume, in addition, that

$$\limsup_{|x|\to\infty} \underline{u}(x) = \lim_{|x|\to\infty} \overline{u}(x) = 1, \tag{2.1}$$

and that

$$0 \leqslant \underline{u} \leqslant \overline{u} \leqslant 1 \quad \text{in } \mathbb{R}^N \setminus K. \tag{2.2}$$

Then, there exists $u \in L^\infty(\mathbb{R}^N \setminus K)$ such that

$$\begin{cases} Lu + f(u) = 0 & \text{in } \mathbb{R}^N \setminus K, \\ \underline{u} \leqslant u \leqslant \overline{u} & \text{in } \mathbb{R}^N \setminus K. \end{cases}$$

Although the proof of Lemma 2.1 relies on rather standard arguments it is not that straightforward. For this reason, we will give a detailed proof (which is postponed to the Appendix at the end of the paper).

Next, we recall a regularity result for nonlocal equations of the form

$$\int_{\Omega\setminus K} J(x-y)u(y)\mathrm{d}y - \mathcal{J}(x)u(x) + f(u(x)) = 0 \quad \text{in } \Omega \setminus K, \tag{2.3}$$

where

$$\mathcal{J}(x) := \int_{\mathbb{R}^N \setminus K} J(x-y)\mathrm{d}y. \tag{2.4}$$

Precisely,

Lemma 2.2. *Assume that $f \in C^1([0,1])$ and that J satisfies (1.6). Let $\Omega \subset \mathbb{R}^N$ be an open set having C^1 boundary. Suppose that $K \subset \Omega$ is a compact set and that*

$$\max_{[0,1]} f' < \inf_{\Omega \setminus K} \mathcal{J}. \tag{2.5}$$

Let $u \in L^\infty(\Omega \setminus K, [0,1])$ be a solution to (2.3) a.e. in $\Omega \setminus K$. Then, u can be redefined up to a negligible set and extended as a uniformly continuous function in $\overline{\Omega \setminus K}$.

For a detailed proof, we refer to [7, Lemma 3.2] (see also [1,3]).

Remark 2.3. Note that Ω need not be bounded. In particular, Lemma 2.2 holds when $\Omega = \mathbb{R}^N$.

Remark 2.4. In some cases, condition (2.5) turns out to be also necessary. For example, if $\Omega = \mathbb{R}^N$ and if $K = \emptyset$, then (2.5) reads $\max_{[0,1]} f' < 1$ and it was shown in [1,22] that if this latter condition is violated, then the stationary problem $J * u - u + f(u) = 0$ in $\mathbb{R}^N$ admits a solution with jump discontinuities. From the biological perspective, (2.5) can be interpreted as a sufficient (and, as pointed out above, close to being necessary) condition ensuring that in a uniform environment the individuals of a given species are distributed continuously in space.

Finally, we recall a result on the asymptotic behaviour of positive solutions to (2.3).

Lemma 2.5 *(Uniform asymptotic behaviour of positive solutions). Let $K \subset \mathbb{R}^N$ be a compact set and suppose that J and f satisfy (1.6) and (1.7). Assume further that J is compactly supported and that $J \in L^2(\mathbb{R}^N)$. Let $u \in C(\mathbb{R}^N \setminus K, [0,1])$ be a solution to*

$$\begin{cases} Lu + f(u) = 0 \ \ \text{in } \overline{\mathbb{R}^N \setminus K}, \\ \sup\limits_{\mathbb{R}^N \setminus K} u = 1. \end{cases} \tag{2.6}$$

Then, $u(x) \to 1$ as $|x| \to \infty$.

The proof may be found in [7, Lemma 7.2].

Remark 2.6. The above results still hold when $J(x-y)$ is replaced by $\widetilde{J}(d_g(x,y))$. For the validity of Lemma 2.1 in this case, we refer to Remark A.2 in the Appendix. On the other hand, a careful inspection of the proof of [7, Lemma 3.2] shows that the condition (2.5) with $\mathcal{J}$ replaced by

$$\widetilde{\mathcal{J}}(x) := \int_{\mathbb{R}^N \setminus K} \widetilde{J}(d_g(x,y))\mathrm{d}y, \tag{2.7}$$

still implies the continuity of solutions to

$$\int_{\Omega \setminus K} \widetilde{J}(d_g(x,y))u(y)\mathrm{d}y - \widetilde{\mathcal{J}}(x)u(x) + f(u(x)) = 0,$$

in $\overline{\Omega \setminus K}$. Similarly, Lemma 2.5 holds as well with L_g (as given by (1.11)) instead of L since its proof requires only estimates on convex regions on which it trivially holds that $d_g(x,y) = |x-y|$.

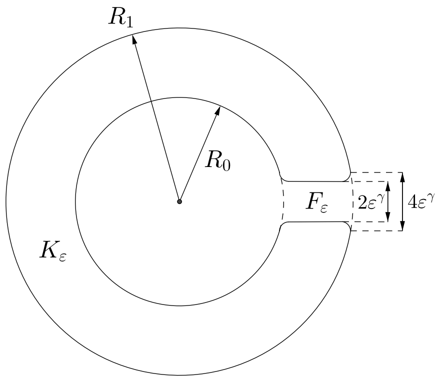

Fig. 1. Illustration of K_ε in dimension 2.

3. Construction of a family of obstacles

This section is devoted to the construction of an appropriate family of obstacles $(K_\varepsilon)_{0<\varepsilon<1}$. Our construction will depend on the interplay with the datum (J, f). As mentioned in the introduction, we will assume that J satisfies (1.6) and (1.8) and that f satisfies (1.7). However, before constructing $(K_\varepsilon)_{0<\varepsilon<1}$, we need to define some important quantities depending on f and J. We will call $C_0 > 0$ and $M_2(J) > 0$ the constants respectively defined by

$$\begin{cases} C_0 := \max_{s\in[0,1]} f(s), & \text{(a)} \\ M_2(J) := \int_{\mathbb{R}^N} J(z)|z|^2 \mathrm{d}z. & \text{(b)} \end{cases} \tag{3.1}$$

Note that the assumptions (1.6) and (1.7) guarantee that these two numbers are well-defined. Furthermore, we introduce two quantities, $C_{N,J}$ and R_0^*, respectively defined by

$$\begin{cases} C_{N,J} := \dfrac{\pi^2 M_2(J)}{32N}, & \text{(a)} \\ R_0^*(J, f) := \sqrt{\dfrac{\theta C_{N,J}}{5C_0}}. & \text{(b)} \end{cases} \tag{3.2}$$

Let us now start the construction of the obstacle. Let $0 < R_0 < R_0^*(J, f)$ (where $R_0^*(J, f)$ is as in (3.2)(b)) and fix some $R_1 > \sup\{2, R_0^*\}$. Let $0 < \varepsilon < 1$ be a small parameter and set $\gamma := \frac{N}{N-1}$. We call $\mathcal{A}$ the annulus $\mathcal{A} := \mathcal{A}(R_0, R_1)$ and we consider a smooth compact simply connected set $K_\varepsilon \subset \overline{\mathcal{A}}$ satisfying the following properties:

(i) $\overline{\mathcal{A}} \cap \{x \in \mathbb{R}^N ;\ x_1 \leqslant 0\} \subset K_\varepsilon$,
(ii) $\overline{\mathcal{A}} \cap \{x \in \mathbb{R}^N ;\ x_1 > 0, |x'| > 2\varepsilon^\gamma\} \subset K_\varepsilon$,
(iii) $K_\varepsilon \subset \left(\overline{\mathcal{A}} \cap \{x \in \mathbb{R}^N ;\ x_1 \leqslant 0\}\right) \cup \left(\overline{\mathcal{A}} \cap \{x \in \mathbb{R}^N ;\ x_1 > 0, |x'| \geqslant \varepsilon^\gamma\}\right)$,
(iv) $\mathcal{A}(R_0 + \varepsilon^\gamma/4, R_1 - \varepsilon^\gamma/4) \cap \{x \in \mathbb{R}^N ; x_1 > 0, |x'| \geqslant \varepsilon^\gamma\} \subset K_\varepsilon$,

where $x = (x_1, x')$ and $x' = (x_2, \dots, x_N)$ (see Fig. 1 for a visual illustration). The presence of the exponent γ is for technical purpose, its relevance will be made clear later on (see Remark 4.6). Furthermore, we define the following open set:

$$F_\varepsilon := \mathcal{A} \setminus K_\varepsilon.$$

We will refer to $(K_\varepsilon)_{0<\varepsilon<1}$ as the *family of obstacles associated to the pair* (J, f).

Let us also list in this section a preparatory lemma.

Proposition 3.1. *Let $N \geqslant 2$. Suppose that f and J are such that (1.6), (1.7) and (1.8) hold true. Assume further that $(K_\varepsilon)_{0<\varepsilon<1}$ is the family of obstacles associated to the pair (J, f). Let*

$$f_\varepsilon(s) := \varepsilon^2 f(s) \text{ and } J_\varepsilon(z) := \frac{1}{\varepsilon^N} J\left(\frac{z}{\varepsilon}\right). \tag{3.3}$$

Then, there exists some $\varepsilon_0 > 0$ depending only on N, R_0, J and f', such that

$$\max_{[0,1]} f'_\varepsilon < \inf_{x\in\mathbb{R}^N\setminus K_\varepsilon} \int_{\mathbb{R}^N\setminus K_\varepsilon} J_\varepsilon(x-y)\mathrm{d}x, \text{ for all } \varepsilon \in (0, \varepsilon_0). \tag{3.4}$$

Proposition 3.1 will play an important role in the sequel. Inter alia, it guarantees that the solutions of some nonlocal equations defined in the sequel are *continuous*.

Proof. By assumption (1.8), up to rescale J, we may assume without loss of generality that

$$\operatorname{supp}(J) = B_{1/2}.$$

Let $0 < \varepsilon < \varepsilon_1 := \min\{1, R_0/2\}$ and $x \in \mathbb{R}^N \setminus K_\varepsilon$. Define

$$\widetilde{F}_\varepsilon := \{z \in \mathbb{R}^N;\ R_0 < z_1 < R_1,\ |z'| < \varepsilon^\gamma\} \text{ and } \Lambda_\varepsilon(x) := B_{\varepsilon/2} \cap (\widetilde{F}_\varepsilon - x).$$

We will estimate from below the integral in the right-hand side of (3.4). For it, we will treat separately the case where $x \in \widetilde{F}_\varepsilon$ and the case where $x \in \mathbb{R}^N \setminus (K_\varepsilon \cup \widetilde{F}_\varepsilon)$.

Step 1: Lower bound in $\widetilde{F}_\varepsilon$

Let $x \in \widetilde{F}_\varepsilon$. Since J_ε is radially non-increasing, non-negative and supported in $B_{\varepsilon/2}$, there is some $\widetilde{J}_\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ such that $J_\varepsilon(z) = \widetilde{J}_\varepsilon(|z|)$ and $\operatorname{supp}(\widetilde{J}_\varepsilon) = [0, \varepsilon/2)$. Thus, passing to polar coordinates, the mass carried by $J_\varepsilon(x - \cdot)$ in $\widetilde{F}_\varepsilon$ can be rewritten as

$$\int_{\widetilde{F}_\varepsilon} J_\varepsilon(x-y)\mathrm{d}y = \int_{\Lambda_\varepsilon(x)} J_\varepsilon(y)\mathrm{d}y = \int_{\mathbb{S}^{N-1}} \left(\int_0^{\varepsilon/2} \mathbb{1}_{\Lambda_\varepsilon(x)}(\sigma t) \widetilde{J}_\varepsilon(t) t^{N-1} \mathrm{d}t \right) \mathrm{d}\mathscr{H}^{N-1}(\sigma).$$

Notice that $\Lambda_\varepsilon(x)$ is a convex set and that $0 \in \Lambda_\varepsilon(x)$. In particular, both $t \mapsto \mathbb{1}_{\Lambda_\varepsilon(x)}(\sigma t)$ and $t \mapsto \widetilde{J}_\varepsilon(t)$ are non-increasing functions. Hence, using Chebyshev's integral inequality (see e.g. [14, Theorem 2.5.10, p. 40]), we have

$$\int_{\widetilde{F}_\varepsilon} J_\varepsilon(x-y)\mathrm{d}y \geqslant \frac{N}{(\varepsilon/2)^N} \int_{\mathbb{S}^{N-1}} \left(\int_0^{\varepsilon/2} \mathbb{1}_{\Lambda_\varepsilon(x)}(\sigma t) t^{N-1} \mathrm{d}t \int_0^{\varepsilon/2} \widetilde{J}_\varepsilon(t) t^{N-1} \mathrm{d}t \right) \mathrm{d}\mathscr{H}^{N-1}(\sigma).$$

Since J_ε has unit mass and $\operatorname{supp}(J_\varepsilon) = B_{\varepsilon/2}$, one has

$$\int_0^{\varepsilon/2} \widetilde{J}_\varepsilon(t) t^{N-1} \mathrm{d}t = \sigma_N^{-1} = \big(N|B_1|\big)^{-1},$$

where $\sigma_N = \mathscr{H}^{N-1}(\mathbb{S}^{N-1})$. Ergo,

$$\begin{aligned}\int_{\widetilde{F}_\varepsilon} J_\varepsilon(x-y)\mathrm{d}y &\geqslant \frac{1}{|B_{\varepsilon/2}|} \int_{\mathbb{S}^{N-1}} \left(\int_0^{\varepsilon/2} \mathbb{1}_{\Lambda_\varepsilon(x)}(\sigma t) t^{N-1} \mathrm{d}t \right) \mathrm{d}\mathscr{H}^{N-1}(\sigma) \\ &= \frac{1}{|B_{\varepsilon/2}|} \int_{B_{\varepsilon/2}} \mathbb{1}_{\Lambda_\varepsilon(x)}(y)\mathrm{d}y.\end{aligned}$$

Since $\widetilde{F}_\varepsilon \subset \mathbb{R}^N \setminus K_\varepsilon$ and $\Lambda_\varepsilon(x) = B_{\varepsilon/2} \cap (\widetilde{F}_\varepsilon - x)$, we get

$$\int\limits_{\mathbb{R}^N \setminus K_\varepsilon} J_\varepsilon(x-y)\mathrm{d}y \geqslant \frac{|B_{\varepsilon/2}(x) \cap \widetilde{F}_\varepsilon|}{|B_{\varepsilon/2}|}, \text{ for any } x \in \widetilde{F}_\varepsilon. \tag{3.5}$$

Let us now estimate the quantity $|B_{\varepsilon/2}(x) \cap \widetilde{F}_\varepsilon|$. Observe that for ε small enough, say when $0 < \varepsilon < \varepsilon_2 := 4^{-(N-1)}$, one has $\varepsilon/2 > 2\varepsilon^\gamma$. In particular, this implies that $B_{\varepsilon/2}(x) \cap \widetilde{F}_\varepsilon$ always contains an hyper-rectangle of the form $\mathscr{T}\big((0, \varepsilon/4) \times (0, 2\varepsilon^\gamma) \times \cdots \times (0, 2\varepsilon^\gamma)\big)$ for some translation $\mathscr{T}$ of $\mathbb{R}^N$, so that

$$|B_{\varepsilon/2}(x) \cap \widetilde{F}_\varepsilon| \geqslant (\varepsilon/4) \times (2\varepsilon^\gamma)^{N-1} = 2^{N-3}\varepsilon^{N+1}.$$

Therefore, recalling (3.5), we obtain that, for all $0 < \varepsilon < \varepsilon_2$ and all $x \in \widetilde{F}_\varepsilon$, it holds

$$\int\limits_{\mathbb{R}^N \setminus K_\varepsilon} J_\varepsilon(x-y)\mathrm{d}y \geqslant C_1\varepsilon, \tag{3.6}$$

for some $C_1 > 0$ depending on N only.

Step 2: Lower bound in $\mathbb{R}^N \setminus (K_\varepsilon \cup \widetilde{F}_\varepsilon)$

Let us now consider the case where $x \in \mathbb{R}^N \setminus (K_\varepsilon \cup \widetilde{F}_\varepsilon)$. For it, we first note that, since $0 < \varepsilon^\gamma < \varepsilon < R_0/2$ (remember $0 < \varepsilon < \varepsilon_1$), the point $x_0 := (R_0, \varepsilon^\gamma, 0, \cdots, 0) \in \partial\widetilde{F}_\varepsilon$ satisfies

$$|x_0|^2 = R_0^2 + \varepsilon^{2\gamma} < R_0^2 + R_0\varepsilon^\gamma/2 < (R_0 + \varepsilon^\gamma/4)^2,$$

which implies that $\widetilde{F}_\varepsilon \cap B_{R_0+\varepsilon^\gamma/4} \neq \varnothing$. On the other hand, it is clear from the definition of $\widetilde{F}_\varepsilon$ that $\widetilde{F}_\varepsilon \setminus B_{R_1-\varepsilon^\gamma/4} \neq \varnothing$. A consequence of this is that

$$\widetilde{F}_\varepsilon \cap \mathcal{A}(R_0 + \varepsilon^\gamma/4, R_1 - \varepsilon^\gamma/4) = \mathcal{A}(R_0 + \varepsilon^\gamma/4, R_1 - \varepsilon^\gamma/4) \cap \big\{z \in \mathbb{R}^N; z_1 > 0, |z'| < \varepsilon^\gamma\big\}.$$

Whence, recalling properties (i) and (iv) in the definition of K_ε, we deduce that

$$\mathcal{A}(R_0 + \varepsilon/4, R_1 - \varepsilon/4) \subset \mathcal{A}(R_0 + \varepsilon^\gamma/4, R_1 - \varepsilon^\gamma/4) \subset K_\varepsilon \cup \widetilde{F}_\varepsilon,$$

where, in the left-hand side, we have used the fact that $\varepsilon^\gamma < \varepsilon$. In turn, this implies that

$$x \in \mathbb{R}^N \setminus \mathcal{A}(R_0 + \varepsilon/4, R_1 - \varepsilon/4).$$

In particular, since $0 < \varepsilon < R_0/2 < R_0$, we may find a point $z \in \mathbb{R}^N$ such that

$$|x - z| = \frac{3\varepsilon}{8} \text{ and } B_{\varepsilon/8}(z) \subset B_{\varepsilon/2}(x) \setminus \overline{\mathcal{A}} \subset \mathbb{R}^N \setminus K_\varepsilon. \tag{3.7}$$

Indeed, when $x \in \mathbb{R}^N \setminus B_{R_1-\varepsilon/4}$, this follows from the convexity of B_{R_1}; and, when $x \in B_{R_0+\varepsilon/4}$, the constraint $0 < \varepsilon < R_0/2$ allows one to choose z on the diagonal of $B_{R_0+\varepsilon/4}$ containing x. On account of this, we may write

$$\int\limits_{\mathbb{R}^N \setminus K_\varepsilon} J_\varepsilon(x-y)\mathrm{d}y \geqslant \int\limits_{B_{\varepsilon/8}(z)} J_\varepsilon(x-y)\mathrm{d}y = \int\limits_{B_{\varepsilon/8}(z-x)} J_\varepsilon(y)\mathrm{d}y = \int\limits_{B_{1/8}\left(\frac{z-x}{\varepsilon}\right)} J(y)\mathrm{d}y.$$

Now, by (3.7), we have $(z - x)/\varepsilon \in \partial B_{3/8}$. Thus,

$$\int\limits_{\mathbb{R}^N \setminus K_\varepsilon} J_\varepsilon(x-y)\mathrm{d}y \geqslant \int\limits_{B_{1/8}(e_x)} J(y)\mathrm{d}y =: M_J(e_x) \text{ for some } e_x \in \partial B_{3/8}.$$

Notice that $B_{1/8}(e_x) \subset B_{1/2} = \mathrm{supp}(J)$ (because $e_x \in \partial B_{3/8}$) which implies $M_J(e_x) > 0$. Moreover, since J is radially symmetric, the quantity $M_J(e_x)$ does not depend on the choice of $e_x \in \partial B_{3/8}$, namely

$$M_J(e_x) = M_J(e) \equiv M_J > 0, \text{ for every } e \in \partial B_{3/8}, \tag{3.8}$$

and some constant M_J depending on J only.

Therefore, for any $0<\varepsilon<\varepsilon_1$ and $x\in\mathbb{R}^N\setminus(K_\varepsilon\cup\widetilde{F}_\varepsilon)$, it holds

$$\int_{\mathbb{R}^N\setminus K_\varepsilon} J_\varepsilon(x-y)\,\mathrm{d}y \geqslant M_J>0. \tag{3.9}$$

Step 3: Conclusion

Since $\mathbb{R}^N\setminus K_\varepsilon=\widetilde{F}_\varepsilon\cup(\mathbb{R}^N\setminus(\widetilde{F}_\varepsilon\cup K_\varepsilon))$, by (3.6) and (3.9), we obtain

$$\inf_{x\in\mathbb{R}^N\setminus K_\varepsilon}\int_{\mathbb{R}^N\setminus K_\varepsilon} J_\varepsilon(x-y)\,\mathrm{d}y\geqslant \min\{M_J,C_1\}\,\varepsilon,$$

for any $0<\varepsilon<\varepsilon_3:=\min\{\varepsilon_1,\varepsilon_2\}$. Whence, letting

$$\varepsilon_0:=\min\left\{\varepsilon_3,\frac{\min\{M_J,C_1\}}{\max_{[0,1]}f'}\right\},$$

and recalling that $f_\varepsilon(s)=\varepsilon^2 f(s)$, we obtain

$$\max_{[0,1]} f'_\varepsilon<\inf_{x\in\mathbb{R}^N\setminus K_\varepsilon}\int_{\mathbb{R}^N\setminus K_\varepsilon} J_\varepsilon(x-y)\,\mathrm{d}x \quad\text{for any } \varepsilon\in(0,\varepsilon_0),$$

which is the desired inequality. □

Remark 3.2. Since J_ε is radially non-increasing and satisfies (1.6), there is some non-increasing $\widetilde{J}_\varepsilon\in L^1_{\mathrm{loc}}(0,\infty)$ satisfying $J_\varepsilon(z)=\widetilde{J}_\varepsilon(|z|)$. In particular, since $d_{\mathrm{g}}(x,y)\geqslant|x-y|$, it holds that

$$\sup_{x\in\mathbb{R}^N\setminus K_\varepsilon}\int_{\mathbb{R}^N\setminus K_\varepsilon}\widetilde{J}_\varepsilon(d_{\mathrm{g}}(x,y))\,\mathrm{d}y\leqslant\sup_{x\in\mathbb{R}^N\setminus K_\varepsilon}\int_{\mathbb{R}^N\setminus K_\varepsilon} J_\varepsilon(x-y)\,\mathrm{d}y=1,$$

thus implying that $\widetilde{J}_\varepsilon$ satisfies (1.12). Moreover, Proposition 3.1 still holds when $J_\varepsilon(x-y)$ is replaced by $\widetilde{J}_\varepsilon(d_{\mathrm{g}}(x,y))$, i.e. we still have

$$\max_{[0,1]} f'_\varepsilon<\inf_{x\in\mathbb{R}^N\setminus K_\varepsilon}\int_{\mathbb{R}^N\setminus K_\varepsilon}\widetilde{J}_\varepsilon(d_{\mathrm{g}}(x,y))\,\mathrm{d}y. \tag{3.10}$$

Indeed, since $\widetilde{F}_\varepsilon\subset\mathbb{R}^N\setminus K_\varepsilon$ is convex, we have $\widetilde{J}_\varepsilon(d_{\mathrm{g}}(x,y))=J_\varepsilon(x-y)$ for all $x,y\in\widetilde{F}_\varepsilon$, so that

$$\int_{\mathbb{R}^N\setminus K_\varepsilon}\widetilde{J}_\varepsilon(d_{\mathrm{g}}(x,y))\,\mathrm{d}y\geqslant\int_{\widetilde{F}_\varepsilon}\widetilde{J}_\varepsilon(d_{\mathrm{g}}(x,y))\,\mathrm{d}y=\int_{\widetilde{F}_\varepsilon}J_\varepsilon(x-y)\,\mathrm{d}y, \tag{3.11}$$

for any $x\in\widetilde{F}_\varepsilon$. Now, if $x\in\mathbb{R}^N\setminus(K_\varepsilon\cup\widetilde{F}_\varepsilon)$, the situation is similar as in Proposition 3.1 with some minor additional subtleties (because we now need to "secure" a region in $\mathbb{R}^N\setminus K_\varepsilon$ that is starshaped with respect to x due to the presence of the geodesic distance). The only delicate part is when x lies in the "corners", i.e. when $x\in F_\varepsilon\setminus\widetilde{F}_\varepsilon$. In this case, given the size of the channel pierced in the obstacle, we may always find a hypercube of the form $\mathscr{R}=\mathscr{T}([0,\varepsilon^\gamma/4]\times\cdots\times[0,\varepsilon^\gamma/4])$ for some translation $\mathscr{T}$ of $\mathbb{R}^N$ such that $\mathscr{R}\subset\mathbb{R}^N\setminus K_\varepsilon$ and x is one of the vertices of $\mathscr{R}$, i.e. $x\in\mathscr{T}(\partial[0,\varepsilon^\gamma/4]\times\cdots\times\partial[0,\varepsilon^\gamma/4])$. Since $\mathscr{R}$ has sidelength $\varepsilon^\gamma/4$, we may then find a cone of revolution $\mathscr{C}_\varepsilon(x)$ with vertex x and cross section a triangle with slant height ε, tangent to one of the faces of $\mathscr{R}$ and with an opening angle ϑ independent of ε and γ. In fact, we can take ϑ to be

$$\vartheta=\arctan\left(\frac{\varepsilon^\gamma/4}{\varepsilon^\gamma/4}\right)=\frac{\pi}{4}.$$

If x lies on the outside of the annulus $\mathcal{A}$, we always have $\mathscr{C}_\varepsilon(x)\subset\mathbb{R}^N\setminus K_\varepsilon$ (because B_{R_1} is convex). If x lies on the inside of the annulus $\mathcal{A}$, up to take ε small, say $0<\varepsilon<\varepsilon_{R_0}$ for some $\varepsilon_{R_0}>0$ depending on R_0, the curvature of B_{R_0}

will be negligible so to ensure that $\mathscr{C}_\varepsilon(x) \subset \mathbb{R}^N \setminus K_\varepsilon$. Consequently, for any $0 < \varepsilon < \varepsilon_{R_0}$, we may find a point $\bar{z} \in \mathbb{R}^N$ such that

$$|x - \bar{z}| = \frac{3\varepsilon}{8} \text{ and } [y, x] \subset \mathbb{R}^N \setminus K_\varepsilon \text{ for any } y \in B_{\varepsilon/100}(\bar{z}) \text{ and } 0 < \varepsilon < \varepsilon_{R_0},$$

which enforces in particular that $B_{\varepsilon/100}(\bar{z}) \subset \mathrm{supp}(\widetilde{J}_\varepsilon(d_{\mathrm{g}}(x,\cdot))) \subset B_{\varepsilon/2}(x) \setminus K_\varepsilon$ and that

$$\int\limits_{\mathbb{R}^N \setminus K_\varepsilon} \widetilde{J}_\varepsilon(d_{\mathrm{g}}(x,y))\mathrm{d}y \geqslant \int\limits_{B_{\varepsilon/100}(\bar{z})} \widetilde{J}_\varepsilon(d_{\mathrm{g}}(x,y))\mathrm{d}y = \int\limits_{B_{\varepsilon/100}(\bar{z})} J_\varepsilon(x-y)\mathrm{d}y. \tag{3.12}$$

With the same arguments as in the proof of Proposition 3.1, it follows that the right-hand side of (3.12) is equal to a positive constant $\widetilde{M}_J$ independent of z and ε. This and (3.11) allow us to conclude exactly as in the proof of Proposition 3.1, thus enforcing (3.10).

4. Construction of a global super-solution

In this section we construct a global super-solution to (1.9). Precisely, given a pair (J, f) satisfying (1.6), (1.7) and (1.8) and given the family of obstacles $(K_\varepsilon)_{0<\varepsilon<1}$ associated to (J, f) (as defined in Section 3), we construct a global super-solution $\bar{u}_\varepsilon$ to

$$\int\limits_{\mathbb{R}^N \setminus K_\varepsilon} J_\varepsilon(x-y)(\bar{u}_\varepsilon(y) - \bar{u}_\varepsilon(x))\mathrm{d}y + f_\varepsilon(\bar{u}_\varepsilon(x)) \leqslant 0 \quad \text{for } x \in \mathbb{R}^N \setminus K_\varepsilon, \tag{4.1}$$

that further satisfies

$$\bar{u}_\varepsilon \equiv 1 \quad \text{for } x \in \mathbb{R}^N \setminus B_R, \tag{4.2}$$

for some large $R > 0$, where f_ε and J_ε are as in (3.3). More precisely, we prove the following

Lemma 4.1. *Let $N \geqslant 2$ and let (J, f) be a pair satisfying (1.6), (1.7) and (1.8). Let $(K_\varepsilon)_{0<\varepsilon<1}$ be the family of obstacles associated to the pair (J, f) (as defined in Section 3). Let f_ε and J_ε be as in (3.3). Then, there exists $R^* > 0$ and $\varepsilon^* > 0$ such that, for all $0 < \varepsilon < \varepsilon^*$ and all $R \geqslant R^*$, there is a continuous positive nonconstant function $\bar{u}_\varepsilon$ satisfying (4.1) and (4.2).*

The proof of Lemma 4.1 follows essentially two steps. In the first step, we construct a positive solution to a suitable auxiliary problem defined in $B_R \setminus K_\varepsilon$ for some large R. Then, in a second step, we regularise this solution to obtain a super-solution that satisfies both (4.1) and (4.2). To simplify the presentation each step of the proof corresponds to a subsection.

4.1. An auxiliary problem in $B_R \setminus K_\varepsilon$

Let us first construct an adequate auxiliary problem. To do so, we define a new nonlinearity, $\widetilde{f}$, satisfying

$$\widetilde{f}(s) := \begin{cases} -\kappa s & \text{for } s \leqslant \frac{3\theta}{4}, \\ f_0(s) & \text{for } \frac{3\theta}{4} < s < \theta, \\ f(s) & \text{for } \theta \leqslant s \leqslant 1, \\ f'(1)(s-1) & \text{for } s > 1, \end{cases} \tag{4.3}$$

where $\theta \in (0, 1)$ is as in (1.7), $\kappa > 0$ is a small number and f_0 is a smooth function such that $\widetilde{f} \in C^1(\mathbb{R})$. From (1.7), we can choose $\kappa > 0$ and f_0 such that

$$f \leqslant \widetilde{f} \text{ in } [0,1], \ \max_{[0,1]} \widetilde{f}(s) = \max_{[0,1]} f \text{ and } \sup_{\mathbb{R}} \widetilde{f}' \leqslant \sup_{[0,1]} f'. \tag{4.4}$$

Now, for $R > R_1 + 2$, we let $L_{R,\varepsilon}$ be the operator given by

$$L_{R,\varepsilon}w(x) := \int_{B_R\setminus K_\varepsilon} J_\varepsilon(x-y)(w(y)-w(x))\mathrm{d}y, \tag{4.5}$$

and we consider the following problem

$$L_{R,\varepsilon}u_{\varepsilon,R}(x) + c_\varepsilon(x)(1-u_{\varepsilon,R}(x)) + \widetilde{f}_\varepsilon(u_{\varepsilon,R}(x)) = 0 \text{ for all } x \in \overline{B_R \setminus K_\varepsilon}, \tag{4.6}$$

where

$$\widetilde{f}_\varepsilon(s) = \varepsilon^2 \widetilde{f}(s) \text{ for } s \in \mathbb{R} \text{ and } c_\varepsilon(x) := \int_{\mathbb{R}^N\setminus B_R} J_\varepsilon(x-y)\mathrm{d}y \text{ for } x \in B_R \setminus K_\varepsilon. \tag{4.7}$$

Our goal in this step is to show that, for each $\varepsilon \in (0,1)$ small enough, there exists a continuous function $u_{\varepsilon,R} : \overline{B_R \setminus K_\varepsilon} \to (0,1)$ satisfying (4.6).

Remark 4.2. Observe that, by construction (remember (4.4)), the function

$$\widehat{u}_{\varepsilon,R} := \begin{cases} u_{\varepsilon,R} & \text{in } \overline{B_R \setminus K}, \\ 1 & \text{in } \mathbb{R}^N \setminus \overline{B}_R, \end{cases}$$

provides a *discontinuous* super-solution to (4.1) satisfying (4.2). We are thus on the right track to construct the required super-solution.

For it, we observe that, by setting $v_{\varepsilon,R} := 1 - u_{\varepsilon,R}$, (4.6) rewrites

$$L_{R,\varepsilon}v_{\varepsilon,R}(x) - c_\varepsilon(x)v_{\varepsilon,R}(x) + g_\varepsilon(v_{\varepsilon,R}(x)) = 0 \text{ for } x \in \overline{B_R \setminus K_\varepsilon}, \tag{4.8}$$

with $g_\varepsilon(s) := -\varepsilon^2\widetilde{f}(1-s)$. Therefore, to construct $u_{\varepsilon,R}$ it suffices to construct a positive solution $v_{\varepsilon,R} : \overline{B_R \setminus K} \to (0,1)$ to (4.8). As in [4], this will be done using a variational argument. To do so, we define

$$g(s) := -\widetilde{f}(1-s), \quad G(t) := \int_0^t g(s)\mathrm{d}s \quad \text{and} \quad G_\varepsilon(t) := \varepsilon^2 G(t),$$

for all $s, t \in \mathbb{R}$ and $\varepsilon \in (0,1)$. Now, for any $\varepsilon \in (0,1)$ and any domain $\Omega \subset B_R \setminus K_\varepsilon$, we consider the following energy functional

$$\mathcal{E}_{\varepsilon,\Omega}(w) := \frac{1}{4}\int_\Omega\int_\Omega J_\varepsilon(x-y)(w(x)-w(y))^2\mathrm{d}x\mathrm{d}y + \frac{1}{2}\int_\Omega c_\varepsilon(x)w^2(x)\mathrm{d}x - \int_\Omega G_\varepsilon(w(x))\mathrm{d}x, \tag{4.9}$$

for $w \in L^2(\Omega)$. Observe that for any $\varepsilon > 0$ and any domain $\Omega \subset B_R \setminus K_\varepsilon$, the null function $w \equiv 0$ is a *global* minimiser of $\mathcal{E}_{\varepsilon,\Omega}$. Therefore, we have to construct a *local* minimiser. However, unlike its local analogue, the energy functional $\mathcal{E}_{\varepsilon,\Omega}$ does *not* posses strong compactness properties, rendering this type of approach very delicate to implement.

With this in mind, we will show that, for the family K_ε constructed in Section 3 and ε small enough, the above energy has indeed a nontrivial local minimiser when $\Omega = B_R \setminus K_\varepsilon$.

Following the scheme of construction introduced in [4], we first show that the function $w_0 := \mathbb{1}_{B_{R_0}}$ is a strict minimiser of the functional $\mathcal{E}_{\varepsilon,B_{R_0}}$ when $\varepsilon \in (0,1)$ is small enough.

More precisely,

Proposition 4.3. *Let $N \geqslant 2$, $0 < R_0 < R_0^*(J,f)$ (where $R_0^*(J,f)$ is given by (3.2)(b)) and let $w_0 := \mathbb{1}_{B_{R_0}}$. Then, there exists $\kappa_0 > 0$, $0 < \varepsilon_1(J,N,R_0) < 1$ and $0 < \delta_0(R_0) < |B_{R_0}|^{1/2}$ such that, for each $0 < \varepsilon < \varepsilon_1$, it holds that*

$$\mathcal{E}_{\varepsilon,B_{R_0}}(w) - \mathcal{E}_{\varepsilon,B_{R_0}}(w_0) \geqslant \kappa_0\varepsilon^2\|w - w_0\|^2_{L^2(B_{R_0})},$$

for all $w \in L^2(B_{R_0})$ such that $\|w - w_0\|_{L^2(B_{R_0})} \leqslant \delta_0$.

Proof. Let us begin with some preliminary observations. First, we notice that since g is linear around 1 (because $\widetilde{f}$ is linear around 0), the function G_ε is smooth in a neighbourhood of 1. In particular, there exists $\tau_0(\widetilde{f}) > 0$ such that

$$G_\varepsilon(t) = G_\varepsilon(1) + G'_\varepsilon(1)(t-1) + \frac{1}{2}G''_\varepsilon(1)(t-1)^2 \quad \text{for any } |t-1| < \tau_0.$$

But since $G'_\varepsilon(1) = \varepsilon^2 G'(1) = \varepsilon^2 g(1) = 0$ and $G''_\varepsilon(1) = \varepsilon^2 G''(1) = \varepsilon^2 g'(1) = -\varepsilon^2 \widetilde{f}'(0) = -\varepsilon^2\kappa$, this expansion can be rewritten as

$$G_\varepsilon(t) = \varepsilon^2 G(1) - \frac{\kappa\varepsilon^2}{2}(t-1)^2 \quad \text{for any } |t-1| < \tau_0. \tag{4.10}$$

Using the number τ_0, we define

$$\delta_0 := \min\left\{\frac{\theta}{4}, \frac{C_0}{\kappa}, \frac{\tau_0}{2}\right\} |B_{R_0}|^{1/2}, \tag{4.11}$$

where θ, C_0 and κ are as in (1.7), (3.1)(a) and (4.3); and we let $w \in L^2(B_{R_0})$ be such that

$$\|w - w_0\|_{L^2(B_{R_0})} \leqslant \delta_0. \tag{4.12}$$

Second, denoting by $\langle w_0\rangle := \text{span}_{L^2(B_{R_0})}(w_0)$ the vector space spanned by w_0 and letting $\langle w_0\rangle^\perp$ be its orthogonal with respect to the standard scalar product of $L^2(B_{R_0})$, we can write the space $L^2(B_R)$ as the direct sum $L^2(B_{R_0}) = \langle w_0\rangle \oplus \langle w_0\rangle^\perp$. This means that we may always find a constant $\alpha \in \mathbb{R}$ and a function $h \in \langle w_0\rangle^\perp$ such that w decomposes as $w = \alpha w_0 + h$. In particular, the orthogonality of h with respect to w_0 implies that

$$\int_{B_{R_0}} h(x)\mathrm{d}x = 0 \ \text{ and } \ \|w - w_0\|^2_{L^2(B_{R_0})} = (1-\alpha)^2\|w_0\|^2_{L^2(B_{R_0})} + \|h\|^2_{L^2(B_{R_0})}. \tag{4.13}$$

In view of this, assumption (4.12) gives

$$-\frac{\delta_0}{|B_{R_0}|^{1/2}} \leqslant (1-\alpha) \leqslant \frac{\delta_0}{|B_{R_0}|^{1/2}} \ \text{ and } \ \|h\|_{L^2(B_{R_0})} \leqslant \delta_0. \tag{4.14}$$

This fact will be abundantly used in the sequel.

This being said, we are now in position to prove Proposition 4.3. For it, we observe that, since $w_0 \equiv 1$ in B_{R_0}, we have that

$$\mathcal{E}_{\varepsilon, B_{R_0}}(w_0) = -\int_{B_{R_0}} G_\varepsilon(w_0(x))\mathrm{d}x = -G_\varepsilon(1)|B_{R_0}| = -\varepsilon^2 G(1)|B_{R_0}|.$$

Furthermore, thanks to $R > R_0 + 2$ and $\text{supp}(J_\varepsilon) \subset B_{\frac{\varepsilon}{2}}$, we have that $c_\varepsilon(x) \equiv 0$ in B_{R_0}, for any $0 < \varepsilon < 1$. Consequently, $\mathcal{E}_{\varepsilon, B_{R_0}}(w)$ rewrites

$$\mathcal{E}_{\varepsilon, B_{R_0}}(w) = \underbrace{\frac{1}{4}\int_{B_{R_0}}\int_{B_{R_0}} J_\varepsilon(x-y)(w(x)-w(y))^2\mathrm{d}x\mathrm{d}y}_{II} - \underbrace{\int_{B_{R_0}} G_\varepsilon(w(x))\mathrm{d}x}_{I}.$$

Let us first estimate *II*. In view of the Bourgain-Brezis-Mironescu representation of $H^1(B_{R_0})$ (see [6]), one can interpret *II* as a nonlocal approximation of $\|\nabla w\|^2_{L^2(B_{R_0})}$. The crux of our strategy is that, as shown by Ponce [18, Theorem 1.1], this nonlocal approximation enjoys a Poincaré-type inequality. Let us now proceed. Let $(\rho_\varepsilon)_{0<\varepsilon<1}$ be the family of radially symmetric mollifiers defined by

$$\rho_\varepsilon(z) := M_2(J)^{-1} J_\varepsilon(z)|z|^2\varepsilon^{-2} \ \text{ for } \varepsilon \in (0,1),$$

where $M_2(J)$ is given by (3.1)(b). Notice that, by construction, it satisfies

$$\rho_\varepsilon \geqslant 0 \text{ a.e. in } \mathbb{R}^N, \quad \int_{\mathbb{R}^N} \rho_\varepsilon(z)\mathrm{d}z = 1 \text{ and } \lim_{\varepsilon\to 0^+} \int_{|z|\geqslant\tau} \rho_\varepsilon(z)\mathrm{d}z = 0,$$

for each $0<\varepsilon<1$ and each $\tau>0$. Moreover, II can be rewritten as

$$II = \varepsilon^2 \frac{M_2(J)}{4} \int_{B_{R_0}} \int_{B_{R_0}} \rho_\varepsilon(x-y) \frac{|w(x)-w(y)|^2}{|x-y|^2} \mathrm{d}x\mathrm{d}y.$$

Now, by [18, Theorem 1.1], we know that there exists some $\varepsilon_1 = \varepsilon_1(J, N, R_0) > 0$ such that the following Poincaré-type inequality

$$\left\| w - \fint_{B_{R_0}} w \right\|^2_{L^2(B_{R_0})} \leqslant \frac{2A_0}{K_{2,N}} \int_{B_{R_0}} \int_{B_{R_0}} \rho_\varepsilon(x-y) \frac{|w(x)-w(y)|^2}{|x-y|^2} \mathrm{d}x\mathrm{d}y \left(= \frac{8A_0\varepsilon^{-2}}{K_{2,N} M_2(J)} \times II \right),$$

holds for all $\varepsilon \in (0, \varepsilon_1)$ and all $w \in L^2(B_{R_0})$. Here,

$$K_{2,N} := \int_{\mathbb{S}^{N-1}} (\sigma\cdot e_1)^2 \,\mathrm{d}\mathcal{H}^{N-1}(\sigma) = \frac{1}{N},$$

and $A_0 > 0$ is the smallest constant such that the standard Poincaré-Wirtinger inequality holds. That is, A_0 is the smallest positive constant such that

$$\left\| w - \fint_{B_{R_0}} w \right\|^2_{L^2(B_{R_0})} \leqslant A_0 \|\nabla w\|^2_{L^2(B_{R_0})},$$

holds for any $w \in H^1(B_{R_0})$. In our case, A_0 satisfies the upper bound:

$$A_0 \leqslant \frac{\mathrm{diam}(B_{R_0})^2}{\pi^2} = \frac{4R_0^2}{\pi^2},$$

see [2, Theorem 3.2] (see also [16]). In particular, this gives

$$\varepsilon^2 \frac{\pi^2 M_2(J)}{32 N R_0^2} \left\| w - \fint_{B_{R_0}} w \right\|^2_{L^2(B_{R_0})} \leqslant II.$$

Now, since $w = \alpha w_0 + h$, since $w_0 \equiv 1$ on B_{R_0} and since h is integral free (by (4.13)) we have

$$II \geqslant \varepsilon^2 \frac{C_{N,J}}{R_0^2} \|h\|^2_{L^2(B_{R_0})}, \tag{4.15}$$

where $C_{N,J}$ is given by (3.2)(a). We are now left to estimate I. For it, we rewrite I as follows

$$I = \mathcal{E}_{\varepsilon, B_{R_0}}(w_0) - \int_{B_{R_0}} \big[G_\varepsilon(w(x)) - G_\varepsilon(w_0(x))\big]\mathrm{d}x. \tag{4.16}$$

To estimate the last integral, we split it into two parts, I_1 and I_2, where

$$I_1 := -\int_{B_{R_0}} \big[G_\varepsilon(w_0+(\alpha-1)w_0+h) - G_\varepsilon(w_0+(\alpha-1)w_0)\big],$$

$$I_2 := -\int_{B_{R_0}} \big[G_\varepsilon(w_0+(\alpha-1)w_0) - G_\varepsilon(w_0)\big].$$

Let us first estimate I_2. Using (4.11), (4.12) and (4.14) we have in particular that $|1-\alpha| < \tau_0$. This, together with (4.10), gives

$$I_2 = -\int_{B_{R_0}} \left[G_\varepsilon(w_0+(\alpha-1)w_0)-G_\varepsilon(w_0)\right] = \frac{\kappa}{2}\varepsilon^2|B_{R_0}|(\alpha-1)^2.$$

Therefore, recalling (4.16), we get

$$I = \mathcal{E}_{\varepsilon,B_{R_0}}(w_0) + \frac{\kappa}{2}\varepsilon^2|B_{R_0}|(\alpha-1)^2 + I_1. \tag{4.17}$$

Let us now estimate I_1. On account of (4.14), we may write

$$\alpha = 1-\eta \quad \text{for some } |\eta| \leqslant \frac{\delta_0}{|B_{R_0}|^{1/2}}. \tag{4.18}$$

Then, a standard change of variables yields

$$I_1 = -\int_{B_{R_0}} \int_{\alpha}^{\alpha+h(x)} g_\varepsilon(\tau)\mathrm{d}\tau\mathrm{d}x = \varepsilon^2 \int_{B_{R_0}} \int_{1-\eta}^{1-\eta+h(x)} \widetilde{f}(1-\tau)\mathrm{d}\tau\mathrm{d}x = -\varepsilon^2 \int_{B_{R_0}} \int_{0}^{-h(x)} \widetilde{f}(\tau+\eta)\mathrm{d}\tau\mathrm{d}x.$$

Now, we set

$$\Sigma := \left\{x \in B_{R_0};\, -h(x) > \frac{\theta}{2}\right\},$$

and we decompose I_1 as

$$I_1 = -\varepsilon^2 \left(\int_{\Sigma} \int_{0}^{-h(x)} \widetilde{f}(\tau+\eta)\mathrm{d}\tau\mathrm{d}x + \int_{B_{R_0}\setminus\Sigma} \int_{0}^{-h(x)} \widetilde{f}(\tau+\eta)\mathrm{d}\tau\mathrm{d}x \right). \tag{4.19}$$

We will estimate these two integrals separately. In view of (4.11) and (4.18), we have that $|\eta| \leqslant \theta/4$. In turn, this implies that

$$-h(x) + |\eta| \leqslant \frac{3\theta}{4} \quad \text{for any } x \in B_{R_0} \setminus \Sigma.$$

Since, by construction, $\widetilde{f}$ is linear in $(-\infty, 3\theta/4]$, we get

$$\begin{aligned}
\int_{B_{R_0}\setminus\Sigma} \int_{0}^{-h(x)} \widetilde{f}(\tau+\eta)\mathrm{d}\tau\mathrm{d}x &= -\kappa \int_{B_{R_0}\setminus\Sigma} \left(\int_{0}^{\eta-h(x)} \tau\,\mathrm{d}\tau - \int_{0}^{\eta} \tau\,\mathrm{d}\tau \right)\mathrm{d}x \\
&= -\frac{\kappa}{2} \int_{B_{R_0}\setminus\Sigma} h^2(x)\mathrm{d}x + \kappa\eta \int_{B_{R_0}\setminus\Sigma} h(x)\mathrm{d}x \\
&= -\frac{\kappa}{2} \int_{B_{R_0}\setminus\Sigma} h^2(x)\mathrm{d}x - \kappa\eta \int_{\Sigma} h(x)\mathrm{d}x,
\end{aligned}$$

where, in the last equality, we have used the fact that h is integral free, that is:

$$\int_{B_{R_0}\setminus\Sigma} h(x)\mathrm{d}x + \int_{\Sigma} h(x)\mathrm{d}x = 0.$$

Using now the Cauchy-Schwarz inequality, we get

$$\int_{B_{R_0}\setminus\Sigma}\int_0^{-h(x)}\widetilde{f}(\tau+\eta)\mathrm{d}\tau\mathrm{d}x \leqslant -\frac{\kappa}{2}\int_{B_{R_0}\setminus\Sigma} h^2(x)\mathrm{d}x + \kappa|\eta|\sqrt{|\Sigma|}\|h\|_{L^2(\Sigma)}.$$

By the Bienaymé-Chebyshev inequality, we have

$$|\Sigma| \leqslant \left(\frac{2}{\theta}\right)^2 \|h\|^2_{L^2(\Sigma)}, \tag{4.20}$$

and thus

$$\int_{B_{R_0}\setminus\Sigma}\int_0^{-h(x)}\widetilde{f}(\tau+\eta)\mathrm{d}\tau\mathrm{d}x \leqslant -\frac{\kappa}{2}\int_{B_{R_0}\setminus\Sigma} h^2(x)\mathrm{d}x + \frac{2\kappa|\eta|}{\theta}\|h\|^2_{L^2(\Sigma)}. \tag{4.21}$$

Thanks to (4.11) and (4.18), (4.21) reduces to

$$\int_{B_{R_0}\setminus\Sigma}\int_0^{-h(x)}\widetilde{f}(\tau+\eta)\mathrm{d}\tau\mathrm{d}x \leqslant -\frac{\kappa}{2}\int_{B_{R_0}\setminus\Sigma} h^2(x)\mathrm{d}x + \frac{2C_0}{\theta}\|h\|^2_{L^2(\Sigma)}. \tag{4.22}$$

Let us now estimate the first integral on the right-hand side of (4.19). For it, we observe that

$$\tau+\eta \geqslant -|\eta| \geqslant \frac{-\delta_0}{\sqrt{|B_{R_0}|}} \geqslant -\frac{C_0}{\kappa} \quad \text{for any } x\in\Sigma \text{ and any } \tau\in(0,-h(x)).$$

Recalling (4.4), we then obtain

$$\sup_{x\in\Sigma}\sup_{\tau\in(0,-h(x))}\widetilde{f}(\tau+\eta) \leqslant \sup_{s\geqslant -\frac{C_0}{\kappa}}\widetilde{f}(s) = \max_{s\in[0,1]}\widetilde{f}(s) = C_0.$$

This, together with the Cauchy-Schwarz inequality, gives

$$\int_\Sigma\int_0^{-h(x)}\widetilde{f}(\tau+\eta)\mathrm{d}\tau\mathrm{d}x \leqslant C_0\int_\Sigma |h(x)|\mathrm{d}x \leqslant C_0\sqrt{|\Sigma|}\|h\|_{L^2(\Sigma)}.$$

Using the Bienaymé-Chebyshev inequality (4.20), we finally get

$$\int_\Sigma\int_0^{-h(x)}\widetilde{f}(\tau+\eta)\mathrm{d}\tau\mathrm{d}x \leqslant \frac{2C_0}{\theta}\|h\|^2_{L^2(\Sigma)}. \tag{4.23}$$

Collecting (4.15), (4.17), (4.22) and (4.23), we obtain that

$$\begin{aligned}&\mathcal{E}_{\varepsilon,B_{R_0}}(w)-\mathcal{E}_{\varepsilon,B_{R_0}}(w_0)\\ &\qquad\geqslant \varepsilon^2\left(\frac{\kappa}{2}|B_{R_0}|(\alpha-1)^2+\frac{\kappa}{2}\|h\|^2_{L^2(B_{R_0}\setminus\Sigma)}-\frac{4C_0}{\theta}\|h\|^2_{L^2(\Sigma)}+\frac{C_{N,J}}{R_0^2}\|h\|^2_{L^2(B_{R_0})}\right),\end{aligned}$$

for all $0<\varepsilon<\varepsilon_1$ and all $w\in L^2(B_{R_0})$ with $\|w-w_0\|_{L^2(B_{R_0})}\leqslant\delta_0$. Recalling that $0<R_0\leqslant R_0^*(J,f)$ and using (3.2)(b), we have $C_{N,J}/R_0^2\geqslant 5C_0/\theta$. This, together with the above inequality, yields

$$\mathcal{E}_{\varepsilon,B_{R_0}}(w)-\mathcal{E}_{\varepsilon,B_{R_0}}(w_0)\geqslant\varepsilon^2\left(\frac{\kappa}{2}|B_{R_0}|(\alpha-1)^2+\frac{C_0}{\theta}\|h\|^2_{L^2(B_{R_0})}\right).$$

Therefore, letting

$$\kappa_0 := \inf\left\{\frac{\kappa}{2},\frac{C_0}{\theta}\right\},$$

and recalling (4.13), we obtain

$$\mathcal{E}_{\varepsilon,B_{R_0}}(w)-\mathcal{E}_{\varepsilon,B_{R_0}}(w_0)\geqslant \varepsilon^2\kappa_0\|w-w_0\|^2_{L^2(B_{R_0})},$$

for all $0<\varepsilon<\varepsilon_1$ and all $w\in L^2(B_{R_0})$ with $\|w-w_0\|_{L^2(B_{R_0})}\leqslant\delta_0$. □

Remark 4.4. Note that the proof of Proposition 4.3 relies only on elementary L^2-estimates and on a Poincaré-type inequality. Remarkably, this allows to adapt straightforwardly our arguments to the local analogue of $\mathcal{E}_{\varepsilon,B_{R_0}}$.

Using Proposition 4.3, we now prove the following

Proposition 4.5. *Let $N\geqslant 2$, and let $\mathcal{E}_{\varepsilon,R}$ be the energy functional defined by (4.9) with $\Omega=B_R\setminus K_\varepsilon$. Then, there exists $C^*>0$, $0<\delta_0<|B_{R_0}|^{1/2}$ and $0<\varepsilon_{\delta_0}<1$ such that, for any $0<\varepsilon<\varepsilon_{\delta_0}$ and any $w\in L^2(B_R\setminus K_\varepsilon)$ with $\|w-w_0\|_{L^2(B_R\setminus K_\varepsilon)}=\delta_0$, it holds that*

$$\mathcal{E}_{\varepsilon,R}(w)-\mathcal{E}_{\varepsilon,R}(w_0)>C^*\varepsilon^2.$$

Proof. Let us first notice that our assumptions on $\widetilde{f}$ imply that there is some $\kappa_1>0$ such that

$$-G(t)\geqslant\kappa_1 t^2 \text{ for every } t\in\mathbb{R}. \tag{4.24}$$

Let us now compute the energy of w_0. Since $\operatorname{supp}(J_\varepsilon)=B_{\varepsilon/2}$ and $R_1-R_0>\varepsilon$, a straightforward calculation yields

$$\mathcal{E}_{\varepsilon,R}(w_0)=\mathcal{E}_{\varepsilon,B_{R_0}}(w_0)+\frac{1}{2}\int_{F_\varepsilon}\int_{B_{R_0}}J_\varepsilon(x-y)\mathrm{d}x\mathrm{d}y.$$

In addition, elementary computations yield

$$\frac{1}{2}\int_{F_\varepsilon}\int_{B_{R_0}}J_\varepsilon(x-y)\mathrm{d}x\mathrm{d}y=\frac{1}{2}\int_{F_\varepsilon\cap B_{R_0+\frac{\varepsilon}{2}}}\left(\int_{B_{R_0}}J_\varepsilon(x-y)\mathrm{d}x\right)\mathrm{d}y\leqslant\frac{|F_\varepsilon\cap B_{R_0+\frac{\varepsilon}{2}}|}{2}\leqslant C\varepsilon^{N+1},$$

for come constant $C=C(N)>0$. As a consequence, we obtain

$$\mathcal{E}_{\varepsilon,R}(w_0)\leqslant\mathcal{E}_{\varepsilon,B_{R_0}}(w_0)+C\varepsilon^{N+1}. \tag{4.25}$$

Next, developing $\mathcal{E}_{\varepsilon,R}(w)$, we get

$$\begin{aligned}\mathcal{E}_{\varepsilon,R}(w)&=\mathcal{E}_{\varepsilon,B_{R_0}}(w)+\mathcal{E}_{\varepsilon,F_\varepsilon}(w)+\mathcal{E}_{\varepsilon,B_R\setminus B_{R_1}}(w)\\&\quad+\frac{1}{2}\int_{F_\varepsilon}\left(\int_{B_{R_0}}+\int_{B_R\setminus B_{R_1}}\right)J_\varepsilon(x-y)(w(x)-w(y))^2\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Using (4.24) we obtain that $\mathcal{E}_{\varepsilon,\Omega}(w)\geqslant\kappa_1\varepsilon^2\|w\|^2_{L^2(\Omega)}$ for any domain $\Omega\subset B_R\setminus K_\varepsilon$. In particular, since $w_0=0$ in $F_\varepsilon\cup B_R\setminus B_{R_1}$ we have

$$\mathcal{E}_{\varepsilon,R}(w)\geqslant\mathcal{E}_{\varepsilon,B_{R_0}}(w)+\kappa_1\varepsilon^2\|w-w_0\|^2_{L^2(F_\varepsilon\cup B_R\setminus B_{R_1})}. \tag{4.26}$$

Gluing together (4.25) and (4.26), we obtain

$$\mathcal{E}_{\varepsilon,R}(w)-\mathcal{E}_{\varepsilon,R}(w_0)\geqslant\mathcal{E}_{\varepsilon,B_{R_0}}(w)-\mathcal{E}_{\varepsilon,B_{R_0}}(w_0)+\kappa_1\varepsilon^2\|w-w_0\|^2_{L^2(F_\varepsilon\cup B_R\setminus B_{R_1})}-C\varepsilon^{N+1}. \tag{4.27}$$

Now, by Proposition 4.3, there exists $\kappa_0>0$, $0<\delta_0<|B_{R_0}|^{1/2}$ and $\varepsilon_1>0$ such that, for any $0<\varepsilon<\varepsilon_1$ and any $w\in L^2(B_{R_0})$ with $\|w-w_0\|_{L^2(B_{R_0})}\leqslant\delta_0$, we have

$$\mathcal{E}_{\varepsilon, B_{R_0}}(w) - \mathcal{E}_{\varepsilon, B_{R_0}}(w_0) \geqslant \kappa_0 \varepsilon^2 \|w - w_0\|^2_{L^2(B_{R_0})}. \tag{4.28}$$

Letting $\bar{\kappa} := \min\{\kappa_1, \kappa_0\}$ and combining (4.28) and (4.27), we obtain

$$\mathcal{E}_{\varepsilon,R}(w) - \mathcal{E}_{\varepsilon,R}(w_0) \geqslant \varepsilon^2 \bar{\kappa} \|w - w_0\|^2_{L^2(B_R \setminus K_\varepsilon)} - C\varepsilon^{N+1} = \varepsilon^2 \left(\bar{\kappa}\delta_0^2 - C\varepsilon^{N-1}\right), \tag{4.29}$$

for all $0 < \varepsilon < \varepsilon_1$ and all $w \in L^2(B_R \setminus K_\varepsilon)$ with $\|w - w_0\|_{L^2(B_R\setminus K_\varepsilon)} = \delta_0$. The conclusion now follows from (4.29) and the choice

$$C^* = \frac{\bar{\kappa}\delta_0^2}{2} \quad \text{and} \quad \varepsilon_{\delta_0} := \min\left\{\varepsilon_1, \left(\frac{\bar{\kappa}\delta_0^2}{2C}\right)^{\frac{1}{N-1}}\right\}.$$

The proof is thereby complete. □

Remark 4.6. The role of the exponent γ (which arises in the definition of the family of obstacles $(K_\varepsilon)_{0<\varepsilon<1}$ constructed at Section 3) is crucial here. It allows us to obtain the exponent $N+1$ in the right-hand side of (4.25) which can then be absorbed in (4.29) by the term with exponent 2. However, this technicality is needed only when $N = 2$. Indeed, if we take $\gamma = 1$ in the definition of K_ε, we would have an exponent N in (4.25) which can be absorbed by the exponent 2 in (4.29) for any value of $N \geqslant 3$.

We are now in position to construct a positive solution to (4.8).

Proposition 4.7. *Let $N \geqslant 2$ and let (J, f) be a pair satisfying (1.6), (1.7) and (1.8). Let $(K_\varepsilon)_{0<\varepsilon<1}$ be the family of obstacles associated to the pair (J, f) (as defined in Section 3). Let $\tilde{f}$ be the extension of f given by (4.3) and let $\tilde{f}_\varepsilon$ and J_ε be respectively given by (4.7) and (3.3). Then, there exists $\bar{\varepsilon} > 0$ such that, for all $0 < \varepsilon < \bar{\varepsilon}$, there is a function $v_{\varepsilon,R} \in C(\overline{B_R \setminus K_\varepsilon})$ satisfying (4.8) and $0 < v_{\varepsilon,R} < 1$ in $\overline{B_R \setminus K_\varepsilon}$.*

Proof. Let $w_0 := \mathbb{1}_{B_{R_0}}$ and let $0 < \delta_0 < |B_{R_0}|^{1/2}$ and $0 < \varepsilon_{\delta_0} < 1$ be quantities constructed in the proof of Proposition 4.5, namely such that

$$\mathcal{E}_{\varepsilon,R}(w) - \mathcal{E}_{\varepsilon,R}(w_0) > C^* \varepsilon^2,$$

holds for some constant $C^* > 0$ and for any $0 < \varepsilon < \varepsilon_{\delta_0}$ and any $w \in L^2(B_R \setminus K_\varepsilon)$ with $\|w - w_0\|_{L^2(B_R\setminus K_\varepsilon)} = \delta_0$. Let us fix $0 < \varepsilon < \bar{\varepsilon} := \min\{\varepsilon_0, \varepsilon_{\delta_0}\}$ where ε_0 is as in Proposition 3.1. Further, we denote by $\mathbb{B}_{\delta_0}(w_0)$ the following set:

$$\mathbb{B}_{\delta_0}(w_0) := \left\{w \in L^2(B_R \setminus K_\varepsilon);\ \|w - w_0\|_{L^2(B_R\setminus K_\varepsilon)} \leqslant \delta_0\right\},$$

and we define

$$m := \inf_{w \in \mathbb{B}_{\delta_0}(w_0)} \mathcal{E}_{\varepsilon,R}(w).$$

Note that m is well-defined since $\mathcal{E}_{\varepsilon,R}$ is a non-negative continuous functional in $L^2(B_R \setminus K_\varepsilon)$.

Using Lemma 4.5, we will show that there is a *local minimum* $v_{\varepsilon,R}$ of the energy $\mathcal{E}_{\varepsilon,R}$ in the ball $\mathbb{B}_{\delta_0}(w_0)$ which is also a solution to (4.8). However, it must be noted that $\mathcal{E}_{\varepsilon,R}$ lacks of strong compactness properties and passing to the limit along a subsequence is not straightforward. So let us first show that m is achieved in $\mathbb{B}_{\delta_0}(w_0)$.

Take a minimising sequence $(v_j)_{j\in\mathbb{N}} \subset \mathbb{B}_{\delta_0}(w_0)$. Notice that $|w| \in \mathbb{B}_{\delta_0}(w_0)$ for all $w \in \mathbb{B}_{\delta_0}(w_0)$. Moreover, a straightforward computation shows that $\mathcal{E}_{\varepsilon,R}(|v_j|) \leqslant \mathcal{E}_{\varepsilon,R}(v_j)$ for all $j \geqslant 0$. Thus, we may assume that the v_j's are a.e. non-negative for every $j \geqslant 0$. By (4.24), we have $-G_\varepsilon(t) \geqslant \kappa_1 \varepsilon^2 t^2$ for all $t \in \mathbb{R}$. In particular, $\mathcal{E}_{\varepsilon,R}(v_j) \geqslant \kappa_1 \varepsilon^2 \|v_j\|^2_{L^2(B_R\setminus K_\varepsilon)}$ for all $j \geqslant 0$. Therefore $(v_j)_{j\in\mathbb{N}}$ is bounded in $L^2(B_R \setminus K_\varepsilon)$. Whence, up to extract a subsequence, we obtain that v_j converges weakly in $L^2(B_R \setminus K_\varepsilon)$ to some $v_{\varepsilon,R} \in \mathbb{B}_{\delta_0}(w_0)$ (notice that $\mathbb{B}_{\delta_0}(w_0)$ is *closed* in $L^2(B_R \setminus K_\varepsilon)$). Let us check that $v_{\varepsilon,R}$ is indeed a minimiser of $\mathcal{E}_{\varepsilon,R}$ in $\mathbb{B}_{\delta_0}(w_0)$. To this end, we shall introduce the following notations

$$\mathcal{J}_\varepsilon(x) := \int_{\mathbb{R}^N \setminus K_\varepsilon} J_\varepsilon(x - y)\,\mathrm{d}y \quad \text{and} \quad H_\varepsilon(x, s) := \int_0^s \big(\mathcal{J}_\varepsilon(x)\tau - g_\varepsilon(\tau)\big)\,\mathrm{d}\tau.$$

Since $0 < \varepsilon < \varepsilon_0$, by Proposition 3.1, we have

$$\max_{[0,1]} f'_\varepsilon < \inf_{\mathbb{R}^N \setminus K_\varepsilon} \mathcal{J}_\varepsilon.$$

Therefore, from the construction of g_ε (remember (4.4)), we have

$$g'_\varepsilon(s) = \varepsilon^2 \widetilde{f}'(1-s) \leqslant \max_{\mathbb{R}} \widetilde{f}'_\varepsilon \leqslant \max_{[0,1]} f'_\varepsilon < \inf_{\mathbb{R}^N \setminus K_\varepsilon} \mathcal{J}_\varepsilon \quad \text{for any } s \in \mathbb{R}. \tag{4.30}$$

Whence, $H_\varepsilon(x, \cdot)$ is convex for each fixed x. Developing the terms involved in the definition of $\mathcal{E}_{\varepsilon,R}$ we arrive at

$$\mathcal{E}_{\varepsilon,R}(w) = -\frac{1}{2} \int_{B_R \setminus K_\varepsilon} \int_{B_R \setminus K_\varepsilon} J_\varepsilon(x-y) w(x) w(y) \mathrm{d}x\mathrm{d}y + \int_{B_R \setminus K_\varepsilon} H_\varepsilon(x, w(x)) \mathrm{d}x.$$

Using the weak convergence of $(v_j)_{j\in\mathbb{N}}$ towards $v_{\varepsilon,R}$ and the dominated convergence theorem, we can pass to the limit in the double integral and get that

$$\lim_{j\to+\infty} \int_{B_R \setminus K_\varepsilon} \int_{B_R \setminus K_\varepsilon} J_\varepsilon(x-y) v_j(x) v_j(y) \mathrm{d}x\mathrm{d}y = \int_{B_R \setminus K_\varepsilon} \int_{B_R \setminus K_\varepsilon} J_\varepsilon(x-y) v_{\varepsilon,R}(x) v_{\varepsilon,R}(y) \mathrm{d}x\mathrm{d}y.$$

Moreover, since $H_\varepsilon(x, \cdot)$ is convex, we have

$$\int_{B_R \setminus K_\varepsilon} \left[H_\varepsilon(x, v_j(x)) - H_\varepsilon(x, v_{\varepsilon,R}(x))\right] \mathrm{d}x \geqslant \int_{B_R \setminus K_\varepsilon} \partial_s H_\varepsilon(x, v_{\varepsilon,R}(x))(v_j(x) - v_{\varepsilon,R}(x)) \mathrm{d}x.$$

From the definition of H_ε, g_ε and from (4.30) a quick computation shows that $|\partial_s H_\varepsilon(x,s)| = |\mathcal{J}_\varepsilon(x)s - g_\varepsilon(s)| \leqslant A|s|$ for all $s \in \mathbb{R}$ and some constant $A > 0$. Since $v_{\varepsilon,R} \in L^2(B_R \setminus K_\varepsilon)$, it follows that $\partial_s H_\varepsilon(\cdot, v_{\varepsilon,R}(\cdot)) \in L^2(B_R \setminus K_\varepsilon)$. Therefore, using the previous two displayed formulas and the weak convergence of v_j towards $v_{\varepsilon,R}$, we obtain $\lim_{j\to\infty}[\mathcal{E}_{\varepsilon,R}(v_j) - \mathcal{E}_{\varepsilon,R}(v_{\varepsilon,R})] \geqslant 0$. Since, on the other hand, $\lim_{j\to\infty} \mathcal{E}_{\varepsilon,R}(v_j) = m \leqslant \mathcal{E}_{\varepsilon,R}(v_{\varepsilon,R})$, we finally obtain

$$\mathcal{E}_{\varepsilon,R}(v_{\varepsilon,R}) = m = \inf_{w\in\mathbb{B}_{\delta_0}(w_0)} \mathcal{E}_{\varepsilon,R}(w) \leqslant \mathcal{E}_{\varepsilon,R}(w_0).$$

Now, thanks to Proposition 4.5, we deduce that $v_{\varepsilon,R} \in \mathbb{B}_{\delta_0}(w_0)$ is a local minimiser and, as such, $v_{\varepsilon,R}$ solves (4.8) almost everywhere in $\overline{B_R \setminus K_\varepsilon}$.

Let us now check that $v_{\varepsilon,R}$ is a continuous solution to (4.8) in the whole set $\overline{B_R \setminus K_\varepsilon}$. Since $J_\varepsilon \in L^2(\mathbb{R}^N)$ and $v_{\varepsilon,R} \in L^2(B_R \setminus K_\varepsilon)$, it follows from the equation (4.8) satisfied by $v_{\varepsilon,R}$ that $N_\varepsilon(\cdot, v_{\varepsilon,R}(\cdot)) \in L^\infty(B_R \setminus K_\varepsilon)$ where $N_\varepsilon(x,s) := \mathcal{J}_\varepsilon(x)s - g_\varepsilon(s)$. By (4.30), the map $N_\varepsilon(x,\cdot)$ is bijective and thus $v_{\varepsilon,R} \in L^\infty(B_R \setminus K_\varepsilon)$. Using now Lemma 2.2 and (4.30) we may further infer that $v_{\varepsilon,R}$ is continuous in $\overline{B_R \setminus K_\varepsilon}$.

To complete the proof it remains to show that $0 < v_{\varepsilon,R} < 1$. Let us first prove that $v_{\varepsilon,R} < 1$. Suppose, by contradiction, that $\|v_{\varepsilon,R}\|_\infty \geqslant 1$. Then, by continuity of $v_{\varepsilon,R}$, there must be a point $\bar{x} \in \overline{B_R \setminus K_\varepsilon}$ at which $v_{\varepsilon,R}$ attains its maximum, i.e. $v_{\varepsilon,R}(\bar{x}) = \|v_{\varepsilon,R}\|_\infty$. Using now the equation satisfied by $v_{\varepsilon,R}$, we have

$$0 \geqslant \int_{B_R \setminus K_\varepsilon} J_\varepsilon(\bar{x}-y)(v_{\varepsilon,R}(y) - v_{\varepsilon,R}(\bar{x})) \mathrm{d}y = c_\varepsilon(\bar{x}) v_{\varepsilon,R}(\bar{x}) - g_\varepsilon(v_{\varepsilon,R}(\bar{x})) \geqslant 0.$$

Thus, since $\mathrm{supp}(J_\varepsilon) = B_{\varepsilon/2}$, we have $v_{\varepsilon,R}(y) = v_{\varepsilon,R}(\bar{x})$ for any $y \in B_{\varepsilon/2}(\bar{x}) \cap \overline{B_R \setminus K_\varepsilon}$. Note that $B_{\varepsilon/2}(\bar{x}) \cap \overline{B_R \setminus K_\varepsilon}$ is nonempty whence we may iterate this reasoning over again and obtain that $v_{\varepsilon,R} \equiv v_{\varepsilon,R}(\bar{x}) \geqslant 1$. Now choose $x_0 \in \Omega_\varepsilon$ such that $c_\varepsilon(x_0) > 0$. Then, evaluating (4.8) at x_0, one obtains

$$0 = \int_{B_R \setminus K_\varepsilon} J_\varepsilon(x_0-y)(v_{\varepsilon,R}(y) - v_{\varepsilon,R}(x_0)) \mathrm{d}y = c_\varepsilon(x_0) v_{\varepsilon,R}(x_0) - g_\varepsilon(v_{\varepsilon,R}(x_0)) \geqslant c_\varepsilon(x_0) > 0,$$

which is a contradiction.

Therefore $v_{\varepsilon,R} < 1$. Since, by construction, we have that $v_{\varepsilon,R} \geqslant 0$, it remains to check that $v_{\varepsilon,R}$ cannot cancel. Assume, by contradiction, that this is the case, namely that there exists a point $x_0 \in \overline{B_R \setminus K_\varepsilon}$ such that $v_{\varepsilon,R}(x_0) = 0$. Then, by (4.8), we have that

$$\int_{B_R\setminus K_\varepsilon} J_\varepsilon(x_0-y)(v_{\varepsilon,R}(y)-v_{\varepsilon,R}(x_0))\mathrm{d}y=0,$$

and, as above, this implies that $v_{\varepsilon,R}\equiv 0$. However, since $v_{\varepsilon,R}\in\mathbb{B}_{\delta_0}(w_0)$ and $\delta_0<|B_{R_0}|^{1/2}$, we have $\delta_0\geqslant\|v_{\varepsilon,R}-w_0\|_{L^2(B_R\setminus K_\varepsilon)}=\|w_0\|_{L^2(B_R\setminus K_\varepsilon)}=|B_{R_0}|^{1/2}>\delta_0$, which is a contradiction. The proof of Proposition 4.7 is thereby complete. □

From now on (and until the end of Section 4), ε will be fixed and taken so small that $0<\varepsilon<\bar\varepsilon$, where $\bar\varepsilon$ is as defined in Proposition 4.7.

4.2. *An extension procedure*

Let us now complete the proof of Lemma 4.1. We will modify the function $v_{\varepsilon,R}$ constructed above in order to get a continuous super-solution to (4.1) satisfying (4.2). Let us briefly explain our strategy. Since, by construction, $v_{\varepsilon,R}$ satisfies (4.8), the function $u_{\varepsilon,R}=1-v_{\varepsilon,R}$ verifies (4.6) and, as already noted above, extending the function $u_{\varepsilon,R}$ by 1 outside B_R, we obtain a (discontinuous) super-solution to (4.1) that satisfies (4.2). The aim of this section is to find the right extension of $u_{\varepsilon,R}$ that provides the desired super-solution.

To do so, we first introduce some useful notations. Given $R>0$ and $x\in\mathbb{R}^N$, we let $\mathcal{P}_R(x)$ be the projection of x to the ball $\overline{B_R}$, that is

$$\mathcal{P}_R(x)\in\overline{B_R}\ \text{ and }\ |x-\mathcal{P}_R(x)|=\mathrm{dist}(x,B_R)=\min_{y\in\overline{B_R}}|x-y|.$$

For $\sigma>0$, we let $\overline{u}_{\varepsilon,\sigma}\in C(\overline{\mathbb{R}^N\setminus K_\varepsilon})$ be the following function

$$\overline{u}_{\varepsilon,\sigma}(x):=\min\left\{u_{\varepsilon,R}(\mathcal{P}_R(x))+\sigma^{-1}|x-\mathcal{P}_R(x)|,1\right\}. \tag{4.31}$$

We shall see that, for well-chosen σ, the function $\overline{u}_{\varepsilon,\sigma}$ will satisfy

$$L_\varepsilon\overline{u}_{\varepsilon,\sigma}(x)+\widetilde{f}_\varepsilon(\overline{u}_{\varepsilon,\sigma}(x))\leqslant 0\quad\text{for all } x\in\mathbb{R}^N\setminus K_\varepsilon, \tag{4.32}$$

where L_ε is the nonlocal operator given by

$$L_\varepsilon w(x):=\int_{\mathbb{R}^N\setminus K_\varepsilon} J_\varepsilon(x-y)(w(y)-w(x))\mathrm{d}y. \tag{4.33}$$

Namely, we claim

Claim 4.8. *There exists* $\sigma_\varepsilon>0$ *such that* $\overline{u}_{\varepsilon,\sigma}$ *satisfies* (4.32) *for all* $0<\sigma<\sigma_\varepsilon$.

Observe that by proving Claim 4.8, we end the proof of Lemma 4.1. Indeed, by construction, we have $f\leqslant\widetilde{f}$ so that $\overline{u}_{\varepsilon,\sigma}$ trivially satisfies (4.1). As for condition (4.2) it is also satisfied (by construction of $\overline{u}_{\varepsilon,\sigma}$) provided that R is taken sufficiently large.

Proof. Define $\mathcal{A}_R:=\mathbb{R}^N\setminus\overline{B_R}$. As in the previous section, we set

$$\mathcal{J}_\varepsilon(x)=\int_{\mathbb{R}^N\setminus K_\varepsilon} J(x-y)\mathrm{d}y\ \text{ and }\ c_\varepsilon(x)=\int_{\mathbb{R}^N\setminus B_R} J_\varepsilon(x-y)\mathrm{d}y.$$

Then, in view of (4.31), we have

$$L_\varepsilon\overline{u}_{\varepsilon,\sigma}(x)+\widetilde{f}_\varepsilon(\overline{u}_{\varepsilon,\sigma}(x))\leqslant\int_{B_R\setminus K_\varepsilon} J_\varepsilon(x-y)(u_{\varepsilon,R}(y)-\overline{u}_{\varepsilon,\sigma}(x))\mathrm{d}y$$
$$+c_\varepsilon(x)(1-\overline{u}_{\varepsilon,\sigma}(x))+\widetilde{f}(\overline{u}_{\varepsilon,\sigma}(x)). \tag{4.34}$$

Since $\overline{u}_{\varepsilon,\sigma}(x) = u_{\varepsilon,R}(x)$ for all $x \in \overline{B_R \setminus K_\varepsilon}$, using (4.6) we easily get that

$$L_\varepsilon \overline{u}_{\varepsilon,\sigma}(x) + \widetilde{f}_\varepsilon(\overline{u}_{\varepsilon,\sigma}(x)) \leqslant 0 \quad \text{for } x \in \overline{B_R \setminus K_\varepsilon}. \tag{4.35}$$

To complete the proof, it remains to show that $\overline{u}_{\varepsilon,\sigma}$ satisfies (4.32) in the set $\mathcal{A}_R$. We shall consider two sub-domains, Π^+ and Π^-, defined as follows

$$\Pi^- := \mathcal{A}_R \cap \{\overline{u}_{\varepsilon,\sigma} < 1\},$$
$$\Pi^+ := \mathcal{A}_R \cap \{\overline{u}_{\varepsilon,\sigma} = 1\}.$$

Note that since $\overline{u}_{\varepsilon,\sigma}(x) = 1$ for all $x \in \Pi^+$, it follows directly from (4.34) that

$$L_\varepsilon \overline{u}_{\varepsilon,\sigma}(x) + \widetilde{f}_\varepsilon(\overline{u}_{\varepsilon,\sigma}(x)) = \int\limits_{\mathbb{R}^N \setminus K_\varepsilon} J_\varepsilon(x-y)(u_{\varepsilon,R}(y) - 1)\mathrm{d}y \leqslant 0 \quad \text{for any } x \in \Pi^+. \tag{4.36}$$

Thus, to conclude the proof we need only to check that (4.36) still holds in Π^-. To this end, for any $x \in \Pi^-$ and any $s \in [0, 1]$, we set

$$g_R(x,s) := \mathcal{J}_\varepsilon(\mathcal{P}_R(x))s - \widetilde{f}_\varepsilon(s). \tag{4.37}$$

Now, since $0 < \varepsilon < \varepsilon_0$, it follows from Proposition 3.1 that there exists a $\iota > 0$ such that

$$\inf_{z \in \mathbb{R}^N \setminus K_\varepsilon} \min_{s \in [0,1]} \partial_s g_R(z,s) > \iota. \tag{4.38}$$

Next, since $J \in W^{1,1}(\mathbb{R}^N)$ (by (1.6)) we may set

$$\sigma_\varepsilon := \varepsilon\iota \times \left(\int\limits_{\mathbb{R}^N} |\nabla J(z)| \mathrm{d}z \right)^{-1} > 0. \tag{4.39}$$

Let us also set

$$s(x) := u_{\varepsilon,R}(\mathcal{P}_R(x)) \ \text{ and } \ \tau(x) := \mathrm{dist}(x, B_R) = |x - \mathcal{P}_R(x)| > 0.$$

Then, $\overline{u}_{\varepsilon,\sigma}$ rewrites $\overline{u}_{\varepsilon,\sigma}(x) = s(x) + \sigma^{-1}\tau(x)$ and

$$0 < s(x) + \sigma^{-1}\tau(x) < 1 \quad \text{for any } x \in \Pi^-. \tag{4.40}$$

On the other hand, in view of (4.31) and by definition of L_ε, we can rewrite $L_\varepsilon \overline{u}_{\varepsilon,\sigma}(x)$ as

$$\begin{aligned} L_\varepsilon \overline{u}_{\varepsilon,\sigma}(x) = L_\varepsilon \overline{u}_{\varepsilon,\sigma}(\mathcal{P}_R(x)) &+ \int\limits_{\mathbb{R}^N \setminus K_\varepsilon} [J_\varepsilon(x-y) - J_\varepsilon(\mathcal{P}_R(x) - y)](\overline{u}_{\varepsilon,\sigma}(y) - \overline{u}_{\varepsilon,\sigma}(x))\mathrm{d}y \\ &- \frac{\tau(x)}{\sigma} \int\limits_{\mathbb{R}^N \setminus K_\varepsilon} J_\varepsilon(\mathcal{P}_R(x) - y)\mathrm{d}y. \end{aligned}$$

Since $\mathcal{P}_R(x) \in \overline{B_R \setminus K_\varepsilon}$, since $J \in W^{1,1}(\mathbb{R}^N)$ and since $J \geqslant 0$ a.e. in $\mathbb{R}^N$, by (4.35) we obtain

$$\begin{aligned} L_\varepsilon \overline{u}_{\varepsilon,\sigma}(x) &\leqslant -\frac{\tau(x)}{\sigma}\mathcal{J}_\varepsilon(\mathcal{P}_R(x)) - \widetilde{f}_\varepsilon(s(x)) + \int\limits_{\mathbb{R}^N \setminus K_\varepsilon} |J_\varepsilon(x-y) - J_\varepsilon(\mathcal{P}_R(x) - y)|\,\mathrm{d}y. \\ &\leqslant -\frac{\tau(x)}{\sigma}\mathcal{J}_\varepsilon(\mathcal{P}_R(x)) - \widetilde{f}_\varepsilon(s(x)) + \int\limits_{\mathbb{R}^N} |J_\varepsilon(x-y) - J_\varepsilon(\mathcal{P}_R(x) - y)|\,\mathrm{d}y. \\ &\leqslant -\frac{\tau(x)}{\sigma}\mathcal{J}_\varepsilon(\mathcal{P}_R(x)) - \widetilde{f}_\varepsilon(s(x)) + \frac{\tau(x)}{\varepsilon} \int\limits_{\mathbb{R}^N} |\nabla J(z)|\mathrm{d}z. \end{aligned}$$

Therefore, we get

$$L_\varepsilon \overline{u}_{\varepsilon,\sigma}(x) + \widetilde{f}_\varepsilon(s(x)+\sigma^{-1}\tau(x)) \leqslant \Big(\widetilde{f}_\varepsilon(s(x)+\sigma^{-1}\tau(x)) - \widetilde{f}_\varepsilon(s(x))\Big) - \frac{\tau(x)}{\sigma}\mathcal{J}_\varepsilon(\mathcal{P}_R(x)) + \frac{\tau(x)}{\varepsilon}\int_{\mathbb{R}^N} |\nabla J(z)|\mathrm{d}z.$$

By adding and subtracting $s(x)\mathcal{J}_\varepsilon(\mathcal{P}_R(x))$ on the right hand side of the above inequality and recalling (4.37), we obtain

$$L_\varepsilon \overline{u}_{\varepsilon,\sigma}(x) + \widetilde{f}_\varepsilon(\overline{u}_{\varepsilon,\sigma}(x)) \leqslant \Big(g_R(x,s(x)) - g_R\big(x, s(x)+\sigma^{-1}\tau(x)\big)\Big) + \iota\sigma_\varepsilon^{-1}\tau(x),$$

where we have used (4.39). By (4.38), (4.40) and the mean value theorem, we deduce that there exists some

$$\xi \in \Big[s(x), s(x)+\sigma^{-1}\tau(x)\Big] \subset [0,1],$$

such that

$$g_R(x,s(x)) - g_R\Big(x, s(x)+\sigma^{-1}\tau(x)\Big) = -\partial_s g_R(x,\xi)\sigma^{-1}\tau(x) \leqslant -\iota\sigma^{-1}\tau(x).$$

Therefore, for every $0<\sigma<\sigma_\varepsilon$, we obtain that

$$L_\varepsilon \overline{u}_{\varepsilon,\sigma}(x) + \widetilde{f}_\varepsilon(\overline{u}_{\varepsilon,\sigma}(x)) \leqslant \iota\tau(x)\left(\frac{1}{\sigma_\varepsilon} - \frac{1}{\sigma}\right) < 0 \quad \text{for any } x \in \Pi^-.$$

The proof of Claim 4.8 is thereby complete. □

Remark 4.9. An analogue version of Lemma 4.1 holds when $J_\varepsilon(x-y)$ is replaced by $\widetilde{J}_\varepsilon(d_g(x,y))$ where $\widetilde{J}_\varepsilon$ is a locally integrable function such that $\widetilde{J}_\varepsilon(|z|) = J_\varepsilon(z)$ and $d_g(x,y)$ is the geodesic distance on $\overline{\mathbb{R}^N \setminus K_\varepsilon}$. Indeed, the only places where the structure of the radial kernel J_ε came into place is when we used the Poincaré-type inequality [18, Theorem 1.1] in Proposition 4.5, when we asserted that the solutions to (4.8) satisfying $\max_{[0,1]} f_\varepsilon' < \inf_{B_R\setminus K_\varepsilon} \mathcal{J}_\varepsilon$ are continuous and when we made our extension procedure. But the Poincaré inequality was only needed in the ball B_{R_0} and, by convexity, it trivially holds that $J_\varepsilon(x-y) = \widetilde{J}_\varepsilon(d_g(x,y))$ for any $(x,y) \in B_{R_0}\times B_{R_0}$. Similarly, the extension procedure required only to evaluate the new function on the annulus $B_{R+\sigma}\setminus B_R$ but, since $R-R_1>0$ is large and ε is small, it still holds that $J_\varepsilon(x-y) = \widetilde{J}_\varepsilon(d_g(x,y))$ for any $x\in B_{R+\sigma}\setminus B_R$ and any $y \in \mathbb{R}^N\setminus K_\varepsilon$. Moreover, as already noted in Remark 2.6, condition (3.10) still implies the continuity of solutions to the corresponding auxiliary problem:

$$\int_{B_R\setminus K_\varepsilon} \widetilde{J}_\varepsilon(d_g(x,y))(v_{\varepsilon,R}(y) - v_{\varepsilon,R}(x))\mathrm{d}y - \widetilde{c}_\varepsilon(x)v_{\varepsilon,R} + g_\varepsilon(v_{\varepsilon,R}(x)) = 0 \quad \text{for } x \in \overline{B_R\setminus K_\varepsilon},$$

where, by analogy, we have set

$$\widetilde{c}_\varepsilon(x) := \int_{\mathbb{R}^N\setminus B_R} \widetilde{J}_\varepsilon(d_g(x,y))\mathrm{d}y.$$

In fact, the only place where some care should be taken is when justifying that if

$$\int_{B_R\setminus K_\varepsilon} \widetilde{J}_\varepsilon(d_g(\bar{x},y))(v_{\varepsilon,R}(y) - v_{\varepsilon,R}(\bar{x}))\mathrm{d}y = 0, \tag{4.41}$$

where $\bar{x} \in \overline{B_R\setminus K_\varepsilon}$ is a point at which $v_{\varepsilon,R}$ reaches an extremum, then it holds that $v_{\varepsilon,R}(y) \equiv v_{\varepsilon,R}(\bar{x})$ for any $y \in \overline{B_R\setminus K_\varepsilon}$ (which is needed to establish the analogue of Proposition 4.7). But, fortunately, the geometry of K_ε is simple enough to ensure that this is still the case. Indeed, (4.41) implies that $v_{\varepsilon,R}(y) \equiv v_{\varepsilon,R}(\bar{x})$ for any $y \in \Pi_1(\bar{x}) := \{z \in \overline{B_R\setminus K_\varepsilon};\, d_g(\bar{x},z) < \varepsilon/2\}$. By iteration, one finds that $v_{\varepsilon,R}(y) \equiv v_{\varepsilon,R}(\bar{x})$ for any $y \in \Pi_j(\bar{x})$ and any $j \geqslant 1$, where $\Pi_j(\bar{x})$ is given by

$$\Pi_{j+1}(\bar{x}) := \bigcup_{y\in\Pi_j(\bar{x})} \left\{z \in \overline{B_R \setminus K_\varepsilon};\ d_g(y,z) < \varepsilon/2\right\},\ \text{for any } j \geqslant 1.$$

Then, one can show that, for some $j_0 \geqslant 1$ (independent of $\bar{x}$), it holds that $B_{\varepsilon/4}(\bar{x}) \cap \overline{B_R \setminus K_\varepsilon} \subset \Pi_{j_0}(\bar{x})$. Whence, iterating the same reasoning over again, one gets that $v_{\varepsilon,R}(y) \equiv v_{\varepsilon,R}(\bar{x})$ for any $y \in B_{k\varepsilon/4}(\bar{x}) \cap \overline{B_R \setminus K_\varepsilon}$ and any $k \in \mathbb{N}$; which then gives the desired result.

5. Construction of continuous global solutions

In this final section we construct a positive nonconstant solution to (1.9). Our goal will be to find an ordered pair of global continuous sub- and super-solution. That is, given $0 < \varepsilon < \varepsilon^*$ (where ε^* has the same meaning as in Lemma 4.1), we aim to construct two functions, $\underline{u}_\varepsilon$ and $\overline{u}_\varepsilon$, such that

$$\begin{cases} L_\varepsilon \overline{u}_\varepsilon + f_\varepsilon(\overline{u}_\varepsilon) \leqslant 0 & \text{in } \mathbb{R}^N \setminus K_\varepsilon, \\ L_\varepsilon \underline{u}_\varepsilon + f_\varepsilon(\underline{u}_\varepsilon) \geqslant 0 & \text{in } \mathbb{R}^N \setminus K_\varepsilon, \\ 0 \leqslant \underline{u}_\varepsilon \leqslant \overline{u}_\varepsilon \leqslant 1 & \text{in } \mathbb{R}^N \setminus K_\varepsilon, \end{cases}$$

(where L_ε is as in (4.33)) and which further satisfy

$$\lim_{x_1\to+\infty} \underline{u}_\varepsilon(x) = 1 \quad \text{and} \quad \lim_{|x|\to+\infty} \overline{u}_\varepsilon(x) = 1. \tag{5.1}$$

Here, $x_1 = x \cdot e_1$ where $e_1 := (1, 0, \cdots, 0) \in \mathbb{S}^{N-1}$. Then, by Lemmata 2.1 and 2.2 we automatically obtain the existence of a continuous solution u_ε to

$$L_\varepsilon u_\varepsilon + f_\varepsilon(u_\varepsilon) = 0 \quad \text{in } \mathbb{R}^N \setminus K_\varepsilon, \tag{5.2}$$

satisfying $0 \leqslant \underline{u}_\varepsilon \leqslant u_\varepsilon \leqslant \overline{u}_\varepsilon \leqslant 1$. This, together with (5.1), yields a continuous solution to (5.2) satisfying $0 < u_\varepsilon < 1$ and $u_\varepsilon(x) \to 1$ as $x_1 \to \infty$. In particular, we have $\sup_{x\in\mathbb{R}^N\setminus K_\varepsilon} u_\varepsilon(x) = 1$. Since (1.6), (1.7) are satisfied, u_ε is continuous, J_ε is compactly supported and $J_\varepsilon \in L^2(\mathbb{R}^N)$ (by (1.8)), we may apply Lemma 2.5 and we obtain that $\lim_{|x|\to+\infty} u_\varepsilon(x) = 1$, which proves that u_ε satisfies the requirements of Theorem 1.2 and thus Theorem 1.1 is proved.

Therefore, to complete the proof of Theorem 1.2, we need only to prove the following lemma.

Lemma 5.1. *Let (J, f) be a pair satisfying (1.6), (1.7) and (1.8). Let $(K_\varepsilon)_{0<\varepsilon<1}$ be the family of obstacles associated to the pair (J, f) (as defined in Section 3). Let $(J_\varepsilon, f_\varepsilon)$ be as in (3.3) and let $\varepsilon^* > 0$ be as in Lemma 4.1. Then, there exists $r_0 > 0$ such that, for all $0 < \varepsilon < \varepsilon^*$, there is*

(i) *a continuous global sub-solution $\underline{u}_\varepsilon$ to (5.2) satisfying $\underline{u}_\varepsilon \equiv 0$ in $\{x_1 \leqslant r_0\}$ and $\underline{u}_\varepsilon(x) \to 1$ as $x_1 \to \infty$,*
(ii) *a continuous global nonconstant super-solution $\overline{u}_\varepsilon$ to (5.2) satisfying $\overline{u}_\varepsilon \equiv 1$ in $\mathbb{R}^N \setminus B_{r_0}$ and $0 < \overline{u}_\varepsilon \leqslant 1$.*

In particular, $0 \leqslant \underline{u}_\varepsilon < \overline{u}_\varepsilon \leqslant 1$.

Proof. By Lemma 4.1, we know that there exists some $R^* > 0$ and some $0 < \varepsilon^* < 1$ such that, for all $0 < \varepsilon < \varepsilon^*$, there is a nonconstant super-solution $\overline{u}_\varepsilon \in C(\overline{\mathbb{R}^N \setminus K_\varepsilon})$ to (5.2) that satisfies $\overline{u}_\varepsilon \equiv 1$ in $\mathbb{R}^N \setminus B_{R^*}$. So, we are left to prove that there exists a sub-solution $\underline{u}_\varepsilon$ to (5.2) satisfying (i) and such that $\underline{u}_\varepsilon \leqslant \overline{u}_\varepsilon$.

To do so, let us extend f outside $[0, 1]$ by $f'(0)s$ when $s \leqslant 0$ and $f'(1)(s - 1)$ for $s \geqslant 1$. For simplicity, we still denote by f this extension. Now, we take $\delta \in (0, 1)$ and we let f_δ be a C^1 function defined in $\mathbb{R}$ such that

$$\begin{cases} f_\delta \leqslant f \text{ in } \mathbb{R}, \text{ and } f_\delta(s) = f(s) \text{ for } s \geqslant \theta, \\ f_\delta \text{ has only one zero, } \theta_\delta = \theta, \text{ in } (-\delta, 1), \\ f_\delta(-\delta) = 0, \quad f_\delta(1) = 0, \\ f'_\delta(s) < 1 \text{ for any } s \in [-\delta, 1] \text{ and } f'_\delta(-\delta),\ f'_\delta(1) < 0, \\ \displaystyle\int_{-\delta}^{1} f_\delta(s)\mathrm{d}s > 0. \end{cases}$$

Since $f \in C^1(\mathbb{R})$ satisfies (1.7) such a function $f_\delta \in C^1(\mathbb{R})$ always exists provided that δ is taken sufficiently small, say if $0 < \delta < \delta_1$ for some small $\delta_1 > 0$.

Let $f_{\varepsilon,\delta}(s) := \varepsilon^2 f_\delta(s)$ and let $L_{\mathbb{R}^N}$ be the operator given by

$$L_{\mathbb{R}^N} u(x) := \int_{\mathbb{R}^N} J_\varepsilon(x-y)(u(y)-u(x))\mathrm{d}y. \tag{5.3}$$

Since J_ε is radially symmetric (because J is), using the results obtained in [1,8,9,22], we know that, for any $0<\varepsilon<1$, there exists an increasing function $\phi_{\varepsilon,\delta} \in C^1(\mathbb{R})$ and a number $c_{\varepsilon,\delta} > 0$ such that the function $\varphi_{\varepsilon,\delta}(x) := \phi_{\varepsilon,\delta}(x \cdot e_1)$ satisfies

$$\begin{cases} L_{\mathbb{R}^N}\varphi_{\varepsilon,\delta}(x) + f_{\varepsilon,\delta}(\varphi_{\varepsilon,\delta}(x)) = c_{\varepsilon,\delta}\phi'_{\varepsilon,\delta}(x_1) \geqslant 0 \text{ for all } x \in \mathbb{R}^N, \\ \lim\limits_{x_1\to-\infty} \varphi_{\varepsilon,\delta}(x) = -\delta, \quad \lim\limits_{x_1\to\infty} \varphi_{\varepsilon,\delta}(x) = 1 \ \text{ and } \ \varphi_{\varepsilon,\delta} = 0 \text{ in } H_{e_1}, \end{cases} \tag{5.4}$$

where H_{e_1} is the hyperplane $H_{e_1} := \{x_1 = 0\}$. Now, for any $r_0 > 0$, we let $\varphi_{\varepsilon,\delta,r_0}$ be the function defined by

$$\varphi_{\varepsilon,\delta,r_0}(x) := \phi_{\varepsilon,\delta}(x \cdot e_1 - r_0).$$

By construction, for every $r_0 > 0$, we have

$$L_{\mathbb{R}^N}\varphi_{\varepsilon,\delta,r_0} + f_\varepsilon(\varphi_{\varepsilon,\delta,r_0}) \geqslant L_{\mathbb{R}^N}\varphi_{\varepsilon,\delta,r_0} + f_{\varepsilon,\delta}(\varphi_{\varepsilon,\delta,r_0}) \geqslant 0 \quad \text{in } \mathbb{R}^N. \tag{5.5}$$

Now, we set

$$\underline{u}_\varepsilon(x) := \max\left\{0, \varphi_{\varepsilon,\delta,r_0}(x)\right\} \quad \text{and} \quad H_* := \left\{x \in \mathbb{R}^N;\ x_1 \geqslant r_0\right\}.$$

Note that, for all $0 < \varepsilon < \varepsilon^*$, it holds that $K_\varepsilon \subset \mathbb{R}^N \setminus H_*$ provided that r_0 is chosen sufficiently large. Let us now prove that, for r_0 large enough, $\underline{u}_\varepsilon$ is a sub-solution to (5.2).

First, if $x \in \mathbb{R}^N \setminus (K \cup H_*)$, then $\underline{u}_\varepsilon(x) = 0$ and

$$L_\varepsilon \underline{u}_\varepsilon(x) + f_\varepsilon(\underline{u}_\varepsilon(x)) = \int_{\mathbb{R}^N\setminus K_\varepsilon} J_\varepsilon(x-y)\underline{u}_\varepsilon(y)\mathrm{d}y \geqslant 0. \tag{5.6}$$

Next, if $x \in H_*$, then, since J_ε is compactly supported, we have

$$\bigcup_{x\in H_*} \big(x + \mathrm{supp}(J_\varepsilon)\big) \subset \mathbb{R}^N \setminus K_\varepsilon,$$

provided that r_0 is chosen sufficiently large. From this and (5.5), we deduce that

$$\begin{aligned} L_\varepsilon \underline{u}_\varepsilon(x) + f_\varepsilon(\underline{u}_\varepsilon(x)) &= \int_{\mathbb{R}^N\setminus K_\varepsilon} J_\varepsilon(x-y)(\underline{u}_\varepsilon(y) - \varphi_{\varepsilon,\delta,r_0}(x))\mathrm{d}y + f_\varepsilon(\varphi_{\varepsilon,\delta,r_0}(x)) \\ &\geqslant \int_{\mathbb{R}^N\setminus K_\varepsilon} J_\varepsilon(x-y)(\varphi_{\varepsilon,\delta,r_0}(y) - \varphi_{\varepsilon,\delta,r_0}(x))\mathrm{d}y + f_\varepsilon(\varphi_{\varepsilon,\delta,r_0}(x)) \\ &= L_{\mathbb{R}^N}\varphi_{\varepsilon,\delta,r_0}(x) + f_\varepsilon(\varphi_{\varepsilon,\delta,r_0}(x)) \geqslant 0. \end{aligned}$$

Together with (5.6), we obtain that $\underline{u}_\varepsilon$ is a global sub-solution to (5.2) which, by (5.4), satisfies $\underline{u}_\varepsilon(x) \to 1$ as $x_1 \to \infty$ and $\underline{u}_\varepsilon(x) = 0$ if $x_1 \leqslant r_0$. By increasing r_0 to R^* (if necessary) we then achieve $\underline{u}_\varepsilon < \overline{u}_\varepsilon$ when $0 < \varepsilon < \varepsilon^*$. The proof of Lemma 5.1 is thereby complete. □

Remark 5.2. Observe that, on account of Remarks 2.6, 3.2 and 4.9, the *same* proof as above yields an analogous result with L_g in place of L. To see this, it suffices to notice that our arguments are essentially focused on what is happening *far away* from K and, since the kernel we consider is compactly supported, the operator L_g will then coincide with L (possibly up to take R sufficiently large). In like manner, as already mentioned in Remark 2.6, the fact that "$\sup_{\mathbb{R}^N\setminus K_\varepsilon} u = 1$" implies that "$\lim_{|x|\to\infty} u(x) = 1$" still holds with L_g in place of L since, here as well, the proof relies only on estimates of the behaviour of u far away from K_ε.

Declaration of competing interest

The authors declare to have no conflict of interests of any kind.

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Appendix A

In this appendix, we prove Lemma 2.1. Our strategy closely follows [7,10] and relies on the well-known monotone iterative method. Before doing so, we first state a preliminary lemma.

Lemma A.1. *Let $K \subset \mathbb{R}^N$ be a compact set and assume that J satisfies (1.6). Let $k > 0$ and let $w \in C(\mathbb{R}^N \setminus K)$ be such that*

$$Lw - kw \geqslant 0 \quad \text{in } \mathbb{R}^N \setminus K, \tag{A.1}$$

and that

$$\limsup_{|x|\to\infty} w(x) \leqslant 0. \tag{A.2}$$

Then,

$$w \leqslant 0 \quad \text{in } \mathbb{R}^N \setminus K.$$

Proof. Suppose, by contradiction, that $\sup_{\mathbb{R}^N\setminus K} w > 0$. Then, by assumption (A.2), there exists a number $r > 0$ with $K \subset B_r$ and a sequence $(x_j)_{j\geqslant 0} \subset B_r \setminus K$ such that

$$\lim_{j\geqslant 0} w(x_j) = \sup_{B_r\setminus K} w = \sup_{\mathbb{R}^N\setminus K} w > 0. \tag{A.3}$$

Since $(x_j)_{j\geqslant 0}$ is bounded, up to extraction of a subsequence, there exists a point $\bar{x} \in \overline{B_r \setminus K}$ such that $x_j \to \bar{x}$ as $j \to \infty$. Moreover, since w is continuous and (A.1) is satisfied everywhere in $\mathbb{R}^N \setminus K$, it makes sense to evaluate (A.1) at x_j for any $j \geqslant 0$. That is, we have

$$\int_{\mathbb{R}^N\setminus K} J(x_j - y)(w(y) - w(x_j))\mathrm{d}y \geqslant k w(x_j) \quad \text{for any } j \geqslant 0.$$

But, since $k > 0$, using (A.3) and the dominated convergence theorem, we obtain

$$0 \geqslant \int_{\mathbb{R}^N\setminus K} J(\bar{x} - y)\left(w(y) - \sup_{\mathbb{R}^N\setminus K} w\right)\mathrm{d}y \geqslant k \sup_{\mathbb{R}^N\setminus K} w > 0,$$

which is a contradiction. The proof is thereby complete. □

We are now in position to prove Lemma 2.1.

Proof of Lemma 2.1. Let us first observe that, from the assumptions made on J, the operator L is linear and continuous on $(C_0(\mathbb{R}^N \setminus K), \|\cdot\|_\infty)$, where

$$C_0(\mathbb{R}^N \setminus K) := \left\{ w \in C(\mathbb{R}^N \setminus K);\ \lim_{|x|\to\infty} w(x) = 0 \right\}.$$

Indeed, this is because, given any $w \in C_0(\mathbb{R}^N \setminus K)$, we have

$$Lw(x) = \int_{\mathbb{R}^N} J(y) \left(\mathbb{1}_{x-\mathbb{R}^N\setminus K}(x) w(x-y)\right) \mathrm{d}y - \mathcal{J}(x) w(x),$$

where $\mathcal{J}$ is as in (2.4), and, by the dominated convergence theorem, we have that $Lw(x) \to 0$ as $|x| \to \infty$. The continuity of Lw is a mere consequence of the continuity of translations in $L^1(\mathbb{R}^N)$ and of the continuity of w, as is easily seen from the (trivial) inequality

$$|Lw(x_1) - Lw(x_2)| \leqslant 2\|w\|_\infty \int_{\mathbb{R}^N} |J(y + x_1 - x_2) - J(y)| \mathrm{d}y + |w(x_1) - w(x_2)|, \tag{A.4}$$

which holds for any $x_1, x_2 \in \mathbb{R}^N \setminus K$. So that L indeed maps $C_0(\mathbb{R}^N \setminus K)$ into itself. Moreover, the continuity of the operator L follows from the fact that

$$\|Lw\|_\infty \leqslant 2\|w\|_\infty \quad \text{for any } w \in C_0(\mathbb{R}^N \setminus K).$$

Next, we let $k > 0$ be a number large enough so that the map $s \mapsto -ks - f(s)$ is decreasing in $[0,1]$ and that $k \in \rho(L)$ where $\rho(L)$ denotes the resolvent of the operator L.

Let $\underline{u}$ and $\overline{u}$ be continuous global sub- and super-solutions to

$$Lu + f(u) = 0 \quad \text{in } \mathbb{R}^N \setminus K, \tag{A.5}$$

satisfying (2.1) and (2.2).

We will construct a solution u to (A.5) satisfying $\underline{u} \leqslant u \leqslant \overline{u}$ using a monotone iterative scheme. That is, we will construct u as the limit of an appropriate sequence of functions. The main tool behind our construction is the comparison principle Lemma A.1. To this end, we have to make sure that the sequence we construct has the right asymptotic behaviour as $|x| \to \infty$ (as required by Lemma A.1). With this aim in mind, we first construct an appropriate sequence of auxiliary functions. Namely, we define $v_0 \equiv 0$ and, for $j \geqslant 0$, we let

$$Lv_{j+1}(x) - kv_{j+1}(x) = -kv_j(x) - f(\overline{u}(x) + v_j(x)) - L\overline{u}(x) \quad \text{for } x \in \mathbb{R}^N \setminus K. \tag{A.6}$$

Let us check that the v_j's are well-defined elements of $C_0(\mathbb{R}^N \setminus K)$. Since $k \in \rho(L)$ and $0 \equiv v_0 \in C_0(\mathbb{R}^N \setminus K)$, v_1 is a well-defined element of $C_0(\mathbb{R}^N \setminus K)$ as soon as

$$f(\overline{u}(\cdot)) + L\overline{u}(\cdot) \in C_0(\mathbb{R}^N \setminus K),$$

which is the case since $f(1) = 0$, f is continuous, $\overline{u}(x) \to 1$ as $|x| \to \infty$ and $L\overline{u} \in C_0(\mathbb{R}^N \setminus K)$ (because $\overline{u} \in C(\mathbb{R}^N \setminus K)$) and

$$L\overline{u}(x) = \int_{\mathbb{R}^N} J(y) \mathbb{1}_{x-\mathbb{R}^N\setminus K}(y)(\overline{u}(x-y) - \overline{u}(x)) \mathrm{d}y \underset{|x|\to\infty}{\longrightarrow} \int_{\mathbb{R}^N} J(y)(1-1)\mathrm{d}y = 0.$$

Similarly, if, for some $j \geqslant 0$, it holds that $v_j \in C_0(\mathbb{R}^N \setminus K)$, then, given that $k \in \rho(L)$ and that $L\overline{u} \in C_0(\mathbb{R}^N \setminus K)$, v_{j+1} is a well-defined element of $C_0(\mathbb{R}^N \setminus K)$ as soon as

$$f(\overline{u}(\cdot) + v_j(\cdot)) \in C_0(\mathbb{R}^N \setminus K),$$

which trivially holds since f is continuous, $f(1) = 0$ and $\overline{u}(x) \to 1$, $v_j(x) \to 0$ as $|x| \to \infty$. Whence, by induction, we infer that the v_j's are, indeed, well-defined elements of $C_0(\mathbb{R}^N \setminus K)$.

Let us now define a sequence $(u_j)_{j\geqslant 0} \subset C(\mathbb{R}^N \setminus K)$ by setting $u_j := \overline{u} + v_j$. Then, by construction, we have

$$Lu_{j+1}(x) - ku_{j+1}(x) = -ku_j(x) - f(u_j(x)) \quad \text{for any } x \in \mathbb{R}^N \setminus K \text{ and } j \geqslant 0, \tag{A.7}$$

and the u_j's satisfy the limit condition

$$\lim_{|x|\to\infty} u_j(x) = 1 \quad \text{for any } j \geqslant 0. \tag{A.8}$$

We will show that the desired solution to (A.5) can be obtained as the pointwise limit of $(u_j)_{j\geqslant 0}$. Let us proceed step by step. First, when $j = 0$, we have

$$Lu_1(x) - ku_1(x) = -ku_0(x) - f(u_0(x)) \quad \text{for } x \in \mathbb{R}^N \setminus K. \tag{A.9}$$

We claim that $\underline{u} \leqslant u_1 \leqslant u_0 = \overline{u}$ in $\mathbb{R}^N \setminus K$. Indeed, we have

$$\begin{cases} L(u_1 - u_0)(x) - k(u_1 - u_0) = -Lu_0(x) - f(u_0(x)), \\ L(u_1 - \underline{u})(x) - k(u_1 - \underline{u}) \leqslant f(\underline{u}(x)) + k\underline{u}(x) - f(u_0(x)) - ku_0(x). \end{cases}$$

Since $u_0 = \overline{u}$ is a super-solution to (A.5), $\underline{u} \leqslant \overline{u}$ and $s \mapsto -ks - f(s)$ is decreasing, we obtain that

$$\begin{cases} L(u_1 - u_0)(x) - k(u_1 - u_0) \geqslant 0, \\ L(u_1 - \underline{u})(x) - k(u_1 - \underline{u}) \leqslant 0. \end{cases} \tag{A.10}$$

By construction of u_1 (remember (2.1) and (A.8)), we have

$$\lim_{|x|\to\infty} (u_1 - u_0)(x) = 0 \ \text{ and } \ \liminf_{|x|\to\infty} (u_1 - \underline{u})(x) \geqslant 0. \tag{A.11}$$

This, together with Lemma A.1, then gives that $\underline{u} \leqslant u_1 \leqslant u_0 = \overline{u}$ in $\overline{\mathbb{R}^N \setminus K}$. Similarly, by (A.7), the function $u_2 \in C(\mathbb{R}^N \setminus K)$ solves (A.9) with u_2 in place of u_1 and u_1 in place of u_0. Thus, from (2.1), (A.8) and the monotonicity of $s \mapsto -ks - f(s)$, we deduce that (A.10) and (A.11) still hold with u_2 instead of u_1 and u_1 instead of u_0. We may then apply the comparison principle Lemma A.1 and we deduce that $\underline{u} \leqslant u_2 \leqslant u_1 \leqslant u_0 = \overline{u}$ in $\mathbb{R}^N \setminus K$. By induction, we infer that the u_j's satisfy the monotonicity relation

$$\underline{u} \leqslant \cdots \leqslant u_{j+1} \leqslant u_j \leqslant \cdots \leqslant u_2 \leqslant u_1 \leqslant u_0 = \overline{u}.$$

Since $(u_j)_{j\geqslant 0}$ is non-increasing and bounded from below by $\underline{u}$, the function

$$u(x) := \lim_{j\to\infty} u_j(x) \in \left[\underline{u}(x), \overline{u}(x)\right], \tag{A.12}$$

is well-defined for any $x \in \mathbb{R}^N \setminus K$. In particular, since $0 \leqslant \underline{u} \leqslant \overline{u} \leqslant 1$, it follows from (A.12) that $u \in L^\infty(\mathbb{R}^N \setminus K)$. It remains only to check that the function u is a solution to (A.5). For it, it suffices to let $j \to \infty$ in (A.7) (using the dominated convergence theorem), which then gives

$$Lu(x) + f(u(x)) = 0 \quad \text{for any } x \in \mathbb{R}^N \setminus K.$$

The proof is thereby complete. □

Remark A.2. The same arguments also apply when the operator L is replaced by L_g provided that $J = \widetilde{J}(|\cdot|)$ satisfies (1.8), since it still holds that if $w(x) \to \ell \in \mathbb{R}$ as $|x| \to \infty$, then $L_g w(x) \to 0$ as $|x| \to \infty$. Moreover, the continuity of w still implies the continuity of $L_g w$ but the proof is less obvious since one can no longer rely on the continuity of translations in $L^1(\mathbb{R}^N)$. For the sake of completeness, we state a last lemma below which justifies why this is true.

Lemma A.3. *Let $K \subset \mathbb{R}^N$ be a compact set and assume that $\widetilde{J}$ satisfies (1.12) and that $\widetilde{J}$ is supported in $[0, r]$ for some $r > 0$. Let $w \in C(\mathbb{R}^N \setminus K)$. Then, $L_g w \in C(\mathbb{R}^N \setminus K)$.*

Proof. Let $x_1, x_2 \in \mathbb{R}^N \setminus K$ with x_1 fixed and x_2 arbitrarily close to x_1. For $w \in C(\mathbb{R}^N \setminus K)$, the analogue of (A.4) is here:

$$|L_g w(x_1) - L_g w(x_2)| \leqslant 2\|w\|_\infty \left| \int_{\mathbb{R}^N} [\widetilde{J}(d_g(x_1, y)) - \widetilde{J}(d_g(x_2, y))] \mathrm{d}y \right| + \|\widetilde{\mathcal{J}}\|_\infty |w(x_1) - w(x_2)|,$$

where $\widetilde{\mathcal{J}}$ is as in (2.7). Since $w \in C(\mathbb{R}^N \setminus K)$, the delicate part is to show that the first term on the right-hand side vanishes as $x_2 \to x_1$. This can be done as follows. Let $\delta > 0$ be small enough so that $x_2 \in B_{\delta/2}(x_1) \subset B_\delta(x_1) \subset \mathbb{R}^N \setminus K$. Then, we may write

$$\begin{aligned}&\left|\int_{\mathbb{R}^N\setminus K} [\widetilde{J}(d_g(x_1,y)) - \widetilde{J}(d_g(x_2,y))]\mathrm{d}y\right| \\ &\quad\leqslant \int_{\mathbb{R}^N\setminus(B_\delta(x_1)\cup K)} |\widetilde{J}(d_g(x_1,y)) - \widetilde{J}(d_g(x_2,y))|\mathrm{d}y + \int_{B_\delta(x_1)} |\widetilde{J}(d_g(x_1,y)) - \widetilde{J}(d_g(x_2,y))|\mathrm{d}y \\ &\quad=: I_1(x_1,x_2) + I_2(x_1,x_2).\end{aligned}$$

Since $d_g(x_i,y) = |x_i - y|$ for any $i \in \{1,2\}$ and $y \in B_\delta(x_1)$, we have

$$I_2(x_1,x_2) \leqslant \|J(\cdot + x_1 - x_2) - J\|_{L^1(\mathbb{R}^N)} \underset{x_2\to x_1}{\longrightarrow} 0.$$

On the other hand, since J is radially symmetric, $\mathrm{supp}(J) = B_r$ and $J \in W^{1,1}(B_r)$, by [12, Theorems 1.1 and 2.3], we have that $\widetilde{J} \in W^{1,1}((0,r), t^{N-1})$, $\widetilde{J}$ is almost everywhere equal to a continuous function, $\widetilde{J}'$ exists almost everywhere and

$$\int_0^r |\widetilde{J}'(t)|t^{N-1}\mathrm{d}t \leqslant C_1 \int_{B_r} |\nabla J(z)|\mathrm{d}z. \tag{A.13}$$

Therefore, using the fact that $d_g(x_i,y) \geqslant |x_i - y| \geqslant \delta/2$ for any $y \in \mathbb{R}^N \setminus (B_\delta(x_1) \cup K)$, we have

$$\begin{aligned}I_1(x_1,x_2) &\leqslant \int_{\mathbb{R}^N\setminus(B_\delta(x_1)\cup K)} \int_{d_g(x_2,y)}^{d_g(x_1,y)} |\widetilde{J}'(t)|\mathrm{d}t\mathrm{d}y \\ &\leqslant \left(\frac{2}{\delta}\right)^{N-1} \int_{\mathbb{R}^N\setminus(B_\delta(x_1)\cup K)} \int_{d_g(x_2,y)}^{d_g(x_1,y)} |\widetilde{J}'(t)|t^{N-1}\mathrm{d}t\mathrm{d}y. \end{aligned}\tag{A.14}$$

Now, since $x_1, x_2 \in B_{\delta/2}(x_1) \subset \mathbb{R}^N \setminus K$ and $d_g(\cdot,\cdot)$ is a distance, we have

$$|d_g(x_1,y) - d_g(x_2,y)| \leqslant d_g(x_1,x_2) = |x_1 - x_2| \underset{x_2\to x_1}{\longrightarrow} 0$$

Therefore, using (A.13), (A.14) and the dominated convergence theorem, we obtain that

$$I_1(x_1,x_2) \to 0 \quad \text{as } x_2 \to x_1.$$

This completes the proof. □

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