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A higher speed type II blowup for the five dimensional energy critical heat equation

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Abstract

This paper is concerned with blow-up solutions of the five dimensional energy critical heat equation $u_t = \Delta u + |u|^{\frac{4}{3}}u$. A goal of this paper is to show the existence of type II blowup solutions which behave as $\|u(t)\|_\infty \sim (T-t)^{-3k}$ ($k = 2, 3, \cdots$). These solutions are the same ones formally derived by Filippas, Herrero and Velázquez [8]. We find a mistake in their blowup rate and correct it.

Keywords: Semilinear heat equation; Energy critical; Type II blowup; Matched asymptotic expansion

1. Introduction

This paper is concerned with blowup solutions for the semilinear heat equation.

$$u_t = \Delta u + |u|^{p-1}u \qquad \text{in } \mathbb{R}^n \times (0,T). \tag{1.1}$$

This problem is a simple model of nonlinear diffusion problems. Various complex and interesting phenomena have been found for these 30 years. Our concern in this paper is a blowup caused by a concentration. A local solvability of this problem is well understood, and a blow up occurs at $t = T$ if $\limsup_{t\to T} \|u(t)\|_\infty = \infty$. For a blowup solution, the blowup is called type I if $\limsup_{t\to T}(T-t)^{\frac{1}{p-1}}\|u(t)\|_\infty < \infty$, and type II if $\limsup_{t\to T}(T-t)^{\frac{1}{p-1}}\|u(t)\|_\infty = \infty$. A typical type I blowup solution is given by

$$u(x,t) = (p-1)^{-\frac{1}{p-1}}(T-t)^{-\frac{1}{p-1}}. \tag{1.2}$$

In the study of blowup problems, there are two important critical values of p defined by

$$p_S = \frac{n+2}{n-2}, \qquad p_{JL} = \begin{cases} \infty & \text{if } n \le 10, \\ 1 + \dfrac{4}{n-4-2\sqrt{n-1}} & \text{if } n \ge 11. \end{cases}$$

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For the case $1 < p < p_S$, it is well known that every blowup solution is locally approximated by (1.2), namely type I (see [9]). On the other hand, for the case $p \geq p_S$, different types of blowup behavior are observed. A type II blowup solution is first discovered by Herrero and Velázquez [12,13] (see also [17]). They construct type II blowup solutions with the exact blowup rates for the case $p > p_{JL}$. Very recently Seki [19] proves the existence of type II blowup solutions for the case $p = p_{JL}$. In the middle range of $p \in (p_S, p_{JL})$, Matano and Merle [15] exclude the occurrence of a type II blowup under a radial setting. Another example of a type II blowup is found by Filippas, Herrero and Velázquez [8] for the energy critical case $p = p_S$. They formally obtain a type II blowup by using the matched asymptotic expansion approach. The blowup rate of their solutions are given by

$$\|u(t)\|_\infty \sim \begin{cases} (T-t)^{-k} & n=3, \\ (T-t)^{-k}|\log(T-t)|^{\frac{2k}{2k-1}} & n=4, \\ (T-t)^{-3k} & n=5 \quad \text{(corrected)}, \\ (T-t)^{-\frac{5}{2}}|\log(T-t)|^{\frac{15}{4}} & n=6 \quad \text{(corrected)}, \end{cases} \tag{1.3}$$

where $k = 1, 2, 3, \cdots$. This list is a corrected version, not the original one obtained in [8] (see Appendix for details). As for a higher dimensional case $n \geq 7$, the possibility of a type II blowup near the ground states is ruled out by Collot, Merle and Raphaël [1]. The first rigorous proof of the existence of a type II blowup for the energy critical case is given by Schweyer [20]. He constructs a type II blowup solution for $n = 4$ by adapting the energy method developed in the study of geometrical dispersive problems ([18,16]) to the problem (1.1). The blowup rate of his solution coincides with $k = 1$ in (1.3). Very recently del Pino, Musso and Wei [6] obtain a type II blowup for $n = 5$ with the same blowup rate as $k = 1$ in (1.3). They apply so-called the inner-outer gluing method developed in [2–4]. Furthermore a new type of type II blowup not listed in (1.3) is found by del Pino, Musso and Wei [5] for $n \geq 7$. In this paper, we prove the existence of type II blowup solutions for $n = 5$ with a higher blowup speed. The solutions constructed here give the first example for $k \geq 2$ in (1.3).

2. Main result

Let $\mathsf{Q}_\lambda(x)$ be the positive radial stationary solution given by

$$\mathsf{Q}_\lambda(x) = \lambda^{-\frac{n-2}{2}}\left(1 + \frac{1}{n(n-2)}\frac{|x|^2}{\lambda^2}\right)^{-\frac{n-2}{2}} \qquad \text{(ground state)}.$$

Theorem 1. *Let $n = 5$ and $p = p_S$. For any integer $l \geq 1$ and any two constants $A > 0$, $\kappa \in (0, 1)$, there exist $T > 0$ and a radial solution $u(x, t) \in C(\mathbb{R}^5 \times [0, T)) \cap C^{2,1}(\mathbb{R}^5 \times (0, T))$ of (1.1) such that*

$$u(x,t) = \mathsf{Q}_{\lambda(t)}(x) + v(x,t),$$

where $\lambda(t)$ and $v(x, t)$ satisfy

$$\left|\lambda(t) - A(T-t)^{2l+2}\right| < \kappa A(T-t)^{2l+2} \qquad \text{and} \qquad v(x,t) \in L^\infty(\mathbb{R}^5 \times (0,T)).$$

Remark 1. The blowup rate of the solution obtained in Theorem 1 is given by

$$\|u(t)\|_\infty \sim \|\mathsf{Q}_{\lambda(t)}\|_\infty \sim \lambda(t)^{-\frac{n-2}{2}} \sim (T-t)^{-3(l+1)} \qquad (l = 1, 2, 3, \cdots).$$

This solution gives an example for $k = l + 1 \geq 2$ in (1.3).

Remark 2. Our strategy is based on so-called the inner-outer gluing method used in [6]. We look for a solution of the form

$$u(x,t) = \mathsf{Q}_{\lambda(t)} + \underbrace{\Theta(x,t) + \lambda^{-\frac{n-2}{2}}\epsilon(y,t) + w(x,t)}_{=v(x,t)}, \qquad y = \frac{x}{\lambda}.$$

The function $\Theta(x, t)$ is a particular solution of $\Theta_t = \Delta_x \Theta$. In [6], they consider a simpler case, where $\Theta(x, t)$ is chosen to be a constant function. In this paper, we try other types of a particular solution $\Theta(x, t)$ satisfying $\Theta(x, t) \sim (T - t)^l$

for $|x| \sim \sqrt{T-t}$. This contributes to the blowup rate. Generally the function $\Theta(x,t)$ can not be chosen arbitrarily, since it must satisfy a certain matching condition (see Section 4). The functions $\epsilon(y,t)$ and $w(x,t)$ represent the remainders in the inner region $|x| \sim \lambda(t)$ and in the self-similar region $|x| \sim \sqrt{T-t}$ respectively. Since the inner part of our solution behaves in the same way as [6], namely $u(x,t) \sim \mathsf{Q}_{\lambda(t)}(x)$, the inner part can be treated in the same manner. On the other hand, we improve their argument in the selfsimilar part. In fact, since our solution $\Theta(x,t)$ is not stable, we introduce additional parameters and choose them carefully. Furthermore to obtain such a solution, we prepare a more elaborate functional space for the fixed point argument (see Section 5.2). Under this setting, we prove that the selfsimilar part of the remainder $w(x,t)$ is sufficiently small. In the selfsimilar region, $w(x,t)$ is governed by the linear heat equation with a certain error term $G(x,t)$.

$$w_t = \Delta_x w + G(x,t)$$

This equation is decomposed into two parts. One represents the effect from the inner part and the other the errors from the approximation procedure. We use a technique given in [6] to deal with the first part, and establish different types of estimates for the second part.

Remark 3. However we obtain the same blowup rate as [8] (see (1.3)), we do not understand the relation between these two methods.

3. Preliminary

Throughout this paper, $\chi(\xi) \in C^\infty(\mathbb{R})$ stands for a standard cut off function satisfying

$$\chi(\xi) = \begin{cases} 1 & \text{if } \xi < 1, \\ 0 & \text{if } \xi > 2. \end{cases}$$

Furthermore we write

$$f(u) = |u|^{p-1}u, \qquad p = \frac{n+2}{n-2}.$$

3.1. Linearization aound the ground state

Let us consider the eigenvalue problem related to a linearization around the ground state $\mathsf{Q}(y) = \mathsf{Q}_\lambda(y)|_{\lambda=1}$.

$$-H_y\psi = \mu\psi \qquad \text{in } \mathbb{R}^n, \tag{3.1}$$

where the operator H_y is defined by

$$H_y = \Delta_y + V(y), \qquad V(y) = f'(\mathsf{Q}(y)) = p\mathsf{Q}(y)^{p-1}.$$

We recall that the operator H_y has a negative eigenvalue $\mu_1 < 0$ and a zero eigenvalu (see Proposition 5.5 p. 37 [7]). We denote by $\psi_1(r)$ a positive radial eigenfunction associated to the negative eigenvalue with $\psi_1(0) = 1$. Furthermore there exists $C > 0$ such that (see p. 18 [2])

$$\psi_1(r) < C\,(1+r)^{-\frac{n-1}{2}}\,e^{-\sqrt{|\mu_1|}\,r}.$$

The eigenfunction associated to a zero eigenvalue is explicitly given by

$$\Lambda_y\mathsf{Q}(y) = \left(\frac{n-2}{2} + y\cdot\nabla_y\right)\mathsf{Q}(y).$$

3.2. Perturbated linearized problem

We next consider the eigenvalue problem (3.1) in a bounded but very large domain.

$$\begin{cases} -H_y\psi = \mu\psi & \text{in } B_R, \\ \psi = 0 & \text{on } \partial B_R, \\ \psi \text{ is radial.} \end{cases} \tag{3.2}$$

We denote the ith eigenvalue of (3.2) by $\mu_i^{(R)}$ and the associated eigenfunction by $\psi_i^{(R)}$. We normalize $\psi_i^{(R)}(r)$ as $\psi_i^{(R)}(0) = 1$. Most of the lemmas stated in this subsection are proved in Section 7 [2]. However for the sake of convenience, we give the proofs. Throughout this subsection, we write

$$k_1 \lesssim k_2 \qquad (k_1, k_2 > 0)$$

if there is a universal constant $c > 0$ independent of R such that $k_1 < ck_2$. These definitions will be changed slightly in Section 5.3.

Lemma 3.1. *It holds that for any $R > 0$*

$$0 < \psi_1^{(R)}(r) \lesssim (1+r)^{-\frac{n-1}{2}} e^{-\sqrt{|\mu_1|}\,r} \qquad \text{for } r \in (0, R).$$

Proof. It is enough to prove $\psi_1^{(R)}(r) < 2\psi_1(r)$. We prove by contradiction. Suppose that there exists $r_1 \in (0, R)$ such that $\psi_1^{(R)}(r) < 2\psi_1(r)$ for $r < r_1$ and $\psi_1^{(R)}(r_1) = 2\psi_1(r_1)$. We now define $\psi_0 \in H_0^1(B_R)$ as

$$\psi_0(r) = \begin{cases} 2\psi_1(r) - \psi_1^{(R)}(r) & \text{for } r < r_1, \\ 0 & \text{for } r_1 < r < R. \end{cases}$$

By the monotonic dependence of the eigenvalue with respect to the domain, it holds that $\mu_1^{(R)} > \mu_1$. Therefore we get

$$\begin{aligned} \int_{B_R} \left(|\nabla_y \psi_0|^2 - V\psi_0^2\right) dy &= \mu_1^{(R)} \int_{B_R} \psi_0^2 dy + (\mu_1 - \mu_1^{(R)}) \int_{B_R} 2\psi_1\psi_0 dy \\ &< \mu_1^{(R)} \int_{B_R} \psi_0^2 dy. \end{aligned}$$

However this contradicts to characterization of $\mu_1^{(R)}$. The proof is completed. □

Lemma 3.2. *There exists $R_1 > 0$ such that if $R > R_1$*

$$|\psi_2^{(R)}(r)| \lesssim (1+r)^{-(n-2)} \qquad \text{for } r \in (0, R).$$

Proof. Put $\psi_2(r) = \Lambda_y \mathsf{Q}(r)/\Lambda_y \mathsf{Q}(0)$. It is clear that ψ_2 gives the eigenfunction of (3.1) associated to a zero eigenvalue and $\psi_2(0) = 1$. Let r_0 and $r_0^{(R)}$ be the unique zero of $\psi_2(r)$ and $\psi_2^{(R)}$ respectively. The Sturm comparison principle implies that $\mu_2^{(R)} > 0$ and $r_0^{(R)} < r_0$. We easily see that $\lim_{R\to\infty} \mu_2^{(R)} = 0$ and $\lim_{R\to\infty} \psi_2^{(R)}(r) = \psi_2(r)$ locally uniformly in $r \in [0, \infty)$. Therefore there exists $R_1 > 0$ such that $\psi_2(r_0+1) < \frac{1}{2}\psi_2^{(R)}(r_0+1) < 0$ if $R > R_1$. We now define $r_1 > r_0$ as

$$r_1 = \sup\left\{ r \in (r_0, r_0+1);\ \psi_2(r) = \frac{1}{2}\psi_2^{(R)}(r) \right\}.$$

From this definition, we immediately see that $\psi_2(r_1) = \frac{1}{2}\psi_2^{(R)}(r_1)$ and

$$\psi_2(r) < \frac{1}{2}\psi_2^{(R)}(r) \quad \text{for } r_1 < r < r_0 + 1.$$

We now suppose that there exists $r_2 > r_1$ such that $\psi_2(r_2) = \frac{1}{2}\psi_2^{(R)}(r_2)$ and

$$\psi_2(r) < \frac{1}{2}\psi_2^{(R)}(r) \quad \text{for } r_1 < r < r_2.$$

However this contradicts the Sturm comparison principle. Therefore we obtain

$$\psi_2(r) < \frac{1}{2}\psi_2^{(R)}(r) < 0 \quad \text{for } r_0 + 1 < r < R.$$

Since $|\psi_2(r)| \lesssim (1+r)^{-(n-2)}$, we complete the proof. □

Lemma 3.3 *(Lemma 7.2 [2]). Let $n \geq 5$. There exists $R_2 > 0$ such that if $R > R_2$*

$$\mu_2^{(R)} \gtrsim R^{-(n-2)}.$$

Proof. We recall that $Z_1(r) = \Lambda_y \mathsf{Q}(r)$ gives a solution of $H_y Z = 0$. Let $Z_2(r) = \Gamma(r)$ be another independent solution of $H_y Z = 0$. Since $\psi_2^{(R)}(r)$ is a solution of (3.2) with $\mu = \mu_2^{(R)}$, it is expressed as

$$\begin{aligned}
\psi_2^{(R)}(r) &= kZ_2(r)\int_0^r \mu_2^{(R)}\psi_2^{(R)}(r')Z_1(r')r'^{n-1}dr' + kZ_1(r)\int_r^R \mu_2^{(R)}\psi_2^{(R)}(r')Z_2(r')r'^{n-1}dr' \\
&\quad - k\frac{Z_2(R)}{Z_1(R)}Z_1(r)\int_0^R \mu_2^{(R)}\psi_2^{(R)}(r')Z_1(r')r'^{n-1}dr' \\
&=: \mu_2^{(R)}(A_1 + A_2 - A_3),
\end{aligned}$$

where k is a constant depending on $Z_1(r)$ and $Z_2(r)$. Since $|Z_2(r)| \lesssim 1 + r^{-(n-2)}$, we easily see that

$$\begin{aligned}
\|A_1\|_{L^2(B_R)} &\lesssim \|\psi_2^{(R)}\|_{L^\infty(B_1)}\|Z_1\|_{L^\infty(B_1)} + R^{\frac{n}{2}}\|\psi_2^{(R)}\|_{L^2(B_R)}\|Z_1\|_{L^2(B_R)}, \\
\|A_2\|_{L^2(B_R)} &\lesssim \|Z_1\|_{L^2(B_R)}\|\psi_2^{(R)}\|_{L^\infty(B_1)} + R^{\frac{n}{2}}\|Z_1\|_{L^2(B_R)}\|\psi_2^{(R)}\|_{L^2(B_R)}, \\
\|A_3\|_{L^2(B_R)} &\lesssim \left|\frac{Z_2(R)}{Z_1(R)}\right| \|Z_1\|_{L^2(B_R)}^2\|\psi_2^{(R)}\|_{L^2(B_R)}.
\end{aligned}$$

Since $\lim_{R\to\infty}\psi_2^{(R)}(r) = \psi_2(r)$ uniformly in $r \in [0, 1]$ (see Lemma 3.2), it follows that $\|\psi_2^{(R)}\|_{L^\infty(B_1)} \lesssim 1$. Therefore when $n \geq 5$, we get from Lemma 3.2 that

$$\|\psi_2^{(R)}\|_{L^2(B_R)} \lesssim \mu_2^{(R)}\left(1 + R^{\frac{n}{2}} + R^{n-2}\right).$$

Since $\lim_{R\to\infty}\|\psi_2^{(R)}\|_{L^2(B_R)} = \|\Lambda_y \mathsf{Q}\|_{L_2(\mathbb{R}^n)}$ if $n \geq 5$ (see Lemma 3.2), we complete the proof. □

3.3. Behavior of the Laplace equation with a perturbation term

Consider a radial solution of

$$\Delta_y p + (1 - \chi_M)V(y)p = 0 \qquad \text{in } \mathbb{R}^n,$$

where $\chi_M(y) = \chi(\frac{|y|}{M})$. Let $p_M(r)$ be a radial solution of this problem satisfying $p_M(r) = 1$ for $r < M$.

Lemma 3.4 *(see proof of Lemma 7.3 [2]). There exist $k \in (0, 1)$ and $M_1 > 0$ such that if $M > M_1$*

$$k < p_M(r) \leq 1 \qquad \text{for } r \in (0, \infty).$$

Proof. Since $V(r) \sim \frac{1}{r^4}$ for $r > 1$, we can take $M_1 > 0$ such that

$$V(r) < \frac{n-3}{r^2} \qquad \text{for } r > M_1.$$

Let $\bar{p}(r) = \frac{M_1^{n-3}}{r^{n-3}}$. It satisfies

$$\Delta_y \bar{p} - \frac{n-3}{r^2}\bar{p} = 0.$$

Therefore by the Sturm comparison principle, it holds that

$$p_{M_1}(r) > \bar{p}(r) = \frac{M_1^{n-3}}{r^{n-3}} \qquad \text{for } r > M_1. \tag{3.3}$$

Let $Z_1(r)$ and $Z_2(r)$ be given in the proof of Lemma 3.3. Since $p_{M_1}(r)$ satisfies $H_y p_{M_1} = 0$ for $r > 2M_1$, there exist two constants $c_1, c_2 \in \mathbb{R}$ such that $p_{M_1}(r) = c_1 Z_1(r) + c_2 Z_2(r)$ for $r > 2M_1$. We recall that $\lim_{r\to\infty} Z_2(r) = \alpha \neq 0$. Since $p_{M_1}(r)$ satisfies (3.3), $c_2\alpha$ must be positive. Therefore it holds that $k := \inf_{r>0} p_{M_1}(r) > 0$. For the case $M > M_1$, by the Sturm comparison principle, we conclude $p_M(r) \geq p_{M_1}(r) \geq k$ for $r > 0$. Since $p_M(r)$ is positive, we easily check that $\partial_r p_M(r) \leq 0$, which implies $p_M(r) < 1$. The proof is completed. □

3.4. The Schauder estimate for parabolic equations

Put $Q = B_1 \times (0, 1)$. For $\alpha \in (0, 1)$, we define the Hölder norm.

$$\|u\|_{C^\alpha(Q)} := \|u\|_{L^\infty(Q)} + [u]_{C^\alpha(Q)}, \qquad [u]_{C^\alpha(Q)} := \sup_{(x_1,t_1),(x_2,t_2)\in Q} \frac{|u(x_1,t_1) - u(x_2,t_2)|}{|x_1 - x_2|^\alpha + |t_1 - t_2|^{\frac{\alpha}{2}}}.$$

We recall the local Höder estimate for parabolic equations.

Lemma 3.5 *(Theorem 4.8 p. 56 [14]). Let $\alpha \in (0, 1)$ and $V(x,t) \in L^\infty(Q)$ satisfy $\|V\|_{L^\infty(Q)} < M$. There exists $c_M > 0$ such that if $u(x,t), \nabla_x u(x,t) \in C^\alpha(Q)$ and $u(x,t)$ satisfies*

$$u_t = \Delta_x u + V(x,t)u + f(x,t) \qquad \text{in } B_1 \times (0, 1),$$

then

$$\|u\|_{C^\alpha(B_{\frac{1}{2}} \times (\frac{1}{2},1))} + \|\nabla_x u\|_{C^\alpha(B_{\frac{1}{2}} \times (\frac{1}{2},1))} < c_M \left(\|u\|_{L^\infty(Q)} + \|f\|_{L^\infty(Q)}\right).$$

3.5. Local behavior of the heat equation

Consider the heat equation

$$u_t = \Delta u \qquad \text{in } \mathbb{R}^n \times (0, \infty). \tag{3.4}$$

To describe the local behavior of solutions, we use self-similar variables.

$$\theta(z,\tau) = u(x,t), \qquad z = \frac{x}{\sqrt{1-t}}, \qquad 1 - t = e^{-\tau}.$$

This function $\theta(z,\tau)$ solves

$$\theta_\tau = A_z\theta \qquad \text{in } \mathbb{R}^n \times (0, \infty), \tag{3.5}$$

where $A_z = \Delta_z - \frac{z}{2} \cdot \nabla_z$. We define the associated wighted L^2 space by

$$L^2_\rho(\mathbb{R}^n) := \{f \in L^2_{\text{loc}}(\mathbb{R}^n);\ \|f\|_\rho < \infty\}, \qquad \|f\|^2_\rho = \int_{\mathbb{R}^n} f(z)^2 \rho(z) dz, \qquad \rho(z) = e^{-\frac{|z|^2}{4}}.$$

The inner product is denoted by

$$(f_1, f_2)_\rho == \int_{\mathbb{R}^n} f_1(z) f_2(z) \rho(z) dz.$$

Consider the eigenvalue problem

$$-\Delta e_i + \frac{z}{2} \cdot e_i = \lambda_l e_i \qquad \text{in } L^2_{\rho,\text{rad}}(\mathbb{R}^n).$$

It is known that

- $\lambda_i = i$ $(i = 0, 1, 2, \cdots)$ and
- $e_i(z) \in L^2_{\rho,\text{rad}}(\mathbb{R}^n)$ is the $2i$th-degree polynomial.

We normalize the eigenfunction $e_i(z)$ as $e_i(0)=1$, which implies

$$e_i(z)=1+a_1|z|^2+a_2|z|^4+\cdots+a_l|z|^{2l}.$$

The function

$$\Theta_i(x,t)=A(T-t)^i e_i(z),\qquad z=\frac{x}{\sqrt{T-t}}\qquad (i=0,1,2,\cdots)\tag{3.6}$$

gives a solution of (3.4). This function plays a crucial role in our argument. The constant A is chosen to be $A=-1$ later. We denote by $e^{A_z\tau}\theta_0$ a solution of (3.5) with the initial datum θ_0 for $\tau=0$. This is expressed by

$$e^{A_z\tau}\theta_0=\frac{c_n}{(1-e^{-\tau})^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{\frac{|e^{-\frac{\tau}{2}}z-\xi|^2}{4(1-e^{-\tau})}}\theta_0(\xi)d\xi.$$

By using this formula, we can obtain the following parabolic estimate for (3.5) (see proof of Lemma 2.2 in [11]).

Lemma 3.6. *There exists $C>0$ such that*

$$|(e^{A_z\tau}\theta_0)(z)|<C\frac{e^{\frac{e^{-\tau}|z|^2}{4(1+e^{-\tau})}}}{(1-e^{-\tau})^{\frac{n}{4}}}\|\theta_0\|_\rho\qquad \text{for } (z,\tau)\in\mathbb{R}^n\times(0,\infty).$$

We prepare another type of parabolic estimates given in [10].

Lemma 3.7 *(Lemma 2.2 in [10]). For any $l\in\mathbb{N}$, there exists $C_l>0$ such that if $\theta_0=|z|^{2l}$*

$$|(e^{A_z\tau}\theta_0)(z)|<C_l\left(1+e^{-l\tau}|z|^{2l}\right)\qquad \text{for } (z,\tau)\in\mathbb{R}^n\times(0,\infty).$$

We next consider the nonhomogeneous heat equation.

$$\begin{cases}\Phi_\tau=A_z\Phi+\dfrac{e^{-\tau}}{\lambda(\tau)^2}\dfrac{1}{1+|y|^{2+a}} & \text{in } \mathbb{R}^n\times(\tau_1,\infty),\\ \Phi=0 & \text{for } \tau=\tau_1,\end{cases}\qquad y=\frac{e^{-\frac{\tau}{2}}}{\lambda(\tau)}z.\tag{3.7}$$

Lemma 3.8 *(Lemma 4.2 p. 8 [6]). Let $a>0$, $\gamma>\frac{1}{2}$ and $T_1=e^{-\tau_1}$. If $\lambda(\tau)$ satisfies*

$$k_1e^{-\gamma\tau}<\lambda<k_2e^{-\gamma\tau}\qquad\text{and}\qquad\left|\frac{d\lambda}{d\tau}\right|<k_3\lambda,$$

there exists $C>0$ depending on a,γ,k_1,k_2,k_3 such that a solution $\Phi(z,\tau)$ of (3.7) satisfies

$$|\Phi(z,\tau)|<C\left(\frac{1}{1+|y|^a}+T_1^{(\gamma-\frac{1}{2})a}\right),\qquad y=\frac{e^{-\frac{\tau}{2}}}{\lambda}z.$$

Proof. We change variable.

$$\psi(x,t)=\Phi(e^{\frac{\tau}{2}}x,\tau),\qquad T_1-t=-e^{-\tau}.$$

The function $\psi(x,t)$ solves

$$\begin{cases}\psi_t=\Delta_x\psi+\dfrac{1}{\lambda^2}\dfrac{1}{1+|y|^{2+a}} & \text{in } \mathbb{R}^n\times(0,T_1),\\ \psi=0 & \text{for } t=0,\end{cases}\qquad y=\frac{x}{\lambda}.$$

From Lemma 2.2 in [6], there exists $C>0$ such that

$$|\psi(x,t)|<C\left(\frac{1}{1+|y|^a}+T_1^{(\gamma-\frac{1}{2})a}\right).$$

The proof is completed. □

4. Formal derivation of blowup speed

We fix $l \in \mathbb{N}$. Throughout this paper, we write (see (3.6))

$$\Theta(x,t) = \Theta_l(x,t) = A(T-t)^l e_l(z), \qquad z = \frac{x}{\sqrt{T-t}}.$$

We look for solutions of the form

$$u(x,t) = \mathsf{Q}_{\lambda(t)}(x) + \Theta(x,t) + v(x,t), \tag{4.1}$$

where $v(x,t)$ is a remainder term. A function $v(x,t)$ satisfies

$$v_t = \Delta v + f'(\mathsf{Q}_\lambda)(\Theta + v) + \mathsf{N}(v) + \frac{\lambda_t}{\lambda^{\frac{n}{2}}} \Lambda_y \mathsf{Q}(y), \qquad y = \frac{x}{\lambda}.$$

The nonlinear term $\mathsf{N}(v)$ is defined by

$$\mathsf{N}(v) = f(\mathsf{Q}_\lambda + \Theta + v) - f(\mathsf{Q}_\lambda) - f'(\mathsf{Q}_\lambda)(\Theta + v).$$

We write $v(x,t)$ as

$$v(x,t) = \lambda^{-\frac{n-2}{2}} \epsilon(y,t), \qquad y = \frac{x}{\lambda}.$$

The relation (4.1) is rewritten as

$$u(x,t) = \mathsf{Q}_{\lambda(t)}(x) + \Theta(x,t) + \epsilon_{\lambda(t)}(x,t).$$

Under this setting, it is natural to assume that

$$|\epsilon(y,s)| \ll \mathsf{Q}(y). \tag{4.2}$$

The function $\epsilon(y,t)$ satisfies

$$\frac{\epsilon_t}{\lambda^{\frac{n-2}{2}}} = \frac{H_y \epsilon}{\lambda^{\frac{n+2}{2}}} + \frac{V(y)}{\lambda^2} \Theta(x,t) + \mathsf{N}(v) + \frac{\lambda_t}{\lambda^{\frac{n}{2}}} \Lambda_y \mathsf{Q}(y) + \frac{\lambda_t}{\lambda^{\frac{n}{2}}} \Lambda_y \epsilon.$$

Neglecting $\mathsf{N}(v)$ and assuming (4.2), we obtain

$$\lambda^2 \epsilon_t \sim H_y \epsilon + \lambda^{\frac{n-2}{2}} V(y) \Theta(x,t) + \lambda \lambda_t \Lambda_y \mathsf{Q}(y).$$

Since $x = \lambda(t) y$ and $\lim_{t \to T} \lambda(t) = 0$, we here replace $\Theta(x,t)$ by $\Theta(0,t)$.

$$\lambda^2 \epsilon_t \sim H_y \epsilon + \lambda^{\frac{n-2}{2}} V(y) \Theta(0,t) + \lambda \lambda_t \Lambda_y \mathsf{Q}(y).$$

We take the inner product $(\cdot, \Lambda_y \mathsf{Q})_{L_y^2(\mathbb{R}^n)}$ to get

$$\lambda^2 (\epsilon_t, \Lambda_y \mathsf{Q})_2 \sim \lambda^{\frac{n-2}{2}} \Theta(0,t) (V, \Lambda_y \mathsf{Q})_2 + \lambda \lambda_t \|\Lambda_y \mathsf{Q}\|_2^2.$$

In addition to (4.2), we assume that the left-hand side is negligible in the relation. Since $\Theta(0,t) = A(T-t)^l$, we obtain

$$0 \sim \lambda^{\frac{n-2}{2}} A(T-t)^l (V, \Lambda_y \mathsf{Q})_2 + \lambda \lambda_t \|\Lambda_y \mathsf{Q}\|_2^2.$$

By a direct calculation, the first term is computed as

$$\begin{aligned}(V, \Lambda_y \mathsf{Q})_2 = (p\mathsf{Q}^{p-1}, \Lambda_y \mathsf{Q})_2 &= \frac{d}{d\lambda} \int_{\mathbb{R}^n} \mathsf{Q}_{\frac{1}{\lambda}}^p dy \bigg|_{\lambda=1} \\ &= \frac{d}{d\lambda} \lambda^{-\frac{n-2}{2}} \|\mathsf{Q}\|_p^p \bigg|_{\lambda=1} = -\frac{n-2}{2} \|\mathsf{Q}\|_p^p.\end{aligned} \tag{4.3}$$

Therefore we obtain a differential equation for $\lambda(t)$.

$$\frac{\lambda_t}{\lambda^{\frac{n-4}{2}}} \sim \frac{A(n-2)}{2} \frac{\|\mathsf{Q}\|_p^p}{\|\Lambda_y \mathsf{Q}\|_2^2} (T-t)^l.$$

Since $\lambda(t)$ must be positive and $\lim_{t\to T} \lambda(t) = 0$, the constant A must be negative. We finally obtain

$$\lambda(t) \sim \left(\frac{-A(6-n)(n-2)}{4(l+1)} \frac{\|\mathsf{Q}\|_p^p}{\|\Lambda_y \mathsf{Q}\|_2^2} \right)^{\frac{2}{6-n}} (T-t)^{\frac{2(l+1)}{6-n}}.$$

This is the desired result. From now on, we choose

$$A = -1.$$

5. Formulation

In this section, we set up our problem as in the proof of Theorem 1 [6]. To justify the argument in Section 4, we need several corrections.

5.1. Setting

We look for solutions of the form

$$u(x,t) = \mathsf{Q}_{\lambda(t)}(x) + \Theta(x,t)\chi_{\mathrm{out}} + v(x,t),$$

where χ_{out} is a cut off function defined by

$$\chi_{\mathrm{out}} = \chi\Big((T-t)^B |z|\Big), \qquad z = \frac{x}{\sqrt{T-t}}, \qquad B = \frac{l+\frac{1}{2}}{2l+2}.$$

A function $v(x,t)$ satisfies

$$v_t = \Delta v + f'(\mathsf{Q}_\lambda)(\Theta\chi_{\mathrm{out}} + v) + \mathsf{N}(v) + \frac{\lambda_t}{\lambda^{\frac{n}{2}}} \Lambda_y \mathsf{Q}(y) + h_{\mathrm{out}}, \tag{5.1}$$

where

$$h_{\mathrm{out}} = 2\nabla_x \Theta \cdot \nabla_x \chi_{\mathrm{out}} + \Theta \Delta_x \chi_{\mathrm{out}} - \Theta \partial_t \chi_{\mathrm{out}}. \tag{5.2}$$

We decompose $v(x,t)$ as

$$v(x,t) = \lambda^{-\frac{n-2}{2}} \epsilon(y,t)\chi_{\mathrm{in}} + w(x,t), \qquad y = \frac{x}{\lambda}. \tag{5.3}$$

The function $\epsilon(y,t)$ is defined on $(y,t) \in B_{2R} \times (0,T)$ and $\chi_{\mathrm{in}} = \chi(\frac{|y|}{R})$. Plugging this into (5.1), we get

$$\frac{\epsilon_t}{\lambda^{\frac{n-2}{2}}} \chi_{\mathrm{in}} + w_t = \frac{H_y \epsilon}{\lambda^{\frac{n+2}{2}}} \chi_{\mathrm{in}} + \Delta w + \frac{V(y)}{\lambda^2} (\Theta\chi_{\mathrm{out}} + v) + \mathsf{N}(v) + \frac{\lambda_t}{\lambda^{\frac{n}{2}}} \Lambda_y \mathsf{Q}(y) + h_{\mathrm{out}} + h_{\mathrm{in}},$$

where

$$h_{\mathrm{in}} = \frac{1}{\lambda^{\frac{n+2}{2}}} \left(2\nabla_y \epsilon \cdot \nabla_y \chi_{\mathrm{in}} + \epsilon \Delta_y \chi_{\mathrm{in}}\right) + \frac{\lambda_t}{\lambda^{\frac{n}{2}}} \Lambda_y \epsilon \chi_{\mathrm{in}} - \frac{1}{\lambda^{\frac{n-2}{2}}} \epsilon \partial_t \chi_{\mathrm{in}}. \tag{5.4}$$

We introduce a parabolic system of $(\epsilon(y,t), w(x,t))$.

$$\begin{cases} \lambda^2 \epsilon_t = H_y \epsilon + G_{\mathrm{in}}(\lambda, w) & \text{in } B_{2R} \times (0,T), \\ w_t = \Delta_x w + G_{\mathrm{out}}(\lambda, w, \epsilon) & \text{in } \mathbb{R}^5 \times (0,T), \end{cases} \tag{5.5}$$

where

$$G_{\mathrm{in}}(\lambda, w) = \lambda^{\frac{n-2}{2}} \Theta(x,t) V + \lambda^{\frac{n-2}{2}} w(x,t) V + \lambda \lambda_t \Lambda_y \mathsf{Q}, \tag{5.6}$$

$$G_{\mathrm{out}}(\lambda, w, \epsilon) = h_{\mathrm{out}} + h_{\mathrm{in}} + \frac{1}{\lambda^2}(1-\chi_{\mathrm{in}}) V(y)(\Theta\chi_{\mathrm{out}} + w) + \frac{\lambda_t}{\lambda^{\frac{n}{2}}}(1-\chi_{\mathrm{in}}) \Lambda_y \mathsf{Q}(y) + \mathsf{N}(v). \tag{5.7}$$

We can check that $v(x,t)$ defied in (5.3) gives a solution of (5.1), if $(\epsilon(y,t), w(x,t))$ solves (5.5). By a lack of boundary condition in the equation for $\epsilon(y,t)$ in (5.5), the problem may not be uniquely solvable. So we appropriately construct a solution $\epsilon(y,t)$ such that $\epsilon(y,t)$ decays enough in the region $|y| \sim R$ (see (6.16)).

5.2. *Fixed point argument*

To construct a solution of (5.5), we apply a fixed point argument. We put

$$\mathcal{W}(x,t) = \begin{cases} (T-t)^l \left(1+|z|^{2l+2}\right) & \text{for } |z| < (T-t)^{-\frac{l}{2l+2}}, \\ \dfrac{1}{1+|x|^2} & \text{for } |z| > (T-t)^{-\frac{l}{2l+2}}, \end{cases} \qquad z = \frac{x}{\sqrt{T-t}}.$$

We fix two small positive constants δ_0 and σ. Let X_σ be the space of all continuous functions on $\mathbb{R}^5 \times [0, T-2\sigma]$ satisfying

$$|w(x,t)| \le \delta_0 \mathcal{W}(x,t) \quad \text{for } t \in [0, T-2\sigma].$$

The metric in X_σ is defined by

$$d_{X_\sigma}(w_1, w_2) = \|w_1 - w_2\|_{C(\mathbb{R}^5 \times [0, T-2\sigma])}. \tag{5.8}$$

We extend $w(x,t)$ to a continuous function on $\mathbb{R}^5 \times [0,T]$.

$$\bar{w}(x,t) = \begin{cases} w(x,t) & \text{if } (x,t) \in \mathbb{R}^5 \times (0, T-2\sigma), \\ \chi_\sigma(t) w(x, T-2\sigma) & \text{if } (x,t) \in \mathbb{R}^5 \times (T-2\sigma, T), \end{cases}$$

where $\chi_\sigma(t)$ is a cut off function satisfying $\chi_\sigma(t) = 1$ if $t \in [0, T - \frac{3}{2}\sigma]$ and $\chi_\sigma(t) = 0$ if $t \in [T-\sigma, T]$. Furthermore we define

$$\tilde{w}(x,t) = \begin{cases} \bar{w}(x,t) & \text{if } t \in [0, T-2\sigma], \\ \delta_0 \mathcal{W}(x,t) & \text{if } t \in [T-2\sigma, T] \text{ and } \bar{w}(x,t) > \delta_0 \mathcal{W}(x,t), \\ \bar{w}(x,t) & \text{if } t \in [T-2\sigma, T] \text{ and } |\bar{w}(x,t)| \le \delta_0 \mathcal{W}(x,t), \\ -\delta_0 \mathcal{W}(x,t) & \text{if } t \in [T-2\sigma, T] \text{ and } \bar{w}(x,t) < -\delta_0 \mathcal{W}(x,t). \end{cases}$$

From this definition, we see that $\tilde{w}(x,t) \in C(\mathbb{R}^5 \times [0,T])$ and

$$|\tilde{w}(x,t)| \le \begin{cases} \delta_0 \mathcal{W}(x,t) & \text{for } (x,t) \in \mathbb{R}^5 \times (0, T-\sigma), \\ 0 & \text{for } (x,t) \in \mathbb{R}^5 \times (T-\sigma, T). \end{cases} \tag{5.9}$$

For given $w(x,t) \in X_\sigma$, we first determine $\lambda(t)$ by the orthogonal condition (6.1). Next we construct $\epsilon(y,t)$ as a solution of

$$\lambda^2 \epsilon_t = H_y \epsilon + G_{\text{in}}(\lambda, \tilde{w}) \qquad \text{in } B_{2R} \times (0,T).$$

After that we solve the problem

$$W_t = \Delta_x W + G_{\text{out}}(\lambda, \tilde{w}, \epsilon) \qquad \text{in } \mathbb{R}^5 \times (0,T).$$

By using this $W(x,t)$, we define the mapping $w(x,t) \mapsto W(x,t)$. The fixed point of this mapping gives the desired solution of (5.5). Finally we take $\sigma \to 0$ to obtain the solution described in Theorem 1. For the rest of paper, we construct the solution mapping in the above procedure.

5.3. Notations

From now on, we assume $R \gg 1$ and choose T as

$$T = e^{-R}.$$

For any positive constant k_1 and k_2, we write

$$k_1 \lesssim k_2$$

if there is a universal constant $c > 0$ independent of R, δ_0, σ such that $k_1 \leq ck_2$.

6. Inner solution

In this section, we repeat the argument in Lemma 4.1 [6] to define the mapping $w(x,t) \in X_\sigma \mapsto \epsilon(y,t)$ mentioned in Section 5.2. Throughout this section, $\tilde{w}(x,t) \in C(\mathbb{R}^5 \times [0,T])$ represents an extension of $w(x,t) \in X_\sigma$ defined in Section 5.2.

6.1. Choice of $\lambda(t)$

We define $\lambda(t) \in C^1([0,T])$ as the unique solution of

$$(\chi_{4R} G_{\text{in}}(t), \Lambda_y \mathsf{Q})_{L_y^2(B_{8R})} = 0 \quad \text{for } t \in (0,T) \qquad \text{and} \qquad \lambda(T) = 0, \tag{6.1}$$

where $G_{\text{in}}(t) = G_{\text{in}}(\lambda(t), \tilde{w}(\lambda(t)y, t))$ and $\chi_{4R}(y) = \chi(\frac{|y|}{4R})$.

Lemma 6.1. *There exists $K > 1$ independent of R, δ_0, σ such that*

$$\left|\lambda(t) - \alpha_l^2 (T-t)^{2l+2}\right| < K\left(\delta_0 + \frac{T-t}{R^2}\right)(T-t)^{2l+2} \qquad \text{for } t \in (0,T),$$

$$\left|\frac{d\lambda}{dt}(t) - (2l+2)\alpha_l^2 (T-t)^{2l+1}\right| < K\left(\delta_0 + \frac{T-t}{R^2}\right)(T-t)^{2l+1} \qquad \text{for } t \in (0,T),$$

where $\alpha_l = -\frac{(\chi_{4R} V, \Lambda_y \mathsf{Q})_{L_y^2(B_{8R})}}{2(l+1)(\chi_{4R}\Lambda_y \mathsf{Q}, \Lambda_y \mathsf{Q})_{L_y^2(B_{8R})}}$.

Proof. From (5.6), the orthogonal condition (6.1) is explicitly given by

$$\sqrt{\lambda}(\chi_{4R} V \Theta(t), \Lambda_y \mathsf{Q})_{L_y^2(B_{8R})} + \sqrt{\lambda}(\chi_{4R} V \tilde{w}(t), \Lambda_y \mathsf{Q})_{L_y^2(B_{8R})} + \frac{d\lambda}{dt}(\chi_{4R}\Lambda_y \mathsf{Q}, \Lambda_y \mathsf{Q})_{L_y^2(B_{8R})} = 0.$$

Put $C(t) = \sqrt{\lambda(t)}$. The function $C(t)$ satisfies

$$\frac{dC}{dt} = -\frac{(\chi_{4R} V \Theta(x,t), \Lambda_y \mathsf{Q})_{L_y^2(B_{8R})} + (\chi_{4R} V \tilde{w}(x,t), \Lambda_y \mathsf{Q})_{L_y^2(B_{8R})}}{2(\chi_{4R}\Lambda_y \mathsf{Q}, \Lambda_y \mathsf{Q})_{L_y^2(B_{8R})}}, \qquad x = C^2 y. \tag{6.2}$$

We now show the unique solvability of the problem (6.2) in

$$\mathcal{S} = \left\{ C(t) \in C([0,T]);\ 0 \leq C(t) \leq \frac{\sqrt{T-t}}{R} \right\}.$$

Let us consider

$$\frac{dD}{dt} = -\frac{(\chi_{4R} V \Theta(x,t), \Lambda_y \mathsf{Q})_{L_y^2(B_{8R})} + (\chi_{4R} V \tilde{w}(x,t), \Lambda_y \mathsf{Q})_{L_y^2(B_{8R})}}{2(\chi_{4R}\Lambda_y \mathsf{Q}, \Lambda_y \mathsf{Q})_{L_y^2(B_{8R})}}, \qquad x = C^2 y.$$

This differential equation defines the mapping $C(t) \mapsto D(t)$. Put $z = \frac{x}{\sqrt{T-t}}$. Since $e_l(z) = 1 + a_1|z|^2 + a_2|z|^4 + \cdots + a_l|z|^{2l}$, the first term on the right-hand side is written as

$$
\begin{aligned}
(\chi_{4R} V\Theta(x,t),\Lambda_y \mathsf{Q})_{L^2_y(B_{8R})} &= -(T-t)^l (Ve_l(z),\Lambda_y\mathsf{Q})_{L^2_y(B_{8R})}\\
&= -(T-t)^l\left((\chi_{4R}V,\Lambda_y\mathsf{Q})_{L^2_y(B_{8R})} + \sum_{k=1}^{l} a_k \left(\chi_{4R}V|z|^{2k},\Lambda_y\mathsf{Q}\right)_{L^2_y(B_{8R})}\right)\\
&= -(T-t)^l\left(\underbrace{(\chi_{4R}V,\Lambda_y\mathsf{Q})_{L^2_y(B_{8R})}}_{=:b_0} + \sum_{k=1}^{l}\underbrace{a_k\left(\chi_{4R}V\left(\frac{|y|}{R}\right)^{2k},\Lambda_y\mathsf{Q}\right)_{L^2_y(B_{8R})}}_{=:b_k}\frac{R^{2k}C(t)^{4k}}{(T-t)^k}\right).
\end{aligned}
$$

Since $|b_k|\lesssim 1$, we easily see that if $C(t)\in\mathcal{S}$

$$
\left|\sum_{k=1}^{l} b_k\frac{R^{2k}C(t)^{4k}}{(T-t)^k}\right| \lesssim \frac{T-t}{R^2} \qquad \text{for } t\in(0,T).
$$

Furthermore when $|z|=\frac{C(t)^2}{\sqrt{T-t}}|y|$ and $C(t)\in\mathcal{S}$, it holds that $|z|<1$ for $y\in B_{8R}$. Therefore since $\tilde{w}(x,t)$ satisfies (5.9), it follows that if $C(t)\in\mathcal{S}$

$$
|(\chi_{4R}V\tilde{w}(C^2y,t),\Lambda_y\mathsf{Q})_{L^2_y(B_{8R})}| \lesssim \delta_0(T-t)^l|(\chi_{4R}V,\Lambda_y\mathsf{Q})_{L^2_y(B_{8R})}| \lesssim \delta_0(T-t)^l.
$$

Therefore since $b_0<0$ (see (4.3)), we get if $C(t)\in\mathcal{S}$

$$
\frac{dD}{dt} = -\left(1+O(\delta_0)+O\left(\frac{T-t}{R^2}\right)\right)\underbrace{\left(\frac{|b_0|}{2(\chi_{4R}\Lambda_y\mathsf{Q},\Lambda_y\mathsf{Q})_{L^2_y(B_{8R})}}\right)}_{=:(l+1)\alpha_l}(T-t)^l.
$$

This implies

$$
D(t) = \left(1+O(\delta_0)+O\left(\frac{T-t}{R^2}\right)\right)\alpha_l(T-t)^{l+1}. \tag{6.3}
$$

Therefore we proved that $D(t)\in\mathcal{S}$ if $C(t)\in\mathcal{S}$. As a consequence, by a fixed point argument, we obtain a solution $C(t)$ of (6.2) satisfying (6.3). Next we prove the uniqueness for solutions of (6.2) in $\mathcal{S}$. Let $C_1(t),C_2(t)\in\mathcal{S}$ be two solutions of (6.2). Since $\tilde{w}(x,t)=0$ for $t\in(T-\sigma,T)$, it is clear that $C_1(t)=C_2(t)$ for $t\in(T-\sigma,T)$. We write $x_1=C_1^2y$, $x_2=C_2^2y$ and $y'=\frac{C_1^2}{C_2^2}y$. By the change of variables, we see that

$$
\begin{aligned}
\left|\frac{d}{dt}(C_1-C_2)\right| &\lesssim \left|\int_{B_{8R}}\chi_{4R}V\tilde{w}(x_1,t)\Lambda_y\mathsf{Q}dy - \int_{B_{8R}}\chi_{4R}V\tilde{w}(x_2,t)\Lambda_y\mathsf{Q}dy\right|\\
&= \left|\int_{B_{8R}}\chi_{4R}V\tilde{w}(x_1,t)\Lambda_y\mathsf{Q}dy - \left(\frac{C_1}{C_2}\right)^{10}\int_{B_{8(\frac{C_2}{C_1})^2R}}\chi_{4R}(y')V(y')\tilde{w}(x_1,t)\Lambda_y\mathsf{Q}(y')dy\right|\\
&\lesssim \left|1-\left(\frac{C_1}{C_2}\right)^{10}\right|\int_{B_{8R}}\left|\chi_{4R}V\tilde{w}(x_1,t)\Lambda_y\mathsf{Q}\right|dy + \left(\frac{C_1}{C_2}\right)^{10}\left|\int_{B_{8R}}\chi_{4R}V\tilde{w}(x_1,t)\Lambda_y\mathsf{Q}dy\right.
\end{aligned}
$$

$$- \int_{B_{8(\frac{C_2}{C_1})^2 R}} \chi_{4R}(y')V(y')\tilde{w}(x,t)\Lambda_y \mathsf{Q}(y')dy \Bigg|.$$

Repeating the above argument, we can verify that $C_1(t), C_2(t)$ satisfy (6.3). Therefore we find that $C_1(t), C_2(t) > \frac{\alpha_l}{2}\sigma^{l+1}$ for $t \in (0, T-\sigma)$. As a consequence, there exists $c_\sigma > 0$ such that

$$\left|\frac{d}{dt}(C_1 - C_2)\right| < c_\sigma |C_1 - C_2| \qquad \text{for } t \in (0, T-\sigma).$$

This assures the uniqueness of solutions in $t \in (0, T-\sigma)$. Therefore the uniqueness of (6.2) in $\mathcal{S}$ is proved. Since $C(t) = \sqrt{\lambda(t)}$, we obtain the conclusion. □

6.2. Construction of $\epsilon(y,t)$

Throughout this subsection, $\lambda(t)$ represents the function given in Lemma 6.1. For simplicity, we write

$$G_{\text{in}}(t) = G_{\text{in}}(\lambda(t), \tilde{w}(\lambda(t)y, t)).$$

We define a function $g(y,t) \in C(\mathbb{R}^5 \times [0,T])$ as a solution of

$$\begin{cases} -H_y g = \chi_{4R} G_{\text{in}}(t) & \text{in } B_{8R}, \\ g = 0 & \text{on } \partial B_{8R}, \\ g(\cdot, t) \text{ is radial.} \end{cases}$$

Since $(\chi_{4R}G_{\text{in}}(t), \Lambda_y \mathsf{Q})_{L^2_y(B_{8R})} = 0$ for $t \in (0,T)$ (see (6.1)), the radial solution $g(r,t)$ is given by

$$g(r,t) = k\Gamma(r)\int_r^{8R} \Lambda_y \mathsf{Q}\chi_{4R}G_{\text{in}}(t)r'^4 dr' - k\Lambda_y\mathsf{Q}(r)\int_r^{8R} \Gamma\chi_{4R}G_{\text{in}}(t)r'^4 dr'$$

for some constant $k \in \mathbb{R}$ depending on $\Lambda_y \mathsf{Q}(r)$ and $\Gamma(r)$. The function $\Gamma(r)$ is a radial solution of $H_y\psi = 0$ given in the proof of Lemma 3.3. From (5.6) and Lemma 6.1, we verify that

$$\begin{aligned} |G_{\text{in}}| &= \left|\lambda^{\frac{3}{2}}\Theta(x,t)V + \lambda^{\frac{3}{2}}\tilde{w}(x,t)V + \lambda\lambda_t\Lambda_y \mathsf{Q}\right| \\ &\lesssim \frac{\lambda^{\frac{3}{2}}(T-t)^l}{1+|y|^4} + \frac{\lambda|\lambda_t|}{1+|y|^3} < \frac{\lambda^{\frac{3}{2}}\lambda^{\frac{l}{2l+2}}}{1+|y|^4} + \frac{\lambda\lambda^{\frac{2l+1}{2l+2}}}{1+|y|^3} \\ &\lesssim \frac{\lambda^{\frac{4l+3}{2l+2}}}{1+|y|^3} \qquad \text{for } |y| < 8R. \end{aligned}$$

Therefore by a direct computation, we get

$$|g(y,t)| \lesssim \frac{\lambda^{\frac{4l+3}{2l+2}}}{1+|y|} \qquad \text{for } |y| < 8R. \tag{6.4}$$

For simplicity, we put

$$\gamma = \frac{4l+3}{2l+2}.$$

We introduce a new time variable s defined by

$$\frac{ds}{dt} = \frac{1}{\lambda(t)^2} \qquad \text{and} \qquad s(t)|_{t=0} = 0. \tag{6.5}$$

Since $\frac{\alpha_l^2}{2}(T-t)^{2l+2} < \lambda < 2\alpha_l^2(T-t)^{2l+2}$ (see Lemma 6.1), it is expressed in the variable s.

$$\frac{\alpha_l^2}{2}\left(\frac{1}{T^{-(4l+3)}+\frac{4l+3}{4}\alpha_l^2 s}\right)^{\frac{2l+2}{4l+3}} < \lambda < 2\alpha_l^2\left(\frac{1}{T^{-(4l+3)}+4(4l+3)\alpha_l^2 s}\right)^{\frac{2l+2}{4l+3}}. \tag{6.6}$$

Let $\mu_1^{(8R)} < 0$ and $\psi_1^{(8R)}(y) \in H_0^1(B_{8R})$ be defined in Section 3.2. We consider

$$\begin{cases} \partial_s E = H_y E + g(y,s) & \text{in } B_{8R}\times(0,\infty),\\ E=0 & \text{on } \partial B_{8R}\times(0,\infty),\\ E=\dfrac{\mathsf{d}_{\text{in}}}{\mu_1^{(8R)}}\psi_1^{(8R)} & \text{for } s=0. \end{cases}$$

The parameter d_{in} is determined below. The desired solution $\epsilon(y,t)$ mentioned in section 5.2 is obtained by $\epsilon(y,t) = H_y E(y,t)$. Let M_1 be the constant given in Lemma 3.4 and fix a large constant $M > M_1$ such that

$$|y|^2 V(y) + |y|^{\frac{5}{2}} e^{-\sqrt{|\mu_1|}|y|} \ll 1 \qquad \text{for } |y| > 2M. \tag{6.7}$$

We first consider

$$\begin{cases} \partial_s E_1 = \Delta_y E_1 + (1-\chi_M)V(y)E_1 + g(y,s) & \text{in } B_{8R}\times(0,\infty),\\ E_1 = 0 & \text{on } \partial B_{8R}\times(0,\infty),\\ E_1 = 0 & \text{for } s=0. \end{cases} \tag{6.8}$$

Since $g(y,t)=0$ near $\partial B_{8R}\times[0,\infty)$, by a certain approximation procedure, we can verify that there exists $\alpha\in(0,1)$ such that

$$E_1,\ \nabla_y E_1,\ \Delta_y E_1,\ \nabla_y\Delta_y E_1 \in C^\alpha(\bar{B}_{8R}\times[0,\infty)). \tag{6.9}$$

Lemma 6.2. *There exists $K_1 > 1$ independent of R, δ_0, σ such that*

$$|E_1(y,s)| + |\nabla E_1(y,s)| \le K_1 R\lambda^\gamma \qquad \text{for } (y,s)\in B_{8R}\times(0,\infty).$$

Proof. To construct a comparison function, we put

$$p(r) = p_1(r)\int_r^{16R}\frac{dr_1}{p_1(r_1)^2 r_1^{n-1}}\int_0^{r_1}\frac{p_1(r_2)r_2^{n-1}}{1+r_2}dr_2,$$

where $p_1(r)$ is a radial function given in Lemma 3.4. The function $p(r)$ gives a positive radial solution of

$$\begin{cases} \Delta_y p + (1-\chi_M)Vp + \dfrac{1}{1+|y|} = 0 & \text{in } B_{16R},\\ p = 0 & \text{on } \partial B_{16R}. \end{cases}$$

Since $k < p_1(r) < 1$ for $r>0$ (see Lemma 3.4), there exist $k_1, k_2 > 0$ independent of M, R such that

$$k_1 R < p(r) < k_2 R \qquad \text{for } r\in(0,8R). \tag{6.10}$$

We now check that $K\lambda^\gamma p(y)$ gives a super-solution of (6.8). Since $|\lambda\frac{d\lambda}{dt}| < (4l+4)\alpha_l^4(T-t)^{4l+3} < (4l+4)\alpha_l^4 T^{4l+3}$ (see Lemma 6.1), we see from (6.5) and (6.10) that

$$\begin{aligned}(\partial_s - \Delta_y - (1-\chi_M)V)\lambda^\gamma p(y) &= \left(\frac{\gamma}{\lambda}\frac{dt}{ds}\frac{d\lambda}{dt}p(y) + \frac{1}{1+|y|}\right)\lambda^\gamma\\ &> \left(-\gamma\lambda\frac{d\lambda}{dt}p(y) + \frac{1}{1+|y|}\right)\lambda^\gamma\\ &> \left(-(4l+4)\gamma\alpha_l^4 k_2 T^{4l+3}R + \frac{1}{1+|y|}\right)\lambda^\gamma.\end{aligned}$$

Since $T = e^{-R}$, it holds that

$$(4l+4)\gamma\alpha_l^4 k_2 T^{4l+3} R < \frac{\frac{1}{2}}{1+8R} < \frac{\frac{1}{2}}{1+|y|} \qquad \text{for } y \in B_{8R}.$$

Therefore we obtain

$$(\partial_s - \Delta_y - (1-\chi_M)V)K\lambda^\gamma p(y) > \frac{K}{2}\frac{\lambda^\gamma}{1+|y|} \qquad \text{for } y \in B_{8R}.$$

Since $|g(y,s)| \lesssim \frac{\lambda^\gamma}{1+|y|}$ (see (6.4)), by a comparison argument, we obtain if $K \gg 1$

$$|E_1(y,s)| < K\lambda^\gamma p(y) < Kk_2 R\lambda^\gamma \qquad \text{for } (y,s) \in B_{8R} \times (0,\infty).$$

Applying a local parabolic estimate in (6.8), we get from (6.4) that

$$\begin{aligned}|\nabla_y E_1(y,s)| &\lesssim \sup_{\min\{s-1,0\}<s'<s} \sup_{|y'-y|<1} \left(|E_1(y',s')| + |g(y',s')|\right) \\ &\lesssim \sup_{\min\{s-1,0\}<s'<s} \left(R\lambda(s')^\gamma + \frac{\lambda(s')^\gamma}{1+|y|}\right).\end{aligned}$$

From (6.6), we can verify that

$$\sup_{\min\{s-1,0\}<s'<s} \lambda(s') \lesssim \lambda(s). \tag{6.11}$$

Therefore we complete the proof. □

Lemma 6.3. *There exists $K_2 > 1$ independent of R, δ_0, σ such that*

$$|\Delta_y E_1(y,s)| + |y| \cdot |\nabla_y \Delta_y E_1(y,s)| < K_2\left(\frac{\lambda^\gamma}{1+|y|} + \frac{R\lambda^\gamma}{1+|y|^2}\right) \qquad \text{for } (y,s) \in B_{6R} \times (0,\infty).$$

Proof. We put

$$e_1 = \Delta_y E_1 + (1-\chi_M)V(y)E_1.$$

We easily see from (6.8)–(6.9) that $e_1(y,s)$ solves

$$\begin{cases} \partial_s e_1 = \Delta_y e_1 + (1-\chi_M)V(y)e_1 + (\Delta_y + (1-\chi_M)V(y))g(y,s) & \text{in } B_{8R} \times (0,\infty), \\ e_1 = 0 & \text{on } \partial B_{8R} \times (0,\infty), \\ e_1 = 0 & \text{for } s = 0. \end{cases}$$

Since $|(\Delta_y + (1-\chi_M)V(y))g(y,s)| < \frac{\lambda^\gamma}{1+|y|^3}$, by the same argument as in the proof of Lemma 6.2, we verify that

$$e_1(y,s) \lesssim \frac{\lambda^\gamma}{1+|y|} \qquad \text{for } (y,s) \in B_{8R} \times (0,\infty). \tag{6.12}$$

To extend $e_1(y,s)$ to $s < 0$, we put

$$\bar{e}_1(y,s) = \begin{cases} e_1(y,s) & \text{if } s \geq 0, \\ 0 & \text{if } s < 0, \end{cases} \qquad \bar{g}(y,s) = \begin{cases} g(y,s) & \text{if } s \geq 0, \\ 0 & \text{if } s < 0. \end{cases}$$

Since $e_1, \nabla_y e_1 \in C^\alpha(\bar{B}_{8R} \times [0,\infty))$ (see (6.9)), we find that $\bar{e}_1, \nabla_x \bar{e}_1 \in C^\alpha(\bar{B}_{8R} \times (-\infty,\infty))$ and $\bar{e}_1(y,s)$ solves

$$\begin{cases} \partial_s \bar{e}_1 = \Delta_y \bar{e}_1 + (1-\chi_M)V(y)\bar{e}_1 + (\Delta_y + (1-\chi_M)V(y))\bar{g}(y,s) & \text{in } B_{8R} \times (-\infty,\infty), \\ \bar{e}_1 = 0 & \text{on } \partial B_{8R} \times (-\infty,\infty). \end{cases}$$

We fix $(y,s) \in B_{6R} \setminus B_1 \times (0,\infty)$. We write $\rho = |y|$ and define

$$\tilde{e}_1(Y,S) = \bar{e}_1\left(y + \frac{\rho}{3}Y, s + \frac{\rho^2}{9}(S-1)\right).$$

We easily see that $\tilde{e}_1(Y,S)$ solves

$$\partial_S \tilde{e}_1 = \Delta_Y \tilde{e}_1 + \frac{\rho^2}{9}(1-\chi_M)V\tilde{e}_1 + \frac{\rho^2}{9}(\Delta_y + 2(1-\chi_M)V)\bar{g} \qquad \text{in } B_1 \times (0,1).$$

Since $|y| = \rho$, it holds that

$$\frac{\rho^2}{9}(1-\chi_M)V \lesssim \frac{\rho^2}{1+|y+\frac{\rho}{2}Y|^4} \lesssim \frac{\rho^2}{1+\rho^4} \qquad \text{for } |Y|<1.$$

Therefore we get from Lemma 3.5 that

$$\|\tilde{e}_1\|_{C^{\alpha}_{(Y,S)}(B_{\frac{1}{2}}\times(\frac{1}{2},1))} + \|\nabla_Y \tilde{e}_1\|_{C^{\alpha}_{(Y,S)}(B_{\frac{1}{2}}\times(\frac{1}{2},1))}$$
$$\lesssim \|\tilde{e}_1\|_{L^{\infty}_{(Y,S)}(B_1\times(0,1))} + \left\|\frac{\rho^2}{9}(\Delta_y + 2(1-\chi_M)V)\bar{g}\right\|_{L^{\infty}_{(Y,S)}(B_1\times(0,1))}.$$

Since $|(\Delta_y + (1-\chi_M)V(y))g(y,s)| < \frac{\lambda^{\gamma}}{1+|y|^3}$, we see that

$$\left\|\frac{\rho^2}{9}(\Delta_y + 2(1-\chi_M)V)\bar{g}\right\|_{L^{\infty}_{(Y,S)}(B_1\times(0,1))} \lesssim \frac{\rho^2}{1+\rho^3}\left(\sup_{\min\{0,s-\frac{\rho^2}{9}\}<s'<s} \lambda(s')\right)^{\gamma}.$$

From (6.6) and $T = e^{-R}$, we verify that

$$\sup_{\min\{0,s-\frac{\rho^2}{9}\}<s'<s} \lambda(s') \lesssim \lambda(s).$$

Since $\nabla_Y \tilde{e}_1(Y,S) = \frac{\rho}{3}\nabla_y \bar{e}_1(y+\frac{\rho}{3}Y, s+\frac{\rho^2}{9}S)$, we deduce from (6.12) that

$$\frac{\rho}{3}|\nabla_y e_1(y,s)| < \|\nabla_Y \tilde{e}_1\|_{C^{\alpha}_{(Y,S)}(B_{\frac{1}{2}}\times(\frac{1}{2},1))}$$
$$\lesssim \|\tilde{e}_1\|_{L^{\infty}_{(Y,S)}(B_1\times(0,1))} + \frac{\rho^2}{1+\rho^3}\lambda(s)^{\gamma}$$
$$\lesssim \frac{\lambda^{\gamma}}{1+\rho}$$

Therefore it follows that

$$|\nabla_y e_1(y,s)| \lesssim \frac{\lambda(s)^{\gamma}}{1+|y|^2} \qquad \text{for } (y,s) \in B_{6R} \setminus B_1 \times (0,\infty).$$

Combining this estimate and (6.12), we obtain

$$|\nabla_y e_1(y,s)| \lesssim \frac{\lambda^{\gamma}}{1+|y|^2} \qquad \text{for } (y,s) \in B_{6R} \times (0,\infty).$$

From definition of $e_1(y,s)$ and Lemma 6.2, we complete the proof. □

Next we put

$$E_2 = E - E_1.$$

The function $E_2(y,s)$ solves

$$\begin{cases} \partial_s E_2 = H_y E_2 + \chi_M V(y) E_1 & \text{in } B_{8R} \times (0,\infty), \\ E_2 = 0 & \text{on } \partial B_{8R} \times (0,\infty), \\ E_2 = \dfrac{\mathsf{d}_{\text{in}}}{\mu_1^{(8R)}} \psi_1^{(8R)} & \text{for } s = 0. \end{cases}$$

We now take

$$\mathsf{d}_{\text{in}} = -\mu_1^{(8R)} \int_0^\infty e^{\mu_1^{(8R)} s'} \left(\chi_M V E_1(s'), \psi_1^{(8R)} \right)_{L_y^2(B_{8R})} ds'$$

and define $\mathsf{c}(s)$ by

$$\begin{cases} \dfrac{d\mathsf{c}}{ds} = -\mu_1^{(8R)} \mathsf{c} + \left(\chi_M V E_1(s), \psi_1^{(8R)} \right)_{L_y^2(B_{8R})}, \\ \mathsf{c}(0) = \dfrac{\mathsf{d}_{\text{in}}}{\mu_1^{(8R)}}. \end{cases}$$

The function $\mathsf{c}(s)$ is explicitly given by

$$\mathsf{c}(s) = -\int_s^\infty e^{-\mu_1^{(8R)}(s-s')} \left(\chi_M V E_1(s'), \psi_1^{(8R)} \right)_{L_y^2(B_{8R})} ds'.$$

From (6.6) and Lemma 6.2, we easily see that

$$|\mathsf{c}(s)| \lesssim R\lambda^\gamma, \qquad |\mathsf{d}_{\text{in}}| \lesssim R\lambda(s)|_{s=0}^\gamma \lesssim RT^{4l+3}. \tag{6.13}$$

We decompose $E_2(y,s)$ as

$$E_2 = v + \mathsf{c}(s)\psi_1^{(8R)}.$$

The function $v(y,s)$ satisfies

$$\begin{cases} \partial_s v = H_y v + V\chi_M E_1 - \left(\chi_M V E_1, \psi_1^{(8R)} \right)_{L_y^2(B_{8R})} \psi_1^{(8R)} & \text{in } B_{8R} \times (0,\infty), \\ v = 0 & \text{on } \partial B_{8R} \times (0,\infty), \\ v = 0 & \text{for } s = 0. \end{cases} \tag{6.14}$$

Lemma 6.4. *There exists $K_3 > 1$ independent of R, δ_0, σ such that*

$$|v(y,s)| + |y| \cdot |\nabla_y v(y,s)| + |y|^2 \cdot |\Delta_y v(y,s)| + |y|^3 \cdot |\nabla_y \Delta_y v(y,s)| < \frac{K_3 R^4 \lambda^\gamma}{1 + |y|^{\frac{5}{2}}}$$

for $(y,s) \in B_{2R} \times (0,\infty)$.

Proof. Since $(v(s), \psi_1^{(8R)})_{L_y^2(B_{8R})} = 0$ for $s \in (0,\infty)$, from Lemma 3.3, there exists $k > 0$ such that

$$(H_y v(s), v(s))_{L_y^2(B_{8R})} < -\frac{k}{R^3} \|v(s)\|_{L_y^2(B_{8R})} \qquad \text{for } s \in (0,\infty).$$

From this estimate and Lemma 6.2, we get

$$e^{\frac{ks}{R^3}} \|v\|_{L_y^2(B_{8R})}^2 \lesssim R^5 \|\chi_M V\|_{L_y^2(B_{8R})}^2 \int_0^s e^{\frac{ks'}{R^3}} \lambda(s')^{2\gamma} ds'.$$

We calculate the integral using Lemma 6.1 and (6.5).

$$\begin{aligned}\int_0^s e^{\frac{ks'}{R^3}}\lambda(s')^{2\gamma}ds' &< \frac{R^3}{k}e^{\frac{ks}{R^3}}\lambda(s)^{2\gamma} - 2\gamma\frac{R^3}{k}\int_0^s e^{\frac{ks'}{R^3}}\lambda^{2\gamma-1}\frac{d\lambda}{ds}ds' \\ &= \frac{R^3}{k}e^{\frac{ks}{R^3}}\lambda(s)^{2\gamma} - 2\gamma\frac{R^3}{k}\int_0^s e^{\frac{ks'}{R^3}}\lambda^{2\gamma+1}\frac{d\lambda}{dt}ds' \\ &\lesssim R^3 e^{\frac{ks}{R^3}}\lambda(s)^{2\gamma} + R^3\int_0^s e^{\frac{ks'}{R^3}}\lambda^{2\gamma+1}\lambda^{\frac{2l+1}{2l+2}}ds' \\ &\lesssim R^3 e^{\frac{ks}{R^3}}\lambda(s)^{2\gamma} + R^3\lambda(s)|_{s=0}^{\frac{4l+3}{2l+2}}\int_0^s e^{\frac{ks'}{R^3}}\lambda^{2\gamma}ds'.\end{aligned}$$

Since $\lambda(s)|_{s=0} = \lambda(t)|_{t=0} \lesssim T^{2l+2}$ and $T = e^{-R}$, it follows that $R^3\lambda(s)|_{s=0}^{\frac{4l+3}{2l+2}} < \frac{1}{2}$. Therefore we obtain

$$\int_0^s e^{\frac{ks'}{R^3}}\lambda(s')^{2\gamma}ds' \lesssim R^3 e^{\frac{ks}{R^3}}\lambda(s)^{2\gamma}.$$

As a consequence, we deduce that

$$\|v(s)\|_{L_y^2(B_{8R})} \lesssim R^4\lambda^\gamma.$$

Applying a local parabolic estimate in (6.14), we get from (6.11) that

$$\|v(s)\|_{L_y^\infty(B_{8R})} \lesssim R^4\lambda^\gamma. \tag{6.15}$$

We now check that $KR^4\lambda^\gamma|y|^{-\frac{5}{2}}$ becomes a super solution for $y \in B_{8R} \setminus B_{2M}$. From (6.7), Lemma 6.1 and (6.5), we see that

$$\begin{aligned}(\partial_s - H_y)\left(\frac{\lambda^\gamma}{|y|^{\frac{5}{2}}}\right) &= \left(\frac{\gamma}{\lambda}\frac{d\lambda}{dt}\frac{dt}{ds} + \frac{5}{4|y|^2} - V(y)\right)\left(\frac{\lambda^\gamma}{|y|^{\frac{5}{2}}}\right) \\ &> \left(\gamma\lambda\frac{d\lambda}{dt} + \frac{5}{8|y|^2}\right)\left(\frac{\lambda^\gamma}{|y|^{\frac{5}{2}}}\right) \\ &> \left(-(4l+4)\gamma\alpha_l^4 T^{4l+3} + \frac{5}{8|y|^2}\right)\left(\frac{\lambda^\gamma}{|y|^{\frac{5}{2}}}\right)\end{aligned}$$

for $y \in B_{8R} \setminus B_{2M}$. Since $T = e^{-R}$, we note that $(4l+4)\gamma\alpha_l^4 T^{4l+3} < \frac{5}{16|y|^2}$ for $y \in B_{8R}$. Therefore we get

$$(\partial_s - H_y)\left(\frac{KR^4\lambda^\gamma}{|y|^{\frac{5}{2}}}\right) > KR^4\left(\frac{5}{16|y|^2}\right)\left(\frac{\lambda^\gamma}{|y|^{\frac{5}{2}}}\right) \qquad \text{for } y \in B_{8R} \setminus B_{2M}.$$

Furthermore from Lemma 3.1, (6.7) and Lemma 6.2, we see that

$$\left(V\chi_M E_1, \psi_1^{(8R)}\right)_{L_y^2(B_{8R})}\psi_1^{(8R)}(y) \lesssim R\lambda^\gamma\frac{e^{-\sqrt{|\mu_1|}\cdot|y|}}{|y|^2} \lesssim \frac{R\lambda^\gamma}{|y|^{2+\frac{5}{2}}} \qquad \text{for } y \in B_{8R} \setminus B_{2M}.$$

Therefore it holds that if $K \gg 1$

$$(\partial_s - H_y)\left(\frac{KR^4\lambda^\gamma}{|y|^{\frac{5}{2}}}\right) > \left(V\chi_M E_1, \psi_1^{(8R)}\right)_{L_y^2(B_{8R})}\psi_1^{(8R)} \qquad \text{for } y \in B_{8R} \setminus B_{2M}.$$

Combining this estimate and (6.15), by a comparison argument in (6.14), we obtain

$$|v(y,s)| < \frac{KR^4\lambda^\gamma}{|y|^{\frac{5}{2}}} \qquad \text{for } (y,s) \in B_{8R} \setminus B_{2M} \times (0,\infty).$$

By the same scaling argument as in the proof of Lemma 6.3, we get

$$|\nabla_y v(y,s)| \lesssim \frac{R^4\lambda^\gamma}{|y|^{\frac{7}{2}}} \qquad \text{for } (y,s) \in B_{6R} \setminus B_{2M} \times (0,\infty).$$

Next we consider the equation for $\partial_{y_i} v(y,s)$. We again use the same scaling argument as above to get

$$|\partial_{y_i} v(y,s)| + |y| \cdot |\nabla_y \partial_{y_i} v(y,s)| \lesssim \frac{R^4\lambda^\gamma}{|y|^{\frac{7}{2}}} \qquad \text{for } (y,s) \in B_{4R} \setminus B_{2M} \times (0,\infty).$$

We finally consider the equation for $\partial_{y_j}\partial_{y_i} v(y,s)$ and obtain

$$|\partial_{y_j}\partial_{y_i} v(y,s)| + |y| \cdot |\nabla_y \partial_{y_j}\partial_{y_i} v(y,s)| \lesssim \frac{R^4\lambda^\gamma}{|y|^{\frac{9}{2}}} \qquad \text{for } (y,s) \in B_{2R} \setminus B_{2M} \times (0,\infty).$$

Since the constant M is independent of R, the proof is completed. □

We now put

$$\epsilon = -H_y E = -H_y \left(E_1 + v + \mathsf{c}(s)\psi_1^{(8R)} \right).$$

Since $-Hg = G_{\text{in}}$ for $|y| < 2R$, it is clear that $\epsilon(y,s)$ satisfies

$$\begin{cases} \partial_s \epsilon = H_y \epsilon + G_{\text{in}} & \text{in } B_{2R} \times (0,\infty), \\ \epsilon = \mathsf{d}_{\text{in}} \psi_1^{(8R)} & \text{for } s = 0. \end{cases}$$

From Lemma 6.2–Lemma 6.4 and (6.13), we conclude

$$\begin{aligned} |\epsilon(y,s)| + |y| \cdot |\nabla_y \epsilon(y,s)| &\lesssim \frac{\lambda^\gamma}{1+|y|} + \frac{R\lambda^\gamma}{1+|y|^2} + \frac{R^4\lambda^\gamma}{1+|y|^{\frac{9}{2}}} + \frac{R\lambda^\gamma e^{-\sqrt{|\mu_1|}\cdot|y|}}{1+|y|^2} \\ &\lesssim \frac{R^4\lambda^\gamma}{1+|y|^{\frac{9}{2}}} \qquad \text{for } (y,s) \in B_{2R} \times (0,\infty). \end{aligned} \tag{6.16}$$

7. Outer solution

We now handle the outer solution $W(x,t)$. A goal of this section is to show $W(x,t) \in X_\sigma$. We recall that $W(x,t) \in X_\sigma$ is defined by

$$W(x,t) \le \begin{cases} \delta_0 (T-t)^l \left(1+|z|^{2l+2}\right) & \text{for } |z| < (T-t)^{-\frac{l}{2l+2}}, \\ \dfrac{\delta_0}{1+|x|^2} & \text{for } |z| > (T-t)^{-\frac{l}{2l+2}}, \end{cases} \qquad z = \frac{x}{\sqrt{T-t}}.$$

The case $l = 0$ is treated in [6]. We here derive more elaborate decay estimates for the case $l \ge 1$ by using the method in [12,13,17,19]. Throughout this section, $\tilde{w}(x,t) \in C(\mathbb{R}^5 \times [0,T])$ represents an extension of $w(x,t) \in X_\sigma$ defined in Section 5.2, and $(\lambda(t), \epsilon(y,t))$ represents a pair of functions obtained in Section 6.

7.1. Choice of parameters

In this section, we consider

$$\begin{cases} W_t = \Delta_x W + G_{\text{out}}(\lambda, \tilde{w}, \epsilon) & \text{in } \mathbb{R}^5 \times (0, T), \\ W = (\mathbf{d} \cdot \mathbf{e})\chi_{\text{out}} & \text{for } t = 0, \end{cases} \tag{7.1}$$

where $\mathbf{d} = (\mathsf{d}_0, \mathsf{d}_1, \cdots, \mathsf{d}_l) \in \mathbb{R}^{l+1}$ is a parameter and

$$\mathbf{e} = (e_0(z), e_1(z), \cdots, e_l(z)), \qquad z = \frac{x}{\sqrt{T-t}}.$$

We recall that $e_i(z)$ is the eigenfunction defined in Section 3.5. We introduce a self-similar transformation.

$$\varphi(z, \tau) = W(z\sqrt{T-t}, t), \qquad T - t = e^{-\tau}.$$

The function $\varphi(z, \tau)$ solves

$$\begin{cases} \varphi_\tau = A_z \varphi + e^{-\tau} G_{\text{out}}(\lambda, \tilde{w}, \epsilon) & \text{in } \mathbb{R}^5 \times (\tau_0, \infty), \\ \varphi = (\mathbf{d} \cdot \mathbf{e})\chi_{\text{out}} & \text{for } \tau = \tau_0 = -\log T. \end{cases} \tag{7.2}$$

We decompose the initial data to the subspace Y_l =span$\{e_0, e_1, \cdots, e_l\}$ and its orthogonal complement in $L^2_\rho(\mathbb{R}^5)$.

$$(\mathbf{d} \cdot \mathbf{e})\chi_{\text{out}} = \sum_{j,k=0}^{l} \frac{\mathsf{d}_j}{\kappa_k^2}(e_j \chi_{\text{out}}, e_k)e_k + \{(\mathbf{d} \cdot \mathbf{e})\chi_{\text{out}}\}^\perp, \qquad \kappa_j = \sqrt{(e_j, e_j)_\rho}.$$

We define $\Phi(z, \tau)$ as

$$\varphi = \mathbf{b}(\tau) \cdot \mathbf{e} + \Phi,$$

where $\mathbf{b}(\tau) = (\mathsf{b}_0(\tau), \mathsf{b}_1(\tau), \cdots, \mathsf{b}_l(\tau)) \in \mathbb{R}^{l+1}$. We easily see that $\Phi(z, \tau)$ satisfies

$$\begin{cases} \Phi_\tau = A_z \Phi + e^{-\tau} G_{\text{out}}(\lambda, \tilde{w}, \epsilon) - \displaystyle\sum_{k=0}^{l} k \mathsf{b}_k e_k - \sum_{k=0}^{l} \frac{d\mathsf{b}_k}{d\tau} e_k & \text{in } \mathbb{R}^5 \times (\tau_0, \infty), \\ \Phi = \left(\displaystyle\sum_{j,k=0}^{l} \frac{\mathsf{d}_j}{\kappa_k^2}(e_j, \chi_{\text{out}} e_k) - \sum_{k=0}^{l} \mathsf{b}_k(\tau_0) \right) e_k + \{(\mathbf{d} \cdot \mathbf{e})\chi_{\text{out}}\}^\perp & \text{for } \tau = \tau_0. \end{cases} \tag{7.3}$$

To obtain a solution $\Phi(z, \tau)$ satisfying $\|\Phi(\tau)\|_\rho = o(e^{-l\tau})$, we choose $\mathbf{b}(\tau)$ as

$$\mathsf{b}_k(\tau) = -e^{-k\tau} \int_\tau^\infty \frac{e^{(k-1)\tau'}}{\kappa_k^2} (G_{\text{out}}(\lambda, \tilde{w}, \epsilon), e_k)_\rho d\tau'.$$

From Lemma 7.1, we verify that

$$\begin{aligned} |(G_{\text{out}}(\lambda, \tilde{w}, \epsilon), e_k)_\rho| &\lesssim |(G_{\text{out}}(\lambda, \tilde{w}, \epsilon), e_k)_{L^2_\rho(|z|<1)}| + e^{-2l\tau} \\ &\lesssim \frac{e^{-l\tau}}{\lambda^2 R^{\frac{1}{4}}} \left(\mathbf{1}_{|y|<2R} + \frac{\mathbf{1}_{|y|>2R}}{|y|^{\frac{11}{4}}}, |e_k| \right)_{L^2_\rho(|z|<1)} + e^{-pl\tau} + e^{-2l\tau} \\ &\lesssim R^{\frac{19}{4}} \lambda^3 e^{-(l-\frac{5}{2})\tau} + R^{-\frac{1}{4}} \lambda^{\frac{3}{4}} e^{-(l-\frac{11}{8})\tau} + e^{-2l\tau} \\ &\lesssim e^{-2l\tau}. \end{aligned}$$

This implies

$$|\mathsf{b}_k(\tau)| \lesssim e^{-(2l+1)\tau}. \tag{7.4}$$

From definition, the parameter $\mathbf{b}(\tau)$ gives a solution of

$$\frac{d\mathsf{b}_k}{d\tau}=-k\mathsf{b}_k+\frac{e^{-\tau}}{\kappa_k^2}(G_{\mathrm{out}}(\lambda,\tilde{w}(x,t),\epsilon(y,t)),e_k)_\rho.$$

We take $\mathbf{d}=(\mathsf{d}_0,\mathsf{d}_1,\cdots,\mathsf{d}_l)$ as

$$(\mathrm{id}+\mathsf{C})\mathbf{d}=\mathbf{b}(\tau_0),$$

where C is a constant $(l+1)\times(l+1)$ matrix defined by

$$\mathsf{C}_{kj}=\frac{1}{\kappa_k^2}(e_j,(1-\chi_{\mathrm{out}})|_{\tau=\tau_0}e_k)_\rho.$$

Since $\chi_{\mathrm{out}}=\chi(\frac{|z|}{e^{B\tau}})$ with $B=\frac{l+\frac12}{2l+2}$, we easily see that $|\mathsf{C}_{kj}|\ll 1$. Therefore we get from (7.4) that

$$|\mathbf{d}|\lesssim|\mathbf{b}(\tau_0)|\lesssim e^{-(2l+1)\tau_0}. \tag{7.5}$$

By the choice of $\mathbf{d}$ and $\mathbf{b}(\tau)$, the equation (7.3) is rewritten as

$$\begin{cases}\Phi_\tau=A_z\Phi+e^{-\tau}G_{\mathrm{out}}^\perp & \text{in } \mathbb{R}^5\times(\tau_0,\infty),\\ \Phi=\{(\mathbf{d}\cdot\mathbf{e})\chi_{\mathrm{out}}\}^\perp & \text{for } \tau=\tau_0,\end{cases} \tag{7.6}$$

where $G_{\mathrm{out}}^\perp=G_{\mathrm{out}}(\lambda,\tilde{w},\epsilon)-\sum_{k=0}^l\frac{e^{-\tau}}{\kappa_k^2}(G_{\mathrm{out}}(\lambda,\tilde{w},\epsilon),e_k)_\rho e_k$.

7.2. *Estimate of G_{out}*

We here provide the estimate of G_{out}. From (5.2), (5.4) and (5.7), we recall that

$$G_{\mathrm{out}}(\lambda,\tilde{w},\epsilon)=h_{\mathrm{out}}+h_{\mathrm{in}}+\frac{1}{\lambda^2}(1-\chi_{\mathrm{in}})V(y)(\Theta\chi_{\mathrm{out}}+\tilde{w})+\frac{\lambda_t}{\lambda^{\frac52}}(1-\chi_{\mathrm{in}})\Lambda_y\mathsf{Q}(y)+\mathsf{N}(v),$$

where

$$\begin{aligned}
&h_{\mathrm{out}}=2\nabla_x\Theta\cdot\nabla_x\chi_{\mathrm{out}}+\Theta\Delta_x\chi_{\mathrm{out}}-\Theta\partial_t\chi_{\mathrm{out}},\\
&h_{\mathrm{in}}=\frac{1}{\lambda^{\frac72}}\left(2\nabla_y\epsilon\cdot\nabla_y\chi_{\mathrm{in}}+\epsilon\Delta_y\chi_{\mathrm{in}}\right)+\frac{\lambda_t}{\lambda^{\frac52}}\Lambda_y\epsilon\chi_{\mathrm{in}}-\frac{1}{\lambda^{\frac32}}\epsilon\partial_t\chi_{\mathrm{in}},\\
&\chi_{\mathrm{in}}=\chi\left(\frac{|y|}{R}\right),\\
&\chi_{\mathrm{out}}=\chi\left(\frac{|z|}{e^{B\tau}}\right)\quad\text{with}\quad B=\frac{l+\frac12}{2l+2},\\
&\mathsf{N}(v)=f(\mathsf{Q}_\lambda+\Theta\chi_{\mathrm{out}}+\epsilon_\lambda\chi_{\mathrm{in}}+\tilde{w})-f(\mathsf{Q}_\lambda)-f'(\mathsf{Q}_\lambda)(\Theta\chi_{\mathrm{out}}+\epsilon_\lambda\chi_{\mathrm{in}}+\tilde{w})
\end{aligned}$$

and

$$y=\frac{x}{\lambda},\qquad z=\frac{x}{\sqrt{T-t}},\qquad \lambda\sim\alpha_l^2(T-t)^{2l+2}.$$

Let $\mathbf{1}_{z\in\Omega}(z)$ be a function on $\mathbb{R}^5$ defined by $\mathbf{1}_{z\in\Omega}(z)=1$ if $z\in\Omega$ and $\mathbf{1}_{z\in\Omega}(z)=0$ if $z\notin\Omega$.

Lemma 7.1. *Let $w \in X_\sigma$ and $(\tilde{w}(x,t), \lambda(t), \epsilon(y,t))$ be given in Section 6. Then*

$$|G_{\rm out}(\lambda, \tilde{w}, \epsilon)| \lesssim \begin{cases} \frac{e^{-l\tau}}{\lambda^2}\frac{1}{R^{\frac{1}{4}}}\left(\frac{\mathbf{1}_{|y|<2R}}{1+|y|^{\frac{9}{4}}} + \frac{1}{1+|y|^{\frac{11}{4}}}\right) + e^{-pl\tau} & \text{for } |z|<1, \\ e^{-2l\tau}|z|^{4l+4} & \text{for } 1<|z|<e^{\frac{l\tau}{2l+2}}, \\ e^{-(l-\frac{1}{2l+2})\tau}|z|^{2l+2}\mathbf{1}_{e^{\frac{(l+\frac{1}{2})\tau}{2l+2}}<|z|<2e^{\frac{(l+\frac{1}{2})\tau}{2l+2}}} + 1 & \text{for } e^{\frac{l\tau}{2l+2}}<|z|<e^{\frac{\tau}{2}}, \\ \frac{e^{-(3l+2)\tau}}{|x|^3} + \frac{\delta_0^2}{|x|^{2p}} & \text{for } |x|>1. \end{cases}$$

Proof. Since $\Theta(x,t) = -e^{-l\tau}e_l(z)$ and $B = e^{\frac{l+\frac{1}{2}}{2l+2}}$, we see that

$$\begin{aligned} |h_{\rm out}| &\lesssim \left|\frac{\partial z_j}{\partial x_i}\right| \cdot |\nabla_z\Theta \cdot \nabla_z\chi_{\rm out}| + \left|\frac{\partial z_j}{\partial x_i}\right|^2 \cdot |\Theta\Delta_z\chi_{\rm out}| + \left|\frac{d\tau}{dt}\right| \cdot |\Theta\partial_\tau\chi_{\rm out}| \\ &\lesssim e^\tau e^{-l\tau}|z|^{2l}\mathbf{1}_{e^{B\tau}<|z|<2e^{B\tau}} \\ &\lesssim e^{-(l-\frac{1}{2l+2})\tau}|z|^{2l+2}\mathbf{1}_{e^{B\tau}<|z|<2e^{B\tau}}. \end{aligned} \tag{7.7}$$

We next estimate $h_{\rm in}$. Since $\frac{e^{-(4l+3)\tau}}{\lambda^{\frac{3}{2}}} \sim e^{-l\tau}$ and $|\lambda\lambda_t| \sim e^{-(4l+3)\tau}$ (see Lemma 6.1), we get from (6.16) that

$$\begin{aligned} |h_{\rm in}| &\lesssim \frac{1}{\lambda^{\frac{7}{2}}}\left(\frac{|\nabla_y\epsilon|}{R} + \frac{|\epsilon|}{R^2}\right)\mathbf{1}_{R<|y|<2R} + \frac{|\lambda_t|}{\lambda^{\frac{5}{2}}}|\Lambda_y\epsilon|\chi_{\rm in} + \frac{1}{\lambda^{\frac{3}{2}}}|\epsilon|\frac{\lambda_t}{\lambda}\mathbf{1}_{R<|y|<2R} \\ &\lesssim \frac{1}{\lambda^2}\frac{1}{\lambda^{\frac{3}{2}}}\left(\frac{|\nabla_y\epsilon|}{R} + \frac{|\epsilon|}{R^2} + |\lambda\lambda_t|\cdot|\epsilon|\right)\mathbf{1}_{R<|y|<2R} + \frac{1}{\lambda^2}\frac{|\lambda\lambda_t|}{\lambda^{\frac{3}{2}}}|\Lambda_y\epsilon|\chi_{\rm in} \\ &\lesssim \frac{e^{-l\tau}}{\lambda^2}\left(\frac{1}{R^{\frac{5}{2}}} + \frac{e^{-(4l+3)\tau}}{R^{\frac{1}{2}}}\right)\mathbf{1}_{R<|y|<2R} + \frac{e^{-l\tau}}{\lambda^2}\frac{R^4e^{-(4l+3)\tau}}{1+|y|^{\frac{9}{2}}}\mathbf{1}_{|y|<2R}. \end{aligned}$$

Therefore since $R = \tau_0$, we deduce that

$$\begin{aligned} |h_{\rm in}| &\lesssim \frac{e^{-l\tau}}{\lambda^2}\frac{1}{R^{\frac{5}{2}}}\mathbf{1}_{R<|y|<2R} + \frac{e^{-l\tau}}{\lambda^2}\frac{e^{-\tau}}{1+|y|^{\frac{9}{2}}}\mathbf{1}_{|y|<2R} \\ &\lesssim \frac{e^{-l\tau}}{\lambda^2}\frac{1}{R^{\frac{1}{4}}}\frac{\mathbf{1}_{|y|<2R}}{1+|y|^{\frac{9}{4}}}. \end{aligned} \tag{7.8}$$

We estimate the third term. Since $\tilde{w}(x,t)$ satisfies (5.9), we verify that

$$\begin{aligned} \frac{1-\chi_{\rm in}}{\lambda^2}V(y)|\Theta\chi_{\rm out} + \tilde{w}| &\lesssim \frac{e^{-l\tau}\mathbf{1}_{|y|>R}}{\lambda^2|y|^4}\left(\left(1+|z|^{2l}\right)\chi_{\rm out} + \delta_0\left(1+|z|^{2l+2}\right)\mathbf{1}_{|z|<e^{\frac{l\tau}{2l+2}}}\right) \\ &\quad + \frac{1}{\lambda^2|y|^4}\frac{\mathbf{1}_{|z|>\frac{l\tau}{2+2}}}{1+|x|^2} \\ &\lesssim \frac{e^{-l\tau}}{\lambda^2|y|^4}\left(\mathbf{1}_{|y|>R}\mathbf{1}_{|z|<1} + |z|^{2l+2}\mathbf{1}_{1<|z|<2e^{\frac{(l+\frac{1}{2})\tau}{2l+2}}}\right) \\ &\quad + \frac{\lambda^2e^{2\tau}}{|z|^4}\mathbf{1}_{e^{\frac{l\tau}{2l+2}}<|z|<e^{\frac{\tau}{2}}} + \frac{\lambda^2}{|x|^6}\mathbf{1}_{|x|>1} \\ &\lesssim \frac{e^{-l\tau}}{\lambda^2|y|^4}\mathbf{1}_{|y|>R}\mathbf{1}_{|z|<1} + e^{-(5l+2)\tau}|z|^{2l-2}\mathbf{1}_{1<|z|<e^{\frac{(l+\frac{1}{2})\tau}{2l+2}}} \end{aligned}$$

$$+ e^{-(4l+4-\frac{2}{l+1})\tau}\mathbf{1}_{e^{\frac{l\tau}{2l+2}}<|z|<e^{\frac{\tau}{2}}} + \frac{e^{-(4l+4)\tau}}{|x|^6}\mathbf{1}_{|x|>1}. \tag{7.9}$$

The fourth term is easily estimated as

$$\begin{aligned}\left|\frac{\lambda_t}{\lambda^{\frac{5}{2}}}(1-\chi_{\rm in})\Lambda_y\mathsf{Q}(y)\right| &< \frac{1}{\lambda^2}\frac{|\lambda_t|}{\lambda^{\frac{3}{2}}}\frac{1}{|y|^3}\mathbf{1}_{|y|>R} < \frac{e^{-l\tau}}{\lambda^2}\frac{1}{|y|^3}\mathbf{1}_{|y|>R}\\ &< \frac{e^{-l\tau}}{\lambda^2}\frac{1}{|y|^3}\mathbf{1}_{|y|>R}\mathbf{1}_{|z|<1} + \lambda e^{\frac{3}{2}\tau}\frac{e^{-l\tau}}{|z|^3}\mathbf{1}_{|z|>1}\\ &< \frac{e^{-l\tau}}{\lambda^2}\frac{1}{|y|^3}\mathbf{1}_{|y|>R}\mathbf{1}_{|z|<1} + e^{-(3l+\frac{1}{2})\tau}\mathbf{1}_{1<|z|<e^{\frac{\tau}{2}}} + \frac{e^{-(3l+2)\tau}}{|x|^3}\mathbf{1}_{|x|>1}.\end{aligned} \tag{7.10}$$

We finally estimate $\mathsf{N}(v)$. Since

$$|\,|a+b|^{p-1}(a+b) - |a|^{p-1}a - p|a|^{p-1}b\,| \lesssim |a|^{p-2}b^2 + |b|^p \qquad \text{for } a,b\in\mathbb{R},$$

we get

$$\begin{aligned}|\mathsf{N}(v)| &= |f(\mathsf{Q}_\lambda+\Theta\chi_{\rm out}+\epsilon_\lambda\chi_{\rm in}+\tilde{w}) - f(\mathsf{Q}_\lambda) - f'(\mathsf{Q}_\lambda)(\Theta\chi_{\rm out}+\epsilon_\lambda\chi_{\rm in}+\tilde{w})|\\ &\lesssim f''(\mathsf{Q}_\lambda)(\Theta\chi_{\rm out}+\epsilon_\lambda\chi_{\rm in}+\tilde{w})^2 + f(\Theta\chi_{\rm out}+\epsilon_\lambda\chi_{\rm in}+\tilde{w})\\ &\lesssim \frac{1}{\sqrt{\lambda}}\frac{\Theta^2\chi_{\rm out}+\epsilon_\lambda^2\chi_{\rm in}+\tilde{w}^2}{1+|y|} + |\Theta|^p\chi_{\rm out} + |\epsilon_\lambda|^p\chi_{\rm in} + |\tilde{w}|^p\\ &\lesssim \left(\frac{1}{\sqrt{\lambda}}\frac{\Theta^2+\tilde{w}^2}{1+|y|} + |\Theta|^p + |\tilde{w}|^p\right)\mathbf{1}_{|z|<1} + \left(\frac{1}{\sqrt{\lambda}}\frac{\epsilon_\lambda^2}{1+|y|} + |\epsilon_\lambda|^p\right)\mathbf{1}_{|z|<2R}\\ &\quad + \left(\frac{1}{\sqrt{\lambda}}\frac{\Theta^2+\tilde{w}^2}{|y|} + |\Theta|^p + |\tilde{w}|^p\right)\mathbf{1}_{1<|z|<e^{\frac{l\tau}{2l+2}}}\\ &\quad + \left(\frac{1}{\sqrt{\lambda}}\frac{\Theta^2}{|y|} + |\Theta|^p\right)\mathbf{1}_{e^{\frac{l\tau}{2l+2}}<|z|<2e^{\frac{(l+\frac{1}{2})\tau}{2l+2}}} + \left(\frac{1}{\sqrt{\lambda}}\frac{\tilde{w}^2}{|y|} + |\tilde{w}|^p\right)\mathbf{1}_{|z|>e^{\frac{l\tau}{2l+2}}}.\end{aligned}$$

Since $|\Theta(x,t)| \lesssim (T-t)^l(1+|z|^{2l})$ and $\tilde{w}(x,t)$ satisfies (5.9), we see that

$$\begin{aligned}\left(\frac{1}{\sqrt{\lambda}}\frac{\Theta^2+\tilde{w}^2}{1+|y|} + |\Theta|^p + |\tilde{w}|^p\right)\mathbf{1}_{|z|<1} &\lesssim \left(\frac{1}{\sqrt{\lambda}}\frac{e^{-2l\tau}}{1+|y|} + e^{-pl\tau}\right)\mathbf{1}_{|z|<1}\\ &\lesssim \frac{e^{-2l\tau}}{\sqrt{\lambda}}\left(\mathbf{1}_{|y|<1} + \frac{|y|^{\frac{7}{4}}}{|y|^{\frac{11}{4}}}\mathbf{1}_{|y|>1}\mathbf{1}_{|z|<1}\right) + e^{-pl\tau}\mathbf{1}_{|z|<1}.\end{aligned}$$

Here we note that $|y|^{\frac{7}{4}}\mathbf{1}_{|z|<1} = (\lambda^{-1}e^{-\frac{\tau}{2}})^{\frac{7}{4}}|z|^{\frac{7}{4}}\mathbf{1}_{|z|<1} < (\lambda^{-1}e^{-\frac{\tau}{2}})^{\frac{7}{4}}\mathbf{1}_{|z|<1}$. Therefore we deduce that

$$\begin{aligned}\left(\frac{1}{\sqrt{\lambda}}\frac{\Theta^2+\tilde{w}^2}{1+|y|} + |\Theta|^p + |\tilde{w}|^p\right)\mathbf{1}_{|z|<1} &\lesssim \frac{e^{-2l\tau}}{\sqrt{\lambda}}\left(1 + \frac{e^{-\frac{7\tau}{8}}}{\lambda^{\frac{7}{4}}}\frac{1}{|y|^{\frac{11}{4}}}\mathbf{1}_{|y|>1}\right)\mathbf{1}_{|z|<1} + e^{-pl\tau}\mathbf{1}_{|z|<1}\\ &\lesssim \frac{1}{\lambda^2}\frac{e^{-(\frac{3l}{2}+\frac{3}{8})\tau}}{1+|y|^{\frac{11}{4}}}\mathbf{1}_{|z|<1} + e^{-pl\tau}\mathbf{1}_{|z|<1}.\end{aligned}$$

Furthermore from (6.16) and $\tau_0 = R$, we verify that

$$\begin{aligned}\left(\frac{1}{\sqrt{\lambda}}\frac{\epsilon_\lambda^2}{1+|y|} + |\epsilon_\lambda|^p\right)\mathbf{1}_{|y|<2R} &\lesssim \frac{1}{\lambda^2}\left(\frac{\lambda^{\frac{3}{2}}}{1+|y|}\frac{R^8e^{-2l\tau}}{1+|y|^9} + \frac{\lambda^2R^{4p}e^{-pl\tau}}{1+|y|^{\frac{9p}{2}}}\right)\mathbf{1}_{|y|<2R}\\ &\lesssim \frac{e^{-l\tau}}{\lambda^2}\left(\frac{R^8e^{-(4l+3)\tau}}{1+|y|^{10}} + \frac{R^{4p}e^{-((p+3)l+4)\tau}}{1+|y|^{\frac{9p}{2}}}\right)\mathbf{1}_{|y|<2R}\end{aligned}$$

$$\lesssim \frac{e^{-l\tau}}{\lambda^2}\left(\frac{e^{-\tau}}{1+|y|^{10}}+\frac{e^{-\tau}}{1+|y|^{\frac{9p}{2}}}\right)\mathbf{1}_{|y|<2R}.$$

For the case $2<p<3$, there exists $c_p>0$ such that

$$|\xi|^p<c_p(|\xi|^2+|\xi|^3) \qquad \text{for } \xi\in\mathbb{R}. \tag{7.11}$$

From this relation, we see that

$$\begin{aligned}
&\left(\frac{1}{\sqrt{\lambda}}\frac{\Theta^2+\tilde{w}^2}{|y|}+|\Theta|^p+|\tilde{w}|^p\right)\mathbf{1}_{1<|z|<e^{\frac{l\tau}{2l+2}}}\\
&\quad\lesssim\left(\frac{1}{\sqrt{\lambda}}\frac{e^{-2l\tau}(|z|^{4l}+\delta_0^2|z|^{4l+4})}{|y|}+e^{-pl\tau}\left(|z|^{2pl}+\delta_0^p|z|^{2p(l+1)}\right)\right)\mathbf{1}_{1<|z|<e^{\frac{l\tau}{2l+2}}}\\
&\quad\lesssim\left(\frac{1}{\sqrt{\lambda}}\frac{e^{-2l\tau}|z|^{4l+4}}{|y|}+\left(e^{-l\tau}|z|^{2(l+1)}\right)^p\right)\mathbf{1}_{1<|z|<e^{\frac{l\tau}{2l+2}}}\\
&\quad\lesssim\left(\frac{\sqrt{\lambda}e^{\frac{\tau}{2}}}{|z|}e^{-2l\tau}|z|^{4l+4}+e^{-2l\tau}|z|^{4(l+1)}+e^{-3l\tau}|z|^{6(l+1)}\right)\mathbf{1}_{1<|z|<e^{\frac{l\tau}{2l+2}}}\\
&\quad\lesssim e^{-2l\tau}|z|^{4l+4}\mathbf{1}_{1<|z|<e^{\frac{l\tau}{2l+2}}}.
\end{aligned}$$

We use (7.11) again to get

$$\begin{aligned}
\left(\frac{1}{\sqrt{\lambda}}\frac{\Theta^2}{|y|}+|\Theta|^p\right)\mathbf{1}_{e^{\frac{l\tau}{2l+2}}<|z|<2e^{\frac{(l+\frac12)\tau}{2l+2}}}&\lesssim\left(\frac{1}{\sqrt{\lambda}}\frac{e^{-2l\tau}|z|^{4l}}{|y|}+e^{-pl\tau}|z|^{2pl}\right)\mathbf{1}_{e^{\frac{l\tau}{2l+2}}<|z|<2e^{\frac{(l+\frac12)\tau}{2l+2}}}\\
&\lesssim\left(e^{-2l\tau}|z|^{4l}+e^{-3l\tau}|z|^{6l}\right)\mathbf{1}_{e^{\frac{l\tau}{2l+2}}<|z|<2e^{\frac{(l+\frac12)\tau}{2l+2}}}\\
&\lesssim e^{-2l\tau}|z|^{4l}\mathbf{1}_{e^{\frac{l\tau}{2l+2}}<|z|<2e^{\frac{(l+\frac12)\tau}{2l+2}}}\\
&\lesssim e^{-(l+1)\tau}|z|^{2l+2}\mathbf{1}_{e^{\frac{l\tau}{2l+2}}<|z|<2e^{\frac{(l+\frac12)\tau}{2l+2}}}.
\end{aligned}$$

We finally estimate the last term.

$$\begin{aligned}
\left(\frac{1}{\sqrt{\lambda}}\frac{\tilde{w}^2}{|y|}+|\tilde{w}|^p\right)\mathbf{1}_{|z|>e^{\frac{l\tau}{2l+2}}}&\lesssim\left(\frac{1}{\sqrt{\lambda}}\frac{1}{|y|}\frac{\delta_0^2}{1+|x|^4}+\frac{\delta_0^p}{1+|x|^{2p}}\right)\mathbf{1}_{|z|>e^{\frac{l\tau}{2l+2}}}\\
&\lesssim\left(\frac{\sqrt{\lambda}e^{\frac{\tau}{2}}}{|z|}+1\right)\mathbf{1}_{e^{\frac{l\tau}{2l+2}}<|z|<e^{\frac{\tau}{2}}}+\left(\frac{\sqrt{\lambda}}{|x|}\frac{\delta_0^2}{|x|^4}+\frac{\delta_0^p}{|x|^{2p}}\right)\mathbf{1}_{|x|>1}\\
&\lesssim\mathbf{1}_{e^{\frac{l\tau}{2l+2}}<|z|<e^{\frac{\tau}{2}}}+\frac{\delta_0^2}{|x|^{2p}}\mathbf{1}_{|x|>1}.
\end{aligned}$$

From the above estimates, we conclude that

$$\begin{aligned}
|\mathsf{N}(v)|\lesssim{}&\frac{e^{-l\tau}}{\lambda^2}\frac{e^{-\frac38\tau}}{1+|y|^{\frac{11}{4}}}\mathbf{1}_{|z|<1}+e^{-pl\tau}\mathbf{1}_{|z|<1}+e^{-2l\tau}|z|^{4l+4}\mathbf{1}_{1<|z|<e^{\frac{l\tau}{2l+2}}}\\
&+e^{-(l+1)\tau}|z|^{2l+2}\mathbf{1}_{e^{\frac{l\tau}{2l+2}}<|z|<2e^{\frac{(l+\frac12)\tau}{2l+2}}}+\mathbf{1}_{e^{\frac{l\tau}{2l+2}}<|z|<e^{\frac{\tau}{2}}}+\frac{\delta_0^2}{|x|^{2p}}\mathbf{1}_{|x|>1}.
\end{aligned}\tag{7.12}$$

Combining (7.7)–(7.10) and (7.12), we complete the proof. □

7.3. $L^2_\rho(\mathbb{R}^5)$ estimate for $\Phi(z,\tau)$

To derive the estimate for a solution $\Phi(z,\tau)$ of (7.6), we first consider

$$\begin{cases} \partial_\tau\Phi_1 = A_z\Phi_1 & \text{in } \mathbb{R}^5\times(\tau_0,\infty), \\ \Phi_1 = \{(\mathbf{d}\cdot\mathbf{e})\chi_{\text{out}}\}^\perp & \text{for } \tau=\tau_0. \end{cases}$$

Lemma 7.2. *There exists $K_1'>1$ independent of R, δ_0, σ such that*

$$|\Phi_1(z,\tau)| < K_1' e^{-(l+1)\tau}\left(1+|z|^{2l+2}\right) \qquad \text{for } (z,\tau)\in\mathbb{R}^5\times(\tau_0,\infty).$$

Proof. We estimate the initial data.

$$\begin{aligned} \{(\mathbf{d}\cdot\mathbf{e})\chi_{\text{out}}\}^\perp &= (\mathbf{d}\cdot\mathbf{e})\chi_{\text{out}} - \sum_{j,k=0}^{l}\frac{\mathsf{d}_j}{\kappa_k^2}(e_j\chi_{\text{out}},e_k)_\rho e_k \\ &= (\mathbf{d}\cdot\mathbf{e})\chi_{\text{out}} - \sum_{j,k=0}^{l}\frac{\mathsf{d}_j}{\kappa_k^2}(e_j,e_k)_\rho e_k + \sum_{j,k=0}^{l}\frac{\mathsf{d}_j}{\kappa_k^2}(e_j(1-\chi_{\text{out}}),e_k)_\rho e_k \\ &= (\mathbf{d}\cdot\mathbf{e})(\chi_{\text{out}}-1) + \sum_{j,k=0}^{l}\frac{\mathsf{d}_j}{\kappa_k^2}(e_j(1-\chi_{\text{out}}),e_k)_\rho e_k. \end{aligned}$$

Since $\chi_{\text{out}} = \chi(\frac{|z|}{e^{B\tau}})$ with $B=\frac{l+\frac12}{2l+2}$, we see from (7.5) that

$$\|\{(\mathbf{d}\cdot\mathbf{e})\chi_{\text{out}}\}^\perp\|_\rho \lesssim |\mathbf{d}| \lesssim e^{-(2l+1)\tau_0}.$$

Therefore we deduce that

$$\|\Phi_1(\tau)\|_\rho \lesssim \|\{(\mathbf{d}\cdot\mathbf{e})\chi_{\text{out}}\}^\perp\|_\rho e^{-(l+1)(\tau-\tau_0)} \lesssim e^{-(l+1)\tau}. \tag{7.13}$$

We next derive a pointwise estimate. Let $e_{l+1}(z)$ be given in Section 3.5, which is written as

$$e_{l+1}(z) = 1 + a_1|z|^2 + a_2|z|^4 + \cdots + a_{l+1}|z|^{2l+2}.$$

To construct a comparison function, we define

$$\tilde e_{l+1}(z) = k e_{l+1}(z) \qquad \text{with} \quad k = \begin{cases} 1 & \text{if } a_{l+1}>0, \\ -1 & \text{if } a_{l+1}<0. \end{cases}$$

From this definition, there exists $r_{l+1}>0$ such that

$$\tilde e_{l+1}(z) > \frac{|a_{l+1}|}{2}|z|^{2l+2} \qquad \text{for } |z|>r_{l+1}.$$

Therefore we note from (7.13) that there exists $K>1$ such that $|\Phi_1(z,\tau)| < Ke^{-(l+1)\tau}\tilde e_{l+1}(z)$ for $|z|=r_{l+1}$. Furthermore it holds from (7.5) that $|\Phi_1(\tau_0)| < |\mathbf{d}|\cdot|\mathbf{e}(z)| \lesssim e^{-(2l+1)\tau_0}|z|^{2l}$ for $|z|>r_{l+1}$. Therefore a comparison argument shows that there exists $K'>1$ such that

$$|\Phi_1(z,\tau)| < K'e^{-(l+1)\tau}\tilde e_{l+1}(z) \qquad \text{for } |z|>r_{l+1},\ \tau>\tau_0.$$

This completes the proof. □

Next we write Φ as

$$\Phi = \Phi_1 + \Phi_2.$$

The function $\Phi_2(z,\tau)$ solves

$$\begin{cases} \partial_\tau \Phi_2 = A_z \Phi_2 + e^{-\tau} G_{\mathrm{out}}^{\perp} & \text{in } \mathbb{R}^5 \times (\tau_0, \infty), \\ \Phi_2 = 0 & \text{for } \tau = \tau_0. \end{cases} \tag{7.14}$$

We first provide L^2_ρ estimates of G_{out}.

Lemma 7.3. *It holds that*

$$\|G_{\mathrm{out}}\|_\rho \lesssim e^{-(2l-\frac{1}{2})\tau}.$$

Proof. From Lemma 7.1, we easily verify that

$$\begin{aligned} \|G_{\mathrm{out}}\|_\rho &\lesssim \|G_{\mathrm{out}} \mathbf{1}_{|z|<1}\|_\rho + \|G_{\mathrm{out}} \mathbf{1}_{|z|>1}\|_\rho \\ &\lesssim \|G_{\mathrm{out}} \mathbf{1}_{|z|<1}\|_\rho + e^{-2l\tau}. \end{aligned}$$

We estimate the first term.

$$\begin{aligned} \|G_{\mathrm{out}} \mathbf{1}_{|z|<1}\|_\rho &\lesssim \frac{e^{-l\tau}}{\lambda^2 R^{\frac{1}{4}}} \left(\left\| \frac{\mathbf{1}_{|y|<2R}}{1+|y|^{\frac{9}{4}}} \right\|_{L^2(|z|<1)} + \left\| \frac{\mathbf{1}_{|y|<2R} + \mathbf{1}_{|y|>2R}}{1+|y|^{\frac{11}{4}}} \right\|_{L^2(|z|<1)} \right) + e^{-pl\tau} \\ &\lesssim \frac{e^{-l\tau}}{\lambda^2 R^{\frac{1}{4}}} \left(\left(R\lambda e^{\frac{\tau}{2}} \right)^{\frac{5}{2}} + \left\| \frac{\mathbf{1}_{|y|>2R}}{|y|^{\frac{11}{4}}} \right\|_{L^2(|z|<1)} \right) + e^{-pl\tau} \\ &\lesssim \frac{e^{-l\tau}}{\lambda^2 R^{\frac{1}{4}}} \left(\left(R\lambda e^{\frac{\tau}{2}} \right)^{\frac{5}{2}} + \left(\lambda e^{\frac{\tau}{2}} \right)^{\frac{11}{4}} \left(R\lambda e^{\frac{\tau}{2}} \right)^{-\frac{1}{4}} \right) + e^{-pl\tau} \\ &\lesssim R^{\frac{9}{4}} e^{-\frac{\tau_0}{4}} e^{-(2l-\frac{1}{2})\tau} + e^{-pl\tau}. \end{aligned}$$

Since $\tau_0 = R$, we complete the proof. □

From these estimates, we immediately obtain L^2_ρ estimates of $\Phi_2(z,\tau)$.

Lemma 7.4. *There exists $K_2' > 1$ independent of R, δ_0, σ such that*

$$\|\Phi_2(\tau)\|_\rho < K_2' e^{-(l+1)\tau}.$$

Proof. We take the inner product $(\cdot, \Phi_2)_\rho$ in (7.14) to get

$$\frac{1}{2}\frac{d}{d\tau}\|\Phi_2\|_\rho^2 = (A_z \Phi_2, \Phi_2)_\rho + e^{-\tau}(G_{\mathrm{out}}^{\perp}, \Phi_2)_\rho.$$

Since $(A_z\Phi_2, \Phi_2) < -(l+1)\|\Phi_2\|_\rho^2$, we deduce from Lemma 7.3 that

$$\|\Phi_2(\tau)\|_\rho \lesssim e^{-(l+1)\tau}.$$

The proof is completed. □

7.4. Pointwise estimate for $\Phi_2(z,\tau)$ in $|z| < e^{\frac{l\tau}{2l+2}}$

Since $\Phi_2(z,\tau)$ is a solution of (7.14), it is written as an integral form.

$$\Phi_2(z,\tau) = \int_{\tau_0}^{\tau} e^{A_z(\tau-\tau')e^{-\tau'}} G_{\mathrm{out}}^{\perp} d\tau'. \tag{7.15}$$

We here estimate $\Phi_2(z,\tau)$ by the same manner as in [12,13] (see also [17,19] and Section 5.2 [11]). For simplicity we put

$$b = \min\left\{ \frac{2l+\frac{3}{2}}{4}, \frac{2l+1}{2l+2} \right\}.$$

Proposition 7.1. *There exists $K_3' > 1$ independent of R, δ_0, σ such that*

$$\int_{\tau_0}^{\tau} e^{A_z(\tau-\tau')}e^{-\tau'}|G_{\rm out}^{\perp}|d\tau' < K_3'\left(\frac{1}{R^{\frac14}}\frac{e^{-l\tau}}{1+|y|^{\frac14}} + T^b e^{-l\tau} + \frac{e^{-l\tau}}{R^{\frac14}}|z|^{-2l} + e^{-(2l+1)\tau}|z|^{4l+4} + e^{-(l+\frac{2l+1}{2l+2})\tau}|z|^{2l+2}\right) \qquad \text{for } (z,\tau)\in\mathbb{R}^n\times(\tau_0,\infty).$$

As a consequence of this proposition, we obtain

$$\int_{\tau_0}^{\tau} e^{A_z(\tau-\tau')}e^{-\tau'}|G_{\rm out}^{\perp}|d\tau' \lesssim \frac{1}{R^{\frac14}}\frac{e^{-l\tau}}{1+|y|^{\frac14}} + T^b e^{-l\tau} + \frac{e^{-l\tau}}{R^{\frac14}}|z|^{-2l} + e^{-(l+\frac{2l+1}{2l+2})\tau}|z|^{2l+2} \qquad \text{for } |z|<e^{\frac{l\tau}{2l+2}},\ \tau\in(\tau_0,\infty). \tag{7.16}$$

For simplicity of computation, we arrange the estimate in Lemma 7.1 as

$$\begin{aligned}|G_{\rm out}| &\lesssim \frac{e^{-l\tau}}{\lambda^2}\frac{1}{R^{\frac14}}\frac{1}{1+|y|^{\frac94}} + \left(e^{-pl\tau}+e^{-2l\tau}|z|^{4l+4}+e^{-(l-\frac{1}{2l+2})\tau}|z|^{2l+2}\right) \qquad \text{for } z\in\mathbb{R}^5\\ &=: G_{\rm out}^{(1)} + G_{\rm out}^{(2)}.\end{aligned}$$

We first prepare the following lemma.

Lemma 7.5. *There exists $k'>0$ independent of τ_1, R, δ_0, σ such that*

$$\int_{\tau_1}^{\tau} e^{A_z(\tau-\tau')}e^{-\tau'}|G_{\rm out}^{\perp}|d\tau' < k'\left(\frac{1}{R^{\frac14}}\frac{e^{-l\tau_1}}{1+|y|^{\frac14}} + T^b e^{-l\tau_1} + e^{-(2l+1)\tau}|z|^{4l+4} + e^{-(l+\frac{2l+1}{2l+2})\tau}|z|^{2l+2}\right) \qquad \text{for } (z,\tau)\in\mathbb{R}^5\times(\tau_1,\infty).$$

Proof. We write the integral as

$$\int_{\tau_1}^{\tau} e^{A_z(\tau-\tau')}e^{-\tau'}|G_{\rm out}^{\perp}|d\tau' \lesssim \int_{\tau_1}^{\tau} e^{A_z(\tau-\tau')}e^{-\tau'}\left(\left|G_{\rm out}^{(1)}\right|+\left|G_{\rm out}^{(2)}\right|\right)d\tau' + \sum_{k=0}^{l}\int_{\tau_1}^{\tau}\left|e^{A_z(\tau-\tau')}\frac{e^{-\tau'}}{\kappa_k^2}(G_{\rm out},e_k)_\rho e_k\right|d\tau'.$$

From Lemma 3.7, we see that

$$\begin{aligned}\int_{\tau_1}^{\tau} e^{A_z(\tau-\tau')}e^{-\tau'}\left|G_{\rm out}^{(2)}\right|d\tau' &\lesssim \int_{\tau_1}^{\tau}\Big\{e^{-(pl+1)\tau'} + e^{-(2l+1)\tau'}\left(1+e^{-(2l+2)(\tau-\tau')}|z|^{4l+4}\right)\\ &\qquad + e^{-(l+\frac{2l+1}{2l+2})\tau'}\left(1+e^{-(l+1)(\tau-\tau')}|z|^{2l+2}\right)\Big\}d\tau'\\ &\lesssim e^{-(l+\frac{2l+1}{2l+2})\tau_1} + e^{-(2l+1)\tau}|z|^{4l+4} + e^{-(l+\frac{2l+1}{2l+2})\tau}|z|^{2l+2}\\ &\lesssim T^{\frac{2l+1}{2l+2}}e^{-l\tau_1} + e^{-(2l+1)\tau}|z|^{4l+4} + e^{-(l+\frac{2l+1}{2l+2})\tau}|z|^{2l+2}.\end{aligned}$$

Since $|a|^k \lesssim 1+|a|^{l+1}$ for $k=0,1,2,\cdots,l$, we get from Lemma 7.3 that

$$\begin{aligned}\int_{\tau_1}^{\tau}\left|e^{A_z(\tau-\tau')}e^{-\tau'}(G_{\mathrm{out}},e_k)_\rho e_k\right|d\tau' &= \int_{\tau_1}^{\tau}\left|e^{-k(\tau-\tau')}e^{-\tau'}(G_{\mathrm{out}},e_k)_\rho e_k\right|d\tau'\\ &\lesssim e^{-(2l+\frac{1}{2})\tau_1}\left(e^{-(\tau-\tau_1)}\left(1+|z|^2\right)\right)^k\\ &\lesssim e^{-(2l+\frac{1}{2})\tau_1}\left\{1+\left(e^{-(\tau-\tau_1)}\left(1+|z|^2\right)\right)^{l+1}\right\}\\ &\lesssim e^{-(2l+\frac{1}{2})\tau_1}+e^{-(l-\frac{1}{2})\tau_1}e^{-(l+1)\tau}|z|^{2l+2}.\end{aligned}$$

Furthermore Lemma 3.8 implies

$$\int_{\tau_1}^{\tau}e^{A_z(\tau-\tau')}e^{-\tau'}\left|G_{\mathrm{out}}^{(1)}\right|d\tau' \lesssim \frac{1}{R^{\frac{1}{4}}}\frac{e^{-l\tau_1}}{1+|y|^{\frac{1}{4}}}+T^{(2l+\frac{3}{2})\frac{1}{4}}e^{-l\tau_1}.$$

The proof is completed. $\square$

To obtain the estimate in Proposition 7.1, we consider four cases separately.

$$\begin{aligned}&\text{(i)} && z\in\mathbb{R}^n && \text{and} && \tau\in(\tau_0,\tau_0+1),\\ &\text{(ii)} && |z|<4 && \text{and} && \tau\in(\tau_0+1,\infty),\\ &\text{(iii)} && 2<|z|<e^{\frac{\tau-\tau_0}{2}} && \text{and} && \tau\in(\tau_0+1,\infty),\\ &\text{(iv)} && |z|>e^{\frac{\tau-\tau_0}{2}} && \text{and} && \tau\in(\tau_0+1,\infty).\end{aligned}$$

7.4.1. (i) Estimate in $z\in\mathbb{R}^n$ and $\tau\in(\tau_0,\tau_0+1)$

In this case, the estimate follows from Lemma 7.5.

7.4.2. (ii) Estimate in $|z|<4$ and $\tau\in(\tau_0+1,\infty)$

We divide the integral in (7.15) into two parts.

$$\Phi_2(\tau)=\int_{\tau_0}^{\tau-1}e^{A_z(\tau-\tau')}e^{-\tau'}G_{\mathrm{out}}^{\perp}d\tau'+\int_{\tau-1}^{\tau}e^{A_z(\tau-\tau')}e^{-\tau'}G_{\mathrm{out}}^{\perp}d\tau'.$$

We apply Lemma 3.6 and Lemma 7.3 to get

$$\begin{aligned}\left|\int_{\tau_0}^{\tau-1}e^{A_z(\tau-\tau')}e^{-\tau'}G_{\mathrm{out}}^{\perp}d\tau'\right| &\lesssim \int_{\tau_0}^{\tau-1}\left|e^{A_z\frac{1}{2}}e^{A_z(\tau-\tau'-\frac{1}{2})}e^{-\tau'}G_{\mathrm{out}}^{\perp}\right|d\tau'\\ &\lesssim \int_{\tau_0}^{\tau-1}\frac{\frac{e^{-\frac{1}{2}|z|^2}}{4(1+e^{-\frac{1}{2}})}}{(1-e^{-\frac{1}{2}})^{\frac{n}{4}}}\left\|e^{A_z(\tau-\tau'-\frac{1}{2})}e^{-\tau'}G_{\mathrm{out}}^{\perp}\right\|_\rho d\tau'\\ &\lesssim \int_{\tau_0}^{\tau-1}e^{-(l+1)(\tau-\tau')}\left\|e^{-\tau'}G_{\mathrm{out}}^{\perp}\right\|_\rho d\tau'\\ &\lesssim \int_{\tau_0}^{\tau-1}e^{-(l+1)(\tau-\tau')}e^{-(2l+\frac{1}{2})\tau'}d\tau'\\ &\lesssim e^{-(l+1)\tau}\qquad \text{for } |z|<4,\ \tau\in(\tau_0+1,\infty).\end{aligned}$$

The second integral is reduced to Lemma 7.5.

7.4.3. (iii) Estimate in $2<|z|<e^{\frac{\tau-\tau_0}{2}}$ and $\tau\in(\tau_0+1,\infty)$

For $2<|z|<e^{\frac{\tau-\tau_0}{2}}$, we define $\tau_1\in(\tau_0,\tau-1)$ by

$$|z|=e^{\frac{\tau-\tau_1}{2}}.$$

We write $\Phi_2(z,\tau)$ as

$$\Phi_2(\tau)=\int_{\tau_0}^{\tau_1}e^{A_z(\tau-\tau')}e^{-\tau'}G_{\rm out}^{\perp}d\tau'+\int_{\tau_1}^{\tau-1}e^{A_z(\tau-\tau')}e^{-\tau'}G_{\rm out}^{\perp}d\tau'+\int_{\tau-1}^{\tau}e^{A_z(\tau-\tau')}e^{-\tau'}G_{\rm out}^{\perp}d\tau'.$$

Since $\tau_0<\tau_1<\tau-1$ and $|z|=e^{\frac{\tau-\tau_1}{2}}$, we get from Lemma 3.6 and Lemma 7.3 that

$$\begin{aligned}\left|\int_{\tau_0}^{\tau_1}e^{A_z(\tau-\tau')}e^{-\tau'}G_{\rm out}^{\perp}d\tau'\right|&\lesssim\int_{\tau_0}^{\tau_1}\left|e^{A_z(\tau-\tau_1)}e^{A_z(\tau-\tau'+\tau_1)}e^{-\tau'}G_{\rm out}^{\perp}\right|d\tau'\\&\lesssim\int_{\tau_0}^{\tau_1}\frac{e^{\frac{e^{-(\tau-\tau_1)}|z|^2}{4(1+e^{-\frac{1}{2}})}}}{(1-e^{-(\tau-\tau_1)})^{\frac{n}{4}}}\left\|e^{A_z(\tau_1-\tau')}e^{-\tau'}G_{\rm out}^{\perp}\right\|_{\rho}d\tau'\\&\lesssim\int_{\tau_0}^{\tau_1}e^{-(l+1)(\tau_1-\tau')}\left\|e^{-\tau'}G_{\rm out}^{\perp}\right\|_{\rho}d\tau'\\&\lesssim\int_{\tau_0}^{\tau_1}e^{-(l+1)(\tau_1-\tau')}e^{-(2l+\frac{1}{2})\tau'}d\tau'\\&\lesssim e^{-(l+1)\tau}.\end{aligned}$$

We next estimate the second term. We apply Lemma 7.5 to get

$$\int_{\tau_1}^{\tau}e^{A_z(\tau-\tau')}e^{-\tau'}|G_{\rm out}^{\perp}|d\tau'\lesssim\frac{1}{R^{\frac{1}{4}}}\frac{e^{-l\tau_1}}{1+|y|^{\frac{1}{4}}}+T^be^{-l\tau_1}+e^{-(2l+1)\tau}|z|^{4l+4}+e^{-(l+\frac{2l+1}{2l+2})\tau}|z|^{2l+2}.$$

Since $|z|=e^{\frac{\tau-\tau_1}{2}}$, it holds that $e^{-l\tau_1}=e^{-l\tau}|z|^{2l}$. Therefore we deduce that

$$\begin{aligned}\int_{\tau_1}^{\tau}e^{A_z(\tau-\tau')}e^{-\tau'}|G_{\rm out}^{\perp}|d\tau'&\lesssim\frac{1}{R^{\frac{1}{4}}}\frac{e^{-l\tau}|z|^{2l}}{1+|y|^{\frac{1}{4}}}+T^be^{-l\tau}|z|^{2l}+e^{-(2l+1)\tau}|z|^{4l+4}+e^{-(l+\frac{2l+1}{2l+2})\tau}|z|^{2l+2}\\&\lesssim\frac{e^{-l\tau}}{R^{\frac{1}{4}}}|z|^{2l}+e^{-(2l+1)\tau}|z|^{4l+4}+e^{-(l+\frac{2l+1}{2l+2})\tau}|z|^{2l+2}.\end{aligned}$$

The estimate for the last term is derived from Lemma 7.5.

7.4.4. (iv) Estimate in $|z|>e^{\frac{\tau-\tau_0}{2}}$ and $\tau\in(\tau_0+1,\infty)$

We get from Lemma 7.5 that

$$\begin{aligned}\int_{\tau_0}^{\tau}e^{A_z(\tau-\tau')}e^{-\tau'}|G_{\rm out}^{\perp}|d\tau'<c\Bigg(&\frac{1}{R^{\frac{1}{4}}}\frac{e^{-l\tau_0}}{1+|y|^{\frac{1}{4}}}+T^be^{-l\tau_0}+e^{-(2l+1)\tau}|z|^{4l+4}\\&+e^{-(l+\frac{2l+1}{2l+2})\tau}|z|^{2l+2}\Bigg)\qquad\text{for }(z,\tau)\in\mathbb{R}^5\times(\tau_1,\infty).\end{aligned}$$

Since $|z|>e^{\frac{\tau-\tau_0}{2}}$, it holds that $e^{-l\tau_0}<e^{-l\tau}|z|^{2l}$. Therefore we obtain the desired estimate. Combining estimates (i)–(iv), we complete the proof of Proposition 7.1.

7.5. Pointwise estimate for $\Phi(z,\tau)$ in $|z| > e^{\frac{l\tau}{2l+2}}$

We go back to the function $\varphi(z,\tau)$ defined in (7.2).

Lemma 7.6. *There exists $K_4' > 0$ independent of R, δ_0, σ such that*

$$|\varphi(z,\tau)| < K_4' T^{\frac{l}{l+1}} \qquad \text{for } |z| > e^{\frac{l\tau}{2l+2}},\ \tau > \tau_0.$$

Proof. We recall that

$$\varphi(z,\tau) = \mathbf{b}(\tau)\cdot\mathbf{e} + \Phi_1(z,\tau) + \Phi_2(z,\tau).$$

We now check that

$$\bar{\varphi}(z,\tau) = K\left(2e^{-\frac{l\tau_0}{l+1}} - e^{-\frac{l\tau}{l+1}}\right)$$

gives a super solution in $|z| > e^{\frac{l\tau}{2l+2}}$. From (7.4), Lemma 7.2 and (7.16), we note that

$$|\varphi(z,\tau)| \lesssim e^{-(2l+1)\tau}|z|^{2l} + e^{-(l+1)\tau}|z|^{2l+2} + e^{-l\tau}|z|^{2l} + e^{-(l+\frac{2l+1}{2l+2})\tau}|z|^{2l+2}$$

for $|z| > 1$. Therefore we get

$$|\varphi(z,\tau)| \lesssim e^{-\frac{l\tau}{l+1}} \qquad \text{for } |z| = e^{\frac{l\tau}{2l+2}}.$$

Furthermore by (7.5), we see that the initial data satisfies

$$\begin{aligned}|\varphi(\tau_0)| = |\mathbf{d}\cdot\mathbf{e}|\chi_{\text{out}} &\lesssim e^{-(2l+1)\tau_0}\left(1+|z|^{2l}\right)\mathbf{1}_{|z|<2e^{\frac{(l+\frac{1}{2})\tau_0}{2l+2}}} \\ &\lesssim e^{-(2l+1)\tau_0}\left(1+|z|^{2l}\right)\mathbf{1}_{|z|<e^{\frac{\tau_0}{2}}} \lesssim e^{-(l+1)\tau_0}.\end{aligned}$$

Finally we get from Lemma 7.1 that

$$\begin{aligned}e^{-\tau}|G_{\text{out}}| &\lesssim e^{-(l+\frac{2l+1}{2l+2})\tau}|z|^{2l+2}\mathbf{1}_{e^{\frac{(l+\frac{1}{2})\tau}{2l+2}}<|z|<2e^{\frac{(l+\frac{1}{2})\tau}{2l+2}}} + \delta_0^2 e^{-\tau} \\ &\lesssim e^{-\frac{l\tau}{l+1}} \qquad \text{for } |z| > e^{\frac{l\tau}{2l+2}}.\end{aligned}$$

From this estimate, we verify that if $K \gg 1$

$$\bar{\varphi}_\tau - A_z\bar{\varphi} = \frac{lK}{l+1}e^{-\frac{l\tau}{l+1}} \geq e^{-\tau}|G_{\text{out}}| \qquad \text{for } |z| > e^{\frac{l\tau}{2l+2}}.$$

Therefore by a comparison argument, we obtain

$$|\varphi(z,\tau)| \lesssim \bar{\varphi}(z,\tau) \lesssim Ke^{-\frac{l\tau_0}{l+1}} = KT^{\frac{l}{l+1}} \qquad \text{for } |z| > e^{\frac{l\tau}{2l+2}},\ \tau > \tau_0.$$

The proof is completed. □

We now assume that

$$T^{\frac{l}{l+1}} = e^{-\frac{lR}{l+1}} < \delta_0^2.$$

From this relation, Lemma 7.6 is rewritten as

$$|\varphi(z,\tau)| < K_4\delta_0^2 \qquad \text{for } |z| > e^{\frac{l\tau}{2l+2}},\ \tau > \tau_0. \tag{7.17}$$

We finally derive estimates in $|x| > 1$. Let $W(x,t)$ be a solution of (7.1).

Lemma 7.7. *There exists $K_5' > 0$ independent of R, δ_0, σ such that*

$$|W(x,t)| < K_5' \left(\frac{T^{3l+3}}{|x|^3} + \frac{\delta_0^2}{|x|^2} \right) \qquad \text{for } |x| > 1,\ t \in (0,T).$$

Proof. We now claim that

$$\bar{W}(x,t) = \frac{K}{|x|^3} \left(T^{3l+3} - (T-t)^{3l+3} \right) + \frac{K\delta_0^2}{|x|^2}$$

gives a super solution in $|x| > 1$. Since $\varphi(z,\tau) = W(x,t)$ (see Section 7.1), from (7.17), we verify that if $K > K_4$

$$|W(x,t)| < K_4\delta_0^2 \le \bar{W}(x,t) \qquad \text{for } |x| = 1.$$

Furthermore from Lemma 7.1, we recall that

$$|G_{\mathrm{out}}| \lesssim \frac{(T-t)^{3l+2}}{|x|^3} + \frac{\delta_0^2}{|x|^{2p}} \qquad \text{for } |x| > 1.$$

Therefore since $p > 2$, we get if $K \gg 1$

$$\bar{W}_t - \Delta_x \bar{W} = \frac{(3l+3)K}{|x|^3}(T-t)^{3l+2} + \frac{2K\delta_0^2}{|x|^4} \ge |G_{\mathrm{out}}| \qquad \text{for } |x| > 1.$$

Since $W(x,t) = 0$ for $|x| > 1$, $t = 0$ (see (7.1)), by a comparison argument, we conclude

$$|W(x,t)| < \bar{W}(x,t) \qquad \text{for } |x| > 1,\ t \in (0,T).$$

This completes the proof. □

8. Proof of Theorem 1

We perform the argument mentioned in Section 5.2 rigorously. We fix $\delta_0 \in (0,1)$ small enough. Let $\tilde{w}(x,t) \in C(\mathbb{R}^5 \times [0,T])$ be an extension of $w(x,t) \in X_\sigma$ defined in Section 5.2, $(\lambda(t), \epsilon(y,t))$ be a pair of functions constructed in Section 6, and $W(x,t)$ be a function obtained in Section 7. We now define

$$w(x,t) \in X_\sigma \mapsto W(x,t) \in C(\mathbb{R}^5 \times [0,T)) \cap C^{2,1}(\mathbb{R}^5 \times (0,T)).$$

From Proposition 7.1 and Lemma 7.6–Lemma 7.7, there exists $R_0 > 0$ such that $W(x,t) \in X_\sigma$ if $R > R_0$. By the way of construction, the mapping $w(x,t) \in (X_\sigma, d_{X_\sigma}) \mapsto W(x,t) \in (X_\sigma, d_{X_\sigma})$ is continuous, where d_{X_σ} is the metric in X_σ defined in (5.8). Furthermore we note that

$$\frac{\alpha_l^2}{2}(2\sigma)^{2l+2}\lambda(t) < 2\alpha_2^2 T^{2l+2} \qquad \text{and} \qquad \tau_0 < \tau < |\log(2\sigma)| \qquad \text{for } t \in (0, T-2\sigma).$$

Therefore from the local Hölder estimate and the decay of space infinity $W(x,t) < \frac{\delta_0}{1+|x|^2}$ for $|x| > 1$, we find that the mapping is compact. From the Schauder fixed point theorem, we obtain a function $w(x,t) = W(x,t) \in X_\sigma$, which is denoted by $w_\sigma(x,t)$. Since $\tilde{w}_\sigma(x,t) = w_\sigma(x,t)$ for $t \in (0, T-2\sigma)$ (see Section 5.2), a pair of functions $(w_\sigma(x,t), \lambda_\sigma(t), \epsilon_\sigma(y,t))$ gives a solution of (5.5) for $t \in (0, T-2\sigma)$. We take $\sigma \to 0$ to obtain $w(x,t) = \lim_{\sigma\to 0} w_\sigma(x,t)$. This is the desired solution, which has the form

$$u(x,t) = \mathsf{Q}_\lambda + \Theta(x,t)\chi_{\mathrm{out}} + \lambda^{-\frac{n-2}{2}}\epsilon(y,t)\chi_{\mathrm{in}} + w(x,t)$$

with

$$\left|\lambda(t) - \alpha_l^2(T-t)^{2l+2}\right| < K\left(\delta_0 + \frac{T-t}{R^2}\right)(T-t)^{2l+2}.$$

If we repeat the argument for any $A < 0$ (we fixed $A = -1$ in Section 4), we obtain the solution described in Theorem 1.

Declaration of competing interest

There are no competing interests to declare in this paper.

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Appendix A

As is stated in Introduction, we correct the mistake in [8]. Throughout this section, we use their notation. They define a rescaled function $\Phi(y,\tau)$ on p. 2959.

$$\Phi(y,\tau)=(T-t)^{\frac{1}{p-1}}u(x,t),\qquad y=\frac{x}{\sqrt{T-t}},\quad \tau=-\log(T-t). \tag{A.1}$$

Furthermore they introduce inner variables in (3.1) on p. 2961.

$$\Phi(y,\tau)=\epsilon^{-\frac{2}{p-1}}G(\xi,\tau),\qquad \xi=\frac{y}{\epsilon(\tau)}. \tag{A.2}$$

In their construction, $\epsilon(\tau)$ satisfies the following formula ((3.4) on p. 2961)

$$\Phi(0,\tau)=\epsilon(\tau)^{-\frac{2}{p-1}}. \tag{A.3}$$

Case $n=5$

They derive the following equation for $\epsilon(\tau)$ on p. 2971.

$$\frac{d}{d\tau}\epsilon^{\frac{1}{2}}-\frac{1}{4}\epsilon^{\frac{1}{2}}=-\frac{A\phi_l(0)}{15B_2}e^{-(l+\frac{3}{4})\tau}.$$

From this equation, they get

$$\epsilon(\tau)\sim Ke^{-(2l+\frac{3}{2})\tau}.$$

Therefore from this relation and (A.1)–(A.3), we conclude

$$\begin{aligned}u(0,t)&=(T-t)^{-\frac{1}{p-1}}\Phi(0,\tau)\\&=(T-t)^{-\frac{1}{p-1}}\epsilon(0,\tau)^{-\frac{2}{p-1}}\\&\sim(T-t)^{-\frac{3}{4}}\{K(T-t)^{2l+\frac{3}{2}}\}^{-\frac{3}{2}}\\&\sim K^{-\frac{3}{2}}(T-t)^{-(3l+3)}\qquad(l=0,1,2,\cdots).\end{aligned}$$

However their computation gives (p. 2971)

$$u(0,t)\sim K^{-\frac{3}{2}}(T-t)^{-(l+1)}\qquad(l=0,1,2,\cdots).$$

Case $n=6$

From the matched asymptotic expansion technique, they derive the following equation for $\epsilon(\tau)$ on p. 2971.

$$6B_6\left(\frac{2\dot{\epsilon}}{\epsilon}-1\right)=-1-\frac{3}{2\tau}+o\left(\frac{1}{\tau}\right).$$

Since $B_6=\frac{1}{15}$ (p. 2971), they obtain

$$\epsilon(\tau)\sim Ae^{-\frac{3}{4}\tau}\tau^{-\frac{15}{8}}.$$

In the same manner as the case $n=5$, we conclude

$$
\begin{aligned}
u(0,t) &= (T-t)^{-\frac{1}{p-1}}\epsilon(0,\tau)^{-\frac{2}{p-1}} \\
&\sim (T-t)^{-1}\{A(T-t)^{\frac{3}{4}}\tau^{-\frac{15}{8}}\}^{-2} \\
&\sim A^{-2}(T-t)^{-\frac{5}{2}}|\log(T-t)|^{\frac{15}{4}}.
\end{aligned}
$$

However their blowup rate is given by (p. 2972)

$$
u(0,t) \sim A^{-1}(T-t)^{-\frac{1}{4}}|\log(T-t)|^{-\frac{15}{8}}.
$$

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