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Global classical solutions to quadratic systems with mass control in arbitrary dimensions

Klemens Fellner [a], Jeff Morgan [b], Bao Quoc Tang [a,*]

[a] *Institute of Mathematics and Scientific Computing, University of Graz, Heinrichstrasse 36, 8010 Graz, Austria*
[b] *Department of Mathematics, University of Houston, Houston, TX 77204, USA*

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Abstract

The global existence of classical solutions to reaction-diffusion systems in arbitrary space dimensions is studied. The nonlinearities are assumed to be quasi-positive, to have (slightly super-) quadratic growth, and to possess a mass control, which includes the important cases of mass conservation and mass dissipation. Under these assumptions, the local classical solution is shown to be global, and in the case of mass conservation or mass dissipation, to have the L^∞-norm growing at most polynomially in time. Applications include skew-symmetric Lotka-Volterra systems and quadratic reversible chemical reactions.

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1. Introduction and main results

In this paper, we study the global existence of classical solutions to a class of semilinear reaction-diffusion systems. Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded domain, and $u_i : Q_T := \Omega \times (0,T) \to \mathbb{R}$, $i = 1, \dots, N$ be the i-th concentration density. Denote by $u = (u_1, \dots, u_N)$ the vector of concentration densities. We consider the following reaction-diffusion system

$$\begin{cases} \partial_t u_i - d_i \Delta u_i = f_i(u), & (x,t) \in Q_T, \\ \nabla_x u_i \cdot \nu = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u_i(x,0) = u_{i0}(x), & x \in \Omega, \end{cases} \tag{1}$$

where $d_i > 0$ are the diffusion coefficients, and the domain, the initial data and the nonlinearities satisfy the following assumptions:

* Corresponding author.
E-mail addresses: klemens.fellner@uni-graz.at (K. Fellner), jmorgan@math.uh.edu (J. Morgan), quoc.tang@uni-graz.at (B.Q. Tang).

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(A1) (Smooth Domain) Either $\Omega = \mathbb{R}^n$ or $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$ (i.e. $\partial\Omega$ is of C^∞ class), such that Ω lies locally on one side of $\partial\Omega$.[1]

(A2) (Bounded, Nonnegative Initial Data) For all $i = 1 \dots N$: $u_{i0} \in L^1(\Omega) \cap L^\infty(\Omega)$ and $u_{i0}(x) \geq 0$ for a.e. $x \in \Omega$.

(A3) (Mass Control) There exist $K_0 \geq 0$, $K_1 \in \mathbb{R}$ such that

$$\sum_{i=1}^N f_i(u) \leq K_0 + K_1 \sum_{i=1}^N u_i, \qquad \text{for all} \quad u \in \mathbb{R}^N_+ := [0,\infty)^N.$$

(A4) (Local Lipschitz and Quasi-positivity) For all $i = 1, \dots, N$, $f_i : \mathbb{R}^N \to \mathbb{R}$ is locally Lipschitz and

$$f_i(u) \geq 0, \qquad \text{for all} \quad u \in \mathbb{R}^N_+ \quad \text{satisfying} \quad u_i = 0.$$

(A5) ((Super)-quadratic Growth) There exist $\varepsilon > 0$ and $K > 0$ such that

$$|f_i(u)| \leq K(1 + |u|^{2+\varepsilon}), \qquad \text{for all } i = 1, \dots, N, \quad u \in \mathbb{R}^N.$$

The mass control assumption (A3) is a generalisation to the condition of *mass conservation*

$$\sum_{i=1}^N f_i(u) = 0 \tag{A3'}$$

and to the condition of *mass dissipation*

$$\sum_{i=1}^N f_i(u) \leq 0. \tag{A3"}$$

Definition 1.1. We call a function $u = (u_1, \dots, u_N)$ a classical solution to (1) on $(0,T)$ if $\forall p > n$ we have $u_i \in C([0,T); L^p(\Omega) \cap L^\infty(\Omega)) \cap C^{1,2}((0,T) \times \overline{\Omega})$ and u satisfies each equation in (1) pointwise, cf. [28].

The main result of this paper is the following theorem.

Theorem 1.1 *(Global existence of classical solutions).*
Assume the assumptions (A1)–(A4). Then there exists $\varepsilon > 0$ depending on Ω, n, N, d_i, K_0 such that if (A5) is satisfied, (1) possesses a unique global classical solution. Moreover,

(i) if $K_1 \leq 0$ in (A3), then the solution grows at most polynomially in time, i.e.

$$\|u_i(t)\|_{L^\infty(\Omega)} \leq L_0(1 + t^\xi) \qquad \text{for all} \quad t \geq 0 \quad \text{and for all} \quad i = 1, \dots, N,$$

where L_0, ξ depend only on $\Omega, n, N, d_i, K, K_0, \varepsilon$ and $\|u_{i0}\|_{L^\infty(\Omega)}$,

(ii) and if $K_0 = 0$ and $K_1 < 0$ in (A3), the global solution decays to zero exponentially as $t \to \infty$, i.e.

$$\|u_i(t)\|_{L^\infty(\Omega)} \leq L_1 e^{-\mu t} \quad \text{for all} \quad t \geq 0, \quad \text{and} \quad i = 1, \dots, N,$$

where L_1, μ depend only on $\Omega, n, N, d_i, K, K_1, \varepsilon$ and $\|u_{i0}\|_{L^\infty(\Omega)}$.

Remark 1.1 *(Generalisations).*

(i) The results of Theorem 1.1 can be directly generalised to the case where (A3) is replaced by the assumption: There exists $(\alpha_i) \in (0,\infty)^N$ such that

$$\sum_{i=1}^N \alpha_i f_i(u) \leq K_0 + K_1 \sum_{i=1}^N u_i, \qquad \text{for all} \quad u \in \mathbb{R}^N_+. \tag{2}$$

[1] Naturally, the zero-flux boundary condition is only considered in case of a bounded domain.

(ii) (Boundary conditions) The results in Theorem 1.1 are proved for homogeneous Neumann boundary conditions, but the same results can also be obtained for homogeneous Dirichlet boundary conditions. Note that the proof of Theorem 1.1 doesn't apply directly to mixed boundary conditions.

The question of global existence for reaction-diffusion systems is a classical topic, yet still poses a lot of open and challenging problems. A main difficulty in studying global existence for reaction-diffusion systems is the lack of maximum principle estimates or invariant regions, i.e. a structural inapplicability of techniques from scalar equations. Let us mention some classical results from e.g. Rothe [33], Amann [1], Hollis, Martin and Pierre [15] or Morgan [23]. Most of these works assume some technical assumptions on the nonlinearities. Since the work of Pierre and Schmidt [30], more attention has been paid by mathematicians to study systems satisfying the only natural assumptions of mass control (A3) (or mass dissipation (A3")) and positivity preservation (A4).

It was already pointed out in [30] that these two assumptions alone are not enough to prevent blow up (in L^∞-norm), and therefore, some growth restrictions on the nonlinearities are necessary. In particular, the case of quadratic nonlinearities is of interest due to its relevance in many applications, such as chemical reactions or Lotka-Volterra type systems (see Section 5 for more details). Under (A1)–(A5), [6] showed the global existence of weak solutions. More generally, [29] showed the existence of global weak solutions as long as $f_i(u) \in L^1(Q_T)$ for all $i = 1, \dots, N$. The global existence of classical (or strong) solutions to (1) usually requires additional assumptions to (A1)–(A5). Such assumptions are, for instance, small space dimensions ($n = 1, 2$) (see [13,31,37]), or quasi-uniform diffusion coefficients (see [3,11]), or the close-to-equilibrium regimes (see [2,36]). We refer the interested reader also to the excellent review [26]. Interestingly, global existence of classical solution to (1) under Assumptions (A2)-(A3')-(A4)-(A5) in the case $\Omega = \mathbb{R}^n$ was already solved in [16], but it went almost unnoticed until recently. Our main Theorem 1.1 extends the results of [16] to the case of mass control and bounded domains, and moreover gives polynomial bounds for the growth-in-time of the L^∞-norm of solutions in case $K_1 = 0$, and exponential decay in case $K_0 = 0$ and $K_1 < 0$.

Finally, we remark two recent related results of global classical solutions to nonlinear reaction-diffusion equations. Firstly, Caputo, Goudon and Vasseur [4] used De Giorgi's methods to prove global classical solutions when $\Omega = \mathbb{R}^n$ under assumptions (A2)-(A3')-(A4)-(A5), by additionally assuming the *entropy inequality*

$$\sum_{i=1}^{N} f_i(u) \log u_i \leq 0. \tag{3}$$

Secondly, and more recently, Souplet presented in [34] the existence of global smooth solutions for at most quadratic nonlinear systems under the mass dissipation (A3") instead of (A3'), with the entropy dissipation assumption (3) and a simplified proof.

Both our paper and [34] utilise ideas from [16]. In [34], the author used the entropy inequality (3) to compensate for the lack of mass conservation (A3'). Although this entropic structure appears naturally in many applications, it is not always satisfied, for example, in the case of skew-symmetric Lotka-Volterra systems (see Section 5). This inequality seems "marginal", but it plays a crucial role in the analysis of [34] (and also [4]). Our paper removes the assumption (3) completely and allows the more general assumption (A3). Moreover, we give better control on bounds of solutions in time, which helps to determine their long time behaviour.

Let us now describe the main ideas in proving Theorem 1.1. The proof of Theorem 1.1 crucially utilises the following lemma.

Lemma 1.1 *(Regularity Interpolation).*
For some constant $d > 0$, let u be the solution to the inhomogeneous linear heat equation

$$\begin{cases} u_t - d\Delta u = \phi(x,t), & (x,t) \in Q_T, \\ \nabla_x u \cdot \nu = 0, & (x,t) \in \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{4}$$

Assume that

(i) there exists a Hölder exponent $\gamma \in [0,1)$ *such that for all* $x, x' \in \Omega$, *and all* $t \in (0,T)$,

$$|u(x,t) - u(x',t)| \leq H|x - x'|^{\gamma}, \tag{5}$$

(ii) the inhomogeneity satisfies

$$\sup_{Q_T} |\phi(x,t)| \leq F. \tag{6}$$

Then, the following uniform gradient estimate follows:

$$|\nabla_x u(x,t)| \leq C\|u_0\|_{C^1(\Omega)} + BH^{\frac{1}{2-\gamma}} F^{\frac{1-\gamma}{2-\gamma}}, \qquad \text{for all} \quad (x,t) \in Q_T,$$

where $B > 0$ *and* $C > 0$ *are constants depending only on* Ω, n, d *and* γ.

At first glance, Lemma 1.1 seems unrelated to the problem of global classical solutions to systems (1) under Assumptions (A1)–(A5). In fact, a crucial difficulty in showing global classical solutions to (1) is the failure of comparison principles (except in some special cases) and thus the lack of sufficiently strong a-priori estimates. However, maximum principle, and thus L^∞ a-priori estimates, are well known for the scalar equation (4). So how could Lemma 1.1 contribute to showing global classical solutions to system (1)?

In this work, we first consider the case of "equality" and $K_1 = 0$ in (A3), i.e.

$$\sum_{i=1}^{N} f_i(u) = K_0. \tag{A3eq}$$

Under this assumption, we follow closely ideas of Kanel [16], who considered the case $\Omega = \mathbb{R}^n$, in which Lemma 1.1 plays an essential role. We extend his approach to the case of a bounded domain with smooth boundary. In particular, our approach requires careful Hölder regularity estimates up to the boundary for some parabolic equation of non-divergence form in bounded domains (see Lemma 3.5), since the De Giorgi's technique usually gives only local Hölder continuity. As another refinement to [16], our proof keeps track of the involved constants and shows that the L^∞-norm of the solution grows at most polynomially in time.

Such algebraic growth estimates are very useful in interpolation arguments, and allow us, for instance, to prove exponential decay to zero of solutions in case $K_0 = 0$ and $K_1 < 0$. Another example for useful interpolation arguments with algebraic a-priori estimates are systems, which feature exponential L^1-convergence to equilibrium. This is true, for instance for complex balanced chemical reaction networks, see e.g. [9,10,7]. Indeed, for such systems, interpolation of slowly (i.e. algebraically) growing a-priori L^∞-estimates with exponential L^1-convergence to equilibrium yields equilibration estimates in any L^p-norm for $p \in [1,\infty)$ as well as uniform-in-time bounds, see e.g. [5,12].

After proving Theorem 1.1 in the case of (A3eq), the general case of (A3) follows from suitable transformations. See ***Step 2*** below.

In the following, we provide a detailed outline of the proof of Theorem 1.1. In particular, we sketch how to manipulate system (1) under Assumptions (A1)–(A5) such that Lemma 1.1 can be applied.

1.1. Outline of the proof of Theorem 1.1

The proof is divided in two main steps:

***Step 1: The case of condition* (A3eq).** We consider first the case in which the nonlinearities satisfy (A3eq).

(1) The existence of local in time, classical, non-negative solutions $u = (u_1, \dots, u_N)$ on some time interval $[0,T)$ follows from classical references, thanks to Assumptions (A1), (A2) and (A4), see e.g. [28,33,27] and Proposition 3.1 below. The same references also imply global existence of classical solutions provided that for all $T > 0$

$$\sup_{t\in(0,T)} \|u_i(\cdot,t)\|_{L^\infty(\Omega)} < +\infty, \qquad \text{for all} \quad i = 1, \dots, N,$$

which is our aim in the following.

(2) For a constant $d > \max_{i=1,\dots,N}\{d_i\}$, let v_i be the solution to

$$\begin{cases} \partial_t v_i - d\Delta v_i = u_i, & (x,t) \in Q_T, \\ \nabla_x v_i \cdot \nu = 0, & (x,t) \in \partial\Omega \times (0,T), \\ v_i(x,0) = 0, & x \in \Omega. \end{cases} \tag{7}$$

Since $0 \le u_i \in C^\infty((0,T) \times \overline{\Omega}) \cap C([0,T); L^p(\Omega) \cap L^\infty(\Omega))$ for some $p > n$, the existence, uniqueness and nonnegativity of v_i follows from classical results. Note that the solutions v_i can be seen as mollified versions of the u_i. Moreover, by taking the sum over all $i = 1, \dots, N$ of (7), we calculate

$$\underbrace{\sum_{i=1}^N (\partial_t v_i - d_i \Delta v_i)}_{=:z} - \Delta \underbrace{\sum_{i=1}^N (d - d_i) v_i}_{=:v_d} = \sum_{i=1}^N u_i. \tag{8}$$

(3) Concerning the function z, direct calculations (see Lemma 3.1) using assumption (A3eq) and (7) show that z satisfies the linear equation

$$\begin{cases} \partial_t z - d\Delta z = K_0, & (x,t) \in Q_T, \\ \nabla_x z \cdot \nu = 0, & (x,t) \in \partial\Omega \times (0,T), \\ z(x,0) = \sum_{i=1}^N u_{i0}(x), & x \in \Omega, \end{cases} \tag{9}$$

and therefore the following comparison principle estimate holds

$$\sup_{Q_T} |z| \le \sum_{i=1}^N \|u_{i0}\|_{L^\infty(\Omega)} + K_0 T. \tag{10}$$

(4) Secondly, it follows from the definition of v_d in (8) that (see Lemma 3.2)

$$v_d(x,t) = d \underbrace{\int_0^t z(x,s)ds}_{=:\widehat{z}} - \underbrace{\int_0^t \sum_{i=1}^N d_i u_i(x,s)ds}_{=:\widehat{u}}, \tag{11}$$

where the function $\widehat{z}$ solves the following heat equation (see Lemma 3.3)

$$\begin{cases} \partial_t \widehat{z} - d\Delta \widehat{z} = z(0) + K_0 t, \\ \nabla_x \widehat{z} \cdot \nu = 0, \qquad \widehat{z}(0) = 0 \end{cases}$$

and is therefore Hölder continuous.

(5) Next follows the crucial observation of the Hölder continuity of the function $\widehat{u}$ defined in (11), which solves (see Lemma 3.4)

$$\begin{cases} b\,\partial_t \widehat{u} - \Delta \widehat{u} = \sum_{i=1}^N u_{i0} + K_0 t, \\ \nabla_x \widehat{u} \cdot \nu = 0, \qquad \widehat{u}(0) = 0, \end{cases}$$

where the coefficient in front of the time derivative

$$b(x,t) := \frac{\sum_{i=1}^N u_i(x,t)}{\sum_{i=1}^N d_i u_i(x,t)}$$

is uniformly bounded due to its definition and the non-negativity of the u_i:

$$\frac{1}{\max\{d_i\}} \le b(x,t) \le \frac{1}{\min\{d_i\}}, \qquad \text{for all} \quad (x,t) \in Q_T.$$

The boundedness of b permits us to apply a classical result to prove Hölder continuity of $\widehat{u}$, see Lemma 3.5 and its proof in the Appendix. Therefore, by (11) it follows (see Lemma 3.6) that v_d is Hölder continuous with an exponent $\delta \in (0,1)$.

(6) The two final steps prove boundedness of

$$|U| := \sup_{Q_T} \max_{i=1,\dots,N} |u_i| \tag{12}$$

by estimating the right hand side of (8) in terms of regularity estimates of the left hand side terms. This strategy requires, in particular, to control $\sup_{Q_T} |\Delta v_d|$ in terms of $|U|$. First, we observe that v_d satisfies (see Lemma 3.9)

$$\begin{cases} \partial_t v_d - d\Delta v_d = \sum_{i=1}^N (d-d_i)u_i =: u_d, \\ \nabla_x v_d \cdot \nu = 0, \qquad v_d(0) = 0, \end{cases} \tag{13}$$

which allows us to apply Lemma 1.1. At this point, when contemplating why the presented approach is able to succeed, we recall that the lack of maximum principle estimates for (1) stems from the different diffusion coefficients d_i. Note then, that u_d constitutes the difference between a d-weighted sum (or equally average) of the u_i and a d_i-weighted sum (or average). Accordingly, the function v_d has the same interpretation in terms of the v_i, i.e. mollified versions of the u_i. The key observation is that the function v_d satisfies with (13) a nice parabolic equation (rather than a parabolic system). Consequentially, the following estimates on u_d and v_d show that the difference between d-weighted and d_i-weighted sums satisfy the required regularity estimates in terms of $|U|$ (and without a closeness condition between d and the d_i as exploited in [3]).
More precisely, Lemma 1.1 implies for v_d as a δ-Hölder continuous solution of (13) that (see Lemma 3.7)

$$|\nabla_x v_d| \le C_T \sup_{Q_T} |u_d|^{\frac{1-\delta}{2-\delta}} \le C_T |U|^{\frac{1-\delta}{2-\delta}}, \tag{14}$$

where the constant C_T depends at most polynomially on T for any existence interval $[0,T)$.
Analog, from the equations $\partial_t u_i - d_i \Delta u_i = f_i(u)$ and assumption (A5), Lemma 1.1 with $\gamma = 0$ yields (see Lemma 3.8)

$$\begin{aligned} |\nabla_x u_d| &\le C \sum_{i=1}^N |\nabla_x u_i| \le C_T (1 + |U|^{1/2} \sup_{Q_T} \max_{i=1,\dots,N} |f_i(u)|^{1/2}) \\ &\le C_T \left(1 + |U|^{1/2} \left(1 + |U|^{1+\frac{\varepsilon}{2}}\right)\right) \le C_T \left(1 + |U|^{\frac{3+\varepsilon}{2}}\right). \end{aligned} \tag{15}$$

To estimate Δv_d, we use a second order estimate of the heat semigroup $e^{t\Delta}$ to obtain[2] (see Lemma 3.9)

$$\begin{aligned} \sup_{Q_T} |\Delta v_d| &\le C_T \sup_{Q_T} (|v_d| + |\nabla_x v_d|)^{1/2} \sup_{Q_T} (|u_d| + |\nabla_x u_d|)^{1/2} \\ &\le C_T \left(1 + |U|^{\frac{1-\delta}{2-\delta}}\right)^{1/2} \left(1 + |U|^{\frac{3+\varepsilon}{2}}\right)^{1/2} \\ &\le C_T \left(1 + |U|^{\frac{3}{4}+\frac{\varepsilon}{4}+\frac{1-\delta}{2(2-\delta)}}\right). \end{aligned} \tag{16}$$

Note that in the case $\Omega = \mathbb{R}^n$, one can obtain (16) by differentiating (13) and directly apply Lemma 1.1 to the equation of $\partial_{x_j} v_d$ with $\gamma = 0$.

(7) Finally, from (8), (10) and (16), it follows

$$|U| \le C_T \left(1 + |U|^{\frac{3}{4}+\frac{\varepsilon}{4}+\frac{1-\delta}{2(2-\delta)}}\right)$$

and therefore, since $\frac{3}{4} + \frac{\varepsilon}{4} + \frac{1-\delta}{2(2-\delta)} < 1$ for ε sufficiently small,

$$|U| \le C_T, \qquad \text{for all} \quad (x,t) \in Q_T,$$

which implies that the solutions u_i can be extended globally as bounded, and therefore smooth solutions, see e.g. [28,33].

[2] This idea was used in [34].

One important observation here is that the results of ***Step 1*** are still true, except for the fact that the L^∞-norm in this case might grow faster than polynomial, if we replace the constant K_0 by a function $K_0(t)$ which is continuous on $[0,\infty)$. This fact will be helpful in ***Step 2.***

***Step 2: The case of condition* (A3).** The general assumption (A3) can be transformed into condition (A3eq) (where K_0 is replaced by a continuous function $K_0(t)$) by the help of rescaling and the addition of an appropriate equation to the system:

(1) By defining

$$w_i(x,t) = e^{-K_1 t} u_i(x,t) \quad \text{or equivalently} \quad u_i(x,t) = e^{K_1 t} w_i(x,t)$$

we obtain (see Section 4) the following system for $w = (w_1, \dots, w_N)$

$$\begin{cases} \partial_t w_i - d_i \Delta w_i = g_i(w), & (x,t) \in Q_T \quad \text{for} \quad i = 1, \dots, N, \\ \nabla_x w_i \cdot \nu = 0, & (x,t) \in \partial\Omega \times (0,T), \\ w_i(x,0) = u_{i0}(x), & x \in \Omega \end{cases} \tag{17}$$

where the nonlinearities $g_i(w) = e^{-K_1 t}(f_i(u(w)) - K_1 e^{K_1 t} w_i)$. It is obvious that nonlinearities $g_i(w)$ satisfy the assumptions (A4)–(A5), while (A3) is changed to

$$\sum_{i=1}^{N} g_i(w) \le K_0 e^{-K_1 t}. \tag{18}$$

(2) To obtain (A3eq) from (18), we introduce an $(N+1)$-th equation for w_{N+1} as

$$\partial_t w_{N+1} - \Delta w_{N+1} = g_{N+1}(w) := K_0 e^{-K_1 t} - \sum_{i=1}^{N} g_i(w),$$

together with boundary condition $\nabla_x w_{N+1} \cdot \nu = 0$ and initial data $w_{N+1}(x,0) = 0$. It is immediate that $g_{N+1}(w)$ satisfies the conditions (A4) and (A5). We then obtain a new, enlarged system for $\widetilde{w} = (w_1, \dots, w_{N+1})$

$$\begin{cases} \partial_t w_i - d_i \Delta w_i = g_i(\widetilde{w}) := g_i(w), & (x,t) \in Q_T \quad \text{for} \quad i = 1, \dots, N+1, \\ \nabla_x w_i \cdot \nu = 0, & (x,t) \in \partial\Omega \times (0,T), \quad i = 1, \dots, N+1, \\ w_i(x,0) = u_{i0}(x), & x \in \Omega, \quad i = 1, \dots, N, \\ w_{N+1}(x,0) = 0, & x \in \Omega \end{cases} \tag{19}$$

in which the nonlinearities now satisfy

$$\sum_{i=1}^{N+1} g_i(\widetilde{w}) = K_0 e^{-K_1 t}$$

and the problem is thus reduced to ***Step 1.***, taking into account that last remark therein that K_0 could be a continuous function of t. Note that if $K_0 = 0$ and $K_1 < 0$ then $\widetilde{w}$ has at most polynomial growth in time. Thus, from $u_i(x,t) = e^{K_1 t} w_i(x,t)$ we get for $K_1 < 0$ that u_i decays exponentially to zero.

A direct **application** of Theorem 1.1 proves global classical solutions in all space dimensions to nonlinear, skew-symmetric Lotka-Volterra systems with diffusion of the form

$$\begin{cases} \partial_t u_i - d_i \Delta u_i = \left(-\tau_i + \sum_{j=1}^{N} a_{ij} u_j\right) u_i =: f_i(u), & (x,t) \in Q_T, \\ \nabla_x u_i \cdot \nu = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u_i(x,0) = u_{i0}(x), & x \in \Omega, \end{cases} \tag{20}$$

where $A = (a_{ij}) \in \mathbb{R}^{N\times N}$ is skew-symmetric, i.e. $A^\top + A = 0$ and $\tau = (\tau_1, \dots, \tau_N) \in \mathbb{R}^N$. Note that such Lotka-Volterra systems do not satisfy an entropy structure as required by [4,34]. We will show that (20) has a unique global classical solution in all dimensions, and moreover, if $\tau_i > 0$ for all $i = 1, \dots, N$, then the solution decays to zero exponentially (see Section 5).

The organisation of the paper is as follows: In Section 2, we prove the crucial Lemma 1.1. In Section 3, we first present the proof of Theorem 1.1 with condition (A3eq), while the proof of Theorem 1.1 for general condition (A3) is presented in Section 4. Section 5 is devoted to some applications of the main results. Finally, a technical proof of Hölder continuity for $\widehat{u}$ is presented in the Appendix A.

Notations:

- $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}^N_+ = [0, \infty)^N$.
- The usual norm of $L^p(\Omega)$ is denoted by $\|\cdot\|_{L^p(\Omega)}$ for any $1 \le p \le \infty$.
- For $T > 0$, we denote $Q_T = \Omega \times (0, T)$, and for any $1 \le p < +\infty$

$$\|f\|_{L^p(Q_T)} = \left[\int_{Q_T} |f|^p dxdt\right]^{1/p}, \quad \|f\|_{L^\infty(Q_T)} = \operatorname{ess\,sup}_{Q_T} |f|.$$

- The generic constant C_i, $i = 1, \dots, 12$ depends only on the data Ω, N, n, d_i, K, K_0, K_1, ε and $\|u_{i0}\|_{L^\infty(\Omega)}$. In particular, C_i *does not depend on* $T > 0$.

2. Proof of key Lemma 1.1

In case Ω is a bounded domain, we denote by $G(x,t)$ the Green function of the heat equation $\partial_t u - d\Delta u = 0$ subject to a homogeneous Neumann boundary condition. When $\Omega = \mathbb{R}^n$, let $G(x,t)$ be the fundamental solution, i.e. $G(x,t) = \frac{1}{(4d\pi t)^{n/2}} e^{-|x|^2/4dt}$. Let $k > 0$ be a constant which serves as a kind of interpolation parameter, to be specified later. We rewrite (4) as

$$\partial_t u - d\Delta u + ku = \phi(x,t) + ku,$$

and use the representation formula to have

$$u(x,t) = e^{-kt}\widetilde{u}(x,t) + \int_0^t e^{-k(t-s)} \int_\Omega G(x-y, t-s)[\phi(y,s) + ku(y,s)]dyds \tag{21}$$

where $\widetilde{u}(x,t)$ is the solution to

$$\begin{cases} \partial_t \widetilde{u} - d\Delta\widetilde{u} = 0, \\ \nabla_x \widetilde{u} \cdot \nu = 0, \qquad \widetilde{u}(x,0) = u_0(x). \end{cases}$$

Using the property of the heat semigroup, see e.g. [22, Eq. (2.39)] we have

$$\sup_{t>0}\sup_{x\in\Omega} |\nabla_x \widetilde{u}(x,t)| \le C\|u_0\|_{C^1(\Omega)}. \tag{22}$$

By differentiating (21) in the spatial variables and using $\int_\Omega \nabla_x G(x-y, t-s)u(x,s)dy = 0$ thanks to the Neumann boundary condition, we have

$$\nabla_x u(x,t) = e^{-kt}\nabla_x \widetilde{u}(x,t) + \int_0^t e^{-k(t-s)} \int_\Omega \nabla_x G(x-y, t-s)[\phi(y,s) + k(u(y,s) - u(x,s)]dyds. \tag{23}$$

It follows from (23) and (22) that

$$
\begin{aligned}
|\nabla_x u(x,t)| &\le C\|u_0\|_{C^1(\Omega)} + \int_0^t e^{-k(t-s)} \int_\Omega |\nabla_x G(x-y,t-s)|[|\phi(y,s)| + k|u(y,s)-u(x,s)|]dyds \\
&\le C\|u_0\|_{C^1(\Omega)} + F\int_0^t e^{-k(t-s)} \int_\Omega |\nabla_x G(x-y,t-s)|dyds \\
&\quad + kH\int_0^t e^{-k(t-s)} \int_\Omega |\nabla_x G(x-y,t-s)||x-y|^\gamma dyds
\end{aligned} \tag{24}
$$

where we have used the assumptions (5) and (6) in the last step.

We now distinguish the two cases, when Ω is a bounded domain and $\Omega = \mathbb{R}^n$.

Case 1: Let Ω be a bounded domain with smooth boundary. In this case use the following point-wise gradient estimate on the Green function (see e.g. [21,38] or [8,22]),

$$
|\nabla_x G(x-y,t-s)| \le c_n (t-s)^{-\frac{n+1}{2}} e^{-\kappa_n \frac{|x-y|^2}{t-s}}
$$

where positive constants c_n and κ_n depend only on the dimension n, the domain Ω and the diffusion coefficients d. Using this bound, we can estimate further

$$
\begin{aligned}
|\nabla_x u(x,t)| \le C\|u_0\|_{C^1(\Omega)} &+ c_n F \int_0^t \frac{e^{-k(t-s)}}{(t-s)^{\frac{n+1}{2}}} \int_\Omega e^{-\kappa_n \frac{|x-y|^2}{t-s}} dyds \\
&+ c_n kH \int_0^t \frac{e^{-k(t-s)}}{(t-s)^{\frac{n+1}{2}}} \int_\Omega |x-y|^\gamma e^{-\kappa_n \frac{|x-y|^2}{t-s}} dyds.
\end{aligned} \tag{25}
$$

We denote by (I) and (II) the second and last terms on the right hand side of (25). By the change of variables $z = \sqrt{\frac{\kappa_n}{t-s}}(y-x)$, we have for any $\delta \ge 0$

$$
\begin{aligned}
\int_\Omega |x-y|^\delta e^{-\kappa_n \frac{|x-y|^2}{t-s}} dy &= \left(\frac{t-s}{\kappa_n}\right)^{\frac{n+\delta}{2}} \int_{\sqrt{\frac{\kappa_n}{t-s}}(\Omega-x)} |z|^\delta e^{-|z|^2} dz \le \left(\frac{t-s}{\kappa_n}\right)^{\frac{n+\delta}{2}} \int_{\mathbb{R}^n} |z|^\delta e^{-|z|^2} dz \\
&\le \omega_{n-1}\Gamma\left(\frac{n+\delta}{2}\right)\left(\frac{t-s}{\kappa_n}\right)^{\frac{n+\delta}{2}},
\end{aligned} \tag{26}
$$

where ω_{n-1} is the surface area of the $n-1$-dimensional unit sphere and Γ is the Gamma function. Therefore

$$
\text{(I)} \le c_n \kappa_n^{-n/2} F \int_0^t \frac{e^{-k(t-s)}}{\sqrt{t-s}} ds \int_{\mathbb{R}^n} e^{-|z|^2} dz \le c_n \kappa_n^{-n/2} F\, \Gamma(n/2)\frac{\sqrt{\pi}}{\sqrt{k}} =: \frac{B_1 F}{\sqrt{k}} \tag{27}
$$

where

$$
B_1 = c_n \kappa^{-n/2} \Gamma(n/2)\sqrt{\pi}.
$$

For (II), we have

$$
\text{(II)} \le c_n \kappa_n^{-(n+\gamma)/2} kH \int_0^t \frac{e^{-k(t-s)}}{(t-s)^{\frac{1-\gamma}{2}}} ds \int_{\mathbb{R}^n} |z|^\gamma e^{-|z|^2} dz \le B_2 kH \int_0^t \frac{e^{-k(t-s)}}{(t-s)^{\frac{1-\gamma}{2}}} ds \tag{28}
$$

where

$$B_2 = c_n \kappa_n^{-(n+\gamma)/2} \Gamma\Big(\frac{n+\gamma+1}{2}\Big).$$

The last term on the right hand side of (28) is estimated using the change of variable $\tau = \sqrt{k(t-s)}$,

$$\text{(II)} \le B_2 H (\sqrt{k})^{1-\gamma} \int_0^{\sqrt{kt}} e^{-\tau^2} \tau^\gamma d\tau \le B_3 H (\sqrt{k})^{1-\gamma} \tag{29}$$

with

$$B_3 = B_2 \Gamma\Big(\frac{\gamma+1}{2}\Big).$$

From (25), (27) and (29), we have

$$|\nabla_x u(x,t)| \le C \|u_0\|_{C^1(\Omega)} + \frac{B_1 F}{\sqrt{k}} + B_3 H (\sqrt{k})^{1-\gamma}.$$

By choosing

$$\sqrt{k} = \left[\frac{B_1 F}{B_3 H (1-\gamma)}\right]^{\frac{1}{2-\gamma}},$$

we obtain the desired estimate

$$|\nabla_x u(x,t)| \le C \|u_0\|_{C^1(\Omega)} + B F^{\frac{1-\gamma}{2-\gamma}} H^{\frac{1}{2-\gamma}}$$

with

$$B = \left[(1-\gamma)^{\frac{1}{2-\gamma}} + (1-\gamma)^{\frac{\gamma-1}{2-\gamma}}\right] B_1^{\frac{1-\gamma}{2-\gamma}} B_3^{\frac{1}{2-\gamma}}.$$

Case 2: $\Omega = \mathbb{R}^n$. In this case, we use the explicit representation of the fundamental solution to obtain

$$\nabla_x G(x-y, t-s) = \frac{1}{(4\pi d(t-s))^{n/2}} \frac{y-x}{2d(t-s)} e^{-\frac{|x-y|^2}{4d(t-s)}}$$

and consequently get from (24) that

$$\begin{aligned} |\nabla_x u(x,t)| \le C \|u_0\|_{C^1(\mathbb{R}^n)} &+ F \frac{1}{2d(4\pi d)^{n/2}} \int_0^t \frac{e^{-k(t-s)}}{(t-s)^{n/2+1}} \int_{\mathbb{R}^n} |x-y| e^{-\frac{|x-y|^2}{4d(t-s)}} dy ds \\ &+ kH \frac{1}{2d(4\pi d)^{n/2}} \int_0^t \frac{e^{-k(t-s)}}{(t-s)^{n/2+1}} \int_{\mathbb{R}^n} |x-y|^{1+\gamma} e^{-\frac{|x-y|^2}{4d(t-s)}} dy ds. \end{aligned} \tag{30}$$

Denote by (III) and (IV) the second and the last term on the right hand side of (30), respectively. We estimate these two terms similarly as in **Case 1** to have

$$\text{(III)} \le \frac{B_4 F}{\sqrt{k}}, \qquad \text{where} \quad B_4 = \frac{\omega_{n-1}}{\pi^{(n-1)/2}\sqrt{d}} \Gamma\Big(\frac{n+1}{2}\Big)$$

and

$$\text{(IV)} \le B_5 H (\sqrt{k})^{1-\gamma}, \qquad \text{where} \quad B_5 = \frac{\omega_{n-1}}{\pi^{n/2}} 2^{\gamma-1} d^{\frac{\gamma-1}{2}} \Gamma\Big(\frac{1+\gamma}{2}\Big) \Gamma\Big(\frac{n+1+\gamma}{2}\Big).$$

Hence

$$|\nabla_x u(x,t)| \le C \|u_0\|_{C^1(\mathbb{R}^n)} + \frac{B_4 F}{\sqrt{k}} + B_5 H (\sqrt{k})^{1-\gamma},$$

and therefore

$$|\nabla_x u(x,t)| \le C\|u_0\|_{C^1(\mathbb{R}^n)} + BF^{\frac{1-\gamma}{2-\gamma}} H^{\frac{1}{2-\gamma}}$$

with

$$B = \left[(1-\gamma)^{\frac{1}{2-\gamma}} + (1-\gamma)^{\frac{\gamma-1}{2-\gamma}}\right] B_4^{\frac{1-\gamma}{2-\gamma}} B_5^{\frac{1}{2-\gamma}}.$$

3. Proof of Theorem 1.1 with condition (A3eq)

Proposition 3.1 *(Local existence).*
Assume (A1), (A2) *and* (A4). *Then, there exists an interval* $[0,T)$ *(which can be chosen maximal) and a corresponding unique nonnegative classical solution* u *to* (1) *on* $(0,T)$. *Moreover, in order to extend the solution globally, it is sufficient to prove for all* $T > 0$

$$\lim_{t\uparrow T} \|u_i(t)\|_{L^\infty(\Omega)} < +\infty \qquad \forall i = 1,\dots,N, \qquad \Rightarrow \qquad T = +\infty. \tag{31}$$

Proof. Since $u_{i0} \in L^\infty(\Omega)$ and the nonlinearities are locally Lipschitz, the local existence follows from classical results (see e.g. [28,33,1]). The quasi-positivity of the nonlinearity (A4) implies propagation of non-negativity of initial data, e.g. [27,26]. □

Remark 3.1 *(Weaker blow-up criteria).*
The continuation criteria (31) is standard for semilinear reaction-diffusion systems. For the systems considered here, it can be weakened. For instance, with assumptions (A3eq) and (A4), by applying improved duality estimates as in [3], global classical solutions can be shown from the existence of an exponent $p > (1+\varepsilon)(1+n/2)$ such that

$$\limsup_{t\uparrow T} \|u_i\|_{L^p(Q_t)} < +\infty \quad \text{for all} \quad i = 1,\dots,N, \qquad \Rightarrow \qquad T = +\infty.$$

Remark 3.2 *(Smooth initial data).*
Thanks to Definition 1.1 and the smoothing effect of the heat semigroup, it is well-known that the local solutions of Proposition 3.1 are smooth for positive times, i.e. that for all $0 < t_0 < T$, $u_i(t_0) \in C^2(\overline{\Omega})$ for all $i = 1,\dots,N$. This allows us to shift the initial time to t_0 and therefore consider system (1) with smooth initial data in $C^2(\overline{\Omega})$. Note that all parabolic compatibility conditions are satisfied at $t_0 > 0$. Consequentially, from now onwards, we weaken (w.l.o.g.) Assumption (A2) by only considering smooth initial data $u_{i0} \in C^\infty(\overline{\Omega})$ satisfying a homogeneous Neumann boundary condition for all $i = 1,\dots,N$. Consequently, we have $u_i \in C^{1,2}([0,T)\times\overline{\Omega})$.

From now on, because of Remark 3.2, we assume instead of (A2) that the initial data satisfy

(A2') (Smooth Nonnegative Initial Data) For all $i = 1,\dots N$: $0 \le u_{i0} \in C^2(\overline{\Omega})$.

Moreover, we assume

(A6) (Diffusion Coefficient d in (7)) Throughout this section, we fix the diffusion coefficient $d > \max_{i=1,\dots,N}\{d_i\}$. Moreover, let v_i be the unique solution to (7) on Q_T.

Lemma 3.1. *Let* z *be defined in* (8). *Then,* z *is the solution to* (9) *and*

$$\sup_{Q_T} |z| \le M + K_0 T := \sum_{i=1}^N \|u_{i0}\|_{L^\infty(\Omega)} + K_0 T.$$

Proof. First, we show that z solves (9), and the rest follows from the comparison principle. Indeed, by using definition (8), we calculate

$$\begin{aligned}\partial_t z - d\Delta z &= \partial_t\Big(\sum_{i=1}^N(\partial_t v_i - d_i\Delta v_i)\Big) - d\Delta\Big(\sum_{i=1}^N(\partial_t v_i - d_i\Delta v_i)\Big)\\ &= \sum_{i=1}^N\left[\partial_t\left(\partial_t v_i - d\Delta v_i\right) - d_i\Delta\left(\partial_t v_i - d\Delta v_i\right)\right] = \sum_{i=1}^N(\partial_t u_i - d_i\Delta u_i) \qquad \text{(using (7))}\\ &= \sum_{i=1}^N f_i(u) = K_0 \qquad \text{(using (A3eq))}.\end{aligned}$$

Moreover, on $\partial\Omega$, we have $\nabla_x u_i\cdot\nu = \nabla_x v_i\cdot\nu = 0$ and $\partial_t v_i - d\Delta v_i = u_i$, implying $\nabla_x(\Delta v_i)\cdot\nu = 0$. Therefore, the boundary condition $\nabla_x z\cdot\nu = 0$ follows from the definition $z = \sum_{i=1}^N(\partial_t v_i - d_i\Delta v_i)$. For initial data, since $v_i(x,0) = 0$, we have

$$z(x,0) = \sum_{i=1}^N(\partial_t v_i(x,0) - d_i\Delta v_i(x,0)) = \sum_{i=1}^N(\partial_t v_i(x,0) - d\Delta v_i(x,0)) = \sum_{i=1}^N u_i(x,0) = \sum_{i=1}^N u_{i0}(x). \quad \square$$

Lemma 3.2. *The function v_d defined in (8) satisfies*

$$v_d = d\widehat{z} - \widehat{u}$$

with $\widehat{z}$ and $\widehat{u}$ as defined in (11).

Proof. Integrating (7) over $(0,t)$ and using $v_i(x,0) = 0$ yields

$$v_i(x,t) - d\Delta\int_0^t v_i(x,s)ds = \int_0^t u_i(x,s)ds.$$

Hence

$$\begin{aligned}v_d(x,t) &= \sum_{i=1}^N(d-d_i)v_i(x,t) = d\int_0^t\sum_{i=1}^N(d-d_i)\Delta v_i(x,s)ds + \int_0^t\sum_{i=1}^N(d-d_i)u_i(x,s)ds\\ &= d\int_0^t\left[d\Delta\sum_{i=1}^N v_i(x,s) + z(x,s) - \sum_{i=1}^N\partial_s v_i(x,s)\right]ds + \int_0^t\sum_{i=1}^N(d-d_i)u_i(x,s)ds\\ &= d\int_0^t\left[z(x,s) - \sum_{i=1}^N u_i(x,s)\right]ds + \int_0^t\sum_{i=1}^N(d-d_i)u_i(x,s)ds\\ &= d\int_0^t z(x,s)ds - \int_0^t\sum_{i=1}^N d_i u_i(x,s)ds = d\widehat{z} - \widehat{u}. \quad \square\end{aligned}$$

Lemma 3.3. *Define*

$$\widehat{z}(x,t) := \int_0^t z(x,s)ds.$$

Then, $\widehat{z}$ solves

$$\partial_t\widehat{z} - d\Delta\widehat{z} = \sum_{i=1}^N u_{i0} + K_0 t \ \text{ in } Q_T, \qquad \nabla_x\widehat{z}\cdot\nu = 0 \ \text{ on } \partial\Omega\times(0,T), \qquad \widehat{z}(x,0) = 0 \ \text{ in } \Omega$$

and consequently $\widehat{z}$ is Hölder continuous in Q_T with an exponent $\beta\in(0,1)$ depending on Ω, K_0, and d.

Proof. The boundary condition and initial data of $\widehat{z}$ are obvious. Integrating the equation (9) of z over $(0,t)$, we have

$$z(x,t) - d\Delta \int_0^t z(x,s)ds = z(x,0) + K_0 t$$

and hence

$$\partial_t \widehat{z} - d\Delta \widehat{z} = z(x,0) = \sum_{i=1}^N u_{i0} + K_0 t.$$

The comparison principle implies

$$\sup_{Q_T} |\widehat{z}| \le (M + K_0 T)T, \qquad \text{recalling} \quad M := \sum_{i=1}^N \|u_{i0}\|_{L^\infty(\Omega)}. \tag{32}$$

Therefore, Hölder continuity of $\widehat{z}$ follows from classical parabolic theory (see e.g. [24] for $\Omega = \mathbb{R}^n$ or [20, Theorem 4.8] for bounded domains). More precisely, there exists $\beta \in (0,1)$ and C_1 depending only on Ω, n, N and M such that

$$|\widehat{z}(x,t) - \widehat{z}(x',t')| \le C_1(1+T^2)(|x-x'|^\beta + |t-t'|^{\beta/2}) \quad \text{for all} \quad (x,t),(x',t') \in Q_T.$$

The term $1+T^2$ on the right hand side is because of the L^∞ bound of $\widehat{z}$ in (32). □

Lemma 3.4. *Define as in* (11)

$$\widehat{u}(x,t) := \int_0^t \sum_{i=1}^N d_i u_i(x,s)ds. \tag{33}$$

Then, $\widehat{u}$ *solves*

$$b\partial_t \widehat{u} - \Delta \widehat{u} = \sum_{i=1}^N u_{i0} + K_0 t, \qquad \nabla_x \widehat{u} \cdot \nu = 0, \qquad \widehat{u}(x,0) = 0, \tag{34}$$

where

$$b(x,t) = \frac{\sum_{i=1}^N u_i(x,t)}{\sum_{i=1}^N d_i u_i(x,t)}$$

satisfies the bound

$$\frac{1}{\max\{d_i\}} \le b(x,t) \le \frac{1}{\min\{d_i\}}, \qquad \textit{for all} \quad (x,t) \in Q_T. \tag{35}$$

Proof. Summing all equations in (1) and using assumption (A3eq) leads to

$$\partial_t \sum_{i=1}^N u_i - \Delta \sum_{i=1}^N d_i u_i = K_0.$$

Integrating this equation over $(0,t)$ yields

$$\sum_{i=1}^N u_i(x,t) - \Delta \widehat{u} = \sum_{i=1}^N u_{i0}(x) + K_0 t$$

which implies (34) since $\partial_t \widehat{u} = \sum_{i=1}^N d_i u_i$. The bounds of $b(x,t)$ in (35) follows easily from the non-negativity of u_i. □

Lemma 3.5. *The function $\widehat{u}$ defined in (33) is bounded in Q_T. More precisely $\|\widehat{u}\|_{L^\infty(Q_T)} \le d(M+K_0T)T$. Moreover, $\widehat{u}$ is Hölder continuous with some exponent $\alpha \in (0,1)$, i.e.*

$$|\widehat{u}(x,t)-\widehat{u}(x',t')| \le C_2(1+T^2)(|x-x'|^\alpha + |t-t'|^{\alpha/2}) \qquad \text{for all} \quad (x,t),(x',t') \in Q_T, \tag{36}$$

where C_2 and α depend on Ω, M, n, K_0 and d_i.

Remark 3.3. The local Hölder continuity of $\widehat{u}$ follows from a well-known result in [17] since $\widehat{u}$ also solves the following parabolic equation of non-divergence form

$$\partial_t \widehat{u} - \frac{1}{b}\Delta \widehat{u} = \frac{1}{b}\sum_{i=1}^N u_{i0} + \frac{1}{b}K_0 t.$$

The results in [17] are based on a probabilistic approach. Here, we provide an alternative proof using [18] and [25]. A similar result was presented in [4, Proposition 3.1], in which the authors proved Hölder continuity for a homogeneous equation.

Proof of Lemma 3.5. Since $v_i \ge 0$ and $d \ge d_i$ for all $i = 1,\dots,N$, we use $v_d = d\widehat{z} - \widehat{u}$ to get

$$0 \le \widehat{u} = d\widehat{z} - v_d \le d\widehat{z}.$$

Therefore, thanks to Lemma 3.3,

$$\sup_{Q_T} |\widehat{u}| \le d \sup_{Q_T} |\widehat{z}| \le d(M+K_0T)T. \tag{37}$$

The proof of Hölder continuity is technical and lengthy, and therefore, we postpone it to Appendix A. □

Lemma 3.6. *The function $v_d = d\widehat{z} - \widehat{u}$ as defined in (8) is Hölder continuous in Q_T, i.e. there exists $C_5 > 0$ such that*

$$|v_d(x,t)-v_d(x',t')| \le C_5(1+T^2)(|x-x'|^\delta + |t-t'|^{\delta/2}) \quad \text{for all} \quad (x,t),(x',t') \in Q_T, \tag{38}$$

where the Hölder exponent $\delta \in (0,1)$ depends on the parameters Ω, n, N, d_i, K_0 of the problem.

Proof. The Hölder continuity follows clearly from the Lemmas 3.3 and 3.5. In the following, we provided explicit estimates for the constants. First, for $|x-x'|, |t-t'| \le 1$, we estimate

$$\begin{aligned} |v_d(x,t)-v_d(x',t')| &\le d|\widehat{z}(x,t)-\widehat{z}(x',t')| + |\widehat{u}(x,t)-\widehat{u}(x',t')| \\ &\le \max\{dC_1, C_2\}(1+T^2)(|x-x'|^\delta + |t-t'|^{\delta/2}), \end{aligned} \tag{39}$$

where $\delta = \min\{\alpha,\beta\}$ with β and α as in Lemmas 3.3 and 3.5. When $|x-x'| \ge 1$ or $|t-t'| \ge 1$, we estimate

$$\begin{aligned} |v_d(x,t)-v_d(x',t')| &\le 2\|v_d\|_{L^\infty(Q_T)}(|x-x'|^\delta + |t-t'|^{\delta/2}) \\ &\le 2d(M+K_0T)T(|x-x'|^\delta + |t-t'|^{\delta/2}) \end{aligned} \tag{40}$$

since $\|v_d\|_{L^\infty(Q_T)} \le d\|\widehat{z}\|_{L^\infty(Q_T)} + \|\widehat{u}\|_{L^\infty(Q_T)} \le 2d(M+K_0T)T$. From (39) and (40), we get (38) with $\delta = \min\{\alpha,\beta\}$. □

Lemma 3.7. *Let $|U| = \sup_{Q_T} \max_{i=1,\dots,N} |u_i|$. Then, there exists a constant $C_6 > 0$ such that*

$$\sup_{Q_T} |\nabla_x v_d| \le C_6 (1+T^2)^{\frac{1}{2-\delta}} |U|^{\frac{1-\delta}{2-\delta}}, \tag{41}$$

where $\delta \in (0,1)$ is the Hölder exponent introduced in Lemma 3.6.

Proof. From Lemma 3.6, we know that v_d is Hölder continuous. In particular

$$|v_d(x,t)-v_d(x',t)| \le C_5 (1+T^2)|x-x'|^\delta, \qquad \text{for all} \quad (x,t),(x',t) \in Q_T.$$

By the initial data $v_i(x,0) = 0$ and $\sup_{Q_T} |u_i| \le |U|$ (i.e. by the definition of $|U|$), we can apply Lemma 1.1 to (7) and obtain

$$\sup_{Q_T} |\nabla_x v_d| \le B(C_5 (1+T^2))^{\frac{1}{2-\delta}} |U|^{\frac{1-\delta}{2-\delta}},$$

whence (41) with $C_6 = BC_5^{\frac{1}{2-\delta}}$. □

Lemma 3.8. *There exists a constant $C_7 > 0$ such that*

$$\sup_{Q_T} |\nabla_x u_d| \le C_7 \left(1 + |U|^{\frac{3+\varepsilon}{2}}\right),$$

where ε is given in growth assumption (A5).

Proof. Firstly by the definition of u_d in (13) it follows

$$\sup_{Q_T} |\nabla_x u_d| \le \sum_{i=1}^{N} (d - d_i) \sup_{Q_T} |\nabla_x u_i|.$$

In order to apply Lemma 1.1 to equation of u_i we first observe that

$$|u_i(x,t) - u_i(x',t)| \le 2 \sup_{Q_T} |u_i| \le 2|U||x - x'|^0, \qquad \text{for all } (x,t), (x',t) \in Q_T.$$

The right hand sides in (1) can be estimated by using the growth assumption (A5)

$$\sup_{Q_T} |f_i(u)| \le K(1 + \sup_{Q_T} |u|^{2+\varepsilon}) \le K N^{2+\varepsilon} (1 + |U|^{2+\varepsilon}).$$

Now we can apply Lemma 1.1 to the equation of u_i with the exponent $\gamma = 0$ to obtain

$$\begin{aligned} \sup_{Q_T} |\nabla_x u_i| &\le C \|u_{i0}\|_{C^1(\Omega)} + B(2|U|)^{1/2} \left(K N^{2+\varepsilon} (1 + |U|^{2+\varepsilon})\right)^{1/2} \\ &\le C \|u_{i0}\|_{C^1(\Omega)} + C_8 \left(1 + |U|^{\frac{3}{2} + \frac{\varepsilon}{2}}\right) \end{aligned}$$

where C_8 depends on B, K and N. Hence

$$\sup_{Q_T} |\nabla_x u_d| \le \max\{d_i\} \sum_{i=1}^{N} \sup_{Q_T} |\nabla_x u_i| \le C_7 \left(1 + |U|^{\frac{3+\varepsilon}{2}}\right),$$

which is the claim of Lemma 3.8 where C_7 depends on $\|u_{i0}\|_{C^1(\Omega)}$, d_i, N and C_8. □

Lemma 3.9. *There exists a constant $C_9 > 0$ such that*

$$\sup_{Q_T} |\Delta v_d| \le C_9 (1+T) \left(1 + |U|^{\frac{3+\varepsilon}{4} + \frac{1-\delta}{2(2-\delta)}}\right),$$

where ε is given in growth assumption (A5) and $\delta \in (0,1)$ in Lemma 3.6.

Proof. By definition $v_d = \sum_{i=1}^{N} (d - d_i) v_i$, we have

$$\begin{aligned} \partial_t v_d - d\Delta v_d &= \partial_t \sum_{i=1}^{N} (d - d_i) v_i - d\Delta \sum_{i=1}^{N} (d - d_i) v_i = \sum_{i=1}^{N} (d - d_i)(\partial_t v_i - d\Delta v_i) \\ &= \sum_{i=1}^{N} (d - d_i) u_i = u_d. \end{aligned}$$

Let $k > 0$ to be chosen later, and define $\eta(x,t) = e^{kt} v_d(x,t)$. It follows from the equation of v_d that

$$\partial_t \eta - d\Delta\eta = e^{kt}(k v_d + u_d), \qquad \nabla_x \eta \cdot \nu = 0, \qquad \eta(x,0) = v_d(x,0) = 0. \tag{42}$$

By Duhamel's formula

$$\eta(x,t) = \int_0^t e^{(t-s)d\Delta} e^{ks}(k v_d(x,s) + u_d(x,s))ds.$$

Changing back to v_d we get

$$v_d(x,t) = \int_0^t e^{-k(t-s)} e^{(t-s)d\Delta}(k v_d(x,s) + u_d(x,s))ds.$$

We now apply the second order semigroup estimate, see e.g. [22, Eq. (2.39)]

$$\|e^{td\Delta} f\|_{C^2(\Omega)} \le C t^{-1/2} \|f\|_{C^1(\Omega)}, \qquad \text{for} \quad t \in (0,T] \tag{43}$$

to have

$$\sup_{Q_T} |\Delta v_d| \le C \sup_{t\in(0,T)} \|v_d(t)\|_{C^2(\Omega)} \le C \int_0^t e^{-k(t-s)}(t-s)^{-1/2}[k\|v_d(s)\|_{C^1(\Omega)} + \|u_d(s)\|_{C^1(\Omega)}]ds,$$

where in this proof we always denote by C a generic constant depending on Ω, n, and d, but *independent of* T. We estimate

$$\begin{aligned}
\int_0^t e^{-k(t-s)}(t-s)^{-1/2} k\|v_d(s)\|_{C^1(\Omega)} ds &\le \sup_{s\in(0,T)} \|v_d(s)\|_{C^1(\Omega)} k \int_0^t e^{-k(t-s)}(t-s)^{-1/2} ds \\
&\le C \sup_{Q_T}(|v_d| + |\nabla v_d|)\sqrt{k} \int_0^\infty s^{-1/2} e^{-s} ds \\
&\le C \sup_{Q_T}(|v_d| + |\nabla v_d|)\sqrt{k}
\end{aligned}$$

and similarly

$$\int_0^t e^{-k(t-s)}(t-s)^{-1/2}\|u_d(s)\|_{C^1(\Omega)} ds \le C \sup_{Q_T}(|u_d| + |\nabla u_d|)\frac{1}{\sqrt{k}}.$$

By choosing $\sqrt{k} = \sup_{Q_T}(|u_d| + |\nabla u_d|)^{1/2} \sup_{Q_T}(|v_d| + |\nabla v_d|)^{-1/2}$, we obtain from (43)

$$\sup_{Q_T} |\Delta v_d| \le C \sup_{Q_T}(|u_d| + |\nabla u_d|)^{1/2} \sup_{Q_T}(|v_d| + |\nabla v_d|)^{1/2}.$$

Therefore, by using Lemma 3.6, i.e. $\sup_{Q_T} |v_d| \le 2d(M + K_0 T)T$ and Lemmas 3.7, 3.8, we finally get

$$\begin{aligned}
\sup_{Q_T} |\Delta v_d| &\le C\left(|U| + 1 + |U|^{\frac{3+\varepsilon}{2}}\right)^{1/2}\left(1 + T^2 + (1+T^2)^{\frac{1}{2-\delta}}|U|^{\frac{1-\delta}{2-\delta}}\right)^{1/2} \\
&\le C(1+T)\left(1 + |U|^{\frac{3+\varepsilon}{4} + \frac{1-\delta}{2(2-\delta)}}\right). \quad \square
\end{aligned}$$

Remark 3.4. In the case $\Omega = \mathbb{R}^n$, we can apply Lemma 1.1 to the equation of $\partial_{x_j} v_d$ to obtain Lemma 3.9 immediately. This argument seems to not directly work in the case of a bounded domain since the Neumann boundary condition is not satisfied by $\partial_{x_j} v_d$.

We are now ready to prove Theorem 1.1 with (A3eq).

Proof of Theorem 1.1 with (A3eq). From (8) we have

$$|U| = \max_{i=1,\dots,N} \sup_{Q_T} |u_i| \le \sum_{i=1}^{N} \sup_{Q_T} |u_i| \le \sup_{Q_T}(|z| + |\Delta v_d|).$$

By using Lemmas 3.1 and 3.9, it follows

$$\begin{aligned} |U| &\le (M + K_0 T) + C_9(1+T)\left(1 + |U|^{\frac{3+\varepsilon}{4} + \frac{1-\delta}{2(2-\delta)}}\right) \\ &\le C_{11}\left[1 + T + (1+T)|U|^{\frac{3+\varepsilon}{4} + \frac{1-\delta}{2(2-\delta)}}\right], \end{aligned} \tag{44}$$

where C_{11} depends only on N, C_9 and M. We choose ε small enough such that

$$\lambda := \frac{3+\varepsilon}{4} + \frac{1-\delta}{2(2-\delta)} < 1 \quad \text{or equivalently} \quad \varepsilon < \frac{\delta}{2-\delta}.$$

By using Young's inequality we have

$$C_{11}(1+T)|U|^{\lambda} \le \frac{|U|}{2} + (1-\lambda)(2\lambda C_{11})^{\frac{\lambda}{1-\lambda}}(1+T)^{\frac{1}{1-\lambda}}$$

and therefore it follows from (44)

$$|U| \le 2C_{11}[1+T] + 2(1-\lambda)(2\lambda C_{11})^{\frac{\lambda}{1-\lambda}}(1+T)^{\frac{1}{1-\lambda}} \le C_{12}(1+T^{\frac{1}{1-\lambda}}),$$

where C_{12} depends only on C_{11}, ε and δ. The uniqueness follows immediately thanks to the L^∞-bound and the local Lipschitz continuity of the nonlinearities. □

Remark 3.5. It's straightforward that all the arguments of this section are still valid in case K_0 is replaced by a function $K_0(t)$ which is continuous on $[0,\infty)$, except that the L^∞-norm of the solution might grow faster than polynomial.

4. Proof of Theorem 1.1 with condition (A3)

Proof of Theorem 1.1 with (A3). Our main idea is that with a suitable change of unknowns, and especially *adding one more appropriate equation*, we can transform a system with the mass control condition (A3) into a system with condition (A3eq), that keeps the essential features (A4) and (A5).

We define

$$w_i(x,t) = e^{-K_1 t}u_i(x,t) \quad \text{or equivalently} \quad u_i(x,t) = e^{K_1 t}w_i(x,t)$$

and $w = (w_1, \dots, w_N)$. Direct computations give

$$\begin{aligned} \partial_t w_i &= e^{-K_1 t}(\partial_t u_i - K_1 u_i) \\ &= e^{-K_1 t}(d_i \Delta u_i + f_i(u) - K_1 u_i) \\ &= d_i \Delta w_i + g_i(w) \end{aligned}$$

where

$$g_i(w) = e^{-K_1 t}(f_i(u) - K_1 e^{K_1 t} w_i). \tag{45}$$

Note that

$$\sum_{i=1}^{N} g_i(w) = e^{-K_1 t}\sum_{i=1}^{N}(f_i(u) - K_1 u_i) \le e^{-K_1 t}K_0$$

due to the assumption (A3).

Introduce a new unknown $w_{N+1} : \Omega \times (0, T_{\max}) \to \mathbb{R}_+$ which solves

$$\partial_t w_{N+1} - \Delta w_{N+1} = K_0 e^{-K_1 t} - \sum_{i=1}^{N} g_i(w) =: g_{N+1}(w) \geq 0 \tag{46}$$

with homogeneous Neumann boundary condition $\nabla w_{N+1} \cdot \nu = 0$ and zero initial data $w_{N+1}(x,0) = w_{N+1,0}(x) = 0$ for $x \in \Omega$. With a slight abuse of notation we write the new vector of concentrations $\widetilde{w} = (w_1, \dots, w_N, w_{N+1})$ and the nonlinearities $g_i(\widetilde{w}) := g_i(w_1, \dots, w_N)$ for all $i = 1, \dots, N$ while $g_{N+1}(\widetilde{w}) = K_0 e^{-K_1 t} - \sum_{i=1}^{N} g_i(w)$. We have arrived at the following system

$$\begin{cases} \partial_t w_i - d_i \Delta w_i = g_i(\widetilde{w}), & (x,t) \in \Omega \times (0, T_{\max}), \quad i = 1, \dots, N+1, \\ \nabla w_i \cdot \nu = 0, & (x,t) \in \partial\Omega \times (0, T_{\max}), \quad i = 1, \dots, N+1, \\ w_i(x,0) = u_{i,0}(x), & x \in \Omega, \quad i = 1, \dots, N, \\ w_{N+1}(x,0) = w_{N+1,0}(x) = 0, & x \in \Omega. \end{cases} \tag{47}$$

It's obvious to check that the nonlinearities $g_i, i = 1, \dots, N+1$ satisfy the assumption (A4). Moreover, due to the definition $w_i(x,t) = e^{-K_1 t} u_i(x,t)$, it follows from (A5) and (45) the growth control

$$\begin{aligned} |g_i(\widetilde{w})| &\leq e^{-K_1 t}(|f_i(u)| + K_1 e^{K_1 t}|w_i|) \\ &\leq e^{-K_1 t}(K(1 + |u|^{2+\varepsilon}) + K_1 e^{K_1 t}|w_i|) \\ &\leq C e^{(1+\varepsilon)K_1 T_{\max}}(1 + |\widetilde{w}|^{2+\varepsilon}). \end{aligned} \tag{48}$$

Moreover, the nonlinearities of (47) satisfies the condition (A3eq) (with K_0 replaced by a continuous function in t), i.e.

$$\sum_{i=1}^{N+1} g_i(\widetilde{w}) = K_0 e^{-K_1 t} \tag{49}$$

thanks to (46).

Now we can apply the results in Section 3 (Remark 3.5) to get that (47) has a global classical solution $\widetilde{w}$. Changing back to the original unknowns $u_i(x,t) = e^{K_1 t} w_i(x,t)$ for $i = 1, \dots, N$, we obtain finally the global existence of classical solution to (1).

Moreover, in case $K_0 = 0$ and $K_1 < 0$, the growth condition (48) can be estimated further as

$$|g_i(\widetilde{w})| \leq C(1 + |\widetilde{w}|^{2+\varepsilon})$$

and (49) becomes

$$\sum_{i=1}^{N+1} g_i(\widetilde{w}) = 0.$$

Therefore, from Section 3, we know that the solution $\widetilde{w}$ to (47) grows at most polynomially in time, i.e. for all $i = 1, \dots, N+1$

$$\|w_i(t)\|_{L^\infty(\Omega)} \leq C(1 + t^\zeta)$$

for some $C, \zeta > 0$. Therefore, if $K_0 = 0$ and $K_1 < 0$ we get

$$\|u_i(t)\|_{L^\infty(\Omega)} = e^{K_1 t}\|w_i(t)\|_{L^\infty(\Omega)} \leq C(1 + e^{K_1 t} t^\zeta) \leq C(1 + e^{-\mu t})$$

for some $K_1 < -\mu < 0$, which completes the proof of Theorem 1.1. □

5. Applications

5.1. Quadratic reversible reactions

We consider the reversible chemical reaction

$$A_1 + A_2 \leftrightarrows A_3 + A_4$$

and denote by $u_i(x,t)$ the concentrations of the substances A_i at $x \in \Omega$ and $t > 0$. Using the mass-action law, we obtain the following reaction-diffusion system

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2 + u_3 u_4 =: f_1(u), & (x,t) \in Q_T, \\ \partial_t u_2 - d_2 \Delta u_2 = -u_1 u_2 + u_3 u_4 =: f_2(u), & (x,t) \in Q_T, \\ \partial_t u_3 - d_3 \Delta u_3 = +u_1 u_2 - u_3 u_4 =: f_3(u), & (x,t) \in Q_T, \\ \partial_t u_4 - d_4 \Delta u_4 = +u_1 u_2 - u_3 u_4 =: f_4(u), & (x,t) \in Q_T, \\ \nabla_x u_i \cdot \nu = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u_i(x,0) = u_{i0}(x), & x \in \Omega, \end{cases} \tag{50}$$

where we have normalised the forward and backward reaction rate constants for the sake of simplicity. We assume positive diffusion coefficients $d_i > 0$. System (50) was studied extensively in the literature: Weak solutions were shown in all dimensions [6] while the global strong (classical) solutions were shown in [13] or [3] in one or two dimensions. In higher dimensions, [3] showed global classical solutions under the assumption that the diffusion coefficients are sufficiently close to each other (depending on the space dimension). The question of global classical without restriction on diffusion coefficients was solved in [16] for the case $\Omega = \mathbb{R}^n$ and recently reproved in [4] and [34]. We remark that besides satisfying Assumptions (A3), (A4) and (A5), the nonlinearities of (50) have an additional feature, that is an *entropy inequality*, i.e.

$$\sum_{i=1}^{4} f_i(u) \log u_i \le 0 \quad \text{for all} \quad u \in \mathbb{R}^4_+, \tag{51}$$

which plays an important role in the analysis of [4,34]. This entropy property is also a key in studying the convergence to chemical equilibrium for (50). Due to the homogeneous Neumann boundary condition, system (50) possesses precisely three linear independent conservation laws for all $t > 0$

$$\begin{aligned} \int_\Omega [u_1(x,t) + u_3(x,t)]dx &= \int_\Omega [u_{10}(x) + u_{30}(x)]dx =: M_{13}, \\ \int_\Omega [u_2(x,t) + u_3(x,t)]dx &= \int_\Omega [u_{20}(x) + u_{30}(x)]dx =: M_{23}, \\ \int_\Omega [u_2(x,t) + u_4(x,t)]dx &= \int_\Omega [u_{20}(x) + u_{40}(x)]dx =: M_{24}. \end{aligned}$$

Denote by $\boldsymbol{M} = (M_{13}, M_{23}, M_{24})$ the initial mass vector. By straightforward computations, for any fixed positive initial mass vector $\boldsymbol{M} \in \mathbb{R}^3_+$, there exists a unique positive constant equilibrium $u_\infty = (u_{1\infty}, u_{2\infty}, u_{3\infty}, u_{4\infty}) \in \mathbb{R}^4_+$ satisfying

$$u_{1\infty} u_{2\infty} = u_{3\infty} u_{4\infty}, \quad u_{1\infty} + u_{3\infty} = M_{13}, \quad u_{2\infty} + u_{3\infty} = M_{23}, \quad u_{2\infty} + u_{4\infty} = M_{24}.$$

Note that the equilibrium is determined only by the initial mass vector rather than the precise initial data. It was proved in several works, e.g. [5,9,10,32], that any solution (renormalised, weak or strong) to (50) with initial mass $\boldsymbol{M}$ converges exponentially in L^1-norm to the equilibrium u_∞ defined above, i.e.

$$\sum_{i=1}^{4} \|u_i(t) - u_{i\infty}\|_{L^1(\Omega)} \le C e^{-\lambda t}, \qquad \text{for all} \quad t \ge 0, \tag{52}$$

where $C > 0, \lambda > 0$ are constants which can be computed explicitly. By applying Theorem 1.1, we have

Theorem 5.1. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$. Fix a positive initial mass vector $\boldsymbol{M} \in \mathbb{R}^3_+$.*

Then, for any non-negative initial data $u_0 = (u_{i0})_{i=1\dots4} \in L^\infty(\Omega)^4$ having the initial mass vector $\boldsymbol{M}$, there exists a unique global non-negative classical solution $u = (u_1, u_2, u_3, u_4)$ of (50) which converges exponentially to equilibrium in L^∞-norm, i.e.

$$\sum_{i=1}^{4} \|u_i(t) - u_{i\infty}\|_{L^\infty(\Omega)} \le C_0 e^{-\lambda_0 t}, \qquad \text{for all} \quad t \ge 0, \tag{53}$$

where $C_0 > 0, \lambda_0 > 0$ are positive constants which can be computed explicitly.

Remark 5.1. The global existence of classical solution to (50) on the whole space $\mathbb{R}^n$ was proved in [16] and recently reproved in [4,34]. However, it seems that the results therein do not provide *a priori* estimates in time for solutions, thus such convergence in L^∞-norm (53) does not follow immediately from their results.

Remark 5.2. Results similar to Theorem 5.1 are also valid for complex balanced networks of chemical reactions of substances $A_1, \dots, A_S$ with quadratic nonlinearities, which means that each complex of the networks is of the form $A_i + A_j$ with $i, j = 1, \dots, S$. The global existence and uniqueness of classical solution follows immediately from Theorem 1.1 with (2) replacing (A3), see Remark 1.1. For the exponential convergence we need either the detailed or complex balanced condition, and the absence of boundary equilibria, see [2] and [36] for more details.

Proof of Theorem 5.1. From Theorem 1.1, we have that the L^∞-norm,

$$\|u_i(T)\|_{L^\infty(\Omega)} \le C_T, \qquad \text{for all} \quad T > 0,$$

grows at most polynomially in T. Now using (52) and the interpolation inequality

$$\|f\|_{L^p(\Omega)} \le \|f\|^{(p-1)/p}_{L^\infty(\Omega)} \|f\|^{1/p}_{L^1(\Omega)}$$

for all $p \in (1, \infty)$, we have

$$\begin{aligned} \|u_i(T) - u_{i\infty}\|_{L^p(\Omega)} &\le \|u_i(T) - u_{i\infty}\|^{(p-1)/p}_{L^\infty(\Omega)} \|u_i(T) - u_{i\infty}\|^{1/p}_{L^1(\Omega)} \le C_T^{(p-1)/p} C^{1/p} e^{-(\lambda/p)t} \\ &\le C_p e^{-\lambda_p T} \end{aligned} \tag{54}$$

for some explicit constants $C_p > 0$ and $0 < \lambda_p < \lambda/p$. It follows that for each $p \in (1, \infty)$ there exists $M_p > 0$ such that

$$\sup_{t>0} \|u_i(t)\|_{L^p(\Omega)} \le M_p, \qquad \text{for all} \quad i = 1, \dots, 4.$$

Let $S_i(t) = e^{d_i \Delta t}$ be the heat semigroup with Neumann boundary condition. We use the $L^p - L^\infty$ estimate

$$\|S_i(t) f\|_{L^\infty(\Omega)} \le C t^{-n/(2p)} \|f\|_{L^p(\Omega)},$$

the Duhamel formula

$$u_i(t+1) = S(1)u(t) + \int_0^1 S(1-s) f_i(u(t+s)) ds,$$

and the quadratic growth $|f_i(u)| \le C|u|^2$, to estimate

$$\begin{aligned} \|u_i(t+1)\|_{L^\infty(\Omega)} &\le C\|u_i(t)\|_{L^p(\Omega)} + C \int_0^1 (1-s)^{-n/(2p)} \|u(t+s)\|^2_{L^{2p}(\Omega)} ds \\ &\le C M_p + C M_{2p}^2 \int_0^1 (1-s)^{-n/(2p)} ds. \end{aligned}$$

By choosing $p > n/2$ which leads to $\int_0^1 (1-s)^{-n/(2p)} ds \le C$, we get a uniform in time bound of the L^∞-norm,

$$\|u_i(t)\|_{L^\infty(\Omega)} \le M_\infty, \qquad \text{for all} \quad t > 0 \quad \text{and} \quad i = 1, \dots, 4.$$

To prove exponential convergence in the L^∞-norm, we observe that the equilibrium u_∞ also solves the system (50). Therefore, repeating the previous argument, we get

$$\|u_i(t) - u_{i\infty}\|_{L^\infty(\Omega)} \le C\|u_i(t) - u_{i\infty}\|_{L^p(\Omega)} + C\int_0^1 (1-s)^{-n/(2p)} \|f_i(u(t+s)) - f_i(u_\infty)\|_{L^p(\Omega)} ds.$$

It follows from the uniform in time bound of L^∞-norm that

$$\|f_i(u) - f_i(u_\infty)\|_{L^p(\Omega)} \le C\|u - u_\infty\|_{L^p(\Omega)}.$$

Hence

$$\begin{aligned}
\|u_i(t) - u_{i\infty}\|_{L^\infty(\Omega)} &\le C\|u_i(t) - u_{i\infty}\|_{L^p(\Omega)} + C\int_0^1 (1-s)^{-n/(2p)} \|u(t+s) - u_\infty\|_{L^p(\Omega)} ds \\
&\le CC_p e^{-\lambda_p t} + CC_p \int_0^1 (1-s)^{-n/(2p)} e^{-\lambda_p (t+s)} ds \\
&\le Ce^{-\lambda_p t} \left[1 + \int_0^1 (1-s)^{-n/(2p)} \right] \\
&\le Ce^{-\lambda_p t},
\end{aligned}$$

where we choose $p > n/2$ at the last steps. This finishes the proof of Theorem 5.1. □

Remark 5.3. The convergence in L^∞-norm in Theorem 5.1 can be obtained alternatively by showing that a higher order norm grows polynomially. Indeed, by using the estimate of the Neumann semigroup $S(t)$ (see e.g. [39, Lemma 1.3]) for any $1 \le p \le \infty$,

$$\|\nabla S(t) f\|_{L^p(\Omega)} \le C\left(1 + t^{-\frac{1}{2} + \frac{n}{2p}}\right) \|f\|_{L^\infty(\Omega)} \qquad \text{for all} \quad t > 0,$$

the Duhamel formula $u_i(t) = S_i(t)u_0 + \int_0^t S_i(t-s) f_i(u(s)) ds$, and polynomial growth of the L^∞-norm, we obtain for any $1 \le p < \infty$,

$$\|u_i(t)\|_{W^{1,p}(\Omega)} \le C(1 + t^{\eta_p})$$

for some $\eta_p > 0$. From choosing $p > n$ and using Gagliardo-Nirenberg interpolation inequality

$$\|u_i(t) - u_{i\infty}\|_{L^\infty(\Omega)} \le C\|u_i(t) - u_{i\infty}\|^\alpha_{W^{1,p}(\Omega)} \|u_i(t) - u_{i\infty}\|^{1-\alpha}_{L^1(\Omega)}$$

for some $\alpha > 0$, follows the exponential convergence to equilibrium in $L^\infty(\Omega)$-norm.

5.2. Skew-symmetric Lotka-Volterra systems

A general Lotka-Volterra system with diffusion is of the form (20), i.e.

$$\begin{cases}
\partial_t u_i - d_i \Delta u_i = \left(-\tau_i + \sum_{j=1}^N a_{ij} u_j\right) u_i =: f_i(u), & (x,t) \in Q_T, \\
\nabla_x u_i \cdot \nu = 0, & (x,t) \in \partial\Omega \times (0,T), \\
u_i(x,0) = u_{i0}(x), & x \in \Omega,
\end{cases}$$

where $A = (a_{ij}) \in \mathbb{R}^{N\times N}$ and $\tau = (\tau_1, \dots, \tau_N) \in \mathbb{R}^N$. The system is called skew-symmetric when the matrix A is skew-symmetric, i.e.

$$A^\top + A = 0.$$

Lotka-Volterra systems of the form (20) have been extensively investigated in the literature due to their wide application in biology, see e.g. [14,33,19,35]. In the case of skew-symmetric Lotka-Volterra systems, all existing results concern either weak solutions in all space dimensions or classical solutions in small (one and two) space dimensions. The results of our work imply classical solutions in all space dimensions. By the skew-symmetry of A, we obtain easily

$$\sum_{i=1}^N f_i(u) = -\sum_{i=1}^N \tau_i u_i,$$

thus Assumption (A3) holds. It is straightforward that Assumptions (A4) and (A5) are also satisfied. We remark that (20) does not possess an entropy inequality like (51), and therefore the results of [4] or [34] are not applicable in this case. By applying Theorem 1.1, we have

Theorem 5.2. *Assume that Ω is a bounded domain with smooth boundary $\partial\Omega$. Assume moreover that the system (20) is skew-symmetric, i.e. $A^\top + A = 0$. Then, for any non-negative initial data $u_{i0} \in L^\infty(\Omega)$, there exists a unique global classical solution to (20). Moreover, if $\tau_i > 0$ for all $i = 1, \dots, N$, we have*

$$\|u_i(t)\|_{L^\infty(\Omega)} \le Ce^{-\mu t}$$

for some $\mu > 0$.

Remark 5.4. Obviously, Theorem 5.2 is also applicable in the case of "sub-skew-symmetric" system, i.e. for $A + A^\top \le 0$ in the sense that all elements of $A + A^\top$ are non-positive.

Proof. The global existence and uniqueness of classical solution follows from Theorem 1.1. If $\tau_i > 0$ for all $i = 1, \dots, N$, we can take $K_1 = -\min\{\tau_1, \dots, \tau_N\} < 0$ which implies

$$\sum_{i=1}^N f_i(u) = -\sum_{i=1}^N \tau_i u_i \le K_1 \sum_{i=1}^N u_i$$

and therefore Theorem 1.1 directly applies yielding the exponential decay to zero. □

Declaration of competing interest

We declare that there is no competing interest.

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Appendix A. Proof of Hölder continuity for $\widehat{u}$

In order to show Hölder continuity of $\widehat{u}$ in Lemma 3.5, we follow the approach from [18, Chapter 2]. We remark that this approach gives only local Hölder continuity which is only sufficient in the case $\Omega = \mathbb{R}^n$. In the case of bounded domains $\Omega \subset \mathbb{R}^n$, we need to also prove boundary regularity. This will be done by reflection techniques, which was used e.g. in [25] for elliptic equations.

A.1. Local Hölder continuity

We fix $(x_0, t_0) \in Q_T$ and denote

$$B_\varrho = \{x \in \Omega : |x - x_0| < \varrho\}$$

and

$$Q(\varrho, \tau) = B_\varrho \times (t_0 - \tau, t_0) = \{(x, t) : x \in B_\varrho, t_0 - \tau < t < t_0\}.$$

Lemma A.1 *(Local Hölder continuity).*
For any $t_0 > 0$ and $\widehat{\Omega} \subset\subset \Omega$, there exists $\rho_0 > 0$ with $d(\widehat{\Omega}, \partial\Omega) \geq \rho_0$, and a constant $\alpha \in (0, 1)$, such that for some constant $C > 0$

$$|\widehat{u}(x, t) - \widehat{u}(x', t')| \leq C\|\widehat{u}\|_{L^\infty(Q_T)}(|x - x'|^\alpha + |t - t'|^{\alpha/2}), \qquad \text{for all} \quad (x, t), (x', t') \in \widehat{\Omega} \times [t_0, T].$$

Proof. According to [18, Chapter 2, Section 7], the Hölder continuity of $\widehat{u}$ follows from the boundedness $\|\widehat{u}\|_{L^\infty(Q_T)} \leq CT$ and the energy estimate of the following Lemma A.2. □

Remark A.1. As an alternative proof of local Hölder continuity, note that the energy estimates (55) are sufficient to repeat the arguments in the proof of [4, Proposition 3.1].

Lemma A.2 *(Local energy estimates).*
For any $k > 0$, any $0 < \tau_2 < \tau_1 < 1$ and $0 < \varrho_2 < \varrho_1 < 1$ such that $Q(\varrho_1, \tau_1) \subset Q_T$, the following energy estimate holds

$$\begin{aligned}\sup_{t_0 - \tau_2 < t < t_0} \|(\widehat{u} - k)_+\|^2_{L^2(B_{\varrho_2})} + \int_{t_0 - \tau_2}^{t_0} \|(\widehat{u} - k)_+\|^2_{H^1(B_{\varrho_2})} ds \\ \leq C\left[((\varrho_1 - \rho_2)^{-2} + (\tau_1 - \tau_2)^{-1})\|(\widehat{u} - k)_+\|^2_{L^2(Q(\varrho_1, \tau_1))} + \int_{Q(\varrho_1, \tau_1)} \chi_{\{\widehat{u} > k\}} dxds\right].\end{aligned} \tag{55}$$

Proof. Let $\xi : Q_T \to [0, 1]$ be a smooth cut-off function such that

$$\xi(x, t) = \begin{cases} 1 & \text{if} \quad (x, t) \in Q(\varrho_2, \tau_2) \\ 0 & \text{if} \quad (x, t) \notin Q(\varrho_1, \tau_1). \end{cases}$$

For simplicity we denote by $u^{(k)} = (\widehat{u} - k)_+$. By multiplying equation (34) with $u^{(k)}\xi^2$ and integrating over $B_{\varrho_1} \times (t_0 - \tau_1, t)$, $t_0 - \tau_2 < t < t_0$ and by integration by parts, we calculate

$$\begin{aligned}\int_{t_0 - \tau_1}^{t} \int_{B_{\varrho_1}} b\partial_t \widehat{u} u^{(k)} \xi^2 dxds + \int_{t_0 - \tau_1}^{t} \int_{B_{\varrho_1}} |\nabla_x u^{(k)}|^2 \xi^2 dxds \\ = -2 \int_{t_0 - \tau_1}^{t} \int_{B_{\varrho_1}} (\nabla_x u^{(k)} \cdot \nabla_x \xi) u^{(k)} \xi dxds + \int_{t_0 - \tau_1}^{t} \int_{B_{\varrho_1}} [u_0 + K_0 s] u^{(k)} \xi^2 dxds\end{aligned} \tag{56}$$

where we denote by $u_0 := \sum_{i=1}^N u_{i0}$. Since $\partial_t \widehat{u} = \sum_{i=1}^N d_i u_i \geq 0$, due to (33), and thus

$$\frac{1}{\max d_i} \partial_t \widehat{u} \leq b\partial_t \widehat{u} \leq \frac{1}{\min d_i} \partial_t \widehat{u}.$$

Therefore, with $\nu = \frac{1}{\max d_i}$, we estimate

$$\begin{aligned}\int_{t_0-\tau_1}^{t}\int_{B_{\varrho_1}} b\partial_t\widehat{u}u^{(k)}\xi^2 dxds &\geq \nu\int_{t_0-\tau_1}^{t}\int_{B_{\varrho_1}} \partial_t\widehat{u}u^{(k)}\xi^2 dxds\\ &= \frac{\nu}{2}\int_{B_{\varrho_1}}(u^{(k)}(t))^2\xi(t)^2dx - \nu\int_{t_0-\tau_1}^{t}\int_{B_{\varrho_1}}(u^{(k)})^2\xi\partial_t\xi dxds\\ &\geq \frac{\nu}{2}\int_{B_{\varrho_2}}|u^{(k)}(t)|^2dx - \nu\int_{t_0-\tau_1}^{t}\int_{B_{\varrho_1}}(u^{(k)})^2\xi\partial_t\xi dxds\\ &\geq \frac{\nu}{2}\int_{B_{\varrho_2}}|u^{(k)}(t)|^2dx - \nu\int_{Q(\varrho_1,\tau_1)}|u^{(k)}|^2|\xi_t|dxds,\end{aligned} \tag{57}$$

where we used $\xi(\cdot, t_0-\tau_1) = 0$ at second step and $\xi|_{Q(\rho_2,\tau_2)} \equiv 1$ at the third step. By applying Cauchy-Schwarz's inequality, we get

$$\begin{aligned}-2\int_{t_0-\tau_1}^{t}\int_{B_{\varrho_1}}(\nabla_x u^{(k)}\cdot\nabla_x\xi)u^{(k)}\xi dxds &\leq \frac{1}{2}\int_{t_0-\tau_1}^{t}\int_{B_{\varrho_1}}|\nabla_x u^{(k)}|^2\xi^2dxds + 2\int_{t_0-\tau_1}^{t}\int_{B_{\varrho_1}}|u^{(k)}|^2|\nabla_x\xi|^2dxds\\ &\leq \frac{1}{2}\int_{t_0-\tau_1}^{t}\int_{B_{\varrho_1}}|\nabla_x u^{(k)}|^2\xi^2dxds + 2\int_{Q(\varrho_1,\tau_1)}|u^{(k)}|^2|\nabla_x\xi|^2dxds.\end{aligned} \tag{58}$$

Denoting by $\chi_{\{\widehat{u}>k\}}$ the characteristic function of the set $\{(x,t) : \widehat{u}(x,t) > k\}$, we can estimate

$$\begin{aligned}&\int_{t_0-\tau_1}^{t}\int_{B_{\varrho_1}}[u_0+K_0s]u^{(k)}\xi^2dxds\\ &= \int_{t_0-\tau_1}^{t}\int_{B_{\varrho_1}}[u_0+K_0s]u^{(k)}\xi^2\chi_{\{\widehat{u}>k\}}dxds\\ &\leq \frac{1}{2}\int_{t_0-\tau_1}^{t}\int_{B_{\varrho_1}}|u^{(k)}|^2\xi^2dxds + \frac{1}{2}\int_{t_0-\tau_1}^{t}\int_{B_{\varrho_1}}|u_0+K_0s|^2\xi^2\chi_{\{\widehat{u}>k\}}dxds\\ &\leq \max\left\{1/2; \|u_0\|^2_{L^\infty(\Omega)} + K_0^2t^2\right\}\int_{Q(\varrho_1,\tau_1)}\left(|u^{(k)}|^2 + \chi_{\{\widehat{u}>k\}}\right)ds,\end{aligned} \tag{59}$$

where we have used $|\xi| \leq 1$ in the last step. We now insert the estimates (57), (58) and (59) into (56) to get for all $t \in (t_0-\tau_2, t_0)$

$$\begin{aligned}&\frac{\nu}{2}\|u^{(k)}(t)\|^2_{L^2(B_{\varrho_2})} + \frac{1}{2}\int_{t_0-\tau_1}^{t}\int_{B_{\varrho_1}}|\nabla_x u^{(k)}|^2\xi^2dxds\\ &\leq \max\{\nu; 2; \|u_0\|^2_{L^\infty(\Omega)} + K_0^2t^2\}\left[\int_{Q(\varrho_1,\tau_1)}\left(|u^{(k)}|^2(|\nabla_x\xi|^2 + |\partial_t\xi| + 1) + \chi_{\{\widehat{u}>k\}}\right)dxds\right].\end{aligned} \tag{60}$$

By adding the inequality $\frac{1}{2}\int_{t_0-\tau_1}^{t}\int_{B_{\varrho_1}}|u^{(k)}|^2\xi^2 dxds \le \frac{1}{2}\int_{Q(\varrho_1,\tau_1)}|u^{(k)}|^2 dxds$ and taking the supremum over $t\in(t_0-\tau_2,t_0)$, we have with $\tau_2<\tau_1$

$$\frac{\min\{\nu,1\}}{2}\left(\sup_{t_0-\tau_2<t<t_0}\|u^{(k)}(t)\|^2_{L^2(B_{\varrho_2})}+\int_{t_0-\tau_2}^{t_0}\|u^{(k)}\|^2_{H^1(B_{\varrho_2})}ds\right)$$
$$\le \max\{\nu;2;\|u_0\|^2_{L^\infty(\Omega)}+K_0^2t^2\}\left[\int_{Q(\varrho_1,\tau_1)}\Big(|u^{(k)}|^2(|\nabla_x\xi|^2+|\partial_t\xi|+1)+\chi_{\{\widehat{u}>k\}}\Big)ds\right] \tag{61}$$

Finally, due to the definition of the cut-off function ξ, there exists a constant $C\ge 1$ independent of ϱ_i and τ_i such that $|\nabla_x\xi|\le C(\varrho_1-\varrho_2)^{-1}$ and $|\partial_t\xi|\le C(\tau_1-\tau_2)^{-1}$. Noting also $1\le(\tau_1-\tau_2)^{-1}$ since $\tau_1,\tau_2\in(0,1)$, we get from (61) the energy estimate (55). □

A.2. Boundary Hölder continuity

For this part, we only need $\partial\Omega$ to be Lipschitz continuous. The proof in this part follows from [25], where the author applied it to elliptic equations. For any $z\in\partial\Omega$, we can find an orthogonal matrix $\mathcal{O}$ and a Lipschitz continuous function $\psi:\mathbb{R}^{n-1}\to\mathbb{R}$ and

$$G:=\{(y,\psi(y)+s): y\in B(0,r), s\in(-r,r)\}$$

such that

$$\mathcal{O}(\Omega-z)\cap G=\{(y,\psi(y)+s): y\in B(0,r), s\in(0,r)\}.$$

Without loss of generality, we can assume that $z=0$ and $\mathcal{O}=\mathbb{I}_d$. Define the mapping $T: B(0,r)\times(-r,r)\mapsto G$ by $T(y,s)=(y,\psi(y)+s)$, then T is bi-Lipschitz continuous. Using the mapping $S: G\mapsto G$ with $S(T(y,s))=T(y,-s)$ we can define the reflection mapping of any function $w:(D\subset G)\times(0,T)\mapsto\mathbb{R}$ by

$$\widetilde{w}(x,t):=\begin{cases} w(x,t) & \text{if } x\in G\cap\Omega,\\ w(Sx,t) & \text{if } x\in G\setminus\overline{\Omega}.\end{cases}$$

Lemma A.3 *(Hölder continuity at boundary).*
Let $\widehat{u}$ solve equation (34). Then $\widetilde{u}:=\widetilde{\widehat{u}}$ is the weak solution to

$$\widetilde{b}\partial_t\widetilde{u}-\operatorname{div}(A(x)\nabla\widetilde{u})=\widetilde{u}_0+K_0t \tag{62}$$

where $A(x)\in\mathbb{R}^{n\times n}$ is uniformly bounded and positive definite for all $x\in G$, i.e. $\|A(x)\|\le C$ and there exists $\lambda>0$ such that $\xi^\top A(x)\xi\ge\lambda|\xi|^2$ for all $x\in G$ and $\xi\in\mathbb{R}^n$. Therefore, $\widetilde{u}\in C^\alpha(\widehat{G}\times[t_0,T])$ for any $t_0\in(0,T)$ and $\widehat{G}\subset\subset G$. Moreover,

$$\|\widetilde{u}\|_{C^\alpha(\widehat{G}\times[t_0,T])}\le C_\delta\|\widehat{u}\|_{L^\infty(Q_T)}$$

where C_δ depends on $\delta=\operatorname{dist}(\widehat{G},\partial G)$. Consequently,

$$\|\widehat{u}\|_{C^\alpha(\widehat{G}\cap\Omega\times[t_0,T])}\le C_\delta\|\widehat{u}\|_{L^\infty(Q_T)}.$$

Proof. The validity of (62) can be directly verified using properties of refection functions provided in [25, Lemmas 3.3 and 3.5]. Note that $\widetilde{b}$ always satisfies the bound (35). Therefore, we can repeat the arguments in Subsection A.1 to get the desired local Hölder continuity of $\widetilde{u}$, which in turn implies the local Hölder continuity of $\widehat{u}$ at the boundary. □

A.3. Hölder continuity of $\widehat{u}$

Proof of Hölder continuity in Lemma 3.5. If $\Omega = \mathbb{R}^n$, then the Hölder continuity of $\widehat{u}$ follows directly from Lemma A.1.

When $\Omega \subset \mathbb{R}^n$ bounded with Lipschitz boundary $\partial\Omega$, Hölder continuity (36) up to boundary for $\widehat{u}$ follows from Lemmas A.1 and A.3, thanks to the fact that $\partial\Omega$ is compact and can be covered by finitely many balls with radius ρ_0 (where ρ_0 is in Lemma A.1). □

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