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A conservation law with spatially localized sublinear damping

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Abstract

We consider a general conservation law on the circle, in the presence of a sublinear damping. If the damping acts on the whole circle, then the solution becomes identically zero in finite time, following the same mechanism as the corresponding ordinary differential equation. When the damping acts only locally in space, we show a dichotomy: if the flux function is not zero at the origin, then the transport mechanism causes the extinction of the solution in finite time, as in the first case. On the other hand, if zero is a non-degenerate critical point of the flux function, then the solution becomes extinct in finite time only inside the damping zone, decays algebraically uniformly in space, and we exhibit a boundary layer, shrinking with time, around the damping zone. Numerical illustrations show how similar phenomena may be expected for other equations.

Keywords: One-dimensional conservation laws; Sublinear damping; Finite time extinction; Finite time control; Long time behavior

1. Introduction

We consider a general conservation law on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, in the presence of a sublinear damping, possibly localized in space,

$$\partial_t u + \partial_x \left(f(u) \right) + a(x) \frac{u}{|u|^\alpha} = 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{T}, \tag{1.1}$$

with a smooth flux $f \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R})$, $0 < \alpha \leqslant 1$ and $a = a(x) \geqslant 0$. For the Cauchy problem, we prescribe the initial datum

$$u_{|t=0} = u_0, \quad x \in \mathbb{T}. \tag{1.2}$$

In the case where $a > 0$ is constant, the sublinear nonlinearity is motivated by the effect of friction forces that occur in almost every mechanism with moving parts, this process arising between all surfaces in contact. The first concepts go

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back to the work of Leonardo da Vinci on friction, rediscovered by Amontons [5] at the end of the 17th century, and then developed by Coulomb [17] in the 18th century. The main idea is that the friction is opposed to the movement and that the friction force is independent of the speed v and the contact surface. The friction force, known today as Coulomb friction, is therefore described as $F = F_c \operatorname{sgn}(v)$. Depending on how the sign function is defined, it can be zero or take any value in the interval $[-F_c, F_c]$. In the 19th century, the theory of hydrodynamics was developed leading to expressions for the frictional force caused by the viscosity of lubricants, and is usually modeled by $F = F_v v$. The linearity with respect to speed is not always correct and a more general relation is $F = F_v |v|^{\delta_v} \operatorname{sgn}(v)$ where δ_v depends on the geometry of the application (see e.g. [1,4,26] and references therein). The basic model for the motion of a body lying on a surface is given by the Newton law. It reduces to the ordinary differential equation, for $\alpha \in (0, 1)$,

$$\dot{u} = -\frac{u}{|u|^\alpha}, \quad t \in \mathbb{R}, \qquad u(0) = u_0 \in \mathbb{R}. \tag{1.3}$$

By separating the variables, explicit integration yields, in terms of $\rho = u^2$, since $\rho \geqslant 0$,

$$\dot{\rho} = -2\rho^{1-\alpha/2}, \quad \text{hence } \rho(t) = \begin{cases} (|u_0|^\alpha - \alpha t)^{2/\alpha} & \text{if } t \leqslant |u_0|^\alpha/\alpha, \\ 0 & \text{if } t > |u_0|^\alpha/\alpha. \end{cases} \tag{1.4}$$

Therefore, ρ becomes zero in finite time, and so does u. Note that for $\alpha = 1$, the equation (1.3) should be understood in the sense of Filippov (see [21, Chapter 2]): $u' \in -\operatorname{Sign}(u)$, in which $\operatorname{Sign}(u)$ is defined by

$$\operatorname{Sign}(u) = \begin{cases} \{1\} & \text{if } u > 0, \\ \{-1\} & \text{if } u < 0, \\ [-1, 1] & \text{if } u = 0, \end{cases} \tag{1.5}$$

and the same argument as above still applies. Besides, note that solutions of the above ODE (1.3), whether $\alpha \in (0, 1)$ or $\alpha = 1$, are unique in positive time even if the source term is not $\mathscr{C}^1$ with respect to u, as a consequence of the one-sided Lipschitz condition satisfied by $h_\alpha(u) = -u/|u|^\alpha$, see [21, Chapter 2, Section 10, Theorem 1], which reads as follows: for all $(u, v) \in \mathbb{R}^2$,

$$(u - v)(h_\alpha(u) - h_\alpha(v)) \leqslant 0.$$

Such sublinear damping models have been considered for some partial differential equations: in the case of the wave equation [6,27], in the case of various parabolic equations [2,8–12], and in the case of the Schrödinger equation [13,14]. The aspect that we now wish to investigate is the effect of such a damping when it is localized in space. Typically, the function a in (1.1) can be thought of as an indicating function.

Some of the results that we present can be adapted to the case where the space variable belongs to the whole line $\mathbb{R}$. The reason why we consider the periodic case is the following. On the whole line, the characteristics of the solution of (1.1) may cross the support of a without undergoing such a strong effect as in (1.4), that is, the sublinear damping occurs in too small a region to put u to zero. On the other hand, in a periodic box, and in the case where transport phenomenon is present, the solution will meet the support of a as long as it is not zero. These are typically the possibilities which we want to understand. To our knowledge, the problem that we consider in this paper is not related to a physical or biological model. We rather consider it as a baby model in the study of finite time control with a spatially localized damping.

Assumption 1.1. The function a is nonnegative, $a(x) \geqslant 0$ for all $x \in \mathbb{T}$, bounded, $a \in L^\infty(\mathbb{T})$, and satisfies

$$\sup_{y>0} \frac{1}{y} \int_{\mathbb{T}} |a(x + y) - a(x)| \, dx < \infty.$$

Typically, this condition is satisfied for $a \in BV(\mathbb{T})$, see [19, Chapter 1 Theorem 1.7.1] (in fact, this is nearly equivalent of being in $BV(\mathbb{T})$). In particular, a may be an indicating function, $a(x) = \mathbf{1}_\omega(x)$ for some measurable set $\omega \subset \mathbb{T}$.

1.1. Cauchy problem

The notion of solution, as well as the vanishing viscosity method used to solve the Cauchy problem, follows from standard arguments (which we borrow from [19]). We shall see that the presence of the damping term in (1.1) requires only slight modifications of this approach. We emphasize however that the case $\alpha = 1$ is specific, and can be treated by adapting the approach of Filippov [21].

Definition 1.2 *(Notion of solution, $0 < \alpha < 1$).* Let $\alpha \in (0, 1)$. A bounded measurable function u on $[0, T] \times \mathbb{T}$ is an *admissible weak solution* of (1.1)–(1.2), with $u_0 \in L^\infty(\mathbb{T})$, if the inequality

$$\iint\limits_{0\,\mathbb{T}}^{T} \left(\partial_t \psi \eta(u) + \partial_x \psi q(u) - a\psi \eta'(u) \frac{u}{|u|^\alpha}\right) dxdt + \int\limits_{\mathbb{T}} \psi(0, x)\eta(u_0(x))\, dx \geqslant 0 \tag{1.6}$$

holds for every convex function $\eta \in W^{1,\infty}$, with $q' = f'\eta'$, and all nonnegative Lipschitz continuous test function ψ on $[0, T] \times \mathbb{T}$.

Definition 1.3 *(Notion of solution, $\alpha = 1$).* Let $\alpha = 1$. A bounded measurable function u on $[0, T] \times \mathbb{T}$ is an *admissible weak solution* of (1.1)–(1.2), with $u_0 \in L^\infty(\mathbb{T})$, if there exists $h \in L^\infty((0, T) \times \mathbb{T})$ such that

$$\partial_t u + \partial_x (f(u)) + h = 0, \quad \text{in } \mathscr{D}'((0, T) \times \mathbb{T}),$$

with

$$h(t, x) \in a(x)\,\mathrm{Sign}(u(t, x)), \quad \text{a.e. } (t, x) \in (0, T) \times \mathbb{T},$$

where Sign is defined in (1.5), and such that the inequality

$$\iint\limits_{0\,\mathbb{T}}^{T} \left(\partial_t \psi \eta(u) + \partial_x \psi q(u) - \psi \eta'(u) h(t, x)\right) dxdt + \int\limits_{\mathbb{T}} \psi(0, x)\eta(u_0(x))\, dx \geqslant 0 \tag{1.7}$$

holds for every convex function $\eta \in W^{1,\infty}$, with $q' = f'\eta'$, and all nonnegative Lipschitz continuous test function ψ on $[0, T] \times \mathbb{T}$.

In all that follows, the notion of solution refers either to Definition 1.2 (case $0 < \alpha < 1$), or to Definition 1.3 (case $\alpha = 1$). We show that the Cauchy problem is well-posed, regardless of the value of $\alpha \in (0, 1]$.

Proposition 1.4 *(Cauchy problem). Assume that a satisfies Assumption 1.1 and $\alpha \in (0, 1]$. Let $u_0 \in L^\infty(\mathbb{T})$. There exists a unique, global, admissible weak solution u of (1.1)–(1.2), $u \in \mathscr{C}^0(\mathbb{R}_+; L^1(\mathbb{T}))$.*

We will also need the following comparison result.

Proposition 1.5 *(Comparison principles). Let $\alpha \in (0, 1]$.*

1. Let a satisfying Assumption 1.1, u and v be solutions of (1.1) with respective initial data $u_0, v_0 \in L^\infty(\mathbb{T})$ such that $u_0 \leqslant v_0$. Then

$$u(t, x) \leqslant v(t, x), \quad \forall t \geqslant 0, \ \text{a.e. } x \in \mathbb{T}.$$

Besides,

$$|u(t, x)| \leqslant \|u_0\|_{L^\infty(\mathbb{T})}, \quad \forall t \geqslant 0, \ \text{a.e. } x \in \mathbb{T}.$$

2. Let a_1 and a_2 satisfying Assumption 1.1 such that for almost all $x \in \mathbb{T}$, $a_1(x) \leqslant a_2(x)$. Then, denoting by u_1 and u_2 the respective solutions to (1.1)–(1.2) with the same initial datum $u_0 \in L^\infty(\mathbb{T})$ with $u_0 \geqslant 0$, we have

$$u_1(t, x) \geqslant u_2(t, x) \geqslant 0, \quad \forall t \geqslant 0, \ \text{a.e. } x \in \mathbb{T}.$$

Remark 1.6 *(BV solutions).* As a straightforward consequence of the proof of Proposition 1.5, given in Section A, one can show that if $u_0 \in BV(\mathbb{T})$, then the solution remains in BV, $u \in L^\infty(\mathbb{R}_+; BV(\mathbb{T}))$. However, we shall not use this property in this paper.

The results stated in this subsection are necessary in order to study the dynamics associated to (1.1). However, since they follow from a rather straightforward adaptation of classical results (in particular, [18–20]), we choose to simply state a more general result in an appendix, with assumptions on the nonlinearity that make it easy to adapt the above mentioned results.

1.2. Extinction results

We now focus on the core of this article, and give several results regarding the possible extinction of the solutions u of (1.1)–(1.2). The results depend on the flux f (its behavior near the origin), and the damping coefficient a, which is always assumed to satisfy Assumption 1.1. The first case which we consider is the one corresponding to a damping coefficient acting everywhere.

Proposition 1.7 *(Finite time extinction with damping everywhere). Suppose that there exists $\delta > 0$ such that*

$$a(x) \geqslant \delta > 0, \quad \forall x \in \mathbb{T}. \tag{1.8}$$

Let $u_0 \in L^\infty(\mathbb{T})$. There exists $T > 0$ such that the solution to (1.1)–(1.2) *satisfies*

$$u(t,x) = 0, \quad \forall t \geqslant T,\ a.e.\ x \in \mathbb{T}.$$

Besides, T can be chosen as

$$T = \frac{1}{\alpha\delta}\|u_0\|^\alpha_{L^\infty(\mathbb{T})}. \tag{1.9}$$

The proof of Proposition 1.7 is presented in Section 2 and is based on a Lyapunov approach. More precisely, we derive $L^p(\mathbb{T})$ estimates on the solutions of (1.1)–(1.2), and let then p go to infinity, so that we obtain a differential inequality for the $L^\infty(\mathbb{T})$-norm of the solutions of (1.1)–(1.2), which in turn implies its extinction in finite time.

Next, as motivated above, we consider the case in which the damping coefficient acts only in some part of the domain:

$$\exists \text{ an open interval } \omega \subset \mathbb{T} \text{ and } \delta > 0 \text{ s.t.} \quad a(x) \geqslant \delta, \quad \forall x \in \omega. \tag{1.10}$$

The extinction of the solution of (1.1)–(1.2) in this case will depend on the flux. Namely, we will treat two different cases, depending on whether $f'(0)$ vanishes or not. The easier case corresponds to the presence of transport at the origin,

$$f'(0) \neq 0. \tag{1.11}$$

In this case, one expects that the transport phenomenon will steer the solution through the set ω an arbitrary number of times, so that the strong friction term will make the solution vanish after some finite time. In agreement with these insights, in Section 3 we prove the following result:

Theorem 1.8 *(Finite time extinction by transport). Assume that the flux f is smooth and satisfies* (1.11)*, and the damping profile a satisfies Assumption 1.1 and* (1.10)*. Let K be such that*

$$\inf_{s\in[-K,K]} |f'(s)| > 0. \tag{1.12}$$

Then for any initial datum $u_0 \in L^\infty(\mathbb{T})$ satisfying

$$\|u_0\|_{L^\infty(\mathbb{T})} \leqslant K, \tag{1.13}$$

there exists $T > 0$ such that the solution u of (1.1)–(1.2) *satisfies*

$$u(t,x) = 0, \quad \forall t \geqslant T,\ a.e. x \in \mathbb{T}. \tag{1.14}$$

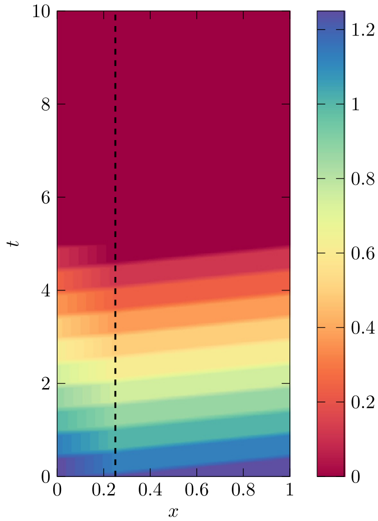

Fig. 1. Evolution of the solution for transport equation.

To illustrate the typical behavior of such a solution, we plot on Fig. 1 the evolution of the solution of the transport equation corresponding to $f(u) = 2u$ with initial datum $u_0(x) = 1.25$ for $(t, x) \in [0, 10] \times (0, 1)$, $a(x) = \mathbf{1}_{(0,1/4)}$ and $\alpha = 1$. The solution is computed using the numerical procedure described in Section 5. The dashed line indicates the position of the support of a. Note that Theorem 1.8 is a semi-global result, as it is valid for any initial datum u_0 whose $L^\infty(\mathbb{T})$-norm is bounded by the constant K, which is chosen in (1.12) to guarantee that the transport phenomenon really occurs. Indeed, if for some $K' > K$, $f'(K') = 0$, then it is easy to check that the solution u of (1.1)–(1.2) corresponding to the initial datum u_0 given by $u_0(x) = K'$ for $x \notin \operatorname{supp}(a)$ and $u_0(x) = 0$ for $x \in \operatorname{supp}(a)$ is stationary, $u(t, x) = u_0(x)$ for all $t \geqslant 0$, showing that the consistency between (1.12) and (1.13) is sharp for the conclusion (1.14) to hold.

Still, we emphasize that K in (1.12) need not be small, and can even be arbitrarily large in the case of a smooth flux without critical point, so that our result is indeed a semi-global result.

Our proof of Theorem 1.8 relies on Lyapunov functionals, in a similar spirit as the one developed in Section 2 to address the proof of Proposition 1.7. However, as the transport phenomenon is now essential to the decay process, we will introduce some weights in space in the functional. This approach is inspired by some recent works on the stabilization of hyperbolic systems of conservation laws, namely [15,16] (see the recent book [7] for further references).

In the case

$$f'(0) = 0, \tag{1.15}$$

corresponding for instance to the celebrated example of Burgers equation

$$f(u) = \frac{u^2}{2}, \tag{1.16}$$

the transport phenomenon competes with the dissipation of the solution, as the smaller the solution is, the slower the characteristics propagate. Our goal thus is to understand the interplay between these phenomena.

Theorem 1.9. *Let f be a smooth function such that*

$$f'(0) = 0 \quad \text{and} \quad \exists K > 0, \quad \inf_{s \in [-K,K]} |f''(s)| > 0. \tag{1.17}$$

Assume that the damping profile $a = a(x)$ satisfies Assumption 1.1 and (1.10). Then, for any initial datum $u_0 \in L^\infty(\mathbb{T})$ satisfying (1.13), the solution u of (1.1)–(1.2) satisfies the following property: There exist a time $t_* > 0$ and a constant $C > 0$ such that for all $t \geqslant t_*$, there exists an open subinterval $\omega(t) \subset \omega$ such that $u(t)_{|\omega(t)} = 0$, and, for all $t \geqslant t_*$,

$$|\omega \setminus \omega(t)| \leqslant \frac{C}{t^{1+\alpha}}, \quad \|u(t)\|_{L^\infty(\mathbb{T})} \leqslant \frac{C}{t}.$$

The proof of Theorem 1.9 is based on a precise description of the solution corresponding to $u_0 = K$, where K is constant (so that (1.17) is satisfied), $\mathbb{T}$ is identified with $(0, 1)$ with periodic boundary conditions, and

$$a(x) = \delta \mathbf{1}_\omega, \text{ with } \omega = (0, A). \tag{1.18}$$

We can then deduce Theorem 1.9 from a simple comparison argument based on Proposition 1.5.

It is clear that there still are cases which are not covered from our results, in particular cases in which the initial datum has an $L^\infty(\mathbb{T})$-norm which is larger than the best K in (1.12) or (1.17). Some particular instance is numerically studied in Section 5, as well as other models for which the complete understanding of a (localized) strong friction on the dynamics of a system is still not well understood.

1.3. Outline of the paper

The proof of Proposition 1.7 (finite time extinction with damping everywhere) is given in Section 2, thanks to suitable Lyapunov functionals. Theorem 1.8 (finite time extinction by transport) is proved in Section 3, by introducing refined Lyapunov functionals. For the case of the generalized Burgers equation, Theorem 1.9, a longer Section 4 is needed, where we first construct a rather explicit solution by following characteristics, which then turns out to be the solution provided by Proposition 1.4. Section 5 provides numerical illustrations in the case of (1.1) studied in this paper, as well as in the case of other equations for which the corresponding analysis turns out to be a challenging issue. The Cauchy problem is addressed in Appendix A, where the proofs of Propositions 1.4 and 1.5 are sketched.

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2. Proof of Proposition 1.7: the case of a damping acting everywhere

Proof of Proposition 1.7. Let $p > 2$. In view of (1.1), we formally have

$$\frac{d}{dt}\int_{\mathbb{T}} |u(t,x)|^p dx = p\int_{\mathbb{T}} |u|^{p-2}u\partial_t u dx = -p\int_{\mathbb{T}} |u|^{p-2}u\partial_x f(u)dx - p\int_{\mathbb{T}} a(x)|u|^{p-\alpha}dx.$$

Writing formally $p|u|^{p-2}u\partial_x f(u) = p|u|^{p-2}uf'(u)\partial_x u$, we have

$$p|u|^{p-2}u\partial_x f(u) = \partial_x g_p(u), \quad \text{with } g_p'(z) = p|z|^{p-2}zf'(z),$$

and so

$$\int_{\mathbb{T}} |u|^{p-2}u\partial_x f(u)dx = 0.$$

As a matter of fact, this reasoning is valid only for sufficiently smooth solutions. However, the conclusion remains true in our context, as can be seen by using Definitions 1.2 and 1.3 with $p > 2$, $\eta(u) = |u|^p$, $\psi(t,x) = \tilde{\psi}(t) \in \mathscr{D}(0,T)$ nonnegative, so that in the sense of distribution:

$$\frac{d}{dt}\int_{\mathbb{T}} |u(t,x)|^p dx \leqslant -p\int_{\mathbb{T}} a(x)|u|^{p-\alpha}dx \leqslant -\frac{p\delta}{\|u(t)\|^\alpha_{L^\infty(\mathbb{T})}}\int_{\mathbb{T}} |u(t,x)|^p dx,$$

that is

$$\frac{d}{dt}\ln\|u(t)\|_{L^p(\mathbb{T})} \leqslant -\frac{\delta}{\|u(t)\|^\alpha_{L^\infty(\mathbb{T})}}.$$

By integration,

$$\|u(t)\|_{L^p(\mathbb{T})} \leqslant \|u_0\|_{L^p(\mathbb{T})} \exp\left(-\delta \int_0^t \frac{ds}{\|u(s)\|^\alpha_{L^\infty(\mathbb{T})}}\right),$$

and by letting $p \to \infty$,

$$\|u(t)\|_{L^\infty(\mathbb{T})} \leqslant \|u_0\|_{L^\infty(\mathbb{T})} \exp\left(-\delta \int_0^t \frac{ds}{\|u(s)\|^\alpha_{L^\infty(\mathbb{T})}}\right). \tag{2.1}$$

Set

$$\Phi(t) = \int_0^t \frac{ds}{\|u(s)\|^\alpha_{L^\infty(\mathbb{T})}}.$$

The above inequality reads

$$\Phi'(t) \geqslant \frac{1}{\|u_0\|^\alpha_{L^\infty(\mathbb{T})}} e^{\alpha\delta\Phi(t)}.$$

Since the solution to $\Psi' = Ce^{\alpha\delta\Psi}$, $\Psi(0) = 0$, is given by

$$\Psi(t) = -\frac{1}{\alpha\delta} \ln(1 - C\alpha\delta t), \quad t \leqslant \frac{1}{C\alpha\delta},$$

we conclude by comparison that $\Phi(t) \to +\infty$ before the time $\|u_0\|^\alpha_{L^\infty(\mathbb{T})}/(\alpha\delta)$, hence the result thanks to (2.1), with

$$T = \frac{\|u_0\|^\alpha_{L^\infty(\mathbb{T})}}{\alpha\delta},$$

which may not be the sharp extinction time, but an upper bound for it. □

Remark 2.1 *(Whole space).* The above argument shows that the conclusion of Proposition 1.7 remains valid if (1.8) is set up on the whole line, $x \in \mathbb{R}$, provided that the solution which we consider goes to zero at $\pm\infty$.

Otherwise, a similar proof can be given, by considering estimates of $u(t)$ in $L^p(A_-(t), A_+(t))$ where

$$A_-(t) = A_-^0 + t \sup_{s\in[-K,K]} f'(s) \quad \text{and} \quad A_+(t) = A_+^0 + t \inf_{s\in[-K,K]} f'(s)$$

for any pair $(A_-^0, A_+^0) \in \mathbb{R}^2$; see e.g. the proof of Lemma 3.2 where the same kind of arguments are developed.

3. Proof of Theorem 1.8: the transport case

The goal of this section is to prove Theorem 1.8. Thus, we consider the setting of Theorem 1.8, and we assume in particular that f is smooth, satisfies (1.11), K satisfies (1.12), and a satisfies Assumption 1.1 and (1.10).

3.1. Strategy

In order to ease the reading of the proof of Theorem 1.8, we decompose it into two lemmas, the first one stating that in the setting of Theorem 1.8 the solutions of (1.1)–(1.2) decay exponentially, while the second one will show that if the initial datum is small enough, then the corresponding solution of (1.1)–(1.2) vanishes in finite time.

Lemma 3.1. *Within the setting of Theorem 1.8, there exist $C > 0$ and $\mu > 0$ such that for any initial datum $u_0 \in L^\infty(\mathbb{T})$ satisfying (1.13), any solution u of (1.1)–(1.2) satisfies*

$$\|u(t)\|_{L^\infty(\mathbb{T})} \leqslant Ce^{-\mu t}\|u_0\|_{L^\infty(\mathbb{T})}. \tag{3.1}$$

Lemma 3.2. *Within the setting of Theorem 1.8, there exist $\varepsilon_0 > 0$ and $T_0 > 0$ such that if*

$$\|u_0\|_{L^\infty(\mathbb{T})} \leqslant \varepsilon_0, \tag{3.2}$$

the solution u of (1.1)–(1.2) satisfies

$$u(t,x) = 0, \quad \forall t \geqslant T_0, \ \text{a.e. } x \in \mathbb{T}. \tag{3.3}$$

The proofs of Lemma 3.1 and Lemma 3.2 are given in Subsections 3.2 and 3.3, respectively. The proof of Theorem 1.8 is then given in Subsection 3.4.

3.2. *Proof of Lemma 3.1*

Proof. We first choose a function $\varphi = \varphi(x)$ such that

$$\varphi \in \mathscr{C}^\infty(\mathbb{T}), \qquad \text{with } \varphi(x) = x, \quad \forall x \in \mathbb{T} \setminus \omega.$$

Note that such a function φ satisfies in particular that

$$\forall x \in \mathbb{T} \setminus \omega, \quad \partial_x \varphi(x) = 1, \qquad \text{and} \qquad \partial_x \varphi \in \mathscr{C}^0(\mathbb{T}).$$

Now, let $u_0 \in L^\infty(\mathbb{T})$ and u the corresponding solution of (1.1)–(1.2). We consider the Lyapunov functionals, indexed by $p \geqslant 2$ and some parameter $\lambda \in \mathbb{R}$ chosen later,

$$E_{p,\lambda}(t) = \int_{\mathbb{T}} \left| e^{-\lambda\varphi(x)} u(t,x) \right|^p dx. \tag{3.4}$$

Formally, these functionals satisfy:

$$\begin{aligned}\frac{dE_{p,\lambda}(t)}{dt} &= p \int_{\mathbb{T}} e^{-p\lambda\varphi(x)} |u(t,x)|^{p-2} u(t,x) \partial_t u(t,x)\, dx \\ &= -p \int_{\mathbb{T}} e^{-p\lambda\varphi(x)} |u(t,x)|^{p-2} u(t,x) \partial_x (f(u(t,x))\, dx \\ &\quad - p \int_{\mathbb{T}} a(x) e^{-p\lambda\varphi(x)} |u(t,x)|^{p-\alpha}\, dx.\end{aligned}$$

If u were smooth, we would write

$$\begin{aligned}p|u(t,x)|^{p-2} u(t,x) \partial_x (f(u(t,x)) &= pf'(u(t,x))|u(t,x)|^{p-2} u(t,x) \partial_x u(t,x) \\ &= \partial_x (g_p(u(t,x))),\end{aligned}$$

where g_p is defined by

$$g_p(s) = p \int_0^s f'(\tau)|\tau|^{p-2} \tau\, d\tau, \tag{3.5}$$

so that we would write:

$$-p \int_{\mathbb{T}} e^{-p\lambda\varphi(x)} |u(t,x)|^{p-2} u(t,x) \partial_x (f(u(t,x))\, dx = -p\lambda \int_{\mathbb{T}} \partial_x \varphi(x) e^{-p\lambda\varphi(x)} g_p(u(t,x))\, dx.$$

Note in passing that g_p has the same sign as $f'(0)$. As solutions u may contain shocks, these estimates should be justified by using the definition of admissible weak solutions, i.e. inequalities (1.6) or (1.7), choosing $p > 2$, $\eta(u) = |u|^p$ and $\psi(t,x) = e^{-p\lambda\varphi(x)} \tilde{\psi}(t)$, with $\tilde{\psi}(t) \in \mathscr{D}(0,T)$ nonnegative (and obviously Lipschitz continuous), so ψ is an admissible test function in the sense of Definition 1.2. One obtains in that way:

$$\frac{dE_{p,\lambda}(t)}{dt} \leqslant -p\lambda \int_{\mathbb{T}} \partial_x\varphi(x)e^{-p\lambda\varphi(x)} g_p(u(t,x))\,dx - p\int_{\mathbb{T}} a(x)e^{-p\lambda\varphi(x)}|u(t,x)|^{p-\alpha}\,dx$$
$$\leqslant -p\lambda \int_{\mathbb{T}} e^{-p\lambda\varphi(x)} g_p(u(t,x))\,dx + p|\lambda|\|\partial_x\varphi - 1\|_{L^\infty(\mathbb{T})}\int_{\omega} e^{-p\lambda\varphi(x)}\left|g_p(u(t,x))\right|dx$$
$$- p\delta\int_{\omega} e^{-p\lambda\varphi(x)}|u(t,x)|^{p-\alpha}\,dx.$$

It is thus natural to study the function g_p in (3.5). In order to do this, we first note that if $\|u_0\|_{L^\infty(\mathbb{T})} \leqslant K$, according to Proposition 1.5, for all time $t \geqslant 0$, the L^∞-norm of $u(t)$ is bounded by K. Therefore, we introduce

$$\beta_- = \inf_{s\in[-K,K]} |f'(s)| \quad \text{and} \quad \beta_+ = \sup_{s\in[-K,K]} |f'(s)|,$$

so that for all $(t,x) \in [0,\infty) \times \mathbb{T}$,

$$\beta_-|u(t,x)|^p \leqslant \left|g_p(u(t,x))\right| \leqslant \beta_+|u(t,x)|^p. \tag{3.6}$$

Therefore, we obtain

$$\frac{dE_{p,\lambda}(t)}{dt} \leqslant -p\lambda \int_{\mathbb{T}} e^{-p\lambda\varphi(x)} g_p(u(t,x))\,dx$$
$$+ p\left(|\lambda|\|\partial_x\varphi - 1\|_{L^\infty(\mathbb{T})}\beta_+ - \delta\|u(t)\|^{-\alpha}_{L^\infty(\mathbb{T})}\right)\int_{\omega} e^{-p\lambda\varphi(x)}|u(t,x)|^p\,dx.$$

In particular, for all $t \geqslant 0$, if

$$|\lambda|\|\partial_x\varphi - 1\|_{L^\infty(\mathbb{T})}\beta_+ \leqslant \delta\|u(t)\|^{-\alpha}_{L^\infty(\mathbb{T})}, \tag{3.7}$$

we have

$$\frac{dE_{p,\lambda}(t)}{dt} \leqslant -p\lambda \int_{\mathbb{T}} e^{-p\lambda\varphi(x)} g_p(u(t,x))\,dx. \tag{3.8}$$

As for all t, $\|u(t)\|_{L^\infty(\mathbb{T})} \leqslant K$, we therefore choose

$$\lambda_* = \operatorname{sign}(f'(0))\frac{\delta K^{-\alpha}}{\|\partial_x\varphi - 1\|_{L^\infty(\mathbb{T})}\beta_+},$$

so that (3.7) is satisfied and (3.8) becomes:

$$\frac{dE_{p,\lambda_*}(t)}{dt} \leqslant -p|\lambda_*|\beta_- E_{p,\lambda*}(t).$$

Therefore, we get that

$$\|e^{-\lambda_*\varphi}u(t)\|_{L^p(\mathbb{T})} \leqslant e^{-|\lambda_*|\beta_- t}\|e^{-\lambda_*\varphi}u_0\|_{L^p(\mathbb{T})}, \quad \forall t \geqslant 0.$$

As λ_* does not depend on p, we can pass to the limit as $p \to \infty$, and obtain:

$$\|e^{-\lambda_*\varphi}u(t)\|_{L^\infty(\mathbb{T})} \leqslant e^{-|\lambda_*|\beta_- t}\|e^{-\lambda_*\varphi}u_0\|_{L^\infty(\mathbb{T})}, \quad \forall t \geqslant 0,$$

and thus, distinguishing the cases $\lambda_* > 0$ and $\lambda_* < 0$, we get

$$\|u(t)\|_{L^\infty(\mathbb{T})} \leqslant e^{|\lambda_*|(\sup\varphi - \inf\varphi) - |\lambda_*|\beta_- t}\|u_0\|_{L^\infty(\mathbb{T})}, \quad \forall t \geqslant 0.$$

This concludes the proof of Lemma 3.1, as $|\lambda_*| > 0$. □

Remark 3.3. Note that one can go further and show that the $L^\infty(\mathbb{T})$ norm of the solutions of (1.1)–(1.2) decays in fact faster than an exponential in time.

Indeed, if we set

$$T_1 = \frac{2}{\beta_-}(\sup\varphi - \inf\varphi),$$

and use the explicit choice

$$\lambda_* = \operatorname{sign}(f'(0))\frac{\delta\|u_0\|_{L^\infty(\mathbb{T})}^{-\alpha}}{\|\partial_x\varphi - 1\|_{L^\infty(\mathbb{T})}\beta_+},$$

which is admissible according to the above proof, one in fact gets

$$\|u(T_1)\|_{L^\infty(\mathbb{T})} \leqslant e^{-|\lambda_*|(\sup\varphi-\inf\varphi)}\|u_0\|_{L^\infty(\mathbb{T})} \leqslant \exp\left(-c_0\|u_0\|_{L^\infty(\mathbb{T})}^{-\alpha}\right)\|u_0\|_{L^\infty(\mathbb{T})},$$

where c_0 is given by

$$c_0 = \delta\frac{(\sup\varphi - \inf\varphi)}{\|\partial_x\varphi - 1\|_{L^\infty(\mathbb{T})}\beta_+}.$$

Starting from there and using the semi-group property, we introduce a sequence of time, indexed by $n \in \mathbb{N}$,

$$T_n = nT_1 \quad \left(= \frac{2n}{\beta_-}(\sup\varphi - \inf\varphi)\right),$$

for which one gets immediately, for all $n \in \mathbb{N}$,

$$\|u(T_{n+1})\|_{L^\infty(\mathbb{T})} \leqslant \exp\left(-c_0\|u(T_n)\|_{L^\infty(\mathbb{T})}^{-\alpha}\right)\|u(T_n)\|_{L^\infty(\mathbb{T})}.$$

It is then easy to check that the sequence $(\|u(T_n)\|_{L^\infty(\mathbb{T})})_{n\in\mathbb{N}}$ goes to 0 faster than any (non-trivial) geometric sequence, which in turn implies that the map $t \mapsto \|u(t)\|_{L^\infty(\mathbb{T})}$ goes to 0 faster than any exponential.

It would be interesting to develop a direct proof of Theorem 1.8 based only on a suitable choice of Lyapunov functionals in the spirit of the one used above.

3.3. *Proof of Lemma 3.2*

Proof. To simplify the presentation, in the proof of this lemma, $\mathbb{T}$ is identified with an interval centered in 0, and ω is identified with an interval of the form $(-A, A)$.

The proof of Lemma 3.2 is divided in two steps. In the first step, we show that if $\varepsilon_0 > 0$ is chosen small enough, then necessarily, the solution u of (1.1)–(1.2) with $u_0 \in L^\infty(\mathbb{T})$ satisfying $\|u_0\|_{L^\infty(\mathbb{T})} \leqslant \varepsilon_0$ vanishes in some part of the domain ω after some (small) time. We then show that this implies that the solution u vanishes everywhere after some time.

• **Step 1.** We introduce the paths

$$A_-(t) = \sup_{[-K,K]}\{f'\}t - \frac{A}{2}, \quad A_+(t) = \inf_{[-K,K]}\{f'\}t + \frac{A}{2}.$$

We fix τ_* such for all $t \in [0, \tau_*)$,

$$-A < A_-(t) < A_+(t) < A,$$

that is

$$\tau_* = \frac{A}{\max\left\{2\inf_{[-K,K]}|f'|,\ \sup_{[-K,K]}|f'| - \inf_{[-K,K]}|f'|\right\}}. \tag{3.9}$$

We then set, for $p \geqslant 2$ and $t \in [0, \tau_*]$,

$$E_{p,\mathrm{loc}}(t) = \int_{A_-(t)}^{A_+(t)} |u(t,x)|^p\,dx.$$

Arguing as in the proof of Proposition 1.7, as $[A_-(t), A_+(t)] \subset \omega$ for all $t \in [0, \tau_*]$ and $\inf f' \leqslant f'(u(t,x)) \leqslant \sup f'$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{T}$, we get, for all $t \geqslant 0$,

$$\frac{d}{dt} E_{p,\mathrm{loc}}(t) \leqslant -\frac{p\delta}{\|u(t)\|^{\alpha}_{L^\infty(A_-(t),A_+(t))}} E_{p,\mathrm{loc}}(t). \tag{3.10}$$

In fact, to prove this estimate rigorously, we use the definition of admissible weak solutions, i.e. the inequalities (1.6) or (1.7), choosing $p > 2$, $\eta(u) = |u|^p$ and $\psi(t,x) = \varphi_\epsilon(t,x)\tilde{\psi}(t)$, $\tilde{\psi}(t) \in \mathscr{D}(0,T)$ nonnegative, where

$$\varphi_\epsilon(t,x) = \varphi_-^0\left(\frac{x - A_-(t)}{\epsilon}\right)\mathbf{1}_{x\in(A_-(t)-\epsilon, A_-(t))} + \mathbf{1}_{x\in(A_-(t),A_+(t))} + \varphi_+^0\left(\frac{x - A_+(t)}{\epsilon}\right)\mathbf{1}_{x\in(A_+(t),A_+(t)+\epsilon)},$$

with φ_-^0, φ_+^0 non-negative monotonic smooth cut-off functions, taking value 1 on $\mathbb{R}_+$ and vanishing on $(-\infty,-1)$ for φ_-^0, taking value 1 on $\mathbb{R}_-$ and vanishing on $(1,\infty)$ for φ_+^0. We then pass to the limit $\epsilon \to 0$ in the inequalities (1.6) or (1.7) to show the estimate (3.10).

This yields, for all $t \in [0, \tau_*]$,

$$\frac{d}{dt}\left(\log(\|u\|_{L^p(A_-(t),A_+(t))})\right) \leqslant -\frac{\delta}{\|u(t)\|^{\alpha}_{L^\infty(A_-(t),A_+(t))}}.$$

Integrating this expression and letting $p \to \infty$, we obtain, similarly as in (2.1), that for all $t \in [0, \tau_*]$,

$$\|u(t)\|_{L^\infty(A_-(t),A_+(t))} \leqslant \|u_0\|_{L^\infty(-A/2,A/2)} \exp\left(-\delta \int_0^t \frac{ds}{\|u(s)\|^{\alpha}_{L^\infty(A_-(s),A_+(s))}}\right).$$

Arguing as in the proof of Proposition 1.7, this implies that $\|u(\tau_0)\|_{L^\infty(A_-(\tau_0),A_+(\tau_0))} = 0$ for

$$\tau_0 = \frac{\varepsilon_0^\alpha}{\alpha\delta}, \tag{3.11}$$

which is smaller than τ_* for $\varepsilon_0 > 0$ small enough, i.e. for small enough initial datum (recall (3.2)).

Using the same argument on the solutions $u(\cdot + t_0)$ for all $t_0 \geqslant 0$, we see that in fact we have obtained

$$\forall t \geqslant \tau_0, \quad \|u(t)\|_{L^\infty(A_-(\tau_0),A_+(\tau_0))} = 0. \tag{3.12}$$

We end up this step by emphasizing that $A_-(\tau_0) < A_+(\tau_0)$, so that (3.12) really implies that $u(t,\cdot)$ vanishes on a constant interval for all time $t \geqslant \tau_0$.

• **Step 2.** In this step, to fix the ideas, we assume that $f'(0) > 0$, as a completely similar proof can be adapted to the case $f'(0) < 0$. We then look at the evolution of the L^2-norm of $u(t)$ on the set $(A_-(\tau_0), B(t))$, where $B(t) = A_+(\tau_0) + \beta_-(t - \tau_0)_+$, with $\beta_- = \inf_{[-K,K]} f'$ (recall that we have assumed $f'(0) > 0$). Recall that $u(t,x) = 0$ for all $t \geqslant \tau_0$ and $x \in [A_-(\tau_0), A_+(\tau_0)]$ according to (3.12). Using the definition of admissible weak solutions, we infer:

$$\frac{d}{dt}\left(\int_{A_-(\tau_0)}^{B(t)} |u(t,x)|^2\,dx\right) \leqslant 0.$$

Indeed, this comes from the definition of admissible weak solutions with $\eta(u) = |u|^2$ and $\psi(t,x) = \varphi_\epsilon(t,x)\tilde{\psi}(t)$, $\tilde{\psi}(t) \in \mathscr{D}(0,T)$ nonnegative, where

$$\varphi_\epsilon(t,x) = \varphi_-^0\left(\frac{x - A_+(\tau_0)}{\epsilon}\right)\mathbf{1}_{x\in(A_+(\tau_0)-\epsilon, A_+(\tau_0))} + \mathbf{1}_{x\in(A_+(\tau_0),B(t))} + \varphi_+^0\left(\frac{x - B(t)}{\epsilon}\right)\mathbf{1}_{x\in(B(t),B(t)+\epsilon)},$$

with φ_-^0, φ_+^0 non-negative monotonic smooth cut-off functions, taking value 1 on $\mathbb{R}_+$ and vanishing on $(-\infty,-1)$ for φ_-^0, taking value 1 on $\mathbb{R}_-$ and vanishing on $(1,\infty)$ for φ_+^0.

Therefore, for all $t \geqslant \tau_0$,

$$\int_{A_-(\tau_0)}^{B(t)} |u(t,x)|^2\,dx = 0.$$

In particular, waiting a time $T_0 > \tau_0$ such that $B(T_0) = A_-(\tau_0) + |\mathbb{T}|$, for all $t \geqslant T_0$, for almost all $x \in \mathbb{T}$, $u(t,x) = 0$. This concludes the proof of Lemma 3.2. □

3.4. *Proof of Theorem 1.8*

Proof. Theorem 1.8 easily follows from Lemma 3.1 and 3.2. Indeed, if one chooses an initial datum u_0 satisfying (1.13), the $L^\infty(\mathbb{T})$-norm of the corresponding solution $u(t)$ of (1.1)–(1.2) decays exponentially. Thus, after some time, it becomes smaller than the parameter ε_0 in Lemma 3.2. It will therefore vanish after some time according to Lemma 3.2. □

3.5. *A control theoretic interpretation of Theorem 1.8*

Let us mention that Theorem 1.8 is closely related to the following control problem: given ω a non-empty subinterval of $\mathbb{T}$, $T > 0$ and $u_0 \in L^2(\mathbb{T})$, find a control function $v \in L^2((0,T) \times \mathbb{T})$ such that the solution u of

$$\begin{cases} \partial_t u + \partial_x (f(u)) + \mathbf{1}_\omega v = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{T}, \\ u_{|t=0} = u_0, & x \in \mathbb{T}, \end{cases} \tag{3.13}$$

satisfies

$$u(T) = 0 \quad \text{in } \mathbb{T}. \tag{3.14}$$

Assuming (1.11) and defining K by (1.12), Theorem 1.8 implies that, for an initial datum $u_0 \in L^\infty(\mathbb{T})$ satisfying (1.13), choosing the control function v under the feedback form

$$v(t,x) = -\delta \frac{u(t,x)}{|u(t,x)|^\alpha}, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{T}, \tag{3.15}$$

for some $\delta > 0$, the controlled trajectory u solving (3.13) will satisfy the controllability requirement (3.14) in some time $T > 0$.

Looking more closely at the proof of Lemma 3.2, we can state the following result:

Proposition 3.4. *Let f be a smooth flux function satisfying (1.11) and define K by (1.12). Let ω a non-empty subinterval of $\mathbb{T}$.*

Given $\gamma > 0$, there exists a parameter δ in (3.15) such that, for any initial datum $u_0 \in L^\infty(\mathbb{T})$ satisfying (1.13), the corresponding solution u of (3.13)–(3.15) vanishes after the time

$$T = (1+\gamma)\frac{|\mathbb{T}|}{\inf_{[-K,K]} |f'|}.$$

Proof. We do the same identifications as in the proof of Lemma 3.2. Indeed, choosing $\gamma > 0$ smaller if necessary, one can assume $\gamma|\mathbb{T}|/\inf|f'| < \tau_*$, with τ_* as in (3.9). Thus, taking

$$\delta = \frac{K^\alpha}{\alpha}\frac{\inf|f'|}{\gamma|\mathbb{T}|},$$

for solutions $u_0 \in L^\infty(\mathbb{T})$ satisfying (1.13), the time τ_0 in (3.11) is smaller than τ_* in (3.9) and than $\gamma|\mathbb{T}|/\inf|f'|$, so that the proof of Lemma 3.2 easily yields that u vanishes after the time $T_0 = (1+\gamma)|\mathbb{T}|/\inf|f'|$. □

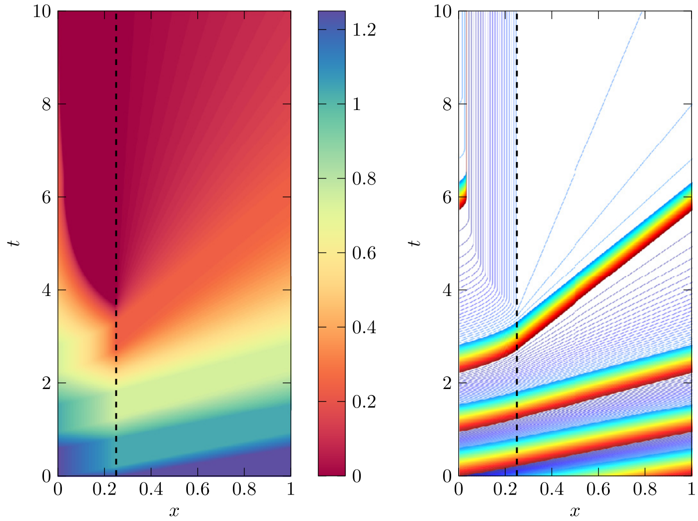

Fig. 2. Evolution of the solution (left) and of the characteristic curves (right) for Burgers flux.

Note that the time of extinction given by Proposition 3.4 can be made arbitrarily close to the critical time expected to control (3.13) when ω is thin, given by $|\mathbb{T}|/\inf_{[-K,K]}|f'|$. In fact, when ω is an interval, the critical time to control (3.13) is given by $|\mathbb{T}\setminus\omega|/\inf_{[-K,K]}|f'|$, and one can check from the proof of Lemma 3.2 that, when increasing δ, the solution u of (3.13) with the feedback law (3.15) will vanish in a time T_δ which is such that $\lim_{\delta\to\infty} T_\delta = |\mathbb{T}\setminus\omega|/\inf_{[-K,K]}|f'|$. In this sense, we have produced a non-linear feedback operator which controls (3.13) in almost sharp time.

4. Proof of Theorem 1.9: the degenerate case

The goal of this section is to discuss the case of a flux satisfying $f'(0)=0$, with $f''(0)\neq 0$, and prove Theorem 1.9. As said in the introduction, we first prove Theorem 1.9 in the case of a constant initial datum and a strictly convex flux satisfying (1.17). We then deduce the other instances of Theorem 1.9 by using symmetry arguments and comparison arguments.

4.1. Computation and estimates for the solution u of (4.2) for a strictly convex flux f with $f'(0)=0$

We first assume that f satisfies, for some $K>0$:

$$f'(0)=0 \quad \text{and} \quad \exists K>0, \quad \inf_{s\in[-K,K]} f''(s)>0, \tag{4.1}$$

and the damping is given by $a(x)=\delta\mathbf{1}_{(0,A)}$.

Let u be the solution of

$$\partial_t u+\partial_x f(u)+\delta\mathbf{1}_{(0,A)}\frac{u}{|u|^\alpha}=0 \text{ in } \mathbb{R}_+\times\mathbb{T}, \quad u_{|t=0}=K \text{ in } \mathbb{T}, \tag{4.2}$$

where K is the positive constant in (4.1), $\delta>0$ and $(0,A)\subset\mathbb{T}$.

Obviously, as $K>0$, the solution u will stay non-negative for all times.

We develop a precise analysis of the characteristics curves of the solution, illustrated in Fig. 2. We depict in Fig. 2 both the evolution of the solution and its characteristic curves for the Burgers equation. The simulations are made following the process described in Section 5. The numerical parameters are $\alpha=1$, $u_0(x)=K=1.25$, $A=0.25$, $\delta=1$, $\delta x=\delta t=5\cdot 10^{-5}$ and the final time is $T_f=10$. The dashed line indicates the location of the support of a. It appears that the solution becomes zero inside the support of a, with corresponding characteristics becoming vertical straight lines. Our goal in the following is to prove rigorously that this observed behavior indeed coincides with the theoretical one.

4.1.1. Formal computation of the solution u of (4.2) for a strictly convex flux f with $f'(0)=0$

We shall compute u using characteristics as if the solution were regular. We will fully justify this assumption later (in Subsection 4.1.3, Lemma 4.1).

For $\alpha < 1$, when u is smooth, for $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{T}$, the characteristics are given by

$$\begin{cases} \dfrac{dX}{dt}(t, t_0, x_0) = f'(u(t, X(t, t_0, x_0))), \quad t \geqslant 0, \\ X(t_0, t_0, x_0) = x_0, \end{cases} \tag{4.3}$$

and the solution u along the characteristics satisfies:

$$\frac{d}{dt}\left(u(t, X(t, t_0, x_0))\right) = -a(X(t, t_0, x_0)) \frac{u(t, X(t, t_0, x_0))}{|u(t, X(t, t_0, x_0))|^\alpha}. \tag{4.4}$$

The solutions of (4.3)–(4.4) are then solutions of

$$\frac{d}{dt}\begin{pmatrix} X(t, t_0, x_0) \\ u(t, X(t, t_0, x_0)) \end{pmatrix} = F_\alpha(X(t, t_0, x_0), u(t, X(t, t_0, x_0))), \quad t \geqslant 0, \tag{4.5}$$

where

$$F_\alpha(X, u) = \begin{pmatrix} f'(u) \\ -a(X)u/|u|^\alpha \end{pmatrix}. \tag{4.6}$$

Of course, when $\alpha = 1$, similar computations can be performed as long as the solution u stays positive, but the corresponding definition of F_α for $\alpha = 1$ should be made more specific when u vanishes. We thus introduce

$$F_1(X, u) = \begin{pmatrix} f'(u) \\ -a(X)\,\mathrm{Sign}(u) \end{pmatrix}, \tag{4.7}$$

where Sign is defined in (1.5), and the corresponding counterpart of (4.5) should then read as

$$\frac{d}{dt}\begin{pmatrix} X(t, t_0, x_0) \\ u(t, X(t, t_0, x_0)) \end{pmatrix} \in F_1(X(t, t_0, x_0), u(t, X(t, t_0, x_0))), \quad t \geqslant 0. \tag{4.8}$$

In the computations given afterward, we will use the fact that as $u_0(x) = K \geqslant 0$, the solution u of (4.2) stays non-negative, so that we can in fact write $u/|u|^\alpha = u^{1-\alpha}$, which will make the various expressions slightly easier.

On the interval $(A, 1)$. As a vanishes on $(A, 1)$, (4.4) implies that the solution u stays constant along the characteristics in $(A, 1)$. Therefore, if we choose $x_0 = A$, we get

$$u(t, X(t, t_0, A)) = u(t_0, A), \qquad \text{for } t \geqslant t_0, \text{ as long as } t \mapsto X(t, t_0, A) \text{ stays smaller than } 1,$$

and therefore

$$X(t, t_0, A) = A + (t - t_0) f'(u(t_0, A)) \qquad \text{for } t \in \left[t_0, t_0 + \frac{1-A}{f'(u(t_0, A))}\right].$$

Thus, we can write, for all $t \geqslant 0$,

$$u\left(t + \frac{1-A}{f'(u(t, A))}, 0\right) = u\left(t + \frac{1-A}{f'(u(t, A))}, 1\right) = u(t, A). \tag{4.9}$$

Let us finally note that easy computations show that

$$\forall t \in \left[0, \frac{1-A}{f'(K)}\right], \quad u(t, 0) = u(t, 1) = K.$$

On the interval $(0, A)$. We deal with this case as before. But now, as long as the characteristic $t \mapsto X(t, t_0, 0)$ defined for $t \geqslant t_0$ stays in $[0, A]$, we have

$$u(t, X(t, t_0, 0)) = \left(u(t_0, 0)^\alpha - \delta\alpha(t - t_0)\right)_+^{1/\alpha}.$$

Therefore, the characteristic $X(t, t_0, 0)$ reaches $x = A$ for the first solution $t \geqslant t_0$ (if any) of

$$\int_0^{t-t_0} f'\left(\left(u(t_0,0)^\alpha - \delta\alpha\tau\right)_+^{1/\alpha}\right) d\tau = A,$$

for which we have

$$u(t,A) = \left(u(t_0,0)^\alpha - \delta\alpha(t-t_0)\right)_+^{1/\alpha}.$$

For $t \geqslant 0$, such that

$$\int_0^{t} f'\left(\left(K^\alpha - \delta\alpha\tau\right)_+^{1/\alpha}\right) d\tau < A,$$

we get

$$u(t,A) = \left(K^\alpha - \delta\alpha t\right)_+^{1/\alpha}.$$

4.1.2. Justification of the above formulae

When the solution u is smooth and strictly positive, all the above computations are fully justified.

Besides, for $\alpha \in (0,1]$, we only have *a priori* existence results for solutions of (4.5) or (4.8) due to Cauchy–Peano Theorem. Due to [21, Chapter 2 Section 10 Theorem 1], we also have forward uniqueness of the solutions of (4.5) as long as X stays in $(0,A)$ or as long as X stays in $(A,1)$, as the function F_α satisfies the following one-sided Lipschitz condition: there exists $C > 0$ such that for all $(X_1,u_1) \in \mathbb{R}^2$, $(X_2,u_2) \in \mathbb{R}^2$ with $|u_1|, |u_2| \leqslant K$, and $(X_1,X_2) \in (0,A)^2 \cup (A,1)^2$,

$$((X_1,u_1) - (X_2,u_2)) \cdot (F_\alpha(X_1,u_1) - F_\alpha(X_2,u_2)) \leqslant C|(X_1,u_1) - (X_2,u_2)|^2. \tag{4.10}$$

The uniqueness across the set $\{X = 0\}$ in our setting will follow from the fact that the solution u of (4.2) with constant initial datum $u_0(x) = K > 0$ stays non-negative for all times, and strictly positive at $x = 0$ for all times (see Section 4.1.3), so the characteristics $t \mapsto X(t,t_0,x_0)$, when meeting $\{X = 0\}$, will simply follow the dynamics in $(0,A)$. This argument can be invoked similarly when characteristics meet the set $\{X = A\}$ while u stays positive. However, as we will see, there will be some time at which $u(t,A)$ vanishes. There, uniqueness should also hold across the set $\{X = A\}$ simply by continuity of $F_\alpha(x,0)$ across $\{X = A\}$, at least for $\alpha \in (0,1)$. This is however less clear to prove, especially when turning to the case $\alpha = 1$.

Thus, to properly justify the above computations, we construct explicitly the solution u of (4.2) using the characteristics formulae above. In turn, this will guarantee the characteristics formulae given above. Note that, strictly speaking, our arguments construct *a* solution of (4.2), but by uniqueness of the admissible solution of (4.2), see Proposition 1.4, this solution is *the* solution of (4.2) with initial datum $u_0(x) = K$.

4.1.3. Regularity of the solution u

The goal of this section is to prove the following regularity result on the solution u of (4.2) with $u_0(x) = K$.

Lemma 4.1. *Let f and K as in* (4.1)*, $\alpha \in (0,1]$, $\delta > 0$ and $(0,A) \subset \mathbb{T}$.*

Then the solution u of (4.2) *satisfies the following regularity properties: $u \in \mathscr{C}^0([0,\infty) \times \mathbb{T})$, u is piecewise $\mathscr{C}^1([0,\infty) \times \mathbb{T})$, and we have the more precise result:*

- *The set $\mathscr{Z} = \{(t,x) \in [0,\infty) \times \mathbb{T},$ s.t. $u(t,x) = 0\}$, if not empty, is a closed set of $[0,\infty) \times (0,A]$ whose boundary is globally Lipschitz and piecewise $\mathscr{C}^1$.*
- *For all bounded open subset Ω such that $\overline{\Omega} \subset ([0,\infty) \times \mathbb{T} \setminus \mathscr{Z})$, there exists a finite number of smooth ($\mathscr{C}^1$) curves $\mathcal{C}_i$, which may intersect only transversally, such that u is $\mathscr{C}^1$ in the adherence of each of the connected component of $\Omega \setminus \mathcal{C}_i$.*
- *For all bounded open subset $\Omega \subset ([0,\infty) \times \mathbb{T})$, there exists a finite number of curves $\mathcal{C}_i$ (globally Lipschitz and piecewise $\mathscr{C}^1$), which may intersect only transversally, such that u is $\mathscr{C}^1$ in each of the connected component of $\Omega \setminus \mathcal{C}_i$.*
- *For all $t \geqslant 0$, $x \mapsto u(t,x)$ is non-decreasing on $[A,1]$.*

The proof of Lemma 4.1 relies on the explicit construction of u using characteristics formula. As we will see in the proof, some curve of strong $\mathscr{C}^1$ singularity (meaning that at each point of the curve, the function u on each side on the curve cannot be both extended as a $\mathscr{C}^1$ function up to the curve) may appear when the characteristic entering the zone in which the damping is active corresponds to a small value of u. In this case, characteristics may become vertical and merge after some time. This does not violate the forward uniqueness of the characteristics in the sense of Filippov. Still, we emphasize that when the characteristics merge, we cannot use them backward in time. This is in fact completely similar to the phenomenon which appears when solving the ODE (1.3).

To be more precise, we introduce $\varepsilon \in (0, K]$ as the solution, if it exists, of

$$\int_0^\infty f'((\varepsilon^\alpha - \delta\alpha\tau)_+^{1/\alpha})\, d\tau = A, \tag{4.11}$$

and T_* as

$$T_* = \inf\{t \in [0,\infty),\ u(t,0) \leqslant \varepsilon\}. \tag{4.12}$$

The role of ε will appear clearly in the proof below. Loosely speaking, when $u(t,0) \leqslant \varepsilon$, the characteristic issued from $(t,0)$ will never reach the set $\{x = A\}$: in other words, the characteristic issued from $(t,0)$ is of too low energy to overpass the damping set.

The curves $\mathcal{C}$ on which $\mathscr{C}^1$ singularities may appear will be constructed with the solution u itself. In fact, these curves will simply be

$$\begin{cases} \mathcal{C}_0 : t \mapsto (t, X(t,0,0)), \\ \mathcal{C}_1 : t \mapsto (t, X(t,0,A)), \\ \mathcal{C}_2 : t \mapsto (t, 0), \\ \mathcal{C}_3 : t \mapsto (t, A), \end{cases} \tag{4.13}$$

to which, if $T_* < \infty$, one should add the boundary of the set $\mathscr{Z}$ (if $\mathscr{Z} \neq \emptyset$), that will be shown to be delimited by

$$\mathcal{C} : t_0 \in (T_*, \infty) \mapsto (t(t_0), x(t_0)),$$
$$\text{where } t(t_0) = t_0 + \frac{(u(t_0,0))^\alpha}{\delta\alpha}, \text{ and } x(t_0) = \int_0^\infty f'((u(t_0,0)^\alpha - \delta\alpha\tau)_+^{1/\alpha})\, d\tau, \tag{4.14}$$

and a part of $\mathcal{C}_3$, namely $\{(t, A),\ \text{for } t \geqslant t_*\}$ for

$$t_* = T_* + \frac{\varepsilon^\alpha}{\delta\alpha}. \tag{4.15}$$

The various discontinuity curves, $\mathscr{Z}$ region and times T_* and t_* are displayed in Fig. 3.

Proof. • **Preliminary computations: Existence and uniqueness of ε in (4.11).** We first emphasize that condition (4.11) is satisfied by at most one parameter $\varepsilon > 0$ as the map

$$g : v \mapsto \int_0^\infty f'((v^\alpha - \delta\alpha\tau)_+^{1/\alpha})\, d\tau \tag{4.16}$$

is well-defined and continuous on $[0, K]$, $g(0) = 0$, and g is strictly increasing. Indeed, if $0 \leqslant v_1 < v_2 \leqslant K$, using the fact that f' is strictly increasing on $[0, K]$ as f is assumed to be strictly convex,

$$g(v_1) = \int_0^\infty f'((v_1^\alpha - \delta\alpha\tau)_+^{1/\alpha})\, d\tau = \int_0^{v_1^\alpha/(\delta\alpha)} f'((v_1^\alpha - \delta\alpha\tau)_+^{1/\alpha})\, d\tau < \int_0^{v_1^\alpha/(\delta\alpha)} f'((v_2^\alpha - \delta\alpha\tau)_+^{1/\alpha})\, d\tau \leqslant g(v_2).$$

Thus, if $g(K) \geqslant A$, there exists a unique $\varepsilon \in [0, K]$ satisfying (4.11). (In case $g(K) < A$, there is no $\varepsilon \in [0, K]$ satisfying (4.11).) Note in particular that, when there exists $\varepsilon \in (0, K]$ satisfying (4.11), for all $v \in (\varepsilon, K]$, $g(v) \geqslant A$, so that

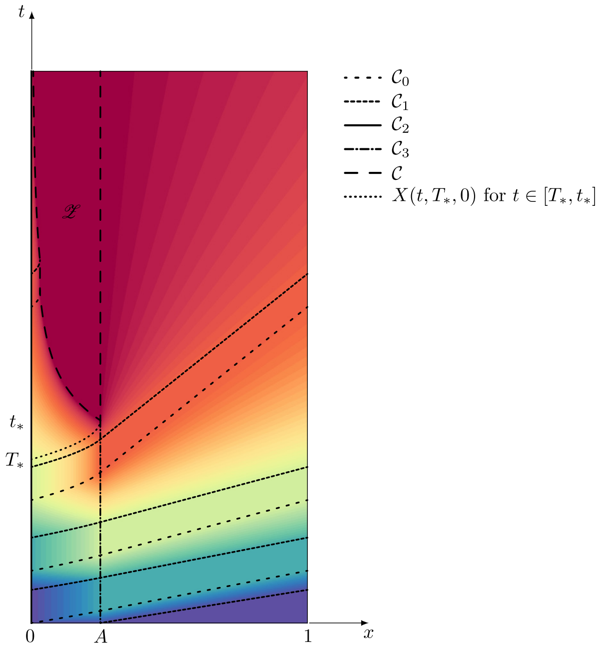

Fig. 3. Discontinuity curves, $\mathscr{Z}$ region and times T_* and t_*.

$$\forall v \in (\varepsilon, K], \quad A \leqslant f'(v)\frac{v^\alpha}{\delta\alpha} \leqslant f'(K)\frac{K^\alpha}{\delta\alpha}. \tag{4.17}$$

• **Construction of** u**.** In our construction below, we distinguish the cases $t < T_*$ and $t \geqslant T_*$. In particular, according to the definition (4.12) of T_*, for all $t \in [0, T_*)$, $u(t, 0) > \varepsilon$.

We restrict ourselves to the case $T_* > 0$, since the case $T_* = 0$ can be easily adapted from the case $T_* > 0$ and $t \geqslant T_*$. Hence, we assume that $T_* > 0$, and deal separately with the cases $t \leqslant T_*$ and $t \geqslant T_*$.

We will not point out below, along the construction of u, that the curves delimiting the $\mathscr{C}^1$ singularities are exactly the ones in (4.13)–(4.14), but it will appear clearly from the construction of u.

As a matter of fact, we construct a sequence of time-space domains which eventually cover $[0, \infty) \times \mathbb{T}$, and a function u on each of these time-space domains, such that u is a globally $\mathscr{C}^0$ function there, and is a piecewise $\mathscr{C}^1$ function, where the curves of $\mathscr{C}^1$ discontinuities are given by (4.13)–(4.14). Besides, apart from these curves of singularities, the solution u is constructed to satisfy the characteristics equations (4.5) in the case $\alpha \in (0, 1)$, or (4.8) in the case $\alpha = 1$ away from the set $\mathscr{Z}$. The regularity of the curves of $\mathscr{C}^1$ discontinuities then allows to check easily that the solution u constructed this way solves (4.2) in the sense of Definition 1.2 for $\alpha \in (0, 1)$ or of Definition 1.3 when $\alpha = 1$.

In the proof below, with a slight abuse of terminology, we call "smooth functions" functions which are $\mathscr{C}^1$.

Case $t \leqslant T_*$**.** As the velocities involved for the solution u should belong to $[0, f'(K)]$, using the light cone of the equation, u should be fully determined in

$$\mathscr{T}_0 = \{(t, x) \text{ with } t \in [0, A/f'(K)] \text{ and } x \in [tf'(K), A]\}$$

by $u(0, \cdot)_{|[0,A]}$. For $(t, x) \in \mathscr{T}_0$, we shall therefore look for x_0 such that $X(t, 0, x_0) = x$, that is

$$x_0 + \int_0^t f'\left((K^\alpha - \delta\alpha\tau)_+^{1/\alpha}\right) d\tau = x.$$

For fixed $t \in [0, A/f'(K)]$, it is easily seen that the map

$$[0, A] \ni x_0 \mapsto k_t(x_0) = x_0 + \int_0^t f'\left((K^\alpha - \delta\alpha\tau)_+^{1/\alpha}\right) d\tau$$

is smooth, and strictly increasing, with image containing $[tf'(K), A]$. Therefore, for all $(t, x) \in \mathscr{T}_0$, there exists a unique $x_0 \in [0, A]$ such that $X(t, 0, x_0) = x$, so that we set $u(t, x) = (K^\alpha - \delta\alpha t)_+^{1/\alpha}$. As $t \leqslant A/f'(K)$, the bound (4.17) allows to guarantee that for all $(t, x) \in \mathscr{T}_0$, $K^\alpha - \delta\alpha t$ is strictly positive, so that $u \in \mathscr{C}^1(\mathscr{T}_0)$ and $t \mapsto u(t, A)$ is smooth on $[0, A/f'(K)]$, non-increasing, and strictly positive.

We then consider the triangle

$$\mathscr{T}_1 = \{(t, x) \text{ with } t \in [0, 1/f'(K)] \text{ and } x \in [tf'(K), 1]\},$$

and construct u in this set. Of course, as we have already shown that u is smooth in the set $\mathscr{T}_0$, we only focus on the set $\mathscr{T}_1 \setminus \mathscr{T}_0$. We determine u in $\mathscr{T}_1 \setminus \mathscr{T}_0$ from $u(\cdot, A)_{|[0,A/f'(K)]}$ and $u(0, \cdot)_{|[A,1]}$. Introducing $X(t, 0, A) = A + tf'(u(0, A)) = A + tf'(K)$, for $(t, x) \in \mathscr{T}_1 \setminus \mathscr{T}_0$, we set $u(t, x) = K$ if $x \in [X(t, 0, A), 1]$, while if $x \in [\max\{tf'(K), A\}, X(t, 0, A)]$, we find a time $t_0 \in [0, A/f'(K)]$ such that $X(t, t_0, A) = x$, and set $u(t, x) = u(t_0, A)$. Indeed, this can be achieved since for $t \in [0, 1/f'(K)]$ and $x \in [\max\{tf'(K), A\}, X(t, 0, A)]$, finding $t_0 \in [0, \min\{t, A/f'(K)\}]$ such that $X(t, t_0, A) = x$ amounts to solving the equation

$$A + (t - t_0) f'(u(t_0, A)) = x.$$

But the map

$$h_t : t_0 \mapsto A + (t - t_0) f'(u(t_0, A)) \tag{4.18}$$

satisfies:

- h_t is strictly decreasing on the interval $[0, \min\{t, A/f'(K)\}]$. This is a consequence of the fact that $t_0 \mapsto u(t_0, A)$ is non-increasing and strictly positive on $[0, A/f'(K)]$ and that f is strictly convex with $f'(0) = 0$. Besides, for all $t_0 \in [0, \min\{t, A/f'(K)\}]$,
$$h_t'(t_0) \leqslant -f'(u(A/f'(K), A)) < 0.$$
- $h_t(0) = A + tf'(K) = X(t, 0, A)$.
- If $t \leqslant A/f'(K)$, $h_t(t) = A$, and if $t \geqslant A/f'(K)$,
$$h_t\left(\frac{A}{f'(K)}\right) = A + \left(t - \frac{A}{f'(K)}\right) f'\left(u\Big(\frac{A}{f'(K)}, A\Big)\right) \leqslant A + \left(t - \frac{A}{f'(K)}\right) f'(K) \leqslant tf'(K).$$
- h_t is smooth.

It then follows that h_t is a diffeomorphism from the interval $[0, \min\{t, A/f'(K)\}]$ to its image, which contains $[\max\{tf'(K), A\}, X(t, 0, A)]$, so that we can write, for $t \in [0, 1/f'(K)]$ and x in the interval $[\max\{tf'(K), A\}, X(t, 0, A)]$, $u(t, x) = u(h_t^{-1}(x), A)$.

These formulas easily show that the function u is smooth for $(t, x) \in \mathscr{T}_1$ with $x \in [\max\{tf'(K), A\}, X(t, 0, A)]$, and it is clear that u is smooth for $(t, x) \in \mathscr{T}_1$ with $x \in [X(t, 0, A), 1]$. However, though it is clear that u is continuous along the curve $t \mapsto (t, X(t, 0, A))$, u may not be locally $\mathscr{C}^1$ in a neighborhood of this curve, even if it can be extended as $\mathscr{C}^1$ functions up to this curve from each side. It follows that u is piecewise $\mathscr{C}^1(\mathscr{T}_1)$, and that $t \mapsto u(t, 1)$ is a piecewise $\mathscr{C}^1$ non-increasing function on $[0, 1/f'(K)]$, which remains strictly positive on the interval $[0, 1/f'(K)]$.

It is also easy to check that for all $t \in [0, A/f'(K)]$, $x \mapsto u(t, x)$ is non-decreasing on $[A, 1]$.

In fact, this is the starting point of an iterative argument showing that for all $n \in \mathbb{N}$ with $T_n = n/f'(K) < T_*$, one can construct a solution u of (4.2) in

$$\mathscr{T}_{2n+1} = \{(t, x) \in [0, (n+1)/f'(K)] \times [0, 1] \text{ with } x \in [\max\{tf'(K) - n, 0\}, 1]\},$$

such that:

- u is globally $\mathscr{C}^0$ and piecewise $\mathscr{C}^1$ on $\mathscr{T}_{2n+1}$.
- $t \mapsto u(t, A)$ is strictly positive and non-increasing on $[0, (n + A)/f'(K)]$.
- $t \mapsto u(t, 1)$ is strictly positive non-increasing on $[0, (n + 1)/f'(K)]$.
- For all $t \in [0, T_n + A/f'(K)]$, $x \mapsto u(t, x)$ is non-decreasing on $[A, 1]$.

Indeed, these properties are already proved for $n = 0$. Let us show that if they hold for $n \in \mathbb{N}$, they also hold for $n + 1$ provided $T_{n+1} < T_*$.

Using the above properties for n, we first consider the equation in the trapezoid

$$\mathscr{T}_{2n+2} = \{(t, x) \in [0, (n + 1 + A)/f'(K)] \times [0, A] \text{ with } x \in [\max\{tf'(K) - (n + 1), 0\}, A]\},$$

and use the boundary condition $u(\cdot, 0)_{|[0,(n+1)/f'(K)]}$ and $u(0, \cdot)_{|[0,A]}$ to construct u in $\mathscr{T}_{2n+2}$. In order to do this, we set, for $t \geqslant 0$,

$$x_0(t) = \min \left\{ \int_0^t f'((K^\alpha - \delta\alpha\tau)_+^{1/\alpha})\, d\tau, A \right\}.$$

Now, let us fix $t \in [0, (n + 1 + A)/f'(K)]$ and $x \in [0, A]$. If $x > x_0(t)$, we define $u(t, x) = (K^\alpha - \delta\alpha t)_+^{1/\alpha}$. If $x < x_0(t)$, we look for $t_0 \in [0, t]$ such that $X(t, t_0, 0) = x$, i.e. such that

$$\int_0^{t-t_0} f'\left((u(t_0, 0)^\alpha - \delta\alpha\tau)_+^{1/\alpha}\right) d\tau = x.$$

We thus define, for $t_0 \in [0, t]$,

$$g_t(t_0) = \int_0^{t-t_0} f'\left((u(t_0, 0)^\alpha - \delta\alpha\tau)_+^{1/\alpha}\right) d\tau.$$

It is not difficult to check that, for $t \leqslant (n + 1 + A)/f'(K)$, g_t enjoys the following properties:

- g_t is strictly decreasing on $[0, \min\{t, T_{n+1}\}]$. This is a consequence of the fact that $t \mapsto u(t, 1)$ $(= u(t, 0))$ is non-increasing and strictly larger than ε on $[0, T_{n+1}]$. Besides, g_t is piecewise $\mathscr{C}^1$ on $[0, \min\{t, T_{n+1}\}]$ and for all t_0 such that $g_t(t_0) \leqslant A$ and for which g_t is differentiable,

  $$g_t'(t_0) \leqslant -f'((u(t_0, 0)^\alpha - \delta\alpha(t - t_0))_+^{1/\alpha}) \leqslant -f'((u(\min\{t, T_{n+1}\}, 0)^\alpha - \delta\alpha(\min\{t, T_{n+1}\} - t_0))_+^{1/\alpha}).$$

  Note that, if $f'(((u(\min\{t, T_{n+1}\}, 0)^\alpha - \delta\alpha(\min\{t, T_{n+1}\} - t_0))_+^{1/\alpha}) = 0$ and $g_t(t_0) \leqslant A$, then

  $$g(u(\min\{t, T_{n+1}\}, 0)) \leqslant A,$$

  which is not compatible with $u(\min\{t, T_{n+1}\}, 0) > \varepsilon$. Thus, there exists $\gamma > 0$ such that for all t_0 such that $g_t(t_0) \leqslant A$ and for which g_t is differentiable, $g_t'(t_0) \leqslant -\gamma$.
- $g_t(t) = 0$.
- $g_t(0) = x_0(t)$ if $\int_0^t f'((K^\alpha - \delta\alpha\tau)_+^{1/\alpha})\, d\tau \leqslant A$, and is larger than A otherwise.

One then easily shows that for $t \leqslant (n + 1 + A)/f'(K)$, the map g_t is a piecewise $\mathscr{C}^1$ function from $[0, \min\{t, T_{n+1}\}]$ to its image, which contains $[0, x_0(t)]$. Besides, there exists a unique $t_0(t, A)$ such that g_t is a piecewise $\mathscr{C}^1$ diffeomorphism from $[t_0(t, A), \min\{t, T_{n+1}\}]$ to $[0, x_0(t)]$. We can then define u for $t \leqslant (n + 1 + A)/f'(K)$ and $[0, x_0(t)]$ by

$$u(t, x) = (u(g_t^{-1}(x), 0)^\alpha - \delta\alpha(t - g_t^{-1}(x)))_+^{1/\alpha},$$

and, for $x \in (x_0(t), A]$, by $u(t, x) = (K^\alpha - \delta\alpha t)_+^{1/\alpha}$.

As g_t depends smoothly on the time parameter t, this defines u as a globally $\mathscr{C}^0$ and piecewise $\mathscr{C}^1$ function in $\mathscr{T}_{2n+2}$.

We then check that $t \mapsto u(t, A)$ is strictly positive, piecewise $\mathscr{C}^1$ and non-increasing on $[0, (n + 1 + A)/f'(K)]$. It is obviously $\mathscr{C}^1$ and decreasing in $\{t, x_0(t) < A\}$, as $u(t, A) = (K^\alpha - \delta\alpha t)_+^{1/\alpha}$, the positivity coming from the fact that $K > \varepsilon$ and $t \leqslant T_{n+1} \leqslant T_*$. For $\{t, x_0(t) = A\}$, which is an interval of the form $[t_A, (n + 1 + A)/f'(K)]$, $u(t, A)$ is given by the formula

$$u(t, A) = (u(g_t^{-1}(A), 0)^\alpha - \delta\alpha(t - g_t^{-1}(A)))_+^{1/\alpha}.$$

Now, one can check that for $t^a, t^b \in [t_A, (n+1+A)/f'(K)]$ such that $t^a < t^b$, defining t_0^a and t_0^b by the formula

$$g_{t^a}(t_0^a) = A = g_{t^b}(t_0^b), \quad \text{i.e.} \quad t_0^a = g_{t^a}^{-1}(A),\ t_0^b = g_{t^b}^{-1}(A),$$

the decay of $t_0 \mapsto u(t_0, 0)$ on $[0, T_n]$ implies that

$$t_0^a \leqslant t_0^b, \quad \text{and} \quad t^a - t_0^a \geqslant t^b - t_0^b.$$

Consequently, since $t_0 \mapsto u(t_0, 0)$ is non-increasing on $[0, T_{n+1}]$, we immediately have that $t \mapsto u(t, A)$ is non-increasing on $[0, (n+1+A)/f'(K)]$. It is also obviously piecewise $\mathscr{C}^1$ on $[0, (n+1+A)/f'(K)]$. The fact that $u(t, A)$ is strictly positive comes from the fact that $t \mapsto u(t, 0)$ stays strictly larger than ε for $t \leqslant T_n$.

We then construct the solution u in the set $\mathscr{T}_{2n+3} \setminus \mathscr{T}_{2n+2}$, using the information given by $u(\cdot, A)_{|[0,(n+1+A)/f'(K)]}$ and $u(0, \cdot)_{|[A,1]}$. Again, as when working in $\mathscr{T}_1$, we construct the solution u using characteristics and setting, for $(t, x) \in \mathscr{T}_{2n+3}$ with $x \in [A, \min\{A + tf'(K), 1\}]$,

$$u(t, x) = u(h_t^{-1}(x), A).$$

The other case, corresponding to $x \in (\min\{A + tf'(K), 1\}, 1]$, lies in fact in $\mathscr{T}_1$, so regularity issues have been dealt with before. We only need to check that for all $t \in [0, T_{n+1} + A/f'(K)]$, $x \mapsto u(t, x)$ is non-decreasing on $[A, 1]$: this is obvious if $x \in [\min\{A + tf'(K), 1\}, 1]$ as $u(t, x) = K$ there; when $x \in [A, \min\{A + tf'(K), 1\}]$, the above formula and the facts that $t \mapsto u(t, A)$ is non-increasing and that h_t^{-1} is strictly decreasing imply that $x \mapsto u(t, x)$ is non-increasing on $[A, \min\{A + tf'(K), 1\}]$. Therefore, all the items in the above property also hold for $n + 1$.

We can thus iterate these arguments while $T_n = n/f'(K) < T_*$. If $T_* = \infty$, this concludes Lemma 4.1. If $T_* < \infty$, we perform a similar iteration in the trapeze

$$\mathscr{T} = \{(t, x) \in [0, T_* + 1/f'(K)] \times [0, 1] \text{ with } x \in [\max\{(t - T_*) f'(K), 0\}, 1]\},$$

constructing a function u in $\mathscr{T}$ such that:

- u is a solution of (4.2), is piecewise $\mathscr{C}^1$ on $\mathscr{T}$ and globally $\mathscr{C}^0$ on $\mathscr{T}$.
- $t \mapsto u(t, A)$ is strictly positive and non-increasing on $[0, T_* + A/f'(K)]$.
- $t \mapsto u(t, 0)$ is strictly positive non-increasing on $[0, T_* + 1/f'(K)]$.
- $u(T_*, 0) = \varepsilon$.
- For all $t \in [0, T_* + A/f'(K)]$, $x \mapsto u(t, x)$ is non-decreasing on $[A, 1]$.

Case $t \geqslant T_*$. The difficulty when $t \geqslant T_*$ is to show that the solution u remains continuous in time-space. In order to do this, we introduce $(x_*(t), u_*(t))$ given by

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} x_* \\ u_* \end{pmatrix} = F_\alpha \begin{pmatrix} x_* \\ u_* \end{pmatrix}, \quad t \geqslant T_*, \text{ if } \alpha \in (0,1), \\ \text{or} \\ \dfrac{d}{dt}\begin{pmatrix} x_* \\ u_* \end{pmatrix} \in F_1 \begin{pmatrix} x_* \\ u_* \end{pmatrix}, \quad t \geqslant T_*, \text{ if } \alpha = 1, \end{cases} \qquad \begin{pmatrix} x_*(T_*) \\ u_*(T_*) \end{pmatrix} = \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix}, \tag{4.19}$$

where F_α is defined in (4.6)–(4.7). Here, x_* corresponds to the characteristics $X(t, T_*, 0)$, $t \geqslant T_*$ and $x_*(t) \leqslant A$, and $u_*(t)$ corresponds to $u(t, X(t, T_*, 0))$. Note that, due to the choice of ε in (4.11), there exists a time t_* ($= T_* + \varepsilon^\alpha/(\delta\alpha)$), such that the solution (x_*, u_*) satisfies

$$\forall t \in [T_*, t_*), \quad x_*(t) \in [0, A), \quad \text{and} \quad \forall t \geqslant t_*, \quad (x_*(t), u_*(t)) = (A, 0), \tag{4.20}$$

thus guaranteeing the uniqueness of the solution of (4.19) according to the one-sided Lipschitz condition (4.10).

We then introduce the following sets (see Fig. 4):

$$\mathscr{R}_0 = \{(t, x) \in [0, \infty) \times [0, A], \text{ such that, if } t \geqslant T_*,\ x \in [x_*(t), A]\},$$

$$\mathscr{R}_1 = \{(t, x) \in [0, \infty) \times [A, 1]\},$$

$$\mathscr{R}_2 = \{(t, x), \text{ with } t \geqslant T_*,\ x \in [0, x_*(t)]\},$$

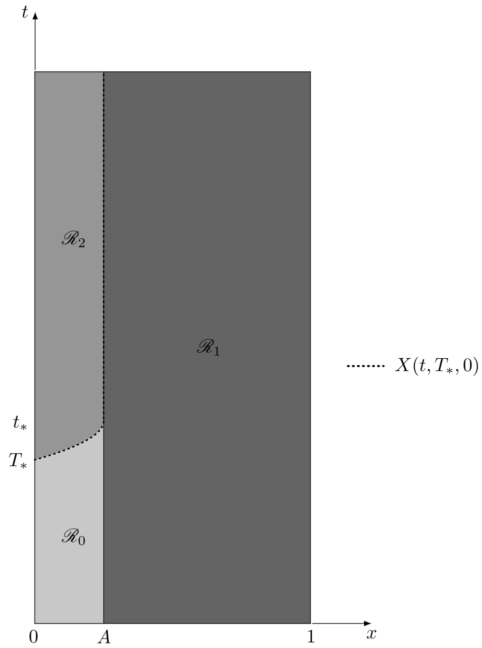

Fig. 4. $\mathscr{R}_0$, $\mathscr{R}_1$ and $\mathscr{R}_2$ regions.

and, similarly as before, for $n \geqslant 0$,

$$\mathscr{T}^*_{2n} = \{(t, x) \in [0, T_* + (n + A)/f'(K)] \times [0, 1] \text{ with } x \in [((t - T_*)f'(K) - n)_+, A]\},$$
$$\mathscr{T}^*_{2n+1} = \{(t, x) \in [0, T_* + (n + 1)/f'(K)] \times [0, 1] \text{ with } x \in [((t - T_*)f'(K) - n)_+, 1]\}.$$

As before, we construct iteratively a solution u of (4.2) in $\mathscr{T}^*_{2n+1} \setminus \mathscr{R}_2$ for all $n \in \mathbb{N}$ such that:

- u is piecewise $\mathscr{C}^1$ and globally $\mathscr{C}^0$ in $\mathscr{T}^*_{2n+1} \setminus \mathscr{R}_2$.
- $t \mapsto u(t, 1)$ is strictly positive non-increasing on $[0, T_* + (n + 1)/f'(K)]$ and piecewise $\mathscr{C}^1$.
- for all $t \geqslant T_*$ $u(t, x_*(t)) = u_*(t)$, where (x_*, u_*) is the solution of (4.19).
- For all $t \in [0, T_* + (n + A)/f'(K)]$, $x \mapsto u(t, x)$ is non-decreasing on $[A, 1]$.

Of course, the previous paragraph shows that this is true for $n = 0$. Let us then assume that these properties are true for some $n \in \mathbb{N}$ and show that they are then true for $n + 1$.

Similarly as before, we work first on $\mathscr{T}^*_{2n+2} \cap \mathscr{R}_0$. The construction of the function u in the set $\mathscr{T}^*_{2n+2} \cap \mathscr{R}_0$ can then be handled as for $\mathscr{T}_{2n+2}$, and following the same arguments, we easily get that the function u there is piecewise $\mathscr{C}^1$ in $\mathscr{T}^*_{2n+2} \cap \mathscr{R}_0$, and that $t \mapsto u(t, A)$ is a non-increasing function on $[0, T_* + (n + 1 + A)/f'(K)]$, strictly positive while $t \leqslant t_*$, and vanishing for $t \geqslant t_*$.

The construction of the solution u in the set $\mathscr{T}^*_{2n+3} \cap \mathscr{R}_1$ can then be done similarly as the one corresponding to $\mathscr{T}_{2n+3}$, and following the same lines, we get that the solution u is piecewise $\mathscr{C}^1$ in $\mathscr{T}^*_{2n+3} \cap \mathscr{R}_1$. We nevertheless make the proof slightly more precise: for $(t, x) \in \mathscr{T}^*_{2n+3} \cap \mathscr{R}_1 \setminus \mathscr{T}_1$, $u(t, x)$ is constructed by

$$u(t, x) = u(h_t^{-1}(x), A),$$

where h_t is given by (4.18). In particular, to establish the strict positivity of $u(t, 1)$ for $t \leqslant T_* + (n + 2)/f'(K)$, we just remark that $t_0 = h_t^{-1}(1)$ is equivalent to $A + (t - t_0)f'(u(t_0, A)) = 1$, so that $u(t, 1) = u(t_0, A)$ cannot be zero.

The continuity of the function u constructed above in $\mathscr{T}^*_{2n+3} \setminus \mathscr{R}_2$ follows easily from the continuity of u across the interfaces of $\mathscr{T}^*_{2n+2} \cap \mathscr{R}_0$, and $\mathscr{T}^*_{2n+3} \cap \mathscr{R}_1$.

The fact that for all $t \in [0, T_* + (n + 1 + A)/f'(K)]$, $x \mapsto u(t, x)$ is non-decreasing in $[A, 1]$ follows as before.

Our goal now is to construct u in $\mathscr{R}_2$ as a piecewise $\mathscr{C}^1$ function such that u is globally continuous on $[0,\infty)\times\mathbb{T}$ and solves (4.2) in $[0,\infty)\times\mathbb{T}$.

As $t\mapsto u(t,0)$ is strictly positive and non-increasing on $[0,\infty)$, we can show, similarly as before that, for $(t,x)\in\mathscr{R}_2$, the map

$$g_t(t_0)=\int_0^{t-t_0} f'\left((u(t_0,0)^\alpha-\delta\alpha\tau)_+^{1/\alpha}\right)d\tau$$

has the following properties:

- g_t is decreasing on $[T_*,t]$, takes values in $[0,x_*(t)]$, and is surjective on $[0,x_*(t)]$,
- For all $(t_0^a,t_0^b)\in[T_*,t]^2$, $g_t(t_0^a)=g_t(t_0^b)$ implies $t_0^a=t_0^b$ or $(u(t_0^a,0)^\alpha-\delta\alpha(t-t_0^a))_+=(u(t_0^b,0)^\alpha-\delta\alpha(t-t_0^b))_+$. (In fact, this case corresponds to the situation where g_t is not strictly decreasing and the characteristics emerging from $(t_0^a,0)$ and from $(t_0^b,0)$ have merged before the time t.)

It thus allows to set, for $(t,x)\in\mathscr{R}_2$,

$$u(t,x)=(u(g_t^{-1}(x),0)^\alpha-\delta\alpha(t-g_t^{-1}(x)))_+^{1/\alpha},\tag{4.21}$$

where $g_t^{-1}(x)$ denotes any $t_0\in[T_*,t]$ such that $g_t(t_0)=x$. Note that, according to the second item above, the definition above does not depend on the choice of t_0 such that $g_t(t_0)=x$. Besides, u defined this way is continuous on $\mathscr{R}_2$ as one can easily check, and if $t\geqslant T_*$, $u(t,x_*(t))=u_*(t)$, where (x_*,u_*) is the solution of (4.19).

We then remark that, for $t\geqslant T_*$, and $T_*\leqslant t_0^a<t_0^b\leqslant t$,

$$\begin{aligned}
g_t(t_0^a)&=\int_0^{t-t_0^a} f'\left((u(t_0^a,0)^\alpha-\delta\alpha\tau)_+^{1/\alpha}\right)d\tau\\
&\geqslant\int_0^{t-t_0^b} f'\left((u(t_0^a,0)^\alpha-\delta\alpha\tau)_+^{1/\alpha}\right)d\tau\\
&\geqslant\int_0^{t-t_0^b} f'\left((u(t_0^b,0)^\alpha-\delta\alpha\tau)_+^{1/\alpha}\right)d\tau=g_t(t_0^b).
\end{aligned}$$

In particular, analyzing the case of equality in the above estimates, we easily get that for $(t,x)\in\mathscr{R}_2$ such that $u(t,x)\neq 0$, $t_0(t,x)=g_t^{-1}(x)$ is uniquely defined and

$$t-t_0(t,x)\leqslant\frac{u(t_0(t,x),0)^\alpha}{\delta\alpha}\leqslant\frac{\varepsilon^\alpha}{\delta\alpha}.\tag{4.22}$$

Besides, g_t is piecewise $\mathscr{C}^1$ locally around $t_0(t,x)$ and

$$\begin{aligned}
g_t'(t_0)=&-f'\left((u(t_0,0)^\alpha-\delta\alpha(t-t_0)^{1/\alpha}\right)\\
&+u(t_0,0)^{\alpha-1}\partial_t u(t_0,0)\int_0^{t-t_0} f'\left((u(t_0,0)^\alpha-\delta\alpha\tau)^{1/\alpha}\right)(u(t_0,0)^\alpha-\delta\alpha\tau)^{1/\alpha-1}\,d\tau.
\end{aligned}$$

As $u(t,x)\neq 0$ and $t_0\mapsto u(t_0,0)$ decays, this implies that

$$g_t'(t_0(t,x))\leqslant -f'\left((u(t_0,0)^\alpha-\delta\alpha(t-t_0(t,x)))^{1/\alpha}\right)=-f'(u(t,x))<0,$$

and the bound is uniform in a neighborhood of t and x. As $t(t_0,x)$ is defined by $g_t(t_0(t,x))=x$, we see that this implies that t_0 is piecewise $\mathscr{C}^1$ for $(t,x)\in\mathscr{R}_2$ such that $u(t,x)\neq 0$, so that the definition (4.21) shows that u is piecewise $\mathscr{C}^1$ for $(t,x)\in\mathscr{R}_2$ such that $u(t,x)\neq 0$.

It is thus also interesting to determine the area $\mathscr{Z} = \{(t, x),\ \text{s.t. } u(t, x) = 0\} \cap \mathscr{R}_2$.

For $(t, x) \in \mathscr{R}_2$, it is clear, from the relation $g_t(t_0(t, x)) = x$ and from the fact that $t_0 \mapsto u(t_0, 0)$ decays, that for $t^a < t^b$ such that (t^a, x) and (t^b, x) are in $\mathscr{R}_2$,

$$t_0(t^a, x) \leqslant t_0(t^b, x), \qquad \text{and} \qquad t^a - t_0(t^a, x) \leqslant t^b - t_0(t^b, x),$$

so that we easily derive from the decay of $t_0 \mapsto u(t_0, 0)$ that

$$u(t^a, x) \leqslant u(t^b, x).$$

It follows that, for all $x \in [0, A]$, the map $t \mapsto u(t, x)$ decays while $(t, x) \in \mathscr{R}_2$.

We can in particular define, for $x \in (0, A]$, the time $t_*(x) = \inf\{t,\ s.t.\, u(t, x) = 0\}$ (which may be infinite). If $t_*(x)$ is finite, for $t < t_*(x)$, $u(t, x) \neq 0$, so $t_0(t, x)$ is well-defined and unique, and is an increasing function of time which is bounded by $t_*(x)$. Thus, the limit of $t_0(t, x)$ as $t \to t_*^-$ exists, and we call it $t_{0,*}(x)$. We then easily get that

$$u(t_{0,*}(x), 0)^\alpha - \delta\alpha(t_*(x) - t_{0,*}(x)) = 0,$$

so that

$$x = g_{t_*(x)}(t_{0,*}(x)) = g\left(u(t_{0,*}(x), 0)\right),$$

where g is defined in (4.16). Let us now note that there exists only one t_0 such that

$$u(t_0, 0)^\alpha - \delta\alpha(t_*(x) - t_0) = 0 \quad \text{and} \quad x = g(u(t_0, 0)). \tag{4.23}$$

Indeed, the second equation determines uniquely $u(t_0, 0)$, and the first equation then determines uniquely t_0. Thus, if t_0 satisfies (4.23), $t_0 = t_{0,*}(x)$ and $t_*(x) = t_0 + u(t_0, 0)^\alpha/(\delta\alpha)$.

This suggests to study the parametric equation

$$\begin{aligned} &\mathcal{C} : t_0 \in (T_*, \infty) \mapsto (t(t_0), x(t_0)), \\ &\text{where } t(t_0) = t_0 + \frac{(u(t_0, 0))^\alpha}{\delta\alpha}, \text{ and } x(t_0) = g(u(t_0, 0)). \end{aligned} \tag{4.24}$$

(This definition of course coincides with the definition of $\mathcal{C}$ in (4.14), recall the definition of g in (4.16).) It is clear that by construction $u(t(t_0), x(t_0)) = 0$ for all $t_0 \geqslant T_*$. Besides, for $(t, x) \in \mathscr{R}_2$, if there exists $t_0 \geqslant T_*$ such that $x = g(u(t_0, 0))$, then

- if $t \geqslant \inf\{t(t_0),\ \text{for } t_0 \text{ s.t. } x = g(u(t_0, 0))\}$, then $u(t, x) = 0$;
- if $t < \inf\{t(t_0),\ \text{for } t_0 \text{ s.t. } x = g(u(t_0, 0))\}$, then $u(t, x) > 0$.

These statements follow immediately from the decay of $t \mapsto u(t, x)$ and the non-negativity of u. It follows that

$$\partial\mathscr{Z} = \mathcal{C} \cup \{(t, A),\ \text{s.t. } t \geqslant t_*\}, \tag{4.25}$$

where t_* is defined in (4.15). We remark that, as $t_0 \mapsto u(t_0, 0)$ is piecewise $\mathscr{C}^1$ and strictly positive, except at singularities (which are in finite number in any bounded interval), we have

$$\frac{d}{dt_0}\left(t_0 + \frac{(u(t_0, 0))^\alpha}{\delta\alpha}, g(u(t_0, 0))\right) = \left(1 + \frac{u(t_0, 0)^{\alpha-1}\partial_t u(t_0, 0)}{\delta}, g'(u(t_0, 0))\partial_t u(t_0, 0)\right),$$

so that

$$\left|\frac{d}{dt_0}\left(t_0 + \frac{(u(t_0, 0))^\alpha}{\delta\alpha}, g(u(t_0, 0))\right)\right| \neq 0.$$

This proves that the tangent, hence the normal, of the curve $\mathcal{C}$ is well-defined except at a locally finite number of points. This indicates that $\mathcal{C}$ is a piecewise $\mathscr{C}^1$ parametric curve with finite limits at singularity points. It is thus a globally Lipschitz and piecewise $\mathscr{C}^1$ parametric curve.

We have thus proved the regularity properties stated in Lemma 4.1. In particular, in all connected component of $([0, \infty) \times \mathbb{T}) \setminus (\cup_{i=1}^4 \mathcal{C}_i \cup \mathcal{C})$, the function u is $\mathscr{C}^1$ and satisfies, by construction

$$\partial_t u + \partial_x(f(u)) + h(t,x) = 0,$$

where $h(t,x) = \delta \mathbf{1}_{(0,A)} u(t,x)/|u(t,x)|^\alpha$ if $u(t,x) > 0$ and $h(t,x) = 0$ if $u(t,x) = 0$. Since u is also continuous in the whole set $[0,\infty) \times \mathbb{T}$, a straightforward application of the integration by parts formula shows that the function u we constructed above is an admissible solution of (4.2) in $[0,\infty) \times \mathbb{T}$. □

4.1.4. Dynamics of the solution u in the case of a strictly convex flux

Lemma 4.2. *Let f and K as in (4.1), $\alpha \in (0,1]$, $\delta > 0$ and $\omega = (0,A) \subset \mathbb{T}$. Then the solution u of (4.2) satisfies:*

(i) T_* defined in (4.12) is finite.
(ii) For all $t \geqslant t_*$ (defined in (4.15)), there exists an open subinterval $\omega(t) \subset \omega$ such that $u(t)_{|\omega(t)} = 0$, and

$$|\omega \setminus \omega(t)| \leqslant \frac{C}{t^{1+\alpha}}, \qquad \|u(t)\|_{L^\infty(\mathbb{T})} \leqslant \frac{C}{t}.$$

Proof of item (i) of Lemma 4.2. The proof of the first item of Lemma 4.2 follows from the analysis of the solution u of (4.2) along the characteristics $t \mapsto X(t,0,0)$, in particular when it crosses the set $\{x = 0\}$.

We thus introduce four sequences $(u_n)_{n\in\mathbb{N}}$, $(t_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$, $(\tau_n)_{n\in\mathbb{N}}$, initialized by $u_0 = K$ and $t_0 = 0$, and defined iteratively for n such that $u_n > \varepsilon$ and $v_n > 0$ as follows:

- $\tau_n > t_n$ is the unique solution of

$$\int_0^{\tau_n - t_n} f'((u_n^\alpha - \delta\alpha\tau)_+^{1/\alpha})\, d\tau = A.$$

- $v_n = (u_n^\alpha - \delta\alpha(\tau_n - t_n))_+^{1/\alpha}$.
- $t_{n+1} = \tau_n + (1-A)/f'(v_n)$.
- $u_{n+1} = v_n$.

These choices are made so that for all $n \in \mathbb{N}$ such that $u_n > \varepsilon$, $u(t_n, 0) = u_n$, and $u(\tau_n, A) = v_n$.

We also set n_0 the first integer (if any) for which $u_n < \varepsilon$ or $v_n = 0$. This index n_0, if any, is in fact such that $u_{n_0} \leqslant \varepsilon$ and $v_{n_0} = 0$. Indeed, if $u_n > \varepsilon$, one easily checks from the definition of ε that $v_n > 0$.

Note that we easily deduce from the formula of v_n that

$$\forall n \in \{0, \cdots, n_0 - 1\}, \qquad \delta\alpha(\tau_n - t_n) \leqslant u_n^\alpha.$$

In order to study these sequences, it will be convenient to have a good estimate on $\tau_n - t_n$ in terms of u_n only. We thus define

$$\beta_- = \inf_{[0,K]} \frac{f'(s)}{s}, \qquad \beta_+ = \sup_{[0,K]} \frac{f'(s)}{s},$$

which are finite as $f'(0) = 0$ and which are both strictly positive as f is strictly convex.

We then recall that $1/\alpha \geqslant 1$. Thanks to the convexity of the function $s \mapsto s^{1/\alpha}$ at the point u^α, its graph on $[0, u^\alpha]$ is below the chord initiated from the origin, i.e. $s \mapsto s u^{1-\alpha}$, and above its tangent $s \mapsto (1 - 1/\alpha)u + s u^{1-\alpha}/\alpha$, which yields the following estimates: for all $u > 0$, $\tau \geqslant 0$,

$$(u - \delta\tau u^{1-\alpha})_+ \leqslant (u^\alpha - \delta\alpha\tau)_+^{1/\alpha} \leqslant (u - \delta\alpha\tau u^{1-\alpha})_+.$$

Combining the above two estimates, we infer

$$\beta_- \int_0^{\tau_n - t_n} (u_n - \delta\tau u_n^{1-\alpha})_+\, d\tau \leqslant A \leqslant \beta_+ \int_0^{\tau_n - t_n} (u_n - \delta\alpha\tau u_n^{1-\alpha})_+\, d\tau,$$

which implies in particular that

$$A \geqslant \begin{cases} \dfrac{\beta_-}{2\delta} u_n^{1+\alpha} & \text{if } u_n^\alpha \leqslant \delta(\tau_n - t_n), \\ \beta_- u_n(\tau_n - t_n) - \dfrac{\delta\beta_-}{2} u_n^{1-\alpha}(\tau_n - t_n)^2 & \text{if } u_n^\alpha \geqslant \delta(\tau_n - t_n), \end{cases}$$

and

$$A \leqslant \begin{cases} \dfrac{\beta_+}{2\alpha\delta} u_n^{1+\alpha} & \text{if } u_n^\alpha \leqslant \alpha\delta(\tau_n - t_n), \\ \beta_+ u_n(\tau_n - t_n) - \dfrac{\alpha\delta\beta_+}{2} u_n^{1-\alpha}(\tau_n - t_n)^2 & \text{if } u_n^\alpha \geqslant \alpha\delta(\tau_n - t_n). \end{cases}$$

We claim that we can deduce from this a lower bound on $\tau_n - t_n$ of the form

$$\forall n \in \{0, \cdots, n_0 - 1\},\ \alpha\delta(\tau_n - t_n) \geqslant u_n^\alpha \min\left\{\alpha, 1 - \sqrt{\left(1 - \frac{2A\alpha\delta}{\beta_+ K^{1+\alpha}}\right)_+}\right\}.$$

Indeed, this is obvious when $\tau_n - t_n \geqslant u_n^\alpha/\delta$. Otherwise, when $\tau_n - t_n \leqslant u_n^\alpha/\delta$, we have $u_n^\alpha \geqslant \alpha\delta(\tau_n - t_n)$ and thus one should have

$$\frac{A\alpha\delta}{2\beta_+ u_n^{1+\alpha}} \leqslant \frac{\alpha\delta}{2}\frac{\tau_n - t_n}{u_n^\alpha} - \left(\frac{\alpha\delta}{2}\frac{\tau_n - t_n}{u_n^\alpha}\right)^2.$$

But $u_n \leqslant K$ as the sequence u_n is non-increasing, so that

$$\frac{A\alpha\delta}{2\beta_+ K^{1+\alpha}} \leqslant \left(\frac{\alpha\delta}{2}\frac{\tau_n - t_n}{u_n^\alpha}\right) - \left(\frac{\alpha\delta}{2}\frac{\tau_n - t_n}{u_n^\alpha}\right)^2.$$

Consequently, if $A\alpha\delta/(2\beta_+ K^{1+\alpha}) > 1/4$, this cannot happen. Besides, if

$$A\alpha\delta/(2\beta_+ K^{1+\alpha}) \leqslant 1/4,$$

we obtain immediately that

$$\frac{\tau_n - t_n}{u_n^\alpha} \geqslant \frac{1}{\alpha\delta}\left(1 - \sqrt{1 - \frac{2A\alpha\delta}{\beta_+ K^{1+\alpha}}}\right).$$

It follows from this estimate and the definition of v_n that there exists $c_0 < 1$ such that for all $n \in \{0, \cdots, n_0 - 1\}$,

$$u_{n+1} = v_n \leqslant c_0 u_n.$$

Therefore,

$$\forall n \in \{0, \cdots, n_0 - 1\}, \quad u_n \leqslant c_0^n K, \text{ and } v_n \leqslant c_0^{n-1} K,$$

and there indeed exists $n_0 \in \mathbb{N}$ such that $u_{n_0} \leqslant \varepsilon$. By construction, and definition of n_0, $t_{n_0} = \tau_{n_0-1} + (1 - A)/f'(v_{n_0-1})$ is finite, and thus T_* is finite. □

Proof of item (ii) of Lemma 4.2. In order to prove item (ii) of Lemma 4.2, we look at the solution u in $[t_*, \infty) \times [A, 1]$, where t_* is given by (4.15).

It follows from Lemma 4.1 that u is piecewise $\mathscr{C}^1$ and continuous in $[t_*, \infty) \times (A, 1)$, non-negative there, and $u(t, A) = 0$ for all $t \geqslant t_*$. Consequently, for all $t \geqslant t_*$, there exists $x_0 \in (A, 1]$ such that

$$u(t, 1) = u(t_*, x_0),$$

and x_0 is the unique solution of

$$x_0 + (t - t_*) f'(u(t_*, x_0)) = 1.$$

In particular, one should have

$$\beta_- u(t_*, x_0) \leqslant f'(u(t_*, x_0)) \leqslant \frac{1}{t - t_*}.$$

Therefore,

$$\forall t \geqslant t_*, \quad u(t,1) \leqslant \frac{1}{\beta_-(t - t_*)}. \tag{4.26}$$

From Lemma 4.1 and the fact that for all $t \geqslant 0$, $x \mapsto u(t,x)$ is non-decreasing on $[A,1]$ and non-negative, we get

$$\forall t \geqslant t_*, \quad \|u(t)\|_{L^\infty(A,1)} \leqslant \frac{1}{\beta_-(t - t_*)}.$$

Now, the set $\mathscr{Z}$ on which $u = 0$ is delimited by the curve $\{(t, A),\ \text{for } t \geqslant t_*\}$ and the curve $\mathcal{C}$ given by (4.14), that we now estimate: for $t_0 \geqslant T_*$, recalling that $u(t_0, 0) = u(t_0, 1)$, (4.26) entails

$$t_0 + \frac{(u(t_0,0))^\alpha}{\delta\alpha} \leqslant t_0 + \frac{1}{\delta\alpha\beta_-^\alpha (t_0 - t_*)^\alpha},$$

and

$$\begin{aligned}\int_0^\infty f'((u(t_0,0)^\alpha - \delta\alpha\tau)_+^{1/\alpha})\, d\tau &= \int_0^{u(t_0,0)^\alpha/(\delta\alpha)} f'((u(t_0,0)^\alpha - \delta\alpha\tau)_+^{1/\alpha})\, d\tau \\ &\leqslant \beta_+ \int_0^{u(t_0,0)^\alpha/(\delta\alpha)} (u(t_0,0) - \delta\alpha\tau u(t_0,0)^{1-\alpha})_+\, d\tau \\ &\leqslant \frac{\beta_+}{2\delta\alpha}(u(t_0,0))^{1+\alpha} \leqslant \frac{\beta_+}{2\delta\alpha\beta_-^{1+\alpha}} \frac{1}{(t_0 - t_*)^{1+\alpha}}.\end{aligned}$$

Recalling that the above left-hand side is equal to $x(t_0)$ defined in (4.14), we deduce that for $t \geqslant t_*$, there exists an open subinterval $\omega(t) \subset \omega$ such that $u(t)_{|\omega(t)} = 0$, and, for some constant C, for all time $t \geqslant t_*$,

$$|\omega \setminus \omega(t)| \leqslant \frac{C}{t^{1+\alpha}}.$$

Besides, for $t \geqslant t_*$, we easily get from (4.21) and (4.22) that

$$\forall x \in [0, A], \quad 0 \leqslant u(t,x) \leqslant \max_{t_0 \in [t - \varepsilon^\alpha/(\delta\alpha), t]} u(t_0, 0) \leqslant \frac{1}{\beta_-(t - t_* - \varepsilon^\alpha/(\delta\alpha))}.$$

The item (ii) of Lemma 4.2 easily follows. □

4.2. *Concave flux and positive constant initial datum*

Let $K > 0$ and f be a strictly concave flux, and consider the solution u of

$$\partial_t u + \partial_x f(u) + a(x)\frac{u}{|u|^\alpha} = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{T}, \quad u_{|t=0} = K. \tag{4.27}$$

Setting

$$v(t,x) = u(t,-x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{T}, \quad \tilde{a}(x) = a(1-x), \quad x \in \mathbb{T},$$

one easily checks that v formally solves

$$\partial_t v + \partial_x g(v) + \tilde{a}(x)\frac{v}{|v|^\alpha} = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{T}, \quad v_{|t=0} = K. \tag{4.28}$$

with $g = -f$, which satisfies $g'(0) = 0$ and $\inf_{[-K,K]} g'' > 0$. The fact that this transformation maps an admissible solution u of (4.27) to an admissible solution v of (4.28) is easy to check.

Therefore, the counterpart of Lemma 4.2 item (ii) also holds for solutions of (4.2) when f only satisfies (1.17):

Lemma 4.3. *Let f and K satisfy (1.17), $\alpha \in (0, 1]$, $\delta > 0$ and $\omega = (0, A) \subset \mathbb{T}$. Then the solution u of (4.2) satisfies the following property: There exists $t_* \geqslant 0$ such that for all $t \geqslant t_*$, there exists an open subinterval $\omega(t) \subset \omega$ such that $u(t)_{|\omega(t)} = 0$, and*

$$|\omega \setminus \omega(t)| \leqslant \frac{C}{t^{1+\alpha}}, \quad \|u(t)\|_{L^\infty(\mathbb{T})} \leqslant \frac{C}{t}.$$

4.3. Negative constant initial datum

Let $K > 0$ and consider the solution u of

$$\begin{cases} \partial_t u + \partial_x f(u) + a(x)\dfrac{u}{|u|^\alpha} = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{T}, \\ u_{|t=0} = -K. \end{cases} \tag{4.29}$$

Then, setting

$$w(t, x) = -u(t, x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{T},$$

w formally solves

$$\begin{cases} \partial_t w + \partial_x h(w) + a(x)\dfrac{w}{|w|^\alpha} = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{T}, \\ w_{|t=0} = K, \end{cases} \tag{4.30}$$

where the flux h is given by

$$\forall s \in [-K, K], \quad h(s) = -f(-s).$$

It is easy to check that $h'(0) = 0$ and $\inf_{[-K,K]} |h''(s)| > 0$. Besides, this transformation also establishes the correspondence between the admissible solution u of (4.29) and the admissible solution w of (4.30).

Therefore, Lemma 4.2 item (ii) also holds when the initial datum is constant $= -K$, under the only condition that the flux f satisfies (1.17).

Lemma 4.4. *Let f and K satisfy (1.17), $\alpha \in (0, 1]$, $\delta > 0$ and $\omega = (0, A) \subset \mathbb{T}$. Then the solutions $u_\pm$ of*

$$\partial_t u_\pm + \partial_x f(u_\pm) + \delta \mathbf{1}_{(0,A)} \frac{u_\pm}{|u_\pm|^\alpha} = 0 \text{ in } \mathbb{R}_+ \times \mathbb{T}, \quad u_\pm\,|_{t=0} = \pm K \text{ in } \mathbb{T}. \tag{4.31}$$

satisfy the following property: There exists $t_* \geqslant 0$ such that for all $t \geqslant t_*$, there exists an open subinterval $\omega_\pm(t) \subset \omega$ such that $u_\pm(t)_{|\omega_\pm(t)} = 0$, and

$$|\omega \setminus \omega_\pm(t)| \leqslant \frac{C}{t^{1+\alpha}}, \quad \|u_\pm(t)\|_{L^\infty(\mathbb{T})} \leqslant \frac{C}{t}.$$

4.4. Proof of Theorem 1.9

The proof of Theorem 1.9 follows by comparing the solution u of (1.1)–(1.2) with initial datum $u_0 \in L^\infty(\mathbb{T})$ with some reference solutions.

Namely, we assume that a satisfies (1.10) for some open interval $\omega \subset \mathbb{T}$. Up to a translation in space, we can assume that $\omega = (0, A)$ for some $A \in (0, 1]$. Therefore, according to the comparison principles stated in Proposition 1.5, the solution u of (1.1)–(1.2) with initial datum $u_0 \in L^\infty(\mathbb{T})$ with $\|u_0\|_{L^\infty} \leqslant K$, K as in (1.12), is sandwiched between the solutions $u_\pm$ of (4.31).

We then immediately conclude Theorem 1.9 from Lemma 4.4.

5. Numerical simulations and open problems

We presented all along the paper various numerical simulations corresponding to the setting of our results presented in Theorems 1.8 and 1.9. We therefore present in this section some numerical experiments for various equations to which our theoretical results do not apply. The main numerical technique relies on the time-splitting scheme as time integrator. If one considers a general evolution equation

$$\begin{cases} \partial_t u = \mathcal{A}u + \mathcal{B}u, & (t,x) \in \mathbb{R}_+ \times \mathbb{T}, \\ u(0,x) = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{5.1}$$

where $\mathcal{A}$ and $\mathcal{B}$ are (possibly non-linear) operators which need not commute. For a given time step $\delta t > 0$, set $t_n = n\delta t$, $n = 0, 1, 2, \dots$ Define the operators $S_{\mathcal{A}}$ and $S_{\mathcal{B}}$ associated respectively to the evolution equations

$$\partial_t u_{\mathcal{A}} = \mathcal{A}u_{\mathcal{A}}, \quad \partial_t u_{\mathcal{B}} = \mathcal{B}u_{\mathcal{B}}, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{T},$$

The operators satisfy the following relations involving the exact solutions of the associated equations:

$$u_{\mathcal{A}}(t+\delta t) = S_{\mathcal{A}}(\delta t)u_{\mathcal{A}}(t) \quad \text{and} \quad u_{\mathcal{B}}(t+\delta t) = S_{\mathcal{B}}(\delta t)u_{\mathcal{B}}(t).$$

The splitting idea (see for example [22]) consists in approximating the continuous flow associated to (5.1) by a composition of operators $S_{\mathcal{A}}$ and $S_{\mathcal{B}}$ in the spirit of Trotter-Kato formula, the key for an efficient implementation being to solve efficiently these two reduced equations. We consider in this paper the second order Strang splitting scheme. Let $u^n(x)$ be the approximation of $u(t_n, x)$. The approximate solution to (5.1) at time t_{n+1} reads

$$u^{n+1} = S_{\mathcal{A}}(\delta t/2)S_{\mathcal{B}}(\delta t)S_{\mathcal{A}}(\delta t/2)u^n. \tag{5.2}$$

5.1. Scalar conservation laws

Let us now describe how it is applied to the equation (1.1). It involves the two reduced equation

$$\partial_t u + \partial_x f(u) = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{T}, \tag{5.3}$$

and

$$\partial_t u = -a(x)\frac{u}{|u|^\alpha}, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{T}. \tag{5.4}$$

The equation (5.3) is a standard nonlinear conservative hyperbolic equation. In the second equation (5.4), the space variable can be considered as a parameter and the equation reduced to an ordinary equation with solution

$$u(t,x) = \operatorname{sign}(u_0(x))\left(|u_0(x)|^\alpha - \alpha a(x)t\right)_+^{1/\alpha}. \tag{5.5}$$

If the flux f is linear, $f(u) = cu$, the solution to (5.3) is obviously

$$u(t,x) = u_0(x - ct).$$

For a general flux, we compute an approximate solution thanks to Rusanov scheme (see for example [25, p.233]). We identify the torus with $(0,1)$ endowed with periodic boundary conditions and choose the spatial mesh size $\delta x > 0$ with $\delta x = 1/J$ with J denoting the number of nodes. The grid points are $x_j = j\delta x$, $j = 0, 1, \cdots, J$. Let u_j^n be the full approximation to $u(t_n, x_j)$. The Rusanov scheme reads

$$u_j^{n+1} = u_j^n - \frac{\delta t}{\delta x}\left(F_{j+1/2}^n - F_{j-1/2}^n\right),$$

where the Rusanov numerical flux is given by

$$F_{j+1/2}^n = F^{\mathrm{Rus}}(u_j^n, u_{j+1}^n) = \frac{f(u_j^n) + f(u_{j+1}^n)}{2} - \frac{\max(|f'(u_j^n)|, |f'(u_{j+1}^n)|)}{2}(u_{j+1}^n - u_j^n).$$

Remark 5.1. For balanced laws and regular nonlinearities (thus ruling out our sublinear damping term), wave front-tracking methods appear to be more efficient numerically (see [3]).

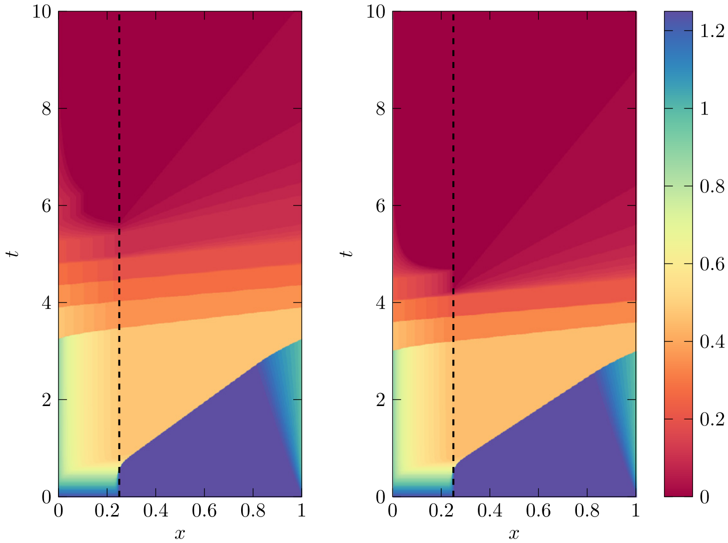

Fig. 5. Evolution of the solution for Buckley-Leverett flux $f_{1/4}^{BL}$ with $\alpha = 0.75$ (left) and $\alpha = 1$ (right).

The theoretical results of previous sections apply to fluxes with assumption $f'(0) \neq 0$ or $f'(0) = 0$ with convexity hypothesis (convex or concave flux). Some fluxes do not satisfy such hypothesis. This is the case of the Buckley-Leverett flux which models two phase fluid flow in a porous medium ([24]). In one space dimension the equation has the standard conservation law form ($k > 0$ is a parameter)

$$f_k^{BL}(u) = \frac{u^2}{u^2 + k(1-u)^2}. \tag{5.6}$$

We compute the evolution of the solution to (1.1) with $f_{1/4}^{BL}$ flux and the damping function $a(x)$ given by (1.18). The numerical parameters are $u_0(x) = K = 1.25$, $A = 1/4$, $\delta = 1$, $\delta t = 10^{-5}$ and $\delta x = 5 \cdot 10^{-5}$. The evolution of the solutions for $\alpha = 3/4$ and $\alpha = 1$ are plotted on Fig. 5 and the evolution of their characteristic curves on Fig. 6. The characteristic curves are computed as the evolution of a vector field with velocity given by the solution to (1.1). We see that, contrary to convex (or concave) fluxes with $f'(0) = 0$, shock waves appear in finite time, in the case of constant initial data (for Burgers equation, shocks may appear for non constant initial data only). Since the domain is a torus, the shock wave initiated from $x = A = 1/4$ propagates until the influence of the damping function a is enough important to annihilate the solution inside the support of a. We then recover a similar process as the one observed on Fig. 2 where characteristic curves become vertical lines in finite time inside the support of the damping function a. The effect of decreasing α is to delay the extinction of the solution in $(0, A)$. The proof of this phenomenon is still missing.

5.2. *Viscous Burgers equations*

We consider here the convection diffusion equation given for $\mu > 0$ by

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = \mu \partial_x^2 u - a(x) \frac{u}{|u|^\alpha}, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{T}. \tag{5.7}$$

We look at the solutions when $a(x) = \delta \mathbf{1}_\omega$, $\omega = (0, A)$, $A < 1$, the torus $\mathbb{T}$ being the circle $(0, 1)$. This equation involves three different operators. We have to apply a second order three-operators splitting scheme which reads for the evolution equation $\partial_t u = (\mathcal{A} + \mathcal{B} + \mathcal{C})u$:

$$u^{n+1} = S_{\mathcal{A}}(\delta t/2) S_{\mathcal{B}}(\delta t/2) S_{\mathcal{C}}(\delta t) S_{\mathcal{B}}(\delta t/2) S_{\mathcal{A}}(\delta t/2) u^n, \tag{5.8}$$

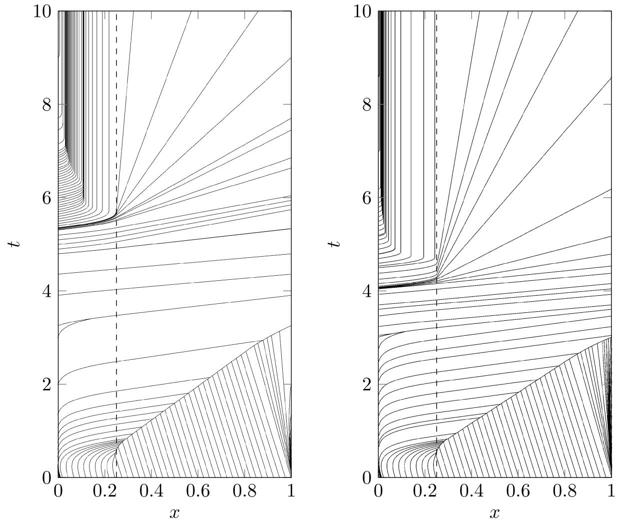

Fig. 6. Evolution of the characteristic curves for Buckley-Leverett flux $f_{1/4}^{BL}$ with $\alpha = 0.75$ (left) and $\alpha = 1$ (right).

where $S_{\mathcal{A}}$, $S_{\mathcal{B}}$ and $S_{\mathcal{C}}$ denote the flows associated to operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. Since we study the equation (5.7) on a torus, we benefit from the periodicity to use fast Fourier transform in order to make space approximation of the solutions of the heat equation

$$\partial_t w = \mu \partial_x^2 w.$$

The spatial mesh size is defined by $\delta x = 1/J$, $J = 2^P$, $P \in \mathbb{N}^*$. Since we discretize the heat equation by the Fourier spectral method, w_j^n and its Fourier transform satisfy the following relations:

$$w_j^n = \frac{1}{J} \sum_{m=-J/2}^{J/2-1} \hat{w}_m^n e^{i\xi_m (x_j - x_\ell)}, \quad j = 0, \cdots, J-1,$$

and

$$\hat{w}_m^n = \sum_{j=0}^{J-1} w_j^n e^{-i\xi_m (x_j - x_\ell)}, \quad m = -\frac{J}{2}, \cdots, \frac{J}{2} - 1,$$

where $\xi_m = 2\pi m$ for all $m = -\frac{J}{2}, \cdots, \frac{J}{2} - 1$. The discrete Laplace operator Δ_P is therefore define by

$$\widehat{(\Delta_P v)}_m = -\xi_m^2 \hat{v}_m, \quad v \in \mathbb{C}^M.$$

We present on Fig. 7 the evolution of the logarithm of the solution. We choose the same numerical parameters used for Buckley-Leverett equation, the only difference relying on the mesh size which is $\delta x = 2^{-14}$. We present the logarithm to show that like in the hyperbolic case, the solution becomes zero on the support of the damping function a after a time T^* which depends on the parameter α. What is more surprising is the fact that after the time T^*, the solution on $(A, 1)$ behaves like the solution of the heat equation with homogeneous Dirichlet boundary conditions associated to the first eigenvalues of the Laplacian. We know that this solution on $(A, 1)$ is

$$v(t, x) = \exp(-\mu\lambda^2 t) \sin(\lambda(x - A)),$$

with $\lambda = \pi/(1 - A)$. We clearly identify this phenomenon by displaying the evolution of the L^∞ norm of the solution with respect to time on Fig. 8. We plot both the L^∞ norm and a dashed line in log-scale with slope $-\mu\lambda^2$. The sin-like behavior of the solution on $[A, 1]$ for time $t = 10$ is also clearly present on Fig. 8. A rigorous mathematical proof of the above observations is, to our knowledge, missing, despite the works [9,12], where conditions for complete extinctions of the solutions are discussed (see also [8,10,11] for related results).

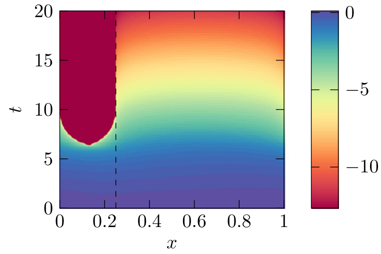

Fig. 7. Evolution of the $\log_{10}$ of the solution to (5.7) with $\alpha = 0.75$.

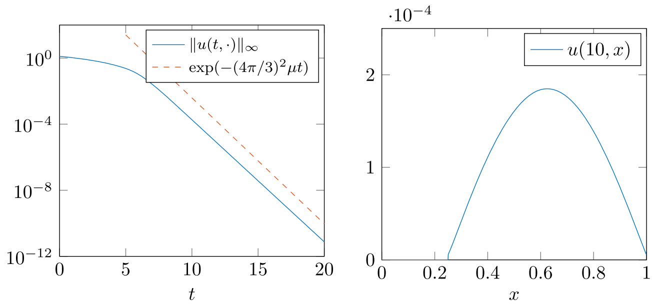

Fig. 8. Evolution of the L^∞ norm of the solution to (5.7) in log-scale (left) and solution at time $t = 10$ (right) for $\alpha = 0.75$.

We end up this paragraph by emphasizing that (5.7) is a viscous approximation of the Burgers equation, which is a conservation law fitting the assumptions of Theorem 1.9. It is thus completely natural to ask the behavior of (5.7) in large times, similarly to what has been done in Theorem 1.9. Though, as our numerical simulations underline, the large-time behavior of (5.7) is very different from the one of the Burgers equations predicted by Theorem 1.9. This is an evidence of the fact that the limit of large times and the limit of small viscosities do not commute, as observed in other contexts for instance in [23].

5.3. *Wave equation*

We consider now the wave equation with homogeneous Dirichlet boundary conditions

$$\begin{cases} \partial_t^2 u - c^2 \partial_x^2 u = -a(x)\dfrac{\partial_t u}{|\partial_t u|^\alpha}, & (t,x) \in \mathbb{R}_+ \times (0,1), \\ u(t,0) = u(t,1) = 0, \end{cases} \tag{5.9}$$

completed with initial conditions $u(0,x) = u_0(x)$ and $\partial_t u(0,x) = u_1(x)$.

To numerically simulate the solution to (5.9), we begin by transforming the equation as the first order hyperbolic system

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ c^2 \partial_x^2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ -a(x) v/|v|^\alpha \end{pmatrix}.$$

We can therefore apply the Strang splitting method (5.2). The solution to the ODE $\partial_t v = -a(x)v/|v|^\alpha$ is given by (5.5) and we approximate the free wave equation with the Newmark scheme ([28])

$$\begin{aligned} u_j^{n+1} &= u_j^n + \delta t v_j^n + \delta t^2 \left[\zeta c^2 w_j^{n+1} + (1/2 - \zeta) c^2 w_j^n \right], \\ v_j^{n+1} &= v_j^n + \delta t \left[(1-\theta) c^2 w_j^n + \theta c^2 w_j^{n+1} \right], \end{aligned} \tag{5.10}$$

with $u_j^0 = u_0(x_j)$, $v_j^0 = u_1(x_j)$ and $w_j^k = (u_{j+1}^k - 2u_j^k + u_{j-1}^k)/(\delta x)^2$. We select for our numerical simulations $\theta = 1/2$ and $\zeta = 1/4$ for which the scheme is both second order in space and time and unconditionally stable.

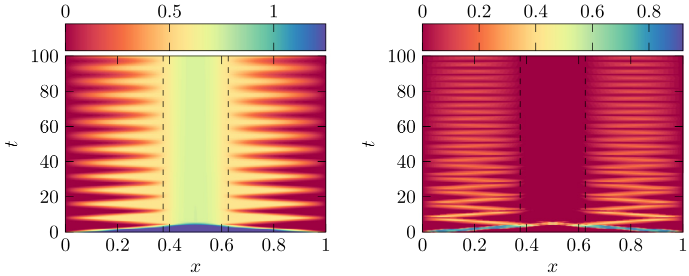

Fig. 9. Evolution of the solution to (5.9), u on the left and $\partial_t u$ on the right.

We select the initial conditions

$$u_0(x) = K \begin{cases} 1 - e^1 \exp(-0.1/(0.1-x)), & \text{if } x < 0.1, \\ 1, & \text{if } 0.1 \leqslant x \leqslant 0.9, \\ 1 - e^1 \exp(-0.1/(x-0.9)), & \text{if } x > 0.9, \end{cases}$$

and $u_1(x) = 0$. The damping function is $a(x) = \delta \mathbf{1}_\omega$, $\omega = (3/8, 5/8)$. The numerical parameters are $\alpha = 1$, $c = 0.1$, $\delta t = 5 \cdot 10^{-4}$, $\delta x = 10^{-3}/3$, $\delta = 1$ and $K = 1.25$.

As can be expected (see Fig. 9), the time derivative of the solution is annihilated on the support of a after a time T^*, the solution u becoming constant for $t > T^*$.

Let us underline that the linear wave equation is the prototype of a 2×2 system of conservation laws, which can be easily seen with the use of characteristics. It is thus natural to consider such models as a generalization of (1.1)–(1.2). Note that the behavior of the solution of (5.9) when the damping acts everywhere in the domain has been studied in [6], or when the damping acts on the boundary [27], but the case of a localized damping term involving $\partial_t u$ still does not seem to be precisely described in the literature. In fact, the interested reader should also notice the close connection of this problem with the non-linear damped oscillator of the form

$$mx'' + \delta \frac{x'}{|x'|^\alpha} + \omega^2 x = 0, \quad t \geqslant 0,$$

with $m > 0$, $\alpha \in (0, 1]$, and $\omega > 0$, whose large time behavior is quite subtle, see e.g. [4,29].

5.4. *Schrödinger equation*

The last equation we consider is the strongly damped cubic nonlinear Schrödinger (NLS) equation, motivated by the works [13,14] in which the damping is effective everywhere. We thus wonder if the previous results for hyperbolic equations can be extended to the Schrödinger equation

$$i\partial_t u + \partial_x^2 u = -q|u|^2 u - ia(x)\frac{u}{|u|^\alpha}, \quad (t, x) \in \mathbb{R} \times \mathbb{T}. \tag{5.11}$$

with initial datum $u(0, x) = u_0(x)$. If the damping function a is zero, then the cubic NLS equation set in the whole space $x \in \mathbb{R}$ admits a special solution known as soliton. This solution is given by

$$u_{\text{sol}}(t, x) = \sqrt{\frac{2k}{q}} \operatorname{sech}\left(\sqrt{k}(x - ct)\right) \exp(i\frac{c}{2}(x - ct)) \exp(i(k + \frac{c^2}{4})t). \tag{5.12}$$

Since this solution is strongly localized, although it does not give an exact solution of (5.11), the effect of the periodic boundary conditions can be neglected. This solution evolves from its initial datum $u_0 = u_{\text{sol}|t=0}$ propagating at velocity c with time phase change. This solution for $c = 20$, $k = 0.81$ and the torus $x \in (-10, 10)$ is plotted on Fig. 10. The numerical scheme again relies on the Strang splitting scheme for three operators (5.8). As for the Burgers heat equation, the space approximation is performed thanks to fast Fourier transform. The complex solution to ODE $\partial_t u = -a(x)u/|u|^\alpha$ is given by

$$u(t, x) = (|u_0|^\alpha - \alpha a(x)t)_+^{1/\alpha} \exp(i \operatorname{Arg}(u_0)),$$

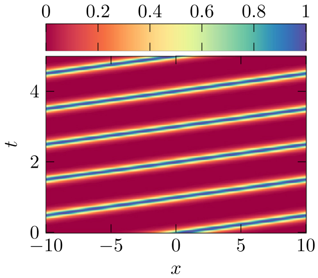

Fig. 10. Evolution of the modulus of the soliton (5.12) for $c = 20$ and $k = 0.81$.

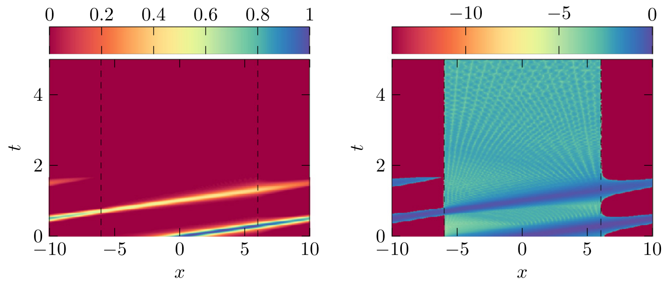

Fig. 11. Evolution of the solution to (5.11) in standard scale (left) and in log scale (right).

whereas the solution to the ODE $i\partial_t u = -q|u|^2 u$ is given by

$$u(t,x) = \exp(itq|u_0(x)|^2)u_0(x).$$

We present the effects of the damping function $a(x) = \delta \mathbf{1}_\omega$, $\omega = (-10,-6) \cup (6,10)$ on the soliton for $\mathbb{T} = (-10,10)$ and $\alpha = 1$. The soliton initial datum overlaps the support of a. The numerical parameters are $\delta t = 5 \cdot 10^{-4}$ and $\delta x = 10 \cdot 2^{-12}$. As expected, the solution begins to propagate to the right direction and then vanishes on the support of a (see Fig. 11). This is more clear on log scale. Again, to our knowledge, this behavior has not been proved rigorously in the literature.

Conflict of interest statement

There is no conflict of interest.

Appendix A. Cauchy problem and comparison principle

We consider more generally the Cauchy problem

$$\partial_t u + \partial_x f(u) + h(x,u) = 0 \text{ in } \mathbb{R}_+ \times \mathbb{T}, \quad u_{|t=0} = u_0 \text{ in } \mathbb{T}, \tag{A.1}$$

with a fairly general semilinear term h, possibly depending on x, in order to generalize the nonlinearity, typically of the form

$$h_\alpha(x,u) = a(x)\frac{u}{|u|^\alpha}, \tag{A.2}$$

where $\alpha \leqslant 1$ corresponds to the case of (1.1), with the modification detailed in Definition 1.3 in the case $\alpha = 1$. We will use the following properties on the source term h, which encompass the framework of Definition 1.2 when $\alpha < 1$.

Assumption A.1. The map $h = h(x,u)$ satisfies:

- $h \in L^\infty_{\text{loc}}(\mathbb{T} \times \mathbb{R})$.
- For all $u \in \mathbb{R}$, $h(x,u)u \geqslant 0$, for almost all $x \in \mathbb{T}$.

- For almost all fixed $x \in \mathbb{T}$, the map $u \mapsto h(x,u)$ is nondecreasing on $\mathbb{R}$.
- For every $R > 0$,

$$\sup_{|u|\leqslant R}\sup_{y>0}\frac{1}{y}\int_{\mathbb{T}} |h(x+y,u)-h(x,u)|\,dx < \infty.$$

In the case (A.2), the first property corresponds to the assumption $a \in L^\infty(\mathbb{T})$, the second property to the fact that h is a damping term, the third property is straightforward (even in the case $\alpha = 1$ with the approach of Filippov), and the last property is a consequence of Assumption 1.1.

Definition A.2 *(Notion of solution).* Let h satisfy Assumption A.1. A bounded measurable function u on $[0,T]\times\mathbb{T}$ is an *admissible weak solution* of (A.1), with $u_0 \in L^\infty(\mathbb{T})$, if the inequality

$$\iint_{0\,\mathbb{T}}^{T} \big(\partial_t\psi\,\eta(u) + \partial_x\psi\, q(u) - \psi\,\eta'(u)h(x,u)\big)\,dxdt + \int_{\mathbb{T}} \psi(0,x)\eta\left(u_0(x)\right)dx \geqslant 0 \tag{A.3}$$

holds for every convex function $\eta \in W^{1,\infty}$, with $q' = f'\eta'$, and all nonnegative Lipschitz continuous test function ψ on $[0,T]\times\mathbb{T}$.

Propositions 1.4 and 1.5 stem from the following result, which is slightly more general in view of Assumption A.1.

Proposition A.3. *Let Assumption A.1 be satisfied.*

(i) *Let $u_0 \in L^\infty(\mathbb{T})$. There exists a unique, global, admissible weak solution u of (A.1), $u \in \mathscr{C}^0(\mathbb{R}_+; L^1(\mathbb{T}))$.*

(ii) *Let u and v be solutions of (A.1) with respective initial data $u_0, v_0 \in L^\infty(\mathbb{T})$, and $u_0(x) \leqslant v_0(x)$ for almost all $x \in \mathbb{T}$. Then*

$$u(t,x) \leqslant v(t,x), \quad \forall t \geqslant 0,\ \text{a.e. } x \in \mathbb{T}. \tag{A.4}$$

Besides,

$$|u(t,x)| \leqslant \|u_0\|_{L^\infty(\mathbb{T})}, \quad \forall t \geqslant 0,\ \text{a.e. } x \in \mathbb{T}. \tag{A.5}$$

(iii) *If $h^{(1)}$ and $h^{(2)}$ satisfy Assumption A.1, and in addition,*

$$(0 \leqslant)\, h^{(1)}(x,u) \leqslant h^{(2)}(x,u), \quad \text{a.e. } (x,u) \in \mathbb{T}\times(0,\infty),$$

then denoting by u_1 and u_2 the respective solutions to (A.1) with the same initial datum $u_0 \in L^\infty(\mathbb{T})$, $u_0 \geqslant 0$, we have

$$u_1(t,x) \geqslant u_2(t,x) \geqslant 0, \quad \forall t \geqslant 0,\ \text{a.e. } x \in \mathbb{T}. \tag{A.6}$$

(iv) *The same properties hold true for solutions of (1.1)–(1.2) for $\alpha = 1$, when considering solutions in the sense of Definition 1.3 and a satisfies Assumption 1.1.*

Main steps of the proof. With Assumption A.1, it is possible to follow very closely the approach of [19, Section 6.3], based on the method of vanishing viscosity. We also refer to [20] for the case of non-smooth balance laws.

• **Uniqueness.** For u and $\underline{u}$ two solutions in the sense of Definition A.2, uniqueness can be established by considering the entropy-entropy flux pair

$$\eta(u,\underline{u}) = |u-\underline{u}|, \quad q(u,\underline{u}) = \operatorname{sign}(u-\underline{u})\left(f(u) - f(\underline{u})\right).$$

This is an entropy-entropy flux pair for u when $\underline{u}$ is fixed, and, conversely, for $\underline{u}$ when u is fixed. By a suitable choice of test functions (the same as in [19]), we find, for any $0 < \tau < T$, and since the third point of Assumption A.1 shows that the contribution of the nonlinear term has a definite sign in this approach,

$$\int_{\mathbb{T}} \eta\left(u_0(x), \underline{u}_0(x)\right) dx \geqslant \int_{\mathbb{T}} \eta\left(u(\tau,x), \underline{u}(\tau,x)\right) dx,$$

that is $\|u(\tau) - \underline{u}(\tau)\|_{L^1(\mathbb{T})} \leqslant \|u_0 - \underline{u}_0\|_{L^1(\mathbb{T})}$, hence uniqueness for solutions in the sense of Definition A.2.

• **Viscous approximation.** For $\mu > 0$, consider the equation

$$\partial_t u_\mu + \partial_x f(u_\mu) + h(x, u_\mu) = \mu \partial_x^2 u_\mu \text{ in } \mathbb{R}_+ \times \mathbb{T}, \quad u_{\mu|t=0} = u_0 \text{ in } \mathbb{T}. \tag{A.7}$$

For a fixed $\mu > 0$, the solution to (A.7) is obtained by a fixed point argument applied to the associated Duhamel's formula.

• **A priori estimate**. The solution is global in time, $u_\mu \in \mathscr{C}^0(\mathbb{R}_+; L^\infty(\mathbb{T}))$, in view of the a priori estimate

$$\|u_\mu(t)\|_{L^\infty(\mathbb{T})} \leqslant \|u_0\|_{L^\infty(\mathbb{T})}, \quad \forall t \geqslant 0, \tag{A.8}$$

which can be established by considering the multiplier $|u_\mu|^{p-2}u_\mu$, using the periodic boundary conditions, and letting $p \to \infty$.

• **Comparison for the viscous solution.** Introduce as in the proof of [19, Theorem 6.3.2] the function η_ε defined for $\varepsilon > 0$ by

$$\eta_\varepsilon(w) = \begin{cases} 0 & \text{for } -\infty < w \leqslant 0, \\ \frac{w^2}{4\varepsilon} & \text{for } 0 < w \leqslant 2\varepsilon, \\ w - \varepsilon & \text{for } 2\varepsilon < w < \infty. \end{cases}$$

If u_μ and $\bar{u}_\mu$ solve (A.7), then by multiplying by $\eta'_\varepsilon(u_\mu - \bar{u}_\mu)$ the equation satisfied by $u_\mu - \bar{u}_\mu$, we compute

$$\begin{aligned} &\partial_t \eta_\varepsilon(u_\mu - \bar{u}_\mu) + \partial_x\left(\eta'_\varepsilon(u_\mu - \bar{u}_\mu)\left(f(u_\mu) - f(\bar{u}_\mu)\right)\right) - \eta''_\varepsilon(u_\mu - \bar{u}_\mu)\left(f(u_\mu) - f(\bar{u}_\mu)\right)\partial_x(u_\mu - \bar{u}_\mu) \\ &\quad = -\eta'_\varepsilon(u_\mu - \bar{u}_\mu)\left(h(x,u_\mu) - h(x,\bar{u}_\mu)\right) + \mu\partial_x^2\eta_\varepsilon(u_\mu - \bar{u}_\mu) - \mu\eta''_\varepsilon(u_\mu - \bar{u}_\mu)\left(\partial_x(u_\mu - \bar{u}_\mu)\right)^2. \end{aligned}$$

The new term compared to the proof of [19, Theorem 6.3.2] is of course the first term of the right-hand side (where h is present). For $0 < s < t < \infty$, integrate over $(s,t) \times \mathbb{T}$, and use Assumption A.1 (third point) to show that the nonlinear term has a non-positive contribution. By letting $s \to 0$,

$$\int_{\mathbb{T}} \left(u_\mu(t,x) - \bar{u}_\mu(t,x)\right)_+ dx \leqslant \int_{\mathbb{T}} (u_0(x) - \bar{u}_0(x))_+ dx. \tag{A.9}$$

Interchanging the roles of u_μ and $\bar{u}_\mu$,

$$\|u_\mu(t) - \bar{u}_\mu(t)\|_{L^1(\mathbb{T})} \leqslant \|u_0 - \bar{u}_0\|_{L^1(\mathbb{T})},$$

and if

$$u_0(x) \leqslant \bar{u}_0(x), \quad \text{a.e. on } \mathbb{T},$$

then we find

$$u_\mu(t,x) \leqslant \bar{u}_\mu(t,x), \quad \forall t \geqslant 0, \ \text{ a.e. } x \in \mathbb{T}.$$

This implies in particular uniqueness for (A.7).

• **Compactness.** To obtain compactness in space, as in [19], we consider $\bar{u}_\mu(t,x) = u_\mu(t,x+y)$. In the case where $h = 0$ (or more generally if h depends on u only), then $\bar{u}_\mu$ is a solution to (A.7), so (A.9) can be used directly. In our case, and precisely because we want to consider spatially localized damping, such $\bar{u}_\mu$ does not solve (A.7), and we have to resume the computations. Essentially, we go back to the previous computations, and replace $\bar{u}_\mu(t,x)$ with $u_\mu(t,x+y)$, noticing that $h(x,\bar{u}_\mu)$ has to be replaced by $h(x+y, u_\mu(t,x+y))$. We have

$$\begin{aligned} &\int_{\mathbb{T}} (u_\mu(t,x) - u_\mu(t,x+y))_+ dx - \int_{\mathbb{T}} (u_0(x) - u_0(x+y))_+ dx \leqslant \\ &\quad \limsup_{\varepsilon \to 0} \left(-\int_0^t \int_{\mathbb{T}} \eta'_\varepsilon(u_\mu - \bar{u}_\mu)\left(h(x, u_\mu(\tau,x)) - h(x+y, u_\mu(\tau,x+y))\right) d\tau dx \right). \end{aligned}$$

In the above integral, insert $\pm h(x+y,u_\mu(\tau,x))$. Invoking the third point in Assumption A.1), we infer

$$\int_{\mathbb{T}} (u_\mu(t,x)-u_\mu(t,x+y))_+dx - \int_{\mathbb{T}} (u_0(x)-u_0(x+y))_+dx \leqslant$$
$$\limsup_{\varepsilon\to 0}\left(-\int_0^t\int_{\mathbb{T}} \eta'_\varepsilon(u_\mu-\bar u_\mu)\left(h(x,u_\mu(\tau,x))-h(x+y,u_\mu(\tau,x))\right)d\tau dx\right),$$

hence

$$\int_{\mathbb{T}} (u_\mu(t,x)-u_\mu(t,x+y))_+dx - \int_{\mathbb{T}} (u_0(x)-u_0(x+y))_+dx$$
$$\leqslant \int_0^t\int_{\mathbb{T}} \left|h(x,u_\mu(\tau,x))-h(x+y,u_\mu(\tau,x))\right| d\tau dx.$$

In view of (A.8) and of the last point in Assumption A.1, we conclude

$$\int_{\mathbb{T}} (u_\mu(t,x)-u_\mu(t,x+y))_+dx \leqslant \int_{\mathbb{T}} (u_0(x)-u_0(x+y))_+dx + \mathcal{O}(y),$$

and

$$\int_{\mathbb{T}} |u_\mu(t,x)-u_\mu(t,x+y)|dx \leqslant \int_{\mathbb{T}} |u_0(x)-u_0(x+y))|dx + \mathcal{O}(y). \tag{A.10}$$

Equicontinuity in time is proved similarly by setting $\bar u_\mu(t,x)=u_\mu(t+\tau,x)$. Since h depends on x and u_μ only, the only extra term that we have to estimate is of the form

$$\left|\int_t^{t+\tau}\int_{\mathbb{T}} h(x,u_\mu(s,x))\phi(x)dsd\tau\right| \leqslant \tau C\left(\|u_0\|_{L^\infty(\mathbb{T})}\right)\|\phi\|_{L^1(\mathbb{T})},$$

where we have used (A.8).

The above properties imply that the sequence $(u_\mu)_\mu$ is uniformly bounded and equicontinuous in $(0,\infty)\times\mathbb{T}$, so there is a subsequence of $(u_\mu)_\mu$, which converges boundedly almost everywhere on $(0,\infty)\times\mathbb{T}$ and strongly in $L^1_{\rm loc}((0,\infty)\times\mathbb{T})$. At this stage, we have proved the item (i), while item (ii) follows from (A.9) and (A.8), after passing to the limit $\mu\to 0$.

Remark A.4. Dividing (A.10) by y yields the propagation of BV regularity mentioned in Remark 1.6.

• **Entropy solution for (A.1).** Up to extracting a subsequence, u_μ converges to an entropy solution (Definition A.2). The above uniqueness result shows that actually, no extraction is needed.

• **Comparison when source terms are ordered.** It remains to prove (iii). Since we assume $u_0\geqslant 0$, we know from (ii) that $u_1(t,x),u_2(t,x)\geqslant 0$ for $(t,x)\in(0,\infty)\times\mathbb{T}$, and so $h^{(2)}(x,u_j)\geqslant h^{(1)}(x,u_j)$ for $j=1,2$. We then consider the viscous approximations $u_{1,\mu}$ and $u_{2,\mu}$ of, respectively, u_1 and u_2. With η_ε as above,

$$\int_{\mathbb{T}} \eta_\varepsilon(u_{2,\mu}(t)-u_{1,\mu}(t)) - \int_{\mathbb{T}} \eta_\varepsilon(u_{2,\mu}(s)-u_{1,\mu}(s))$$
$$\leqslant \int_s^t\int_{\mathbb{T}} \eta''_\varepsilon(u_{2,\mu}-u_{1,\mu})\left(f(u_{2,\mu})-f(u_{1,\mu})\right)\partial_x(u_{2,\mu}-u_{1,\mu})$$
$$-\int_s^t\int_{\mathbb{T}} \eta'_\varepsilon(u_{2,\mu}-u_{1,\mu})\left(h^{(2)}(x,u_{2,\mu})-h^{(1)}(x,u_{1,\mu})\right).$$

Passing to the limits $\varepsilon \to 0$, $s \to 0$, and finally $\mu \to 0$, we infer, since u_1 and u_2 have the same initial datum:

$$\int_{\mathbb{T}} (u_2(t) - u_1(t))_+ \leqslant - \int_0^t \int_{\mathbb{T}} \mathbf{1}_{u_2>u_1} \left(h^{(2)}(x,u_2) - h^{(1)}(x,u_1)\right).$$

Now since the integrand of the right hand side can be decomposed as

$$\underbrace{\mathbf{1}_{u_2>u_1} \left(h^{(2)}(x,u_2) - h^{(2)}(x,u_1)\right)}_{\geqslant 0,\text{ by Assumption A.1}} + \mathbf{1}_{u_2>u_1} \underbrace{\left(h^{(2)}(x,u_1) - h^{(1)}(x,u_1)\right)}_{\geqslant 0,\text{ from above}},$$

we conclude that $\int_{\mathbb{T}} (u_2(t) - u_1(t))_+ = 0$, hence $u_1 \geqslant u_2 \geqslant 0$ as announced.

• **Item** (iv)**: the case** $h(x,u) = a(x)u/|u|$**.**

In this case, the uniqueness of admissible weak solutions in the sense of Definition 1.3 holds without change. The difficulty then is to prove existence of admissible weak solutions. In order to do that, instead of approximating (1.1) by its viscous approximation (A.7), we also add an approximation of the function h. Namely, we consider the approximation given, for $\mu \in (0,1)$, by

$$\partial_t u_\mu + \partial_x f(u_\mu) + h_{1-\mu}(x,u_\mu) = \mu \partial_x^2 u_\mu \text{ in } \mathbb{R}_+ \times \mathbb{T}, \quad u_{\mu|t=0} = u_0 \text{ in } \mathbb{T}, \tag{A.11}$$

where $h_{1-\mu}(x,u)$ is given by (A.2). For each $\mu > 0$, all the computations performed above can be repeated, so that the sequence of solutions $(u_\mu)_\mu$ is uniformly bounded and equicontinuous in $(0,\infty) \times \mathbb{T}$. Therefore, up to some subsequence, it converges boundedly almost everywhere on $(0,\infty) \times \mathbb{T}$ and strongly in $L^1_{\text{loc}}((0,\infty) \times \mathbb{T})$ to some u as $\mu \to 0$, so that a.e. $(t,x) \in (0,\infty) \times \mathbb{T}$,

$$h_{1-\mu}(x, u_\mu(t,x)) \underset{\mu \to 0}{\longrightarrow} h(t,x),$$

where $h(t,x) = a(x)\frac{u(t,x)}{|u(t,x)|}$ if $u(t,x) \neq 0$, and $h(t,x) \in [-1,1]$ if $u(t,x) = 0$.

It follows that u is an admissible weak solution of (1.1)–(1.2) in the sense of Definition 1.3, hence the admissible weak solution in the sense of Definition 1.3 by uniqueness. The comparison results can then be proved as before, by studying them for the solutions of (A.11) and passing to the limit $\mu \to 0$. □

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