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Boltzmann collision operator for the infinite range potential: A limit problem

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Abstract

The conventional Boltzmann collision operator for the infinite range inverse power law model was derived by Maxwell by adopting a collision kernel which is a limit of that for the finite range model by ignoring the glancing angles. Since the interpretation of collision operator for the infinite range potential through limit process to the one with finite range potential is natural in regard to the derivation of the Boltzmann equation. It is the purpose of this paper to clarify the physical meaning of the conventional collision operator for the infinite range inverse power law model through the study of the limiting process of the collision operator as the cutoff radius tends to infinity. We first estimate the extent in which the glancing angles can be ignored in the limiting process. Furthermore we prove that taking limit to collision operator with finite range potential directly will lead to the conventional one with algebraic convergence rate.

Keywords: Boltzmann collision operator; Infinite range potential; Radius cutoff; Fourier integral operator; Pseudodifferential operator

1. Introduction

The Boltzmann equation, Boltzmann [5],

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f, f)\ ,\ \ (t, x, v) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3,$$

is the limiting equation of particle interacting systems under the fixed mean free path and molecular chaos hypotheses. It has been formally derived for particle systems with finite range potential by Harold Grad [9]. On the other hand, the conventional Boltzmann collision operator for the infinite range inverse power law model was derived by Maxwell by adopting a collision kernel which is a limit of that for the finite range model by ignoring the glancing angles. Intermolecular potential of infinite range can be viewed as the limit of cut-off potentials of finite range. With this

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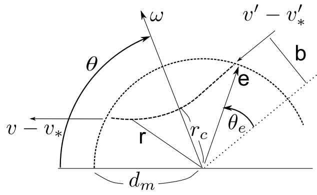

Fig. 1. Binary collision.

limiting process, there are two main steps in the full derivation of the Boltzmann equation. The first step is to study the variation of the collision operator, based on two particles interaction, in the limiting process. The goal of the present article is to analyze this step. Our limiting Boltzmann collision operator in this setting agrees with the one conjectured by Maxwell [18] which is not obvious from analysis point of view. The second step is to show, in the spirit of [9], that the binary interaction is the dominant part of all collisions, and the limiting collision operator has only binary collisions. The second step depends on the quantitative understanding of the first step, especially the interpretation and use of Boltzmann Grad limit $Nd_m^2 = c$ in the infinite range potential setting. The present effort of the study of collision operator for the binary interaction can therefore be viewed as a crucial step in the full derivation of the Boltzmann equation for the infinite range potential, see the upcoming papers by Yoshio Sone [22], [23].

For a potential of finite range d_m, the physical information about the intermolecular potential is condensed in the collision kernel B_{d_m} of collision operator

$$Q_{d_m}(f,f) = \int\limits_{\mathbb{R}^3} \int\limits_{\omega\in S^2} (f' f'_* - f f_*) B_{d_m}(v - v_*, \omega) d\Omega(\omega) dv_*.$$

Here $f' = f(t,x,v')$, $f'_* = f(t,x,v'_*)$, $f_* = f(t,x,v_*)$,

$$v' = v - [\omega\cdot(v - v_*)]\omega\,,\ v'_* = v_* + [\omega\cdot(v - v_*)]\omega\,,\ \omega \in S^2. \tag{1.1}$$

Here ω is the change of direction of the particles under the binary collision and $d\Omega(\omega)$ is the solid angle element in the direction of ω. For an infinite range potential $U(r),\ U(r) > 0,\ U(r) \to 0$ as $r \to \infty$, the corresponding finite range potential $U_{d_m}(r)$ is the one cut off $r = d_m$:

$$U_{d_m}(r) = \begin{cases} U(r), & \text{for } r < d_m, \\ 0, & \text{for } r \ge d_m. \end{cases}$$

For given relative velocity $v' - v'_* = v - v_*,\ \mathcal{V} = |v - v_*|$, and impact parameter b, (Fig. 1), the collision kernel B_{d_m}

$$\begin{aligned} &2B_{d_m}(v - v_*, \omega) d\Omega(\omega) = d_m^2 |(v - v_*)\cdot e| d\Omega(e) \\ &= 2B_{d_m}(\mathcal{V}, \theta) d\theta d\phi = \mathcal{V} b \frac{db}{d\theta} d\theta d\phi\,,\ 0 \le b \le d_m \end{aligned} \tag{1.2}$$

describes one to one correspondence between the incidence unit vector e, and the unit vector ω. Let m be the mass of each molecule. The angle θ represents vector ω by taking $v - v_*$ as the pole and introducing the spherically coordinate $\omega = (\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$. Similarly we use the angle θ_e to represent the vector e. The key to obtain the explicit form of B_{d_m} is to study the effect of the intermolecular potential $U(r)$ on the critical angle θ and the critical distance r_c during the binary collision. The term $db/d\theta$ of (1.2) is found from two particle interaction formula

$$\theta = \sin^{-1}\frac{b}{d_m} + b\int\limits_{r_c}^{d_m} r^{-2}\Big(1 - \frac{4U(r)}{m\mathcal{V}^2} - \frac{b^2}{r^2}\Big)^{-1/2} dr \tag{1.3}$$

where r_c solves $1 - 4U(r)/m\mathcal{V}^2 - b^2/r^2 = 0$.

In this paper, we consider the inverse power law inter-molecular potential:

$$U(r) = c_{\overline{u}} r^{-(n-1)}\,,\ n > 2. \tag{1.4}$$

Define $s = (n-1)^{-1}$. Due to the potential jump $U(d_m)$ at $r = d_m$, the collision kernel B_{d_m} is nothing but the hard sphere model if the relative velocity is sufficient small, $c_u \mathcal{V}^{-1}(d_m)^{-1/2s} \geq 1$ where $c_u = (4c_{\overline{u}} m^{-1})^{1/2}$ is a constant and m is the mass of the particle. For larger relative velocity, $c_u \mathcal{V}^{-1}(d_m)^{-1/2s} < 1$, if the normal component $\mathcal{V}\cos\theta$ of the relative velocity $\mathcal{V}$ is sufficiently small, $\mathcal{V}\cos\theta \leq c_u(d_m)^{-1/2s}$, the incoming particle is reflected as a hard sphere again due to potential jump, otherwise it penetrates into the sphere $r = d_m$. More precisely, the collision kernel B_{d_m} reads (see Section 4)

$$B_{d_m}(\mathcal{V},\theta) = \begin{cases} B^{H_s}_{d_m}(\mathcal{V},\theta) & \text{if } c_u \mathcal{V}^{-1}(d_m)^{-1/2s} \geq 1 \\ B^{H_l}_{d_m}(\mathcal{V},\theta) + B^{I}_{d_m}(\mathcal{V},\theta) & \text{if } c_u \mathcal{V}^{-1}(d_m)^{-1/2s} < 1 \end{cases} \tag{1.5}$$

where

$$B^{H_s}_{d_m}(\mathcal{V},\theta) = \mathcal{V} d_m^2 \cos\theta \sin\theta = \mathcal{V} d_m^2 b^{H_s}(\cos\theta)\sin\theta,\ 0 \leq \theta \leq \pi/2, \tag{1.6}$$

$$B^{H_l}_{d_m}(\mathcal{V},\theta) = \begin{cases} \mathcal{V} d_m^2 b^{H_l}(\cos\theta)\sin\theta, & \text{if } \cos\theta \leq c_u \mathcal{V}^{-1}(d_m)^{-1/2s} \\ 0, & \text{otherwise;} \end{cases} \tag{1.7}$$

$$B^{I}_{d_m}(\mathcal{V},\theta) = \begin{cases} 0, & \text{if } \cos\theta \leq c_u \mathcal{V}^{-1}(d_m)^{-1/2s}, \\ c_{d_m} \mathcal{V}^{\gamma} b^{I}(\cos\theta)\sin\theta, & \text{otherwise,} \end{cases} \tag{1.8}$$

where $b^{H_l}(\cos\theta) = \cos\theta$, $b^I(\cos\theta) = (\cos\theta)^{-(1+2s)}$, c_{d_m} a positive bounded coefficient and $\gamma = (n-5)/(n-1)$. A technical remark is given below. For fixed $\mathcal{V}, d_m$, the bounded coefficient c_{d_m} varies with θ which is small near critical angle $\cos\theta = c_u \mathcal{V}^{-1}(d_m)^{-1/2s}$ so that the value of $B^{H_l}_{d_m}$ from above of critical angle matches that of $B^I_{d_m}$ from below to have a smooth B_{d_m}. Thus we may replace the c_u in (1.7) and (1.8) by $2c_u$ and $c_u/2$ respectively and assume $b^{H_l}(\cos\theta)$ and $b^I(\cos\theta)$ decay to 0 when θ tends to new critical angles. For the convenience of representation, we still use (1.7) and (1.8) without changing c_u but keep in mind that $b^{H_l}(\cos\theta)$ and $b^I(\cos\theta)$ decay to 0 when θ tends to critical angle.

On the other hand, Maxwell derived the conventional collision operator for infinite range by fixing the impact parameter b, or more generally,

$$\lim_{d_m \to \infty} \frac{b}{d_m} = 0 \tag{1.9}$$

and then taking limit of the formula (1.3) to yield

$$\tilde{\theta}_c = b \int_{r_c}^{\infty} r^{-2}\left(1 - \frac{4U(r)}{m\mathcal{V}^2} - \frac{b^2}{r^2}\right)^{-1/2} dr. \tag{1.10}$$

For the inverse power law model, Maxwell's limit yields the collision operator, denoted by Q hereafter, with the kernel

$$\begin{aligned} B(v - v_*, \omega)d\Omega(\omega) &= c_\infty |v - v_*|^{\gamma}(\cos\theta)^{-(1+2s)} \sin\theta d\theta d\phi \\ &= \Phi(|v - v_*|) b(\cos\theta)\sin\theta d\theta d\phi \end{aligned} \tag{1.11}$$

where c_∞ is a constant and $\gamma = (n-5)/(n-1)$, $s = 1/(n-1)$. (See [11] or [7] for more details.) Thus Maxwell's derivation ignores the possibilities of

$$\lim_{d_m \to \infty} \frac{b}{d_m} \neq 0. \tag{1.12}$$

We define the glancing angle be the angle θ with impact parameter b satisfying (1.12) and non-glancing angle θ be the angle with impact parameter b satisfying (1.9) when d_m tends to ∞. For any fixed $\eta > 0$, the contribution from glancing angle to the collision operator is not obvious negligible, Fig. 1 and Fig. 2. With this in mind, comparing (1.5), (1.8) with (1.11) and noting that $Q(f, f)$ was not obtained by taking limit to the $Q_{d_m}(f, f)$ directly, it is necessary to clarify the meaning of the operator $Q(f, f)$. In this paper, the concrete physical meaning of Q is given by following two results. Our first result shows that under suitable conditions, it is legitimate to ignore the contribution from angles with condition (1.12) to Q_{d_m} as d_m tends to infinity. Secondly, we verify that Q can be regarded as the limit of Q_{d_m} on the basis of first result.

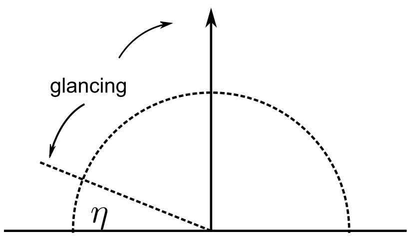

Fig. 2. Glancing angle.

Before stating our results precisely, we define some notations which help us to set up the mathematical problems. Recall the weak formulation

$$\langle Q(g,f),h\rangle = \iiint g(v'_*)f(v')(h(v')-h(v))B(v-v_*)d\Omega(\omega)dv_*dv \tag{1.13}$$

where v, v_*, v', v_* satisfy (1.1). Our results to the limit problems mentioned above will be given in the form of weak formulation as it is often used for non-cutoff model. We also recall that by change of variables

$$\begin{aligned}&\langle Q_{d_m}(f,f),h\rangle \\ &= \iiint (f(v')f(v'_*)-f(v)f(v_*))h(v)B_{d_m}(v-v_*,\omega)d\Omega(\omega)dv_*dv \\ &= \iiint f(v)f(v_*)(h(v')-h(v))B_{d_m}(v-v_*,\omega)d\Omega(\omega)dv_*dv.\end{aligned} \tag{1.14}$$

However the second equality holds if the gain and lost terms are separable which is not true for the infinite range interaction model. It is thus natural to consider

$$\begin{aligned}&\lim_{d_m\to\infty}\langle Q_{d_m}(f,f),h\rangle \\ &= \lim_{d_m\to\infty}\iiint f(v)f(v_*)(h(v')-h(v))B_{d_m}(v-v_*,\omega)d\Omega(\omega)dv_*dv.\end{aligned} \tag{1.15}$$

It would be of independently interesting to study these limit problems without using weak formulation. This is, however, left to the future.

The assumption of glancing or non-glancing angles makes difference when we consider the limit of B_{d_m} as d_m tends to ∞. More precisely, there is a dichotomy to the coefficient c_{d_m} in the second line of (1.8). For the non-glancing angles, the coefficient c_{d_m} tends to a constant c_∞ in (1.11); while for glancing angles, the coefficient c_{d_m} is smaller than previous case and may tends to 0, Lemma 4.1. Therefore the inverse power part of B_{d_m} from glancing angles can be absorbed into that from non-glancing angles. To complete the proof that the contribution from the glancing angles can be ignored, it suffices to prove the contribution form $B^H_{d_m}$ to the collision operator, say $Q^H_{d_m}$ will vanish.

Theorem 1.1. *Consider the inverse power law model with potential cutoff at d_m and let $0 < s < 1$. If $0 < s < 1/2$ and $f \in L^1_{1-2s} \cap H^{1+\rho}_{1-2s}$, $h \in H^{1-\rho}$, $\rho \in [-1, 1]$, then the hard sphere part $Q^H_{d_m}$ of the collision operator decays in the weak sense with the following rates as $d_m \to \infty$:*

$$\langle Q^H_{d_m}(f,f),h\rangle \le O((d_m)^{(1-1/s)})\|f\|_{L^1_{1-2s}}\|f\|_{H^{1+\rho}_{1-2s}}\|h\|_{H^{1-\rho}}. \tag{1.16}$$

The coefficient $O((d_m)^{\frac{1}{2}(1-1/s)})$ grows as s tends to $1/2$. If $1/2 \le s < 1$, $\rho \in [-\frac{1}{2}-s, \frac{1}{2}+s]$ and $f \in L^1 \cap H^{\frac{1}{2}+s+\rho}$, $h \in H^{\frac{1}{2}+s-\rho}$ then

$$\langle Q^H_{d_m}(f,f),h\rangle \le O((d_m)^{N(s)})\|f\|_{L^1}\|f\|_{H^{\frac{1}{2}+s+\rho}}\|h\|_{H^{\frac{1}{2}+s-\rho}}, \tag{1.17}$$

where $N(s) = (1-s^\kappa)(1-1/s) < 0$ and small $\kappa > 0$ satisfies $2s^\kappa - 2s^{1+\kappa} - s > 0$.

It is worth mentioning that if the collision operator is restricted to the case with relative velocity $|v - v_*| \geq 1$, then we have the better convergence rate of $(d_m)^{(2-2/s)}$, $0 < s < 1/2$. This is due to that small relative velocity has less cancellation as expected.

The main result is the following.

Theorem 1.2. *Consider the inverse power law model with potential cut-off at d_m and let $0 < s < 1$. If $0 < s < 1/2$ and $f \in L^1_{1-2s} \cap H^{1+\rho}_{1-2s}$, $h \in H^{1-\rho}$, $\rho \in [-1, 1]$, then we have*

$$|\langle Q_{d_m}(f, f), h\rangle - \langle Q(f, f), h\rangle| \leq O((d_m)^{\frac{1}{2}(1-1/s)})\|f\|_{L^1_{1-2s}}\|f\|_{H^{1+\rho}_{1-2s}}\|h\|_{H^{1-\rho}}; \tag{1.18}$$

The coefficient $O((d_m)^{\frac{1}{2}(1-1/s)})$ grows as s tends to $1/2$. If $1/2 \leq s < 1$ and $f \in L^1 \cap H^{\frac{1}{2}+s+\rho}$, $h \in H^{\frac{1}{2}+s-\rho}$, $\rho \in [-\frac{1}{2} - s, \frac{1}{2} + s]$ then

$$|\langle Q_{d_m}(f, f), h\rangle - \langle Q(f, f), h\rangle| \leq O((d_m)^{N(s)})\|f\|_{L^1}\|f\|_{H^{\frac{1}{2}+s+\rho}}\|h\|_{H^{\frac{1}{2}+s-\rho}}, \tag{1.19}$$

where $N(s) = (1 - s^\kappa)(1 - 1/s) < 0$ and small $\kappa > 0$ satisfies $2s^\kappa - 2s^{1+\kappa} - s > 0$.

Remark 1.1. It is easy to see from the proof that the results of above Theorems can be expressed in terms of $Q(f, g)$ as well. The number 2, $1 + 2s$, sum of exponents of Sobolev spaces in Theorems 1.1, 1.2, is the minimum to consider limit problems for full range $0 < s < 1/2$, $1/2 \leq s < 1$ respectively. When $0 < s < 1/2$, it is easy to check from the proof the increasing regularity 1 can add regularity decay rate $(d_m)^{-1/2s}$ which may be need later in applications. The worse decay rate for $s > 1/2$ is due to the case when the relative velocity is small and the product of relative velocity and its Fourier dual is also small. See the proof for (3.108).

To prove Theorem 1.1 and Theorem 1.2, we need the estimates which enable us to see how the quantity $\langle Q_{d_m}(f, f), h\rangle$ varies as d_m tends to ∞ for f and h lie in suitable norm spaces. The current methods of estimating the upper bound of $\langle Q(g, f), h\rangle$ for non-cutoff Q rely on the Littlewood-Paley theory, [2,3,8,12], or additional assumption on distribution functions, [10]. The application of these methods to the limiting problem will be quite complicated and lengthy due to the appearance of the parameter d_m. In this paper, we consider a structural description of $\langle Q(g, f), h\rangle$ through the understanding of the structure of a specific Radon transform T defined by the formula

$$\langle Q(g, f), h\rangle = \int_v \int_{v_*} g(v_*)f(v)|v - v_*|^{\gamma+2s}[\tau_{-v*} \circ T \circ \tau_{v_*}]h(v)dv_*dv$$

where $\tau_{v_*}h(\cdot) = h(\cdot + v_*)$ is the translation operator. Indeed, the fact that the properties of Q is largely condensed in a class of Radon transforms was observed by Alexandre and Villani [1] and other authors.

Notations

Before proceeding, we fix the notations. Set $l^+ = \max\{l, 0\}$ and $\langle v\rangle = (1 + |v|^2)^{1/2}$, $v \in \mathbb{R}^3$, $(a \cdot b) = \sum_{i=1}^3 a_i b_i$ the scalar product in $\mathbb{R}^3$ and $\langle f, g\rangle = \int_{\mathbb{R}^3} f(x)g(x)dx$ the inner product in $L^2(\mathbb{R}^3)$. The differential operator $\langle D\rangle^s$, $s \in \mathbb{R}$ is expressed through the Fourier transform:

$$\langle D_x\rangle^s f(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{ix\cdot\xi}\langle\xi\rangle^s \widehat{f}(\xi)d\xi,$$

$$\widehat{f}(\xi) = \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x)dx.$$

The weighted Lebesgue, weighted Sobolev spaces are denoted by

$$\|f\|_{L^p_q} = \Big(\int_{\mathbb{R}^3} |f(v)|^p\langle v\rangle^{pq}dv\Big)^{1/p}, \ \|f\|_{H^s_\alpha} = \|\langle v\rangle^\alpha f(v)\|_{H^s}.$$

We use the multiindex notation $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$. A function $p(x,\xi) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ satisfying

$$\forall \alpha, \forall \beta, \ |\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{l-|\beta|} \tag{1.20}$$

for any multi-indices α and β is called a symbol of order l. We also use $S^l_{1,0}$ to denote the set of symbols with order l. For each $p(x,\xi) \in S^l_{1,0}$, the associate operator

$$P(x,D)h(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\widehat{h}(\xi) d\xi$$

is called a pseudodifferential operator of order l. The standard notation $S^{-\infty}_{1,0} = \cap_l S^l_{1,0}, l \in \mathbb{Z}$ is also used. If $p(x,\xi) \in S^{-\infty}_{1,0}$, it is called a symbol of the smooth operator. The operator

$$Sf(x) = \int_{\mathbb{R}^3} e^{i\phi(x,\xi)} a(x,\xi)\widehat{f}(\xi) d\xi$$

with symbol $a(x,\xi)$ of order l and the phase function $\phi(x,\xi)$ satisfies non-degeneracy condition is called a Fourier integral operator of order l. We say a phase function $\phi(x,\xi)$ satisfies the non-degeneracy condition if there is a constant $c > 0$ such that

$$|\det \nabla_x \nabla_\xi \phi(x,\xi)| \ge c > 0 \tag{1.21}$$

for all $(x,\xi) \in \operatorname{supp} a(x,\xi)$ where the matrix

$$\nabla_x \nabla_\xi \phi(x,\xi) \overset{\text{def}}{=} \Big[\partial_{x_i}\partial_{\xi_j}\phi(x,\xi)\Big]. \tag{1.22}$$

A import remark is that we will see that all the symbols in this paper have the derivatives descend order 1 in ξ variable as (1.20) as well as in x variable. For example, a symbol of order 0 in this paper will satisfy (2.5) when $|x| > 4$ and $|xi| > 4$.

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2. Lemma of almost orthogonality

Before the poof of Theorems 1.1, 1.2, we introduce a lemma which proves the L^2 boundedness of the Fourier integral operator (F.I.O.) by the almost orthogonality argument [24].

We consider the cone in the phase space $(x,\xi) \in (\mathbb{R}^3 - \{0\}) \times (\mathbb{R}^3 - \{0\})$ which has the form

$$\begin{aligned} &\{(x,\xi)| \ \frac{\pi}{8} < \arccos(\frac{x\cdot\xi}{|x||\xi|}) < \pi - \frac{\pi}{8}\} \\ &\overset{\text{def}}{=} \Gamma_x \times \Gamma_\xi. \end{aligned} \tag{2.1}$$

For each fixed x, we use $x \times \Gamma_\xi$ to denote the cone whose element $\xi \in \mathbb{R}^3$ satisfies (2.1). The notation Γ_ξ means the cone $x \times \Gamma_\xi$ where the vector x is not specified. Also $\Gamma_x \times x$, Γ_x are defined likewise. To study the F.I.O. whose amplitude function defined on $\Gamma_x \times \Gamma_\xi$, we need a dyadic partition of unity on $\mathbb{R}^3 - \{0\}$. Let $\vartheta \in C^\infty$, $0 \le \vartheta \le 1$ be supported in the interval (2, 8) and satisfy $1 = \sum_{k\in\mathbb{Z}} \vartheta(2^k s)$ for all $s > 0$ with property $\mathrm{supp}\,\vartheta(2^j s) \cap \mathrm{supp}\,\vartheta(2^k s) = \emptyset$ if $|j - k| \geq 2$. For $x \in \mathbb{R}^3$ and $k \in \mathbb{Z}$ we define $\chi_k(x) = \vartheta(2^{-k}|x|)$. The we have dyadic partition of unity

$$1 = \sum_{k\in\mathbb{Z}} \chi_k(x),\ x \neq 0. \tag{2.2}$$

Let

$$A = \max\{ \int_{\mathbb{R}^3 \cap \Gamma_x} \chi_0(x) dx\ ,\ \int_{\mathbb{R}^3 \cap \Gamma_x} \chi_0(x)\chi_1(x) dx\}. \tag{2.3}$$

Lemma 2.1. *Let F be defined by*

$$Fu(x) = \int_{\mathbb{R}^3} e^{i\psi(x,\xi)} p(x,\xi)\, u(\xi) d\xi \tag{2.4}$$

where $\psi(x,\xi)$ is homogeneous of degree 1 in x and ξ, $p(x,\xi) \in C^\infty(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)$ and $\mathrm{supp}\, p(x,\xi) \subset \Gamma_x \times \Gamma_\xi \cap \{|x| > 4, |\xi| > 4\}$ satisfies

$$|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \leq C_{\alpha\beta} |x|^{-|\alpha|} |\xi|^{-|\beta|}, \tag{2.5}$$

for $|\alpha|, |\beta| \leq 5$. When $(x,\xi) \in \mathrm{supp}\, p(x,\xi)$, the phase function $\psi(x,\xi)$ satisfies

$$0 < C_1 < |\det\Big[\partial_x \partial_\xi \psi(x,\xi)\Big]| < C_2 \tag{2.6}$$

and

$$|\partial_x^\alpha \partial_\xi \psi(x,\xi)| \leq C_\alpha\ ,\ |\partial_x \partial_\xi^\beta \psi(x,\xi)| \leq C_\beta\ ,\ 1 \leq |\alpha|, |\beta| \leq 6 \tag{2.7}$$

Then F is L^2 bounded and satisfies

$$\|F\|_{L^2\to L^2} \leq CA \sup_{|\alpha|\leq 5, |\beta|\leq 5} \|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)\|_{L^\infty(\mathbb{R}^3_x\times\mathbb{R}^3_\xi)} \tag{2.8}$$

where C depends on the constants in (2.6) and (2.7).

Proof. Note $\sum_{k=0}^\infty \chi_k(z) = 1$ when $|z| > 4$. Therefore we can decompose F as

$$F = \sum_{(j,l)\in\mathbb{N}\cup\{0\}\times\mathbb{N}\cup\{0\}} F_{(j,l)}$$

where

$$F_{(j,l)} u(x) = \chi_j(x) \int_{\mathbb{R}^3} e^{i\psi(x,\xi)} \chi_l(\xi) p(x,\xi) u(\xi) d\xi$$

The adjoint of $F_{j,l}$, denoted by $F^*_{j,l}$ is

$$F^*_{(j,l)} v(\xi) = \chi_l(\xi) \int e^{-i\psi(y,\xi)} \chi_j(y) \overline{p(y,\xi)} v(y) dy.$$

Then we have

$$F_{(j,l)} F^*_{(k,m)} u(x) = \int K_{(j,l),(k,m)}(x,y) u(y) dy,$$

where

$$K_{(j,l),(k,m)}(x,y) = \chi_j(x)\chi_k(y) \int e^{i(\psi(x,\xi)-\psi(y,\xi))} \chi_l(\xi)\chi_m(\xi) p(x,\xi) \overline{p(y,\xi)} d\xi. \tag{2.9}$$

Since supp $\chi_l(\xi) \cap$ supp $\chi_m(\xi) = \emptyset$ when $|l-m| \geq 2$, we only have to consider $l = m-1, m$ or $m+1$. Without loss of generality, we assume $l = m+1$ and $j \geq k$. Let $\tilde{x} = 2^{-k}x = \tau_1^{-1}x$, $\tilde{y} = 2^{-k}y = \tau_1^{-1}y$, $\tilde{\xi} = 2^{-m}\xi = \tau_2^{-1}\xi$. Since ψ is homogeneous of degree 1 in first and second variables, we have

$$\begin{aligned}
&K_{(j,l)(k,m)}(x,y) \\
&= \tau_2^3 \chi_{j-k}(\tilde{x})\chi_0(\tilde{y}) \int e^{i\tau_1\tau_2(\psi(\tilde{x},\tilde{\xi})-\psi(\tilde{y},\tilde{\xi}))}\chi_0(\tilde{\xi})\chi_1(\tilde{\xi})p(\tau_1\tilde{x},\tau_2\tilde{\xi})\overline{p(\tau_1\tilde{y},\tau_2\tilde{\xi})}d\tilde{\xi} \\
&\stackrel{\text{def}}{=} \tau_2^3\, \mathcal{K}_{(j,l)(k,m)}(\tilde{x},\tilde{y}).
\end{aligned}$$

Hence

$$F_{(j,l)}F^*_{(k,m)}u(\tau_1\tilde{x}) = \int \tau_2^3 \mathcal{K}_{(j,l)(k,m)}(\tilde{x},\tilde{y})u(\tau_1\tilde{y})\tau_1^3 d\tilde{y}. \tag{2.10}$$

Define the operator

$$L_{\tau_1\tau_2} = \frac{1}{i\tau_1\tau_2}\frac{\partial_{\tilde{\xi}}\psi(\tilde{x},\tilde{\xi}) - \partial_{\tilde{\xi}}\psi(\tilde{y},\tilde{\xi})}{|\partial_{\tilde{\xi}}\psi(\tilde{x},\tilde{\xi}) - \partial_{\tilde{\xi}}\psi(\tilde{y},\tilde{\xi})|^2} \cdot \partial_{\tilde{\xi}} \tag{2.11}$$

and observe that

$$L_{\tau_1\tau_2}(e^{i\tau_1\tau_2(\partial_{\tilde{\xi}}\psi(\tilde{x},\tilde{\xi})-\partial_{\tilde{\xi}}\psi(\tilde{y},\tilde{\xi}))}) = e^{i\tau_1\tau_2(\partial_{\tilde{\xi}}\psi(\tilde{x},\tilde{\xi})-\partial_{\tilde{\xi}}\psi(\tilde{y},\tilde{\xi})}.$$

Integration by parts yields

$$\begin{aligned}
&\int e^{i\tau_1\tau_2(\psi(\tilde{x},\tilde{\xi})-\psi(\tilde{y},\tilde{\xi}))}\chi_0(\tilde{\xi})\chi_1(\tilde{\xi})p(\tau_1\tilde{x},\tau_2\tilde{\xi})\overline{p(\tau_1\tilde{y},\tau_2\tilde{\xi})}d\tilde{\xi} \\
&= \int e^{i\tau_1\tau_2(\psi(\tilde{x},\tilde{\xi})-\psi(\tilde{y},\tilde{\xi}))}(L^*_{\tau_1\tau_2})^5(\chi_0(\tilde{\xi})\chi_1(\tilde{\xi})p(\tau_1\tilde{x},\tau_2\tilde{\xi})\overline{p(\tau_1\tilde{y},\tau_2\tilde{\xi})})d\tilde{\xi},
\end{aligned}$$

where $L^*_{\tau_1\tau_2}$ is the transpose of $L_{\tau_1\tau_2}$. Let

$$B = \sup_{|\alpha|,|\beta|\leq 5} \|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)\|_{L^\infty(\mathbb{R}^3_x\times\mathbb{R}^3_\xi)} < \infty. \tag{2.12}$$

Since $p(x,\xi)$ satisfies (2.5), we have

$$\sup_{|\alpha|,|\beta|\leq 5} \|\partial_{\tilde{x}}^\alpha \partial_{\tilde{\xi}}^\beta p(\tau_1\tilde{x},\tau_2\tilde{\xi})\|_{L^\infty(\mathbb{R}^3_{\tilde{x}}\times\mathbb{R}^3_{\tilde{\xi}})} \leq B. \tag{2.13}$$

From the assumption (2.6) we obtain

$$|\partial_{\tilde{\xi}}\psi(\tilde{x},\tilde{\xi}) - \partial_{\tilde{\xi}}\psi(\tilde{y},\tilde{\xi})| \geq C|\tilde{x}-\tilde{y}| \tag{2.14}$$

(C depends on C_1, see [24] p. 397 for obtaining (2.14) from (2.6)) and

$$|\partial_{\tilde{\xi}}^\beta\psi(\tilde{x},\tilde{\xi}) - \partial_{\tilde{\xi}}^\beta\psi(\tilde{y},\tilde{\xi})| \leq C_\beta|\tilde{x}-\tilde{y}| \tag{2.15}$$

for $1 \leq |\beta| \leq 6$ since phase function ψ is homogeneous of degree 1 in first variable. Hence we have

$$|\mathcal{K}_{(j,l)(k,m)}(\tilde{x},\tilde{y})| \leq C\tau_1^{-5}\tau_2^{-5}B^2\frac{\chi_{j-k}(\tilde{x})\chi_0(\tilde{y})}{1+|\tilde{x}-\tilde{y}|^5} \cdot \int_{\mathbb{R}^3} \chi_0|_{y\times\Gamma_\xi}(\tilde{\xi})\chi_1|_{x\times\Gamma_\xi}(\tilde{\xi})d\tilde{\xi} \tag{2.16}$$

where C depends on the constants of (2.6) and (2.7) and $\chi_0|_{y\times\Gamma_\xi}$, $\chi_1|_{x\times\Gamma_\xi}$ are functions $\chi_0(\xi)$, $\chi_1(\xi)$ restricted to cones Γ_ξ determined by x, y respectively. We note that

$$\int_{\mathbb{R}^3} \chi_0|_{y\times\Gamma_\xi}(\tilde{\xi})\chi_1|_{x\times\Gamma_\xi}(\tilde{\xi})d\tilde{\xi} \tag{2.17}$$

attains its maximum when $x = cy$ for $c > 0$. The maximum of (2.17) can be written as

$$\int_{\mathbb{R}^3 \cap \Gamma_{\tilde{\xi}}} \chi_0(\tilde{\xi})\chi_1(\tilde{\xi})d\tilde{\xi}$$

and note that its value depends on the span of cone Γ_ξ. The case $l = m - 1$ clearly has the same maximum. For $l = m$, the corresponding maximum will be

$$\int_{\mathbb{R}^3 \cap \Gamma_{\tilde{\xi}}} \chi_0(\tilde{\xi})\chi_0(\tilde{\xi})d\tilde{\xi}.$$

By definition of $\Gamma_x \times \Gamma_\xi$ and A we have

$$\max\{\int_{\mathbb{R}^3 \cap \Gamma_{\tilde{\xi}}} \chi_0(\tilde{\xi})\chi_0(\tilde{\xi})d\tilde{\xi}\,, \int_{\mathbb{R}^3 \cap \Gamma_{\tilde{\xi}}} \chi_0(\tilde{\xi})\chi_1(\tilde{\xi})d\tilde{\xi}\} \le A.$$

We also note that (2.17) non-vanishes only when the angle spanned by x, y lies in a suitable range determined by the definition of $\Gamma_x \times \Gamma_\xi$. From above and $\tilde{x} \in \operatorname{supp} \chi_{j-k}$, $\tilde{y} \in \operatorname{supp} \chi_0$, we have

$$\begin{aligned}
\sup_{\tilde{x}} \int |\tau_2^3 \mathcal{K}_{(j,l)(k,m)}(\tilde{x}, \tilde{y})\tau_1^3| d\tilde{y} &\le C2^{-(5j-3k)}2^{-2m}AB^2 \int_{\mathbb{R}^3 \cap \Gamma_{\tilde{y}}} \chi_0(\tilde{y})d\tilde{y} \\
&\le C2^{-2j}2^{-2m}A^2B^2 \\
\sup_{\tilde{y}} \int |\tau_2^3 \mathcal{K}_{(j,l)(k,m)}(\tilde{x}, \tilde{y})\tau_1^3| d\tilde{x} &\le C2^{-(5j-3k)}2^{-2m}AB^2 \int_{\mathbb{R}^3 \cap \Gamma_{\tilde{x}}} \chi_{j-k}(\tilde{x})d\tilde{x} \\
&= C2^{-2j}2^{-2m}AB^2 \int_{\mathbb{R}^3 \cap \Gamma_x} \chi_0(x)dx \\
&= C2^{-2j}2^{-2m}A^2B^2.
\end{aligned} \tag{2.18}$$

Let $H(z) : \mathbb{Z} \to \{0, 1\}$ be defined by $H(z) = 1$ if $|z| \le 1$ and $H(z) = 0$ if $|z| > 1$. For the general (j, l), (k, m), the $2^{-2j}2^{-2m}$ in the right hand side of (2.18) should be replaced by $2^{-2\max\{j,k\}}2^{-2\min\{l,m\}}H(l - m)$. By invoking Schur test lemma (Lemma 4.2), we have

$$\|F_{(j,l)}F^*_{(k,m)}\|_{L^2 \to L^2} \le C2^{-2|j-k|}H(l - m)A^2B^2. \tag{2.19}$$

By the same argument, we have

$$\|F^*_{(j,l)}F_{(k,m)}\|_{L^2 \to L^2} \le C2^{-2|l-m|}H(j - k)A^2B^2. \tag{2.20}$$

Then we have

$$\|F_{(j,l)}F^*_{(k,m)}\|_{L^2 \to L^2}, \|F^*_{(j,l)}F_{(k,m)}\|_{L^2 \to L^2} \le CA^2B^2\{\Theta(j - k, l - m)\}^2,$$

where

$$\Theta(j_1, j_2) = \sqrt{\frac{H(j_2)}{2^{2j_1}} + \frac{H(j_1)}{2^{2j_2}}}$$

Since

$$\sum_{(j_1,j_2)\in\mathbb{N}\cup\{0\}\times\mathbb{N}\cup\{0\}} \Theta(j_1, j_2) < \infty,$$

we conclude the result by Coltar-Stein lemma (Lemma 4.3). □

3. Convergence

We first prove Theorem 1.2 by assuming Theorem 1.1 holds. In fact, the proof of Theorem 1.1 can be done by exactly the same argument for 1.2 by modifying set up slightly which will be explained in the end of this section.

Proof of Theorem 1.2. We need to prove that if $0 < s < 1/2$ and $f \in L^1_{1-2s} \cap H^{1+\rho}_{1-2s}$, $h \in H^{1-\rho}$, $\rho \in [-1, 1]$, then

$$|\langle Q_{d_m}(f,f),h\rangle - \langle Q(f,f),h\rangle| \le O((d_m)^{(1-1/s)})\|f\|_{L^1_{1-2s}}\|f\|_{H^{1+\rho}_{1-2s}}\|h\|_{H^{1-\rho}}. \tag{1.18}$$

And if $1/2 \le s < 1$, $\rho \in [-\frac{1}{2} - s, \frac{1}{2} + s]$ and $f \in L^1 \cap H^{\frac{1}{2}+s+\rho}$, $h \in H^{\frac{1}{2}+s-\rho}$, then

$$|\langle Q_{d_m}(f,f),h\rangle - \langle Q(f,f),h\rangle| \le O((d_m)^{N(s)})\|f\|_{L^1}\|f\|_{H^{\frac{1}{2}+s+\rho}}\|h\|_{H^{\frac{1}{2}+s-\rho}}, \tag{1.19}$$

where $N(s) = (1 - s^\kappa)(1 - 1/s) < 0$ and small $\kappa > 0$ satisfies $2s^\kappa - 2s^{1+\kappa} - s > 0$.

To estimate

$$\begin{aligned}&\langle Q(f,f),h\rangle - \langle Q_{d_m}(f,f),h\rangle\\ &= \iiint ff_*(h' - h)(B(v - v_*,\omega) - B_{d_m}(v - v_*,\omega))d\Omega(\omega)dv_*dv,\end{aligned}$$

we first discuss $B - B_{d_m}$. The result of Theorem 1.1 allows us to drop the hard sphere part of B_{d_m}. Hence it is equivalent to consider the truncated collision operator $Q^{tr}_{d_m}$ with following truncated collision kernel

$$B^{tr}_{d_m} \stackrel{\text{def}}{=} B - (B_{d_m} - 1_{\{c_u\mathcal{V}^{-1}(d_m)^{-1/2s}\ge 1\}}\cdot B^{H_s} - 1_{\{c_u\mathcal{V}^{-1}(d_m)^{-1/2s}<1\}}\cdot B^{H_l}).$$

Hence

$$B^{tr}_{d_m}(\mathcal{V},\theta) = \begin{cases} B^{tr1}_{d_m}(\mathcal{V},\theta) & \text{if } c_u\mathcal{V}^{-1}(d_m)^{-1/2s} < 1\\ B^{tr2}_{d_m}(\mathcal{V},\theta) & \text{if } c_u\mathcal{V}^{-1}(d_m)^{-1/2s} \ge 1\end{cases} \tag{3.1}$$

where

$$\begin{aligned}B^{tr1}_{d_m}(\mathcal{V},\theta) &= \begin{cases}\mathcal{V}^\gamma C(\theta)(\cos\theta)^{-(1+2s)}, & \text{when } \cos\theta \le c_u\mathcal{V}^{-1}(d_m)^{-1/2s}\\ 0, & \text{when } \cos\theta > c_u\mathcal{V}^{-1}(d_m)^{-1/2s}\end{cases}\\ &\stackrel{\text{def}}{=} \mathcal{V}^\gamma b^{tr1}_{d_m}(\cos\theta),\end{aligned} \tag{3.2}$$

with $\gamma = 1 - 4s$ and

$$\begin{aligned}B^{tr2}_{d_m}(\mathcal{V},\theta) &= \mathcal{V}^\gamma(\cos\theta)^{-(1+2s)}, \quad 0 \le \theta < \pi/2\\ &\stackrel{\text{def}}{=} \mathcal{V}^\gamma b^{tr2}_{d_m}(\cos\theta).\end{aligned}$$

By the remark after (1.8) about the coefficient c_{d_m} and Lemma 4.1, we know that the coefficient $C(\theta)$ of $b^{tr1}_{d_m}$ is c_∞ of (1.11) as θ tends to $\pi/2$ and decays to 0 as $\cos\theta$ tends to the critical angle $c_u\mathcal{V}^{-1}(d_m)^{-1/2s}$ from above. Hence we consider $b^{tr1}_{d_m}$ a smooth function of θ and decay to 0 when θ tends to the critical angle. We also assume $c_\infty = 1$ from now on.

Let

$$\mathbf{c}(x) = \sum_{k\le 0}\chi_k(x), \quad \bar{\mathbf{c}}(x) = \sum_{k\ge 1}\chi_k(x), \quad x \in \mathbb{R}^3 \tag{3.3}$$

where χ_k is defined in (2.2). Then $c(x)$ equals to 0 when $|x| > 8$ and similarly $\bar{c}(x)$ equals to 0 when $|x| < 4$. And we can write the kinetic factor as sum of two parts

$$\begin{aligned}|v - v_*|^\gamma &= \mathbf{c}(v - v_*)|v - v_*|^\gamma + (\bar{\mathbf{c}}(v - v_*))|v - v_*|^\gamma\\ &\stackrel{\text{def}}{=} |v - v_*|^\gamma_c + |v - v_*|^\gamma_{\bar{c}}.\end{aligned}$$

We also split the collision kernel $B^{tr}_{d_m} = B^{tr}_{d_m,c} + B^{tr}_{d_m,\bar{c}}$ and the collision operator $Q^{tr}_{d_m} = Q^{tr}_{d_m,c} + Q^{tr}_{d_m,\bar{c}}$ accordingly.

Then the estimates (1.18) and (1.19) are the consequence of the following lemmas for the estimates for large and small relative velocity respectively. □

Lemma 3.1. *Let $L \in \mathbb{R}$ be positive. For $0 < s < 1$, $\rho \in [-L, L]$ we have*

$$\langle Q^{tr}_{d_m,\bar{c}}(f,f),h\rangle \le O((d_m)^{(1-1/s)})\|f\|_{L^1_{(1-2s)+}}\|f\|_{H^{L+\rho}_{(1-2s)+}}\|h\|_{H^{L-\rho}}. \tag{3.4}$$

Lemma 3.2. *For $0 < s < 1/2$, $\rho \in [-1, 1]$ we have*

$$\langle Q^{tr}_{d_m,c}(f,f),h\rangle \le O((d_m)^{(1-1/s)})\|f\|_{L^1}\|f\|_{H^{1+\rho}}\|h\|_{H^{1-\rho}}. \tag{3.5}$$

Lemma 3.3. *For $1/2 \le s < 1$, $\rho \in [-\frac{1}{2} - s, \frac{1}{2} + s]$ we have*

$$\langle Q^{tr}_{d_m,c}(f,f),h\rangle \le O((d_m)^{N(s)})\|f\|_{L^1}\|f\|_{H^{\frac{1}{2}+s+\rho}}\|h\|_{H^{\frac{1}{2}+s-\rho}}, \tag{3.6}$$

where $N(s) = (1 - s^\kappa)(1 - 1/s) < 0$ and small $\kappa > 0$ satisfies $2s^\kappa - 2s^{1+\kappa} - s > 0$.

Proof of Lemma 3.1. For the simplicity of the presentation, we let $L = 1$ in the proof and it is easy to check from the proof that this does lose generality.

I. Hard potential $0 < s < 1/2$.

Without loss of generality, we may assume that d_m is large enough so that $B^{tr}_{d_m} = B^{tr1}_{d_m}$ when $\mathcal{V} > 4$. Let

$$(Q^{tr,w}_{d_m,\bar{c}})^t_g(h)(v) = \int_{\mathbb{R}^3}\int_{\omega\in S^2} g(v_*)\langle v_*\rangle^{1-2s}(h(v') - h(v))B^{tr1,w}_{d_m,\bar{c}}(v - v_*\omega)d\Omega(\omega)dv_*\,,$$

where weighted kernel

$$B^{tr1,w}_{d_m,\bar{c}}(v - v_*,\omega) = \frac{|v - v_*|^\gamma_{\bar{c}}}{\langle v\rangle^{1-2s}\langle v_*\rangle^{1-2s}} b^{tr1}_{d_m}(\cos\theta).$$

Note that $\langle D_v\rangle^{1+\rho}$, $\rho \in [-1, 1]$ is a self adjoint operator acting on L^2. By weak form of collision operator and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&|\langle Q^{tr}_{d_m,\bar{c}}(g,f)(v),h(v)\rangle_{L^2}|\\
&= |\langle f(v)\langle v\rangle^{1-2s},(Q^{tr,w}_{d_m,\bar{c}})^t_g(h)(v)\rangle_{L^2}|\\
&= |\langle\langle D_v\rangle^{1+\rho}(f(v)\langle v\rangle^{1-2s}),\langle D_v\rangle^{-(1+\rho)}(Q^{tr,w}_{d_m,\bar{c}})^t_g(h)(v)\rangle_{L^2}|\\
&\le \|f\|_{H^{1+\rho}_{1-2s}}\left\|\langle D_v\rangle^{-(1+\rho)}(Q^{tr,w}_{d_m,\bar{c}})^t_g(h)(v)\right\|_{L^2}
\end{aligned}$$

and it suffices to prove

$$\left\|\langle D_v\rangle^{-(1+\rho)}(Q^{tr,w}_{d_m,\bar{c}})^t_g(h)(v)\right\|_{L^2} \le O((d_m)^{(1-1/s)})\,\|g\|_{L^1_{1-2s}}\,\|h\|_{H^{1-\rho}}\,.$$

Define the translation operator $\tau_m h(\cdot) = h(m + \cdot)$ and note $\gamma = 1 - 4s$. We rewrite

$$\begin{aligned}
&(Q^{tr,w}_{d_m,\bar{c}})^t_g(h)(v)\\
&= \int_{\mathbb{R}^3} g(v_*)\langle v_*\rangle^{1-2s}\frac{|v - v_*|^{1-2s}_{\bar{c}}}{\langle v\rangle^{1-2s}\langle v_*\rangle^{1-2s}}[\tau_{-v_*}\circ T^{tr}_{d_m}\circ\tau_{v_*}]h(v)dv_*
\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}
&T^{tr}_{d_m}\,h(x)\\
&= \int_{\omega\in S^2}|x|^{-2s}b^{tr1}_{d_m}(\cos\theta)(h(x - (x\cdot\omega)\omega) - h(x))d\Omega(\omega),
\end{aligned} \tag{3.8}$$

with $\cos\theta = (x\cdot\omega)/|x|$ and

$$b^{tr1}_{d_m}(\cos\theta) = \begin{cases} C(\theta)(\cos\theta)^{-(1+2s)}, & \text{if } (\frac{\pi}{2}-\theta) \le c_u|x|^{-1}(d_m)^{-1/2s}, \\ 0, & \text{others,} \end{cases}$$

where $C(\theta)$ is 1 when θ is near $\pi/2$ and decays to 0 when θ tends to critical angle $(\frac{\pi}{2}-\theta) = c_u|x|^{-1}(d_m)^{-1/2s}$. For the simplicity, we often write

$$b^{tr1}_{d_m}(\cos\theta) = \begin{cases} (\cos\theta)^{-(1+2s)}, & \text{if, } (\frac{\pi}{2}-\theta) \le c_u|x|^{-1}(d_m)^{-1/2s} \\ 0, & \text{others,} \end{cases} \tag{3.9}$$

but keep in mind that $b^{tr1}_{d_m}$ will not produce any boundary term when applying integration by parts to the integration contains $b^{tr1}_{d_m}$.

Note that $|x|^{-1}(d_m)^{-1/2s}$ can be arbitrary small by selecting large d_m since we have $|x|>4$. Thus for the simplicity of the representation, we use $(\frac{\pi}{2}-\theta) \le c_u|x|^{-1}(d_m)^{-1/2s}$ in (3.9) to replace the condition $\cos\theta \le c_u|x|^{-1}(d_m)^{-1/2s}$ in (3.2). Using formula (3.7) and applying Minkowski inequality, we have

$$\begin{aligned}&\left\| \langle D_v\rangle^{-(1+\rho)}(Q^{tr,w}_{d_m,\bar{c}})^t_g(h)(v)\right\|_{L^2} \\ &\le \int_{\mathbb{R}^3} |g(v_*)|\langle v_*\rangle^{1-2s} \left\| \langle D_v\rangle^{-(1+\rho)}\Big(\frac{|v-v_*|^{1-2s}_{\bar{c}}}{\langle v\rangle^{1-2s}\langle v_*\rangle^{1-2s}}[\tau_{-v_*}\circ T^{tr}_{d_m}\circ\tau_{v_*}]h(v)\Big)\right\|_{L^2} dv_*.\end{aligned}$$

We are reduced to showing that for $\rho\in[-1,1]$

$$\begin{aligned}&\sup_{v_*}\left\| \langle D_v\rangle^{-(1+\rho)}(\frac{|v-v_*|^{1-2s}_{\bar{c}}}{\langle v\rangle^{1-2s}\langle v_*\rangle^{1-2s}}[\tau_{-v_*}\circ T^{tr}_{d_m}\circ\tau_{v_*}]h(v))\right\|_{L^2} \\ &\le O((d_m)^{(1-1/s)})\|h\|_{H^{1-\rho}}.\end{aligned} \tag{3.10}$$

Note the variable v_* is a parameter in above estimate, $H^{1-\rho}$ is translation invariant, and the fact that the term

$$\frac{|v-v_*|^{1-2s}_{\bar{c}}}{\langle v\rangle^{1-2s}\langle v_*\rangle^{1-2s}}$$

itself and its derivatives with respect to v are uniformly bounded. Using these facts, it is easy to check that (3.10) follows from the proof of the estimate

$$\left\| \langle D_x\rangle^{-(1+\rho)} T^{tr}_{d_m,\bar{c}}\, h(x)\right\|_{L^2} \le O((d_m)^{(1-1/s)})\|h\|_{H^{1-\rho}} \tag{3.11}$$

by replacing x with $v-v_*$ where $T^{tr}_{d_m,\bar{c}}$ is modification of (3.8), i.e.,

$$T^{tr}_{d_m,\bar{c}}\, h(x) = \int_{\omega\in S^2} \bar{\mathbf{c}}(x)|x|^{-2s} b^{tr1}_{d_m}(\cos\theta)(h(x-(x\cdot\omega)\omega)-h(x))d\Omega(\omega).$$

The proof of above estimate is given in Lemma 3.4 below.

Before proceeding to the proof of (3.11) for $0<s<1/2$, we consider the reduction of (3.4) for the soft potential case $1/2\le s<1$.

II. Soft potential $1/2\le s<1$.

We can reduce the proof of the estimate (3.4) for $1/2\le s<1$ by following the almost the same process for $0<s<1/2$. Define the same $T^{tr}_{d_m,\bar{c}}$ but replace the weight $|v-v_*|^{1-2s}_{\bar{c}}\langle v\rangle^{-(1-2s)}\langle v_*\rangle^{-(1-2s)}, 0<s<1/2$ with $|v-v_*|^{1-2s}_{\bar{c}}, 1/2\le s<1$. Since $|v-v_*|^{1-2s}_{\bar{c}}, 1/2\le s<1$ itself and its derivatives with respect to v are uniformly bounded, we see that the proof of the soft potential case is equivalent to (3.11) with $1/2\le s<1$. We will prove (3.11) for $1/2\le s<1$ in Lemma 3.4. □

Lemma 3.4. *The estimate (3.11) holds for $0<s<1$.*

Proof. For our purpose, we use inverse Fourier transform to rewrite

$$
\begin{aligned}
&T^{tr}_{d_m,\bar{c}}\, h(x) \\
&= \int\limits_{\omega\in S^2} |x|_{\bar{c}}^{-2s} b^{tr1}_{d_m}(\cos\theta)(h(x-(x\cdot\omega)\omega)-h(x))d\Omega(\omega) \\
&= \frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3}\int\limits_{S^2} |x|_{\bar{c}}^{-2s} b^{tr1}_{d_m}(\cos\theta)e^{ix\cdot\xi}(e^{-i(x\cdot\omega)(\xi\cdot\omega)}-1)\widehat{h}(\xi)d\Omega(\omega)d\xi \\
&= \frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3} e^{ix\cdot\xi}\, a^{tr}_{d_m,\bar{c}}(x,\xi)|\xi|^{2s}\widehat{h}(\xi)d\xi
\end{aligned}
$$

where

$$
\begin{aligned}
a^{tr}_{d_m,\bar{c}}(x,\xi) &= \bar{\mathbf{c}}(x)\, a^{tr}_{d_m}(x,\xi) \\
&= \bar{\mathbf{c}}(x)|x|^{-2s}|\xi|^{-2s}\int\limits_{S^2} b^{tr1}_{d_m}(\cos\theta)(e^{-i(x\cdot\omega)(\xi\cdot\omega)}-1)d\Omega(\omega).
\end{aligned}
\tag{3.12}
$$

We combine $|\xi|^{2s}$ with $\widehat{h}(\xi)$ so that the definition of $a^{tr}_{d_m,\bar{c}}$ is homogeneous in $|x||\xi|$. Before the analysis of $a^{tr}_{d_m,\bar{c}}$ we need a dyadic partition of unity on $\{(x,\xi)|x,\xi\in\mathbb{R}^3-\{0\}\}$. Using (2.2), we have the dyadic partition of unity by

$$
\begin{aligned}
1 &= \sum_{j\in\mathbb{Z}}\chi_j(x)\sum_{k\in\mathbb{Z}}\chi_k(\xi),\ x\neq 0,\ \xi\neq 0 \\
&= \chi_A(x,\xi)+\chi_B(x,\xi)+\chi_C(x,\xi)
\end{aligned}
$$

where

$$
\begin{aligned}
&\chi_A(x,\xi) = \sum_{j\in\mathbb{N}}\chi_j(x)\sum_{k\in\mathbb{N}}\chi_k(\xi), \\
&\chi_B(x,\xi) = \chi_{B,1}(x,\xi)+\chi_{B,2}(x,\xi) \\
&= \sum_{j+k\geq 2,\, j\geq 1,\, k\leq 0}\chi_j(x)\chi_k(\xi)+\sum_{j+k\geq 2,\, j\leq 0}\chi_j(x)\chi_k(\xi), \\
&\chi_C(x,\xi) = \chi_{C,1}(x,\xi)+\chi_{C,2}(x,\xi) \\
&= \sum_{j+k\leq 1,\, j\geq 1}\chi_j(x)\chi_k(\xi)+\sum_{j+k\leq 1,\, j\leq 0}\chi_j(x)\chi_k(\xi).
\end{aligned}
\tag{3.13}
$$

Hence $\operatorname{supp}\chi_A(x,\xi)\subset\{|x|>4,|\xi|>4\}$, $\operatorname{supp}\chi_{B,1}(x,\xi)\subset\{|x||\xi|>16,|x|>4,|\xi|<8\}$, $\operatorname{supp}\chi_{B,2}(x,\xi)\subset\{|x||\xi|>16,|x|<8\}$, $\operatorname{supp}\chi_{C,1}(x,\xi)\subset\{|x||\xi|<128,|x|>4\}$ and $\operatorname{supp}\chi_{C,2}(x,\xi)\subset\{|x||\xi|<128,|x|<8\}$. Using definition of $\bar{c}$, (3.3), we can further decompose it as

$$
\begin{aligned}
a^{tr}_{d_m,\bar{c}} &= a_A+a_B+a_C \\
&\overset{\text{def}}{=} \chi_A\, a^{tr}_{d_m}+\chi_{B,1}\, a^{tr}_{d_m}+\chi_{C,1}\, a^{tr}_{d_m}
\end{aligned}
\tag{3.14}
$$

where $a^{tr}_{d_m}$ is given in (3.12). We also define operator $T^{tr}_{d_m,\bar{c}}=\frac{1}{(2\pi)^3}(T_A+T_B+T_C)$ accordingly. It is reduced to proving that these three operators all satisfy (3.11).

Part 1. Estimate of T_A

The estimate

$$
\left\|\langle D_x\rangle^{-(1+\rho)}T_A\, h(x)\right\|_{L^2}\leq O((d_m)^{(1-1/s)})\|h\|_{H^{1-\rho}}
$$

is equivalent to

$$\left\| \langle D_x \rangle^{-(1+\rho)} T_A^{\rho-1}\, h(x) \right\|_{L^2} \le O((d_m)^{(1-1/s)}) \|h\|_{L^2} \tag{3.15}$$

where

$$T_A^{\rho-1}\, h(x) = \int e^{ix\cdot\xi} a_A(x,\xi)|\xi|^{2s} \langle\xi\rangle^{\rho-1}\, \widehat{h}(\xi) d\xi.$$

We note that $\operatorname{supp} a_A(x,\xi) \subset \{|x| > 4, |\xi| > 4\}$. Fix a small positive number ε which is less than 1 and regard $|x||\xi|$ as a parameter. Let $k(x,\xi) \in \mathbb{N} \cup \{0\}$ be the integer such that $2^k \le |x||\xi| < 2^{k+1}$. Let $\mu(t)$ be a non-negative smooth function defined on $t \ge 0$ with $\mu(t) = 1$ if $t \le \varepsilon$ and $\mu(t) = 0$ if $t \ge 2\varepsilon$. We define

$$\eta(t) = \mu(\frac{|x||\xi|}{2^k} t) - \mu(2\frac{|x||\xi|}{2^k} t),$$

then $\operatorname{supp} \eta \subset (\frac{2^k}{|x||\xi|}\frac{\varepsilon}{2}, \frac{2^k}{|x||\xi|} 2\varepsilon) \subset (\frac{\varepsilon}{4}, 2\varepsilon)$. Let $\eta_j(t) = \eta(2^j t),\ j \in \mathbb{Z}$, so that $1 = \sum_{j\in\mathbb{Z}} \eta_j(t),$ for $t > 0$, is a dyadic partition of unity on $(0,\infty)$. To define a dyadic partition of unity on $[0,\pi/2)$, we discard those η_j with index $j < 0$ and redefine η_0 on $(0,\pi/2]$ by letting $\eta_0 = 1 - \sum_{j>0} \eta_j$, then η_0 is a non-negative smooth function. Therefore $\sum_{j\ge 0} \eta_j$ gives a partition of unity on $(0,\pi/2]$. Finally, we define

$$\begin{aligned} &\beta_{cut}(\theta) + \beta_{can}(\theta) \\ &\stackrel{\text{def}}{=} \sum_{0\le j\le k} \eta_j(\frac{\pi}{2} - \theta) + \sum_{j\ge k+1} \eta_j(\frac{\pi}{2} - \theta) = 1, \end{aligned} \tag{3.16}$$

which is the dyadic partition of unity on $[0,\pi/2)$ as desired. The partition gives

$$\operatorname{supp} \beta_{cut} \subset [\,0, \frac{\pi}{2} - \frac{1}{2}\frac{\varepsilon}{|x||\xi|})\,,\ \operatorname{supp} \beta_{can} \subset (\frac{\pi}{2} - \frac{\varepsilon}{|x||\xi|}, \frac{\pi}{2}). \tag{3.17}$$

For any fixed x and ξ, we split $a_A(x,\xi)$ into two parts as

$$\begin{aligned} a_A(x,\xi) &= \beta_{cut}(\theta) a_A(x,\xi) + \beta_{can}(\theta) a_A(x,\xi) \\ &\stackrel{\text{def}}{=} a_{cut}(x,\xi) + a_{can}(x,\xi) \end{aligned}$$

according to (3.16). We also define the operators corresponding to a_{can}, a_{cut} as

$$\begin{aligned} &T_A^{\rho-1}\, h(x) \\ &= T_{\text{can}} h(x) + T_{\text{cut}} h(x) \\ &= \int_{\mathbb{R}^3} e^{ix\cdot\xi} a_{can}(x,\xi)|\xi|^{2s} \langle\xi\rangle^{\rho-1} \widehat{h}(\xi) d\xi + \int_{\mathbb{R}^3} e^{ix\cdot\xi} a_{cut}(x,\xi)|\xi|^{2s} \langle\xi\rangle^{\rho-1} \widehat{h}(\xi) d\xi. \end{aligned}$$

Next we need to decompose ξ space into two parts smoothly according to d_m. For the simplicity of notations, we just use (3.18) and (3.24) below to separate two different case.

Case 1. If

$$\varepsilon^{-1}|x||\xi| \le c_u^{-1}|x|(d_m)^{1/2s} \tag{3.18}$$

holds, $[\pi/2 - c_u|x|^{-1}(d_m)^{-1/2s}, \pi/2] \subset [\pi/2 - \varepsilon|x|^{-1}|\xi|^{-1}, \pi/2]$ and $a_{cut} = 0$. We use the parametrization

$$\begin{cases} x = |x|(0,0,1) \\ \xi = |\xi|(\cos\varphi_0 \sin\theta_0, \sin\varphi_0 \sin\theta_0, \cos\theta_0) \\ \omega = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)\,,\ \theta \in [0,\pi/2], \varphi \in [0,2\pi]. \end{cases} \tag{3.19}$$

Let $\omega_d = (\cos(\varphi+\pi)\sin\theta, \sin(\varphi+\pi)\sin\theta, \cos\theta)$. From the support property of θ, (3.19) and Taylor expansion, we have

$$\begin{aligned}
&e^{-i(x\cdot\omega)(\xi\cdot\omega)}-1+e^{-i(x\cdot\omega_d)(\xi\cdot\omega_d)}-1\\
&=2\{e^{-i|x||\xi|\cos\theta_0\cos^2\theta}\cos(|x||\xi|\cos(\varphi_0-\varphi)\sin\theta_0\sin\theta\cos\theta)-1\}\\
&=-2\{i|x||\xi|\cos\theta_0\cos^2\theta+|x|^2|\xi|^2\sin^2\theta_0\cos^2(\varphi_0-\varphi)\sin^2\theta\cos^2\theta\}\\
&\quad+\text{remainder terms.}
\end{aligned}\tag{3.20}$$

Using $\sin^2\theta=1-\cos^2\theta$, we see that the remainder term is a series in terms of $i|x||\xi|\cos^2\theta$, $|x|^2|\xi|^2\cos^2\theta$ and their products. The series converges for any $\theta\in[\pi/2-c_u|x|^{-1}(d_m)^{-1/2s},\pi/2]$ if $\varepsilon<1$. The contribution from $|x|^2|\xi|^2\sin^2\theta_0\cos^2(\varphi_0-\varphi)\sin^2\theta\cos^2\theta$ to the $a_{can}(x,\xi)$ is the main term which gives

$$\begin{aligned}
&C\int_{\pi/2-\varepsilon/|x||\xi|}^{\pi/2} b_{d_m}^{tr1}(\cos\theta)|x|^{2-2s}|\xi|^{2-2s}\sin^2\theta_0\cos^2\theta\sin\theta d\theta\\
&=C(\cos\theta_0)\int_{\pi/2-c_u|x|^{-1}(d_m)^{-1/2s}}^{\pi/2}|x|^{2-2s}|\xi|^{2-2s}\cos^{1-2s}\theta\sin\theta d\theta\\
&=C(\cos\theta_0)(d_m)^{(1-1/s)}|\xi|^{2-2s}\ ,\ 0<s<1.
\end{aligned}\tag{3.21}$$

The integration of other terms are of lower order and form a convergent series. Hence we can conclude that for $0<s<1$

$$a_{can}(x,\xi)|\xi|^{2s}\langle\xi\rangle^{\rho-1}=C(\cos\theta_0)(d_m)^{(1-1/s)}|\xi|^2\langle\xi\rangle^{\rho-1}$$

where the bounded coefficient $C(\cos\theta_0)$ depending on $\varepsilon,s,\cos\theta_0$ and bump function μ is a symbol of order 0 (see (3.44) for the derivative of $\cos\theta_0$). By the same C above, we define

$$p^{1+\rho}(x,\xi)=C|\xi|^2\langle\xi\rangle^{\rho-1}.$$

Therefore $T_A=T_{can}$ is a pseudodifferential operator of order $2+\rho-1=1+\rho$ with symbol $O((d_m)^{(1-1/s)})\times p^{1+\rho}(x,\xi)$. From the result of composition of pseudo-differential operators, see for example [16,21] for classical result or [20] for composition of more general symbols,

$$\begin{aligned}
&\langle D_x\rangle^{-(1+\rho)}T_{can}h(x)\\
&=O((d_m)^{(1-1/s)})\langle D_x\rangle^{-(1+\rho)}\Big(\int e^{ix\cdot\xi}p(x,\xi)\widehat{h}(\xi)d\xi\Big)\\
&=O((d_m)^{(1-1/s)})\iint\int e^{i(x-y)\cdot\eta}\langle\eta\rangle^{-(1+\rho)}e^{iy\cdot\xi}p(y,\xi)\widehat{h}(\xi)d\xi dyd\eta\\
&=O((d_m)^{(1-1/s)})\int e^{ix\cdot\xi}q(x,\xi)\,\widehat{h}(\xi)d\xi
\end{aligned}$$

where

$$q(x,\xi)=\iint e^{i(x-y)(\eta-\xi)}\langle\eta\rangle^{-(1+\rho)}p(y,\xi)dyd\eta\in S^0_{1,0}\tag{3.22}$$

has asymptotic expansion

$$q(x,\xi)\sim\sum_{\alpha\in\mathbb{N}_0^3}\frac{i^{-|\alpha|}}{\alpha!}\partial_\xi^\alpha\langle\xi\rangle^{-(1+\rho)}\partial_x^\alpha p(x,\xi)\tag{3.23}$$

in the sense that

$$q(x,\xi)-\sum_{|\alpha|<N}\frac{i^{-|\alpha|}}{\alpha!}\partial_\xi^\alpha\langle\xi\rangle^{-(s+\rho)}\partial_x^\alpha p(x,\xi)\in S^{-N}_{1,0}.$$

Hence $\langle D_x\rangle^{-(1+\rho)}T_{can}$ satisfies the desired estimate by the theory of pseudodifferential operators. Indeed we can also get the desire estimate by using Lemma 2.1 since the symbol $q(x,\xi)$ actually satisfies (2.5).

Case 2. If

$$\varepsilon^{-1}|x||\xi| > c_u^{-1}|x|(d_m)^{1/2s} \tag{3.24}$$

holds, then the interval $[\pi/2 - c_u|x|^{-1}(d_m)^{-1/2s}, \pi/2]$ equals

$$[\pi/2 - c_u|x|^{-1}(d_m)^{-1/2s}, \pi/2 - \varepsilon|x|^{-1}|\xi|^{-1}] \cup [\pi/2 - \varepsilon|x|^{-1}|\xi|^{-1}, \pi/2].$$

Hence $a_{can}(x,\xi)$ is

$$C \int\limits_{\pi/2-\varepsilon|x|^{-1}|\xi|^{-1}}^{\pi/2} |x|^{2-2s}|\xi|^{2-2s}(\cos\theta)^{(1-2s)}\sin\theta d\theta$$
$$= C(\cos\theta_0)$$

and T_{can} a pseudo differential operator of order $0 + 2s + \rho - 1$ whose composition with operator $\langle D_v\rangle^{-(1+\rho)}$ has the principle symbol

$$C\langle\xi\rangle^{-2}|\xi|^{2s}.$$

Using $(d_m)^{1/2s} \le C|\xi|$ we have

$$\langle\xi\rangle^{-2}|\xi|^{2s} \le C(d_m)^{1-1/s} \tag{3.25}$$

and conclude that T_{can} satisfies (3.15). Next we consider T_{cut} and a_{cut}. Note the variable θ is less than $\pi/2 - \varepsilon|x|^{-1}|\xi|^{-1}$. Thus we may separate $a_{cut}(x,\xi)$ into gain and loss terms,

$$\begin{aligned} a_{cut}(x,\xi) &= \chi_A(x,\xi)a_g(x,\xi) - \chi_A(x,\xi)a_l(x,\xi) \\ &= \chi_A(x,\xi)\int\limits_{\omega\in S^2} |x|^{-2s}|\xi|^{-2s}e^{-i(x\cdot\omega)(\xi\cdot\omega)}\beta_{cut}(\theta)b_{d_m}^{tr1}(\cos\theta)d\Omega(\omega) \\ &\quad - \chi_A(x,\xi)\int\limits_{\omega\in S^2} |x|^{-2s}|\xi|^{-2s}(1-\beta_{can}(\theta))b_{d_m}^{tr1}(\cos\theta)d\Omega(\omega). \end{aligned}$$

According to this decomposition, we write T_{cut} as the difference of gain and loss operators, i.e.,

$$\begin{aligned} T_{cut}h(x) &= T_g h(x) - T_l h(x) \\ &= \int e^{ix\cdot\xi}\chi_A(x,\xi)a_g(x,\xi)|\xi|^{2s}\langle\xi\rangle^{\rho-1}\widehat{h}(\xi)d\xi \\ &\quad - \int e^{ix\cdot\xi}\chi_A(x,\xi)a_l(x,\xi)|\xi|^{2s}\langle\xi\rangle^{\rho-1}\widehat{h}(\xi)d\xi \end{aligned}$$

Direct calculation gives that

$$a_l(x,\xi)|\xi|^{2s}\langle\xi\rangle^{\rho-1} = C(|\xi|^{2s}\langle\xi\rangle^{\rho-1} - d_m\langle\xi\rangle^{\rho-1}) \tag{3.26}$$

where C depends on s, ε, $\cos\theta_0$ and bump μ. Then the operator $\langle D_x\rangle^{-(1+\rho)}T_l$ is a pseudo-differential operator with symbol $C(|\xi|^{2s}\langle\xi\rangle^{-2} - d_m\langle\xi\rangle^{-2})$ which is bounded by $C(d_m)^{1-1/s}$, thus $\langle D_x\rangle^{-(1+\rho)}T_l$ satisfies (3.15) again as T_{can}.

We note that, by (3.25), it suffices to show that T_g satisfies

$$\left\|\langle D_x\rangle^{-(1+\rho)}T_g h(x)\right\|_{L^2} \le C\|\langle\xi\rangle^{-2}|\xi|^{2s}\widehat{h}(\xi)\|_{L^2}.$$

The above estimate is equivalent to

$$\left\|\langle D_x\rangle^{-(1+\rho)}T_{g,\langle 1+\rho\rangle}h(x)\right\|_{L^2} \le C\|h\|_{L^2}, \tag{3.27}$$

where

$$T_{g,\langle 1+\rho\rangle}h(x) = \int e^{ix\cdot\xi}\chi_A(x,\xi)a_g(x,\xi)\langle\xi\rangle^{1+\rho}\widehat{h}(\xi)d\xi, \tag{3.28}$$

$$a_g(x,\xi) = \int_{\omega\in S^2} |x|^{-2s}|\xi|^{-2s} e^{-i(x\cdot\omega)(\xi\cdot\omega)} \beta_{cut}(\theta) b^{tr1}_{d_m}(\cos\theta) d\Omega(\omega) \tag{3.29}$$

with

$$\operatorname{supp} \beta_{cut}(\theta) b^{tr1}_{d_m}(\cos\theta) \subset [\pi/2 - c_u|x|^{-1}(d_m)^{-1/2s}, \pi/2 - \varepsilon|x|^{-1}|\xi|^{-1}] \tag{3.30}$$

by (3.9), (3.17) and (3.24).

For the analysis of $a_g(x,\xi)$, we need a partition of unity on (x,ξ) space which relies on a dyadic decomposition in the interval $(0,\pi)$ constructed below. Let $\zeta(\theta) \in C^\infty$ be supported in the interval $(\pi/8, \pi/2)$ and satisfy $\sum_{n\in\mathbb{Z}} \zeta(2^{-n}\theta) = 1$ for all $\theta > 0$. Let $\zeta_n(\theta) = \zeta(2^n\theta)$, if $n > 0$ and $\tilde{\zeta}(\theta) = 1 - \sum_{n>0}\zeta_n(\theta)$. Define $\zeta_0(\theta)$ equals $\tilde{\zeta}(\theta)$, if $0 < \theta \le \pi/2$ and $\zeta_0(\theta) = \zeta_0(\pi - \theta)$, if $\pi/2 < \theta < \pi$. This extension of ζ_0 from $0 < \theta \le \pi/2$ to $0 < \theta < \pi$ by reflection keeps ζ_0 a smooth function since $\tilde{\zeta}$ equals 1 near $\pi/2$. We also define $\zeta_{-n}(\theta) = \zeta_n(\pi - \theta)$ for $n \in \mathbb{N}$. Then we have the dyadic decomposition $1 = \zeta_0(\theta) + \sum_{n\in\mathbb{N}}(\zeta_n(\theta) + \zeta_{-n}(\theta))$ in the interval $\theta \in (0,\pi)$. Abuse the notations, we define

$$\zeta_0(x,\xi) = \zeta_0(\arccos(\frac{x\cdot\xi}{|x||\xi|}))\ ,\ \zeta_{\pm n}(x,\xi) = \zeta_{\pm n}(\arccos(\frac{x\cdot\xi}{|x||\xi|}))\ ,\ n\in\mathbb{N},$$

and note that the supports of $\zeta_0(x,\xi)$, $\zeta_n(x,\xi)$, $\zeta_{-n}(x,\xi)$ lie respectively in the cones

$$\begin{aligned} \Gamma_0 &= \Big\{(x,\xi)|\ \frac{\pi}{8} < \arccos(\frac{x\cdot\xi}{|x||\xi|}) < \pi - \frac{\pi}{8}\Big\}, \\ \Gamma_n &= \Big\{(x,\xi)|\ \frac{\pi}{2^{n+3}} < \arccos(\frac{x\cdot\xi}{|x||\xi|}) < \frac{\pi}{2^{n+1}}\Big\} \\ \Gamma_{-n} &= \Big\{(x,\xi)|\ \pi(1-\frac{1}{2^{n+1}}) < \arccos(\frac{x\cdot\xi}{|x||\xi|}) < \pi(1-\frac{1}{2^{n+3}})\Big\}. \end{aligned} \tag{3.31}$$

Then we have decomposition of $a_g(x,\xi)$ as

$$a_g(x,\xi) = \sum_{j\in\mathbb{Z}} a_j(x,\xi) \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} \zeta_j(x,\xi) a_g(x,\xi).$$

Next we furthermore decompose each $a_j(x,\xi)$, $j \ne 0$ into three part by selecting suitable disjoint subgroups of $(j,k) \in \mathbb{N}\times\mathbb{N}$ in definition of $\chi_A(x,\xi)$ in (3.13) so that each have support lie respectively in

$$\begin{aligned} \Gamma_{\pm n,I} &= \{(x,\xi) \in \Gamma_{\pm n}, |x| > 4\cdot 2^n, |\xi| > 4\cdot 2^n\} \\ \Gamma_{\pm n,II} &= \{(x,\xi) \in \Gamma_{\pm n}, |x||\xi| > 4^2\cdot 2^{2n}, |\xi| < 2\cdot 4\cdot 2^n\} \\ &\quad \cup \{(x,\xi) \in \Gamma_{\pm n}, |x||\xi| > 4^2\cdot 2^{2n}, |x| < 2\cdot 4\cdot 2^n\} \\ \Gamma_{\pm n,III} &= \{(x,\xi) \in \Gamma_{\pm n}, |x||\xi| < 8\cdot 4^2\cdot 2^{2n}\}. \end{aligned}$$

Those functions are denoted by $\chi_{\pm n,I}$, $\chi_{\pm n,II}$ and $\chi_{\pm n,III}$ respectively. We have the decomposition of $a_j(x,\xi)$, $j \ne 0$ as

$$\begin{aligned} a_j(x,\xi) &= \chi_{j,I}(x,\xi)a_j(x,\xi) + \chi_{j,II}(x,\xi)a_j(x,\xi) + \chi_{j,III}(x,\xi)a_j(x,\xi) \\ &\stackrel{\text{def}}{=} a_{j,I}(x,\xi) + a_{j,II}(x,\xi) + a_{j,III}(x,\xi) \end{aligned}$$

and decomposition of $T_{g,\langle 1+\rho\rangle}$ as

$$T_{g,\langle 1+\rho\rangle}\, h(x) = T_0 + \sum_{j\ne 0}(T_{j,I} + T_{j,II} + T_{j,III})h(x).$$

Each operator $T_{j,I}$ is a Fourier integral operator or a pseudo-differential operator whose discrimination will be stated later, $T_{j,II}$ behaves like $T_{j,I}$ after change of variables and $T_{j,III}$ is a pseudo-differential operator. We call $\cup_n\Gamma_{\pm n,I}$ the region I, $\cup_n\Gamma_{\pm n,II}$ the region II and $\cup_n\Gamma_{\pm n,III}$ the region III. Let the angle spanned by x and ξ be θ_0, we have $|x||\xi|\cos^2(\theta_0/2) > C_1 > 1$ on region I and II and $|x||\xi|\cos^2(\theta_0/2) < C_2$ on region III for fixed constants C_1, C_2. The different kinds of behavior of T_g is due to that of $a_g(x,\xi)$. On region I or II, the phase function of $a_g(x,\xi)$ has strong oscillation to disperse the amplitude function except near the critical points. On region III the oscillation is too weak to disperse the amplitude function.

If $|x|, |\xi|$ are large enough and the critical point of phase function of $a_g(x,\xi)$ lies in (3.30), we will calculate $a_g(x,\xi)$ by using the stationary phase formula (see for example [13] or Theorem 4.4 recorded below). The calculation here is similar to that in [17,15]. By (3.19), the phase function in definition of $a_g(x,\xi)$ can be written as

$$(x\cdot\omega)(\xi\cdot\omega) = |x||\xi|\cos\theta(\cos(\varphi-\varphi_0)\sin\theta_0\sin\theta + \cos\theta_0\cos\theta).$$

We regard $|x||\xi|$ as the parameter and define

$$\sigma(x,\xi;\omega) = \cos\theta(\cos(\varphi-\varphi_0)\sin\theta_0\sin\theta + \cos\theta_0\cos\theta) \tag{3.32}$$

be the phase function of integral (3.29) which is obvious smooth over S^2. In order to use the stationary phase formula, we need to find the critical points of the phase function $\sigma(x,\xi;\omega)$ and its Hessians over the critical points. It is known that we can calculate these quantities on the local coordinate or equivalently on manifold by using covariant derivatives (see, for example, [4]). We will use the later approach here. The parametrization (3.19) defines a mapping from (θ,φ) to sphere. Then $\{e_\theta = \frac{\partial}{\partial\theta}, e_\varphi = \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\}$ is a basis on tangent bundle of sphere. Recall that the gradient and Hessian are given by the formulas,

$$\nabla_{S^2} := (\frac{\partial}{\partial\theta}, \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}) \tag{3.33}$$

and

$$H_{S^2} := \begin{bmatrix} \frac{\partial^2}{\partial\theta^2} & \frac{1}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\varphi} - \frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\varphi} \\ \frac{1}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\varphi} - \frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\varphi} & \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} \end{bmatrix}. \tag{3.34}$$

Applying (3.33) to (3.32), we see that the critical points of phase function ψ satisfy the formula

$$(x\cdot\omega)\xi + (\xi\cdot\omega)x = 2(x\cdot\omega)(\xi\cdot\omega)\omega\,,\ \omega\in S^2. \tag{3.35}$$

Note $\beta_{cut}(\theta)\, b^{tr1}_{d_m}(\cos\theta)\sin\theta$ vanishes when $x\cdot\omega = 0$, thus we only have to look for critical points ω such that $x\cdot\omega \neq 0$. By relation (3.35), we have $\xi\cdot\omega\neq 0$. Hence ω belongs to the plane generated by x and ξ. A simple calculation yields four critical points $\omega_+, \omega_-, -\omega_+, -\omega_-$ where

$$\begin{cases} \omega_+ = (x|\xi| + \xi|x|)|x|\xi| + \xi|x||^{-1} \\ \omega_- = (x|\xi| - \xi|x|)|x|\xi| - \xi|x||^{-1} \end{cases} \tag{3.36}$$

if x and ξ are not colinear. We also express them as

$$\begin{aligned} \omega_+ &= (\cos\varphi_0\sin(\theta_0/2), \sin\varphi_0\sin(\theta_0/2), \cos(\theta_0/2)), \\ \omega_- &= (\cos(\pi+\varphi_0)\sin((\pi-\theta_0)/2), \\ &\qquad \sin(\pi+\varphi_0)\sin((\pi-\theta_0)/2), \cos((\pi-\theta_0)/2)). \end{aligned} \tag{3.37}$$

If x and ξ are collinear, the four critical points are $\pm\frac{x}{|x|} = \pm\frac{\xi}{|\xi|}$ and $\pm\omega^\perp$ where $\omega^\perp$ is a unit vector orthogonal to x (and ξ). We observe that if $\frac{|x\cdot\xi|}{|x||\xi|} = 0$ or 1, then for all these four critical points, $\frac{|x\cdot\omega|}{|x|}$ is 0 or 1. The case $x\cdot\omega = 0$ has been excluded, while we may disregard the case $|x\cdot\omega|/|x| = 1$ since $b^{tr1}_{d_m}(\cos\theta)\sin\theta = 0$ at $\theta = 0$. Thus we only have to consider the critical points of the form (3.36).

The Hessians of the phase function $\sigma(x,\xi;\omega)$ at ω_+ and ω_- are

$$\sigma''(x,\xi;\omega_\pm) = \begin{bmatrix} -2 & 0 \\ 0 & -2\sigma_\pm(x,\xi) \end{bmatrix} \tag{3.38}$$

where

$$\sigma_\pm(x,\xi) = \frac{1}{2}(\frac{x\cdot\xi}{|x||\xi|} \pm 1) = \sigma(x,\xi;\omega_\pm).$$

Since

$$|\det \sigma''(x,\xi;\omega_\pm)| = 2^2|\sigma_\pm|\,,\ \operatorname{sgn}\sigma''(\omega_\pm) = \mp 2, \tag{3.39}$$

the critical points do not degenerate.

We first discuss the behavior of $a_g(x,\xi)$ when (x,ξ) belongs to Γ_0. Since $|x|>4$, we assume that d_m is large enough such that

$$\pi/2 - c_u|x|^{-1}(d_m)^{-1/2s} > \pi/2 - \frac{3}{32}\pi.$$

When $(x,\xi)\in\Gamma_0$, the above condition implies

$$|\frac{\partial}{\partial\theta}\sigma(x,\xi;\omega)| > \pi/32,$$

for all θ belong to the support of $\beta_{cut}(\theta)b^{tr1}_{d_m}(\cos\theta)$. Using this fact,

$$(i|x||\xi|\frac{\partial}{\partial\theta}\sigma(x,\xi;\omega))^{-1}\frac{\partial}{\partial\theta}\,e^{i|x||\xi|\sigma(x,\xi;\omega)} = e^{i|x||\xi|\sigma(x,\xi;\omega)},$$

and integration by parts, we know that $a_0(x,\xi)$ is a symbol of smoothing operator on Γ_0 and T_0 is a smoothing operator which clearly satisfies (3.27).

When (x,ξ) belongs to $\cup_{n\in\mathbb{N}}\Gamma_{-n}$, the critical points $\pm\,\omega_+$ may locate at support of $\beta_{cut}(\theta)b^{tr1}_{d_m}(\cos\theta)$. When (x,ξ) belongs to $\cup_{n\in\mathbb{N}}\Gamma_n$, the critical points $\pm\,\omega_-$ may locate at support of $\beta_{cut}(\theta)b^{tr1}_{d_m}(\cos\theta)$. If the stationary phase formula is applied, we only consider two critical points ω_+,ω_- since $(x\cdot\omega)(\xi\cdot\omega)$ and $b^{tr}_{d_m}(\cos\theta)$ are even in ω, the contributions to $a_g(x,\xi)$ from the critical points $-\omega_+,-\omega_-$ are identical respectively to those from ω_+,ω_-. Also by the symmetry, it suffices to discuss $a_g(x,\xi)$ for $(x,\xi)\in\cup_{n\in\mathbb{N}}\Gamma_{-n}$.

First we consider $a_{-n,I}(x,\xi)$ for given (x,ξ) whose critical angle $\theta_0/2$ of ω_+ locates at support of $\beta_{cut}(\theta)b^{tr1}_{d_m}(\cos\theta)$. We split the integration (3.29) by partition of unity on S^2 such that $\kappa_0,\kappa_1\in C^\infty(S^2), 0\le\kappa_i\le 1, \kappa_0+\kappa_1=1$ and $\kappa_0\equiv 1$ in a neighborhood of ω_+ of radius $(|x||\xi|)^{-1/2}$ and vanishes outside radius $2(|x||\xi|)^{-1/2}$. By integration by parts, we see that

$$\int\limits_{\omega\in S^2} |x|_{\overline{c}}^{-2s}|\xi|^{-2s}e^{-i(x\cdot\omega)(\xi\cdot\omega)}\beta_{cut}(\theta)b^{tr1}_{d_m}(\cos\theta)\kappa_1(\omega)d\Omega(\omega)\in S^{-\infty}_{1,0} \tag{3.40}$$

is a symbol of smooth operator. Applying stationary phase formula to above integration with κ_1 replaced by κ_0, we have

$$\begin{aligned} &a_{-n,I}(x,\xi)\\ &= s(x,\xi)\pi|x|^{-2s}|\xi|^{-2s}|\det(|x||\xi|\sigma''(x,\xi,\omega_+))|^{-\frac12}e^{\frac{\pi i}{4}\operatorname{sgn}\sigma''(\omega_+)}\times\\ &\quad e^{-i|x||\xi|\sigma_+(x,\xi)}\Big\{b^{tr1}_{d_m}(\cos(\frac{\theta_0}{2}))\sin(\frac{\theta_0}{2}) + O(|x|^{-1}|\xi|^{-1})\Big\}\\ &\stackrel{\text{def}}{=} e^{-i|x||\xi|\sigma_+(x,\xi)}p_{-n,I}(x,\xi) \end{aligned} \tag{3.41}$$

where $s(x,\xi)$ is a smooth bump function. The first term of $p_{-n,I}(x,\xi)$ is

$$\begin{aligned} &-i\pi\;|x|^{-2s}|\xi|^{-2s}|\det(|x||\xi|\psi''(x,\xi;\omega_+))|^{-\frac12}b^{tr1}_{d_m}(\cos(\theta_0/2))\sin(\theta_0/2)\\ &= \frac{-i\pi\;|x|^{-2s}|\xi|^{-2s}(\cos(\theta_0/2))^{-(1+2s)}\sin(\theta_0/2)}{2|x||\xi||1+x\cdot\xi/|x||\xi||^{1/2}}\\ &= -2^{-3/2}i\pi\;(|x||\xi|)^{-s}(|x||\xi|\cos^2(\theta_0/2))^{-(1+s)}\sin(\theta_0/2), \end{aligned} \tag{3.42}$$

which is bounded by $(|x||\xi|)^{-s}$ since $|x||\xi|\cos^2(\theta_0/2)>1$ on region I and II.

Without loss of generality, we consider $-n=-1$. The term (3.42) is a symbol of order $-s$ as we have $\cos(\theta_0/2)>c>0$ and the fact that the x or ξ derivative of $\cos(\theta_0/2)$ descends one order in x or ξ respectively. For example,

$$\frac{\partial}{\partial\xi_j}\cos(\frac{\theta_0}{2}) = -\frac12\sin(\frac{\theta_0}{2})\frac{\partial}{\partial\xi_j}\theta_0$$

where

$$\begin{aligned}\frac{\partial}{\partial \xi_j}\theta_0 &= \frac{\partial}{\partial \xi_j}\arccos(\frac{x\cdot\xi}{|x||\xi|})\\ &= \frac{-1}{\sqrt{1-(x\cdot\xi/|x||\xi|)^2}}\frac{\partial}{\partial\xi_j}\Big(\frac{x\cdot\xi}{|x||\xi|}\Big)\\ &= \frac{-1}{\sin\theta_0}\frac{x_j-\cos\theta_0|x|\xi_i/|\xi|}{|x||\xi|}.\end{aligned}\tag{3.43}$$

Using (3.19), we see that

$$\frac{\partial}{\partial\xi_j}\theta_0 = \begin{cases} -\cos\theta_0\cos\phi_0|\xi|^{-1}, & j=1\\ -\cos\theta_0\sin\phi_0|\xi|^{-1}, & j=2\\ -\sin\theta_0|\xi|^{-1}, & j=3\end{cases}\tag{3.44}$$

descends one order in ξ. Using the stationary phase formula in the form of the asymptotic series, we find that the lower order k-th term of $p_{-n,I}$ of (3.41), a symbol of order $-k-s$, has the form

$$(|x||\xi|)^{-s}(|x||\xi|\cos^2(\theta_0/2))^{-(1+s)}\sin(\theta_0/2)\frac{q_k(\cos(\theta_0/2),\sin(\theta_0/2))}{(|x||\xi|\cos^2(\theta_0/2)\sin^2(\theta_0/2))^k}\tag{3.45}$$

where $q_k(t,s)$ is a polynomial of (t,s). The remainder term of the stationary phase formula also has the form as above. By the result on the asymptotic summability of symbols, we can conclude that $p_{-1,I}(x,\xi)$ is a symbol of order $-s$, see also [17]. Please note that derivatives of the symbol $p_{-1,I}$ descend order 1 not only in ξ variable but also in x variable. This allows us to use (2.5) in the estimates later.

Therefore

$$T_{-1,I}h(x) = \int e^{i(x\cdot\xi-|x||\xi|\sigma_+(x,\xi))}p_{-1,I}(x,\xi)\langle\xi\rangle^{1+\rho}\widehat{h}(\xi)d\xi$$

is a Fourier integral operator of order $-s+(1+\rho)$ whose phase function $\psi(x,\xi)=x\cdot\xi-|x||\xi|\sigma_+(x,\xi)=(x\cdot\xi-|x||\xi|)/2$ satisfies non-degeneracy condition, i.e.,

$$\begin{aligned}&|\det\partial_x\partial_\xi\Big[\psi(x,\xi))\Big]| = |\det\frac{1}{2}\Big[I-\frac{x}{|x|}\otimes\frac{\xi}{|\xi|}\Big]| = (\frac{1}{2})^3|1-\cos\theta_0|\\ &0<C_1<(\frac{1}{2})^3|1-\cos\theta_0|<C_2\end{aligned}\tag{3.46}$$

on the support of $p_{-1,I}(x,\xi)$ where θ_0 is the angle spanned by x and ξ. By Theorem 2.5 and Remark 2.6 of [20] or Chapter VI of [25] , we know that the composition of the pseudo differential operator $\langle D_x\rangle^{-(1+\rho)}$ with $T_{-1,I}$ is again a Fourier integral operator, i.e.,

$$\langle D_x\rangle^{-(1+\rho)}T_{-1,I}h(x) = \int_{\mathbb{R}^3} e^{i\psi(x,\xi)}c_{-1,I}(x,\xi)\widehat{h}(\xi)d\xi\tag{3.47}$$

where the phase function $\psi(x,\xi)$ is the same as that of $T_{-1,I}$ and symbol $c_{-1,I}(x,\xi)$ is of order $-(1+\rho)-s+(1+\rho)=-s<0$ which have the same support as $p_{-1,I}(x,\xi)\langle\xi\rangle^{1+\rho}$ and their derivatives also descend as fast as $p_{-1,I}(x,\xi)\langle\xi\rangle^{1+\rho}$ respectively. More precisely,

$$c_{-1,I}(x,\xi) = \iint e^{i(\psi(y,\xi)-\psi(x,\xi)+(x-y)\cdot\eta)}\langle\eta\rangle^{-(1+\rho)}p_{-1,I}(y,\xi)\langle\xi\rangle^{1+\rho}dyd\eta\tag{3.48}$$

has asymptotic expansion

$$\sum_{\alpha\in(\mathbb{N}\cup\{0\})^3}\frac{i^{|\alpha|}}{\alpha!}\partial_\xi^\alpha\langle\nabla_x\psi(x,\xi)\rangle^{-(1+\rho)}\partial_y^\alpha\Big[e^{i\Psi(x,y,\xi)}p_{-1,I}(y,\xi)\langle\xi\rangle^{1+\rho}\Big]|_{y=x}\tag{3.49}$$

where

$$\Psi(x,y,\xi)=\psi(y,\xi)-\psi(x,\xi)+(x-y)\cdot\nabla_x\psi(x,\xi).$$

Now we are reduced to proving the L^2 boundedness of the operator given in (3.47). In fact, the L^2 boundedness of such operators only requires finite times derivative of $c_{-1,I}(x,\xi)$ remain bounded, see for example Theorem 2.5 of [19]. We provide another proof of the L^2 boundedness of operators of (3.47) in Lemma 2.1 which will also be used to see that the bounds of $T_{-n,I}$, $j \neq 0$ form a convergent series.

For other symbols $p_{-n,I}$, we note that the value of $\cos(\theta_0/2)$ for $(x,\xi)\in\Gamma_{-(n+1)}$ is about one half of that for $(x,\xi)\in\Gamma_{-n}$. On the other hand, the minimum of $|x|$ or $|\xi|$ on $\Gamma_{-(n+1),I}$ is two times of that on $\Gamma_{-n,I}$. Combining these facts, (3.44) and (3.45), we can conclude that

$$\sup_{(x,\xi)\in\Gamma_{-(n+1),I}} |\partial_x^\alpha\partial_\xi^\beta p_{-(n+1),I}(x,\xi)| \le \sup_{(x,\xi)\in\Gamma_{-n,I}} |\partial_x^\alpha\partial_\xi^\beta p_{-n,I}(x,\xi)|$$

for each pair of (α,β). By (3.48) and (3.49), we have

$$\sup_{(x,\xi)\in\Gamma_{-(n+1),I}} |\partial_x^\alpha\partial_\xi^\beta d_{-(n+1),I}(x,\xi)| \le \sup_{(x,\xi)\in\Gamma_{-n,I}} |\partial_x^\alpha\partial_\xi^\beta d_{-n,I}(x,\xi)|.$$

Since $\theta_0\to\pi$ as $-n\to-\infty$, the non-degeneracy condition (3.46) has uniform upper and lower bounds in Γ_{-n}, $n\ge 1$. These two conditions imply that $T_{-(n+1),I}$ and $T_{-n,I}$ have the bounds (2.8) with the same C. Since the A in (2.8) for $T_{-(n+1),I}$ is about $1/4$ of that for $T_{-n,I}$, we conclude that the series of the bounds of $T_{-n,I}$ is convergent.

Next we consider $a_{-n,I}(x,\xi)$ for given (x,ξ) whose critical angle $\theta_0/2$ of ω_+ does not locate at support of $\beta_{cut}(\theta)b_{d_m}^{tr1}(\cos\theta)$. This happens to $(x,\xi)\in\Gamma_{-n,I}$ for smaller n where $\theta_0/2\le\pi/2-c_u|x|^{-1}(d_m)^{-1/2s}$. Let n_{d_m} be the greatest number among these n such that we can find $(x,\xi)\in\Gamma_{-n_{d_m},I}$ satisfies above relation. As before, we split the integration (3.29) by partition of unity on S^2 such that $\kappa_0\equiv 1$ in a neighborhood of ω_+ of radius $(|x||\xi|)^{-1/2}$ and vanishes outside radius $2(|x||\xi|)^{-1/2}$. When θ support of κ_0 does not intersect the support of $\beta_{cut}(\theta)b_{d_m}^{tr1}(\cos\theta)$, we see that $a_{-n,I}$ is symbol of smooth operator since it equals (3.40). The applies to all $(x,\xi)\in\Gamma_{-n,I}$ for $n<n_{d_m}-1$ by the definition $\Gamma_{-n,I}$. When $(x,\xi)\in\Gamma_{-n_{d_m},I}$ or $\Gamma_{-n_{d_m}-1,I}$ the intersection of θ support of κ_0 with the support of $\beta_{cut}(\theta)b_{d_m}^{tr1}(\cos\theta)$ may not be empty. We need to consider the contribution from

$$\int_{\omega\in S^2} |x|^{-2s}|\xi|^{-2s}e^{-i(x\cdot\omega)(\xi\cdot\omega)}\beta_{cut}(\theta)b_{d_m}^{tr1}(\cos\theta)\kappa_0(\omega)d\Omega(\omega) \tag{3.50}$$

to $a_{-n,I}(x,\xi)$. It is indeed

$$e^{-i|x||\xi|\sigma_+(x,\xi)}q(x,\xi) \tag{3.51}$$

where $q(x,\xi)$ is a symbol of order $-s$. Now we prove the statement above. From (3.50) and (3.32), we know that $q(x,\xi)$ is

$$\int_{\omega\in S^2} |x|^{-2s}|\xi|^{-2s}e^{i(|x||\xi|(\sigma_+(x,\xi)-\sigma(x,\xi;\omega))}\beta_{cut}(\theta)b_{d_m}^{tr1}(\cos\theta)\kappa_0(\omega)d\Omega(\omega). \tag{3.52}$$

Ignoring the phase function, we see that $q(x,\xi)$ is bounded by

$$\begin{aligned}&C(|x||\xi|)^{-1/2}|x|^{-2s}|\xi|^{-2s}(\cos(\theta_0/2))^{-(1+2s)}\sin(\theta_0/2)\\&=C(|x||\xi|)^{-s}(|x||\xi|\cos^2(\theta_0/2))^{-(1+2s)/2}\sin(\theta_0/2)\\&\le C(|x||\xi|)^{-s}.\end{aligned}$$

Since ω_+ is the critical point of $\sigma(x,\xi;\omega)$, we have that

$$\sigma_+(x,\xi)-\sigma(x,\xi;\omega)=O((\theta-\frac{\theta_0}{2})^2+(\varphi-\varphi_0)^2)=O(\frac{1}{|x||\xi|})$$

for $\omega\in\operatorname{supp}\kappa_0$. Thus the phase function of (3.52) doe not oscillate much and we have

$$|\partial_x^\alpha\partial_\xi^\beta q(x,\xi)|\le C_{\alpha,\beta}\langle x\rangle^{-s-|\alpha|}\langle\xi\rangle^{-s-|\beta|}.$$

We remark that this argument also works for (x,ξ) whose critical angle $\theta_0/2$ satisfies

$$C_1(\sqrt{|x||\xi|})^{-1}\le(\frac{\pi}{2}-c_u|x|^{-1}(d_m)^{-1/2s})-\frac{\theta_0}{2}\le C_2(\sqrt{|x||\xi|})^{-1}$$

for fixed $C_2 > C_1 > 2$. Hence for those (x, ξ) satisfy above relation, the function $a_{-n_{d_m},I}$ or $a_{-n_{d_m}-1,I}$, i.e., (3.40), can be regarded either as a symbol only or the from (3.51) which induces a Fourier integral operator. This fact allows us to decompose $a_{-n_{d_m},I}$ or $a_{-n_{d_m}-1,I}$ smoothly and ensure the symbols are differentiable. The additional Fourier integral defined on $\Gamma_{-n_{d_m},I}$ or $\Gamma_{-n_{d_m}-1,I}$ satisfies (3.27) as before. Also those pseudo-differential operators defined on $\Gamma_{-n,I}$ with $1 \le n \le n_{d_m}$ can be estimated by the same way as we did to the Fourier integral operators and the series of bounds is convergent. The ends the discussion on region I.

We continue with the estimates of the operators $T_{j,II}$. It suffices to show that for each $j \in \mathbb{Z} - \{0\}$ the estimate

$$\|\langle D_x\rangle^{-(1+\rho)} T_{j,II} h(x)\|_{L^2} \le C \|\langle D_x\rangle^{-(1+\rho)} T_{j,I} h(x)\|_{L^2} \tag{3.53}$$

holds with C being independent of j. Recall the L^2 boundedness of the right hand side of (3.53) comes from the analysis of $a_{j,I}(x, \xi)$ and the application of the Lemma 2.1. We should show that the analysis of $a_{j,II}(x, \xi)$ can be done as $a_{j,I}(x, \xi)$ after change of variables, then we can compare the norms by checking the proof of Lemma 2.1 again.

By symmetry, we only need to consider $T_{\mathbf{n},II}$, $\mathbf{n} \in \mathbb{N}$. Recall

$$\begin{aligned}
&\langle D_x\rangle^{-(1+\rho)} T_{\mathbf{n},II} h(x) \\
&= \iiint e^{i(x-y)\cdot\eta} \langle \eta\rangle^{-(1+\rho)} e^{iy\cdot\xi} a_{\mathbf{n},II}(y, \xi) \langle \xi\rangle^{1+\rho} \widehat{h}(\xi) d\xi dy d\eta
\end{aligned} \tag{3.54}$$

where

$$\begin{aligned}
a_{\mathbf{n},II}(y, \xi) &= \zeta_{\mathbf{n}}(y, \xi) \chi_{\mathbf{n},II}(y, \xi)\, a_g(y, \xi) \\
&= \zeta_{\mathbf{n}}(y, \xi) \chi_{\mathbf{n},II}(y, \xi) \\
&\quad \times \int\limits_{\omega\in S^2} |y|^{-2s} |\xi|^{-2s} e^{-i(y\cdot\omega)(\xi\cdot\omega)} \beta_{cut}(\theta) b^{tr1}_{d_m}(\cos\theta) d\Omega(\omega).
\end{aligned}$$

Here

$$\chi_{\mathbf{n},II}(y, \xi) = \sum_{(j,l)\in R_{\mathbf{n},II}} \chi_j(y) \chi_l(\xi)$$

and

$$\begin{aligned}
R_{\mathbf{n},II} &= R^1_{\mathbf{n},II} \cup R^2_{\mathbf{n},II} \\
&= \{(j,l) |\ j + l \ge 2(\mathbf{n}+1),\ 1 \le l \le \mathbf{n}\} \\
&\quad \cup \{(j,l) |\ j + l \ge 2(\mathbf{n}+1),\ 1 \le j \le \mathbf{n}\}.
\end{aligned}$$

By the similarity, it is enough to consider only the operator in (3.54) with $T_{\mathbf{n},II}$ restricted to $R^1_{\mathbf{n},II}$, denoted by S, to illustrate the idea. By (3.54) and Plancherel theorem, we can write

$$\begin{aligned}
Su(x) &= \sum_{(j,l)\in R^1_{\mathbf{n},II}} \int e^{i\phi(x,\xi)} c_{(j,l)}(x, \xi) u(\xi) d\xi \\
&\stackrel{\text{def}}{=} \sum_{(j,l)\in R^1_{\mathbf{n},II}} S_{(j,l)} u(x)
\end{aligned}$$

where

$$c_{(j,l)}(x, \xi) = \zeta_{\mathbf{n}}(x, \xi) \iint e^{-i\phi(x,\xi)} e^{i(x-y)\cdot\eta} \langle\eta\rangle^{-(1+\rho)} e^{iy\cdot\xi} \chi_j(y) \chi_l(\xi) a_g(y, \xi) \langle\xi\rangle^{1+\rho} dy d\eta \tag{3.55}$$

and $\phi(x, \xi) = x \cdot \xi$ or $\frac{1}{2}(x \cdot \xi - |x||\xi|)$ to be determined later. Let $S^*_{(j,l)}$ be the adjoint of $S_{(j,l)}$, i.e.,

$$S^*_{(j,l)} v(\xi) = \int e^{-i\phi(y,\xi)} \overline{c_{(j,l)}(y, \xi)} v(y) dy.$$

Then we have

$$S_{(j,l)}S^*_{(k,m)}v(x) = \int K_{(j,l),(k,m)}(x,y)v(y)dy,$$

where

$$\begin{aligned}&K_{(j,l),(k,m)}(x,y)\\&= \int e^{i(\phi(x,\xi)-\phi(y,\xi))}c_{(j,l)}(x,\xi)\overline{c_{(k,m)}(y,\xi)}d\xi.\end{aligned}$$

Since $\operatorname{supp}\chi_l(\xi)\cap\operatorname{supp}\chi_m(\xi)=\emptyset$ when $|l-m|\ge 2$, we only have to consider $l=m-1, m$ or $m+1$. Without loss of generality, we assume $l=m+1$, i.e. $1\le m\le \mathbf{n}-1, 2\le m\le\mathbf{n}$. Let

$$n_+=\mathbf{n}+1-m\ge 2\,,\ \lambda=2^{n_+}\,,\ \xi=\lambda^{-1}\tilde\xi\,,\ x=\lambda\tilde x\,,\ y=\lambda\tilde y.$$

Also let $\eta=\lambda^{-1}\tilde\eta$ in (3.55), then we have

$$\begin{aligned}&c_{(j,\,l)}(x,\xi)\\&=\zeta_{\mathbf{n}}(\tilde x,\tilde\xi)\iint e^{-i\phi(\tilde x,\tilde\xi)}e^{i(\tilde x-\tilde y)\cdot\tilde{\tilde\eta}}\langle\lambda^{-1}\tilde\eta\rangle^{-(1+\rho)}e^{i\tilde y\cdot\tilde\xi}\\&\qquad\chi_{j-n_+}(\tilde y)\chi_{l+n_+}(\tilde\xi)\,a_g(\tilde y,\tilde\xi)\langle\lambda^{-1}\tilde\xi\rangle^{1+\rho}d\tilde y d\tilde\eta\\&\overset{\text{def}}{=} c^\lambda_{(j-n_+,\,l+n_+)}(\tilde x,\tilde\xi).\end{aligned}$$

Since $j-n_+\ge\mathbf{n}+1,\ l+n_+\ge\mathbf{n}+2,\ a_g(\tilde y,\tilde\xi)$ is a function defined on region I of $(\tilde y,\tilde\xi)$ space. By previous calculation, $a_g(\tilde y,\tilde\xi)$ equals to $p(x,\xi)$ or $e^{-i|x||\xi|\sigma(x,\xi)}p(x,\xi)$ where $p(x,\xi)$ is a symbol of order $-s$. We choose $\phi(\tilde x,\tilde\xi)=\tilde x\cdot\tilde\xi$ if $a_g(x,\xi)=p(x,\xi)$, otherwise $\phi(\tilde x,\tilde\xi)=\frac12(\tilde x\cdot\tilde\xi-|\tilde x||\tilde\xi|)$. Similarly

$$c_{(k,m)}(y,\xi)=c^\lambda_{(k-n_+,\,m+n_+)}(\tilde y,\tilde\xi)$$

with $k-n_+\ge\mathbf{n}+1,\ m+n_+\ge\mathbf{n}+1$. Use (3.49), we see that for any α,β

$$|\partial^\alpha_{\tilde x}\partial^\alpha_{\tilde\xi}\,c^\lambda_{(j-n_+,\,l+n_+)}(\tilde x,\tilde\xi)|\le C|\partial^\alpha_{\tilde x}\partial^\alpha_{\tilde\xi}c_{(j-n_+,\,l+n_+)}(\tilde x,\tilde\xi)| \tag{3.56}$$

where $C>0$ is independent of λ and likewise for $c^\lambda_{(k-n_+,\,m+n_+)}$. Then

$$\begin{aligned}&K_{(j,l),(k,m)}(x,y)\\&=\int e^{i(\phi(x,\xi)-\phi(y,\xi))}c_{(j,l)}(x,\xi)\overline{c_{(k,m)}(y,\xi)}\,d\xi\\&=\lambda^{-3}\int e^{i(\phi(\tilde x,\tilde\xi)-\phi(\tilde x,\tilde\xi))}c^\lambda_{(j-n_+,\,l+n_+)}(\tilde x,\tilde\xi)\overline{c^\lambda_{(k-n_+,\,m+n_+)}(\tilde y,\tilde\xi)}\,d\tilde\xi\\&=\lambda^{-3}K^\lambda_{(j-n_+,\,l+n_+),(k-n_+,\,m+n_+)}(\tilde x,\tilde y)\end{aligned}$$

and

$$S_{(j,l)}S^*_{(k,m)}v(\lambda\tilde x)=\int K^\lambda_{(j-n_+,\,l+n_+),(k-n_+,\,m+n_+)}(\tilde x,\tilde y)v(\lambda\tilde y)d\tilde y.$$

From above and (3.56), we have

$$\|S_{(j,l)}S^*_{(k,m)}\|_{L^2_y\to L^2_x}\le C\|S_{(j-n_+,\,l+n_+)}S^*_{(k-n_+,\,m+n_+)}\|_{L^2_{\tilde y}\to L^2_{\tilde x}} \tag{3.57}$$

Note that the right hand side of (3.57) is an operator with kernel defined on region I and thus we can apply Lemma 2.1 to it. From $|(j-n_+)-(k-n_+)|=|j-k|,\ |(l+n_+)-(m+n_+)|=|l-m|$ and estimate (2.19), we see its upper bound has the form as the right hand side of estimate (2.19) depending only on $|j-k|,\ |l-m|$. The estimates $\|S^*_{(j,l)}S_{(k,m)}\|$ can be handled similarly. Thus we conclude the result by Coltar-Stein lemma again.

It remains to discuss the operators $T_{j,III}$ defined on region III. The key observation is that $\sum_{j\neq 0}a_{j,III}$ is a symbol of order $-s$. Using this fact we see that $\sum_{j\neq 0}T_{j,III}$ is a pseudodifferential operator defined on region III. Then the estimate follows by the fact about the composition of the pseudodifferential operators. Now we prove the claim above. From the calculation of $a_g(x,\xi)$ on region I, we know that the it suffices to show that

$$\int_{\omega\in S^2} |x|_{\overline{c}}^{-2s}|\xi|^{-2s}e^{-i(x\cdot\omega)(\xi\cdot\omega)}\beta_{cut}(\theta)b_{d_m}^{tr1}(\cos\theta)\kappa_0(\omega)d\Omega(\omega) \tag{3.58}$$

is a symbol of order $-s$, no more induces a Fourier integral operator, when the critical angle $\theta_0/2$ of ω_+ locals at the support of $\beta_{cut}(\theta)b_{d_m}^{tr1}(\cos\theta)$ or the θ support of κ_0 intersects the support $\beta_{cut}(\theta)b_{d_m}^{tr1}(\cos\theta)$. Again we write (3.58) as

$$e^{-i|x||\xi|\sigma_+(x,\xi)}q(x,\xi)$$

where $q(x,\xi)$ given in (3.52) is a symbol of order $-s$. We only need to show that $e^{-i|x||\xi|\sigma_+(x,\xi)}$ is a symbol of order 0. It simply follows from that

$$|x||\xi||\sigma_+(x,\xi)| = |x||\xi|\cos^2(\theta_0/2) < C_2$$

where the inequality is by the definition of region III.

Part 2. Estimate of T_B

Since $1-\rho\geq 0$, the estimate for T_B is implied by

$$\left\|\langle D_x\rangle^{-(1+\rho)}T_B\, h(x)\right\|_{L^2} \leq O((d_m)^{(1-1/s)})\||\xi|^{1-\rho}\widehat{h}\|_{L^2}.$$

Note that $\|\langle D_x\rangle^{-(1+\rho)}T_B\, h(x)\|_{L^2} \leq \|T_B\, h(x)\|_{L^2}$. Recall that

$$T_B\, h(x) = \int_{\mathbb{R}^3} e^{ix\cdot\xi}a_B(x,\xi)\,|\xi|^{2s}\widehat{h}(\xi)d\xi$$

where $a_B(x,\xi) = \chi_{B,1}(x,\xi)\, a_{d_m}^{tr}(x,\xi)$ and $\operatorname{supp}\chi_{B,1}\subset\{|x||\xi|>16, |\xi|<4\}$.

Thus (3.18) holds and the same calculation as Case 1. of the estimate of T_A gives

$$a_B(x,\xi)\,|\xi|^{2s} = C(\cos\theta_0)(d_m)^{1-1/s}|\xi|^2, |\xi|<4, \tag{3.59}$$

where the bounded coefficient $C(\cos\theta_0)$ depends on $\varepsilon, s, \cos\theta_0$ and bump function μ. Using (3.59), we are able to prove

$$\|T_B\, h(x)\|_{L^2} \leq C(d_m)^{1-1/s}\||\xi|_l^2\widehat{h}(\xi)\|_{L^2} \tag{3.60}$$

here $|\xi|_l$ means the $|\xi|<8$ and it is clearly that $\||\xi|_l^2\widehat{h}(\xi)\|_{L^2} \leq C\||\xi|_l^{1-\rho}\widehat{h}(\xi)\|_{L^2} \leq C\||\xi|^{1-\rho}\widehat{h}(\xi)\|_{L^2}$ for $1-\rho\geq 0$. Again we can combine $|\xi|_l^2$ with $\widehat{h}(\xi)$ and reduce (3.60) to the L^2 boundedness of

$$\mathcal{T}u(x) = \int e^{ix\cdot\xi}C(\cos\theta_0)\,\chi_{B,1}(x,\xi)u(\xi)d\xi.$$

Note the derivatives of C in (3.59) with respect to ξ are unbounded due to $|\xi|<4$. Thus $\mathcal{T}$ is not a pseudo-differential operator. However the condition $|x||\xi|>16$ can help us to obtain the L^2 boundedness of $\mathcal{T}$.

Recall, (3.13),

$$\chi_{B,1}(x,\xi) = \sum_{j+l\geq 2, l\leq 0}\chi_j(x)\chi_k(\xi) = \sum_{j-l\geq 2, l\geq 0}\chi_j(x)\chi_{-l}(\xi).$$

As the Lemma 2.1, we decompose $\mathcal{T}$ as

$$\mathcal{T} = \sum_{j-l\geq 2, l\geq 0}\mathcal{T}_{(j,-l)}$$

where

$$\mathcal{T}_{(j,-l)}u(x)=\int_{\mathbb{R}^3} e^{ix\cdot\xi}a_{(j,-l)}(x,\xi)u(\xi)d\xi$$

and

$$a_{(j,-l)}(x,\xi)=C(\cos\theta_0)\chi_j(x)\chi_{-l}(\xi) \tag{3.61}$$

with the condition

$$j\geq l+2. \tag{3.62}$$

As the proof of Lemma 2.1, the L^2 boundedness of $\mathcal{T}$ can be proved through the estimates of $\|\mathcal{T}_{(j,-l)}\mathcal{T}^*_{(k,-m)}\|_{L^2\to L^2}$ and $\|\mathcal{T}^*_{(j,-l)}\mathcal{T}_{(k,-m)}\|_{L^2\to L^2}$ where $\mathcal{T}^*_{(k,-m)}$ is the adjoint of $\mathcal{T}_{(k,-m)}$. We consider only the estimates of the former since the estimates for the other can be handled similarly. Write

$$\mathcal{T}_{(j,-l)}\mathcal{T}^*_{(k,-m)}u(x)=\int \mathcal{K}_{(j,-l),(k,-m)}(x,y)u(y)dy$$

where

$$\mathcal{K}_{(j,-l),(k,-m)}(x,y)=\int e^{i\xi\cdot(x-y)}a_{(j,-l)}(x,\xi)\overline{a}_{(k,-m)}(y,\xi)d\xi.$$

When $|l-m|\geq 2$ we have $\mathcal{K}_{(j,-l),(k,-m)}(x,y)=0$ by the fact $\gamma_{-l}(\xi)\gamma_{-m}(\xi)=0$ and (3.61). Without loss of generality, we assume $l=m+1$ and $\lambda=2^l$. Let $x=\lambda\tilde{x}$, $y=\lambda\tilde{y}$, $\xi=\lambda^{-1}\tilde{\xi}$. We observe that this change of variables does not change the direction of vectors x,y,ξ, thus we have

$$a_{(j,-l)}(x,\xi)=a_{(j-l,0)}(\tilde{x},\tilde{\xi})\ ,\ \overline{a}_{(k,-m)}(y,\xi)=\overline{a}_{(k-m-1,1)}(\tilde{y},\tilde{\xi}),$$

and

$$\mathcal{K}_{(j,-l),(k,-m)}(x,y)=\lambda^{-3}\mathcal{K}_{(j-l,0),(m-k-1,1)}(\tilde{x},\tilde{y}).$$

Since $j-l,0,k-m-1,1$ are non-negative by relation (3.62), we see that

$$a_{(j-l,0)}(\tilde{x},\tilde{\xi})\ ,\ \overline{a}_{(k-m-1,1)}(\tilde{y},\tilde{\xi})$$

defined on $|\tilde{x}|,|\tilde{\xi}|>2$ are symbols of order 0 which satisfy (2.5) of Lemma 2.1. Thus we have

$$\begin{aligned}\mathcal{T}_{(j,-l)}\mathcal{T}^*_{(k,-m)}u(x)&=\int \mathcal{K}_{(j,-l),(k,-m)}(x,y)u(y)dy\\ &=\int \mathcal{K}_{(j-l,0),(m-k-1,1)}(\tilde{x},\tilde{y})u(\lambda\tilde{y})d\tilde{y}=\mathcal{T}_{(-j,l)}\mathcal{T}^*_{(-k,m)}u(\lambda\tilde{x}).\end{aligned}$$

Hence $\|\mathcal{T}_{(j,-l)}\mathcal{T}^*_{(k,-m)}\|_{L^2_y\to L^2_x}=\|\mathcal{T}^*_{(j-l,0)}\mathcal{T}_{(m-k-1,1)}\|_{L^2_{\tilde{y}}\to L^2_{\tilde{x}}}$. The phase function $\tilde{x}\cdot\tilde{\xi}$ clearly satisfies the non-degeneracy condition (2.6) and we can conclude the result by Lemma 2.1, see the paragraph after (3.57).

Part 3. Estimate of T_C

Similar to the estimate for T_B above, we can drop the operator $\langle D_x\rangle^{-(1+\rho)}$ to simplify the proof of the estimate for T_C by using the fact that supp $\chi_{C,1}(x,\xi)\subset\{|x||\xi|<128, |x|>4\}$ implies that the support of ξ is bounded. However we should prove the estimate for T_C directly for the later use. We note the estimate

$$\left\|\langle D_x\rangle^{-(1+\rho)}T_C\,h(x)\right\|_{L^2}\leq O((d_m)^{(1-1/s)})\|h\|_{H^{1-\rho}}$$

is equivalent to

$$\left\|\langle D_x\rangle^{-(1+\rho)}T_C^{\rho-1}\,h(x)\right\|_{L^2}\leq O((d_m)^{(1-1/s)})\|h\|_{L^2} \tag{3.63}$$

where

$$T_C^{\rho-1}\, h(x) = \int e^{ix\cdot\xi} a_C(x,\xi)|\xi|^{2s}\langle\xi\rangle^{\rho-1}\, h(\xi)d\xi,$$

by Plancherel Theorem and

$$\begin{aligned} a_C(x,\xi) &= \chi_{C,1}(x,\xi)\, a^{tr}_{d_m}(x,\xi) \\ &= \chi_{C,1}(x,\xi)\, |x|^{-2s}|\xi|^{-2s}\int_{S^2} b^{tr1}_{d_m}(\cos\theta)(e^{-i(x\cdot\omega)(\xi\cdot\omega)}-1)d\Omega(\omega). \end{aligned} \tag{3.64}$$

We only consider $\rho\in(-1,1]$ since the special case $\rho=-1$ is easier than others. By definition,

$$\begin{aligned} &\langle D_x\rangle^{-(1+\rho)}(T_C^{\rho-1}h(x)) \\ &= \langle D_x\rangle^{-(1+\rho)}\Big(\int e^{ix\cdot\xi} a_C(x,\xi)|\xi|^{2s}\langle\xi\rangle^{\rho-1}h(\xi)d\xi\Big) \\ &= \iiint e^{i(x-y)\cdot\eta}\langle\eta\rangle^{-(1+\rho)}e^{iy\cdot\xi}a_C(y,\xi)|\xi|^{2s}\langle\xi\rangle^{\rho-1}h(\xi)d\xi dy d\eta \\ &= \int e^{ix\cdot\xi}\, c(x,\xi)\, h(\xi)d\xi = \int K(x,\xi)\, h(\xi)d\xi \end{aligned} \tag{3.65}$$

where

$$c(x,\xi) = \iint e^{i(x-y)\cdot(\eta-\xi)}\langle\eta\rangle^{-(1+\rho)}a_C(y,\xi)|\xi|^{2s}\langle\xi\rangle^{\rho-1}dy d\eta. \tag{3.66}$$

Now we estimate $a_C(y,\xi)$ given in (3.64). Since $|y||\xi|<128$, we may assume d_m is large enough such that

$$\varepsilon^{-1}|y||\xi| \le c_u^{-1}|y|(d_m)^{1/2s} \tag{3.67}$$

holds. Hence the set of θ where $b^{tr1}_{d_m}$ non-vanishing is $[\pi/2 - c_u|y|^{-1}(d_m)^{-1/2s}, \pi/2] \subset [\pi/2-\varepsilon|y|^{-1}|\xi|^{-1},\pi/2]$. Using (3.19), (3.20) and (3.21) as Case 1 of the estimate of T_A, we see the contribution from $|y|^2|\xi|^2\sin^2\theta_0\cos^2(\varphi_0-\varphi)\sin^2\theta\cos^2\theta$ to the $a^{tr}_{d_m}(y,\xi)$ in (3.64) gives

$$\begin{aligned} &C\int_{\pi/2-\varepsilon/|y||\xi|}^{\pi/2} b^{tr1}_{d_m}(\cos\theta)|y|^{2-2s}|\xi|^{2-2s}\cos^2\theta\sin\theta d\theta \\ &= C\int_{\pi/2-c_u|y|^{-1}(d_m)^{-1/2s}}^{\pi/2} |y|^{2-2s}|\xi|^{2-2s}\cos^{1-2s}\theta\sin\theta d\theta \\ &= C'(\cos\theta_0)(d_m)^{(1-1/s)}|\xi|^{2-2s}\ ,\ 0<s<1. \end{aligned} \tag{3.68}$$

The contribution from $|y||\xi|\cos\theta_0\cos^2\theta$ gives

$$C''(\cos\theta_0)(d_m)^{(1-1/s)}|y|^{-1}|\xi|^{1-2s}. \tag{3.69}$$

The bounded coefficients C', C'' depend on $\varepsilon, s, \cos\theta_0$ and bump function μ. Hence we conclude that

$$a_C(y,\xi) = O((d_m)^{(1-1/s)})\,\chi_{C,1}(y,\xi)\, C(\cos\theta_0)\,(|y|^{-1}|\xi|^{1-2s}+|\xi|^{2-2s}). \tag{3.70}$$

The bounded coefficient C, same as C', C'' above, can not be regarded as the symbol of order 0 since its derivatives with respect to ξ do not decay when $|\xi|<1$. Fortunately the y derivatives of C as well as $a_C(y,\xi)$ descend order. Hence the proof that (3.22) has asymptotic expansion (3.23) can be used to estimate (3.66), see for example [20] for the proof of more general formula (3.49), and conclude that $c(x,\xi)$ has the asymptotic expansion

$$|\xi|^{2s}\langle\xi\rangle^{\rho-1}\sum_{\alpha\in\mathbb{N}_0^3}\frac{i^{-|\alpha|}}{\alpha!}\partial_\xi^\alpha\langle\xi\rangle^{-(1+\rho)}\partial_x^\alpha a_C(x,\xi)$$

in the sense that

$$\left|c(x,\xi)-|\xi|^{2s}\langle\xi\rangle^{\rho-1}\sum_{|\alpha|<N}\frac{i^{-|\alpha|}}{\alpha!}\partial_\xi^\alpha\langle\xi\rangle^{-(1+\rho)}\partial_x^\alpha a_C(x,\xi)\right|$$
$$\le C_N\sum_{|\alpha|=N}\frac{i^{-|\alpha|}}{\alpha!}\partial_\xi^\alpha\langle\xi\rangle^{-(1+\rho)}\partial_x^\alpha a_C(x,\xi).$$

In fact, we need the above estimate for N large enough so that the remainder term is integrable with respect to x and ξ instead of full asymptotic formula. From above and (3.70), we conclude that the kernel K in the last line of (3.65) satisfies

$$|K(x,\xi)|=|c(x,\xi)|\le O((d_m)^{(1-1/s)})\chi_{C,1}(x,\xi). \tag{3.71}$$

Let $p(x)=(1+|x|)^{-1}$ and $q(\xi)=|\xi|^{-2}$. For any fixed $|x_0|\ge 4$, using spherically coordinate and (3.71), we have

$$\int\limits_{\{|\xi|\le 128|x_0|^{-1}\}}|K(x_0,\xi)|q(\xi)d\xi$$
$$\le C_1(d_m)^{(1-1/s)}\int\limits_0^{128|x_0|^{-1}}r^{-2}\cdot r^2dr$$
$$\le C_2(d_m)^{(1-1/s)}|x_0|^{-1}\le C_3(d_m)^{(1-1/s)}p(x_0).$$

And for any fixed $|\xi_0|\le 32$, we have

$$\int\limits_{\{2\le|x|\le 128|\xi_0|^{-1}\}}|K(x,\xi_0)|p(x)dx$$
$$\le C_1(d_m)^{(1-1/s)}\int\limits_2^{128|\xi_0|^{-1}}r^{-1}\cdot r^2dr$$
$$\le C_2(d_m)^{(1-1/s)}|\xi_0|^{-2}=C_2(d_m)^{(1-1/s)}q(\xi_0).$$

We conclude (3.63) By Schur's test. □

Proof of Lemma 3.2. Small relative velocity, hard potential.

Let

$$T_{d_m}^{tr}h(x)$$
$$=\int\limits_{\omega\in S^2}|x|^{-2s}b_{d_m}^{tr}(\cos\theta)(h(x-(x\cdot\omega)\omega)-h(x))d\Omega(\omega), \tag{3.72}$$

where $\cos\theta=(x\cdot\omega)/|x|$,

$$b_{d_m}^{tr}(\cos\theta)=\begin{cases}b_{d_m}^{tr1}(\cos\theta) & \text{if } |x|>c_u(d_m)^{-1/2s}\\ b_{d_m}^{tr2}(\cos\theta) & \text{if } |x|\le c_u(d_m)^{-1/2s}\end{cases} \tag{3.73}$$

with

$$b_{d_m}^{tr1}(\cos\theta)=\begin{cases}(\cos\theta)^{-(1+2s)} & \text{if } \cos(\theta)\le c_u|x|^{-1}(d_m)^{-1/2s}\\ 0 & \text{others.}\end{cases} \tag{3.74}$$

and

$$b_{d_m}^{tr2}(\cos\theta)=(\cos\theta)^{-(1+2s)}\ ,\ 0\le\theta<\pi/2. \tag{3.75}$$

Following the proof of Lemma 3.1 with $|v-v_*|_c^{1-2s}\langle v\rangle^{-(1-2s)}\langle v_*\rangle^{-(1-2s)}$ being replaced by $|v-v_*|_c^{1-2s}$, the proof of (3.5) is reduced to proving that for $\rho\in[-1,1]$ and $0<s<1/2$, we have

$$\begin{aligned}&\sup_{v_*}\left\|\langle D_v\rangle^{-(1+\rho)}(|v-v_*|_c^{1-2s}[\tau_{-v_*}\circ T_{d_m}^{tr}\circ\tau_{v_*}]h(v))\right\|_{L^2}\\&\leq O((d_m)^{(1-1/s)})\|h\|_{H^{1-\rho}}.\end{aligned}\tag{3.76}$$

Since v_* is a parameter and L^2 is translation invariant, it is easy to check that the estimate

$$\left\|\langle D_x\rangle^{-(1+\rho)}(|x|_c^{1-2s}T_{d_m}^{tr}h(x))\right\|_{L^2}\leq O((d_m)^{(1-1/s)})\|h\|_{H^{1-\rho}}\tag{3.77}$$

implies (3.76) by replacing x with $v-v_*$ in the proof. We remark that the proof of the case $\rho=-1$ is simpler than others. Hence we only consider $-1<\rho\leq 1$ below. As before, we write

$$T_{d_m}^{tr}\,h(x)=\frac{1}{(2\pi)^3}\int e^{ix\cdot\xi}\,a_{d_m}^{tr}(x,\xi)|\xi|^{2s}\widehat{h}(\xi)d\xi\tag{3.78}$$

where

$$a_{d_m}^{tr}(x,\xi)=|x|^{-2s}|\xi|^{-2s}\int_{S^2}b_{d_m}^{tr}(\cos\theta)(e^{-i(x\cdot\omega)(\xi\cdot\omega)}-1)d\Omega(\omega).\tag{3.79}$$

We should prove (3.77) by reorganizing ingredients in the proof for Lemma 3.4. Noting $\operatorname{supp}|x|_c\subset\{|x|<8\}$ and using dyadic decomposition (3.13), we have the decomposition

$$a_{d_m}^{tr}=a_{B,2}+a_{C,2}\stackrel{\text{def}}{=}\chi_{B,2}\,a_{d_m}^{tr}+\chi_{C,2}\,a_{d_m}^{tr}$$

where $\operatorname{supp}\chi_{B,2}(x,\xi)\subset\{|x||\xi|>16,|x|<8\}$, $\operatorname{supp}\chi_{C,2}(x,\xi)\subset\{|x||\xi|<128,|x|<8\}$ and correspondingly

$$T_{d_m}^{tr}=\frac{1}{(2\pi)^3}(T_{B,2}+T_{C,2}).\tag{3.80}$$

First we prove that the operator $T_{B,2}$ satisfies (3.77). Using (3.16), we further decompose $a_{B,2}(x,\xi)$ as

$$\begin{aligned}a_{B,2}(x,\xi)&=\beta_{can}(\theta)a_{B,2}(x,\xi)+\beta_{cut}(\theta)a_{B,2}(x,\xi)\\&\stackrel{\text{def}}{=}a_{can}(x,\xi)+a_{cut}(x,\xi)\end{aligned}$$

and correspondingly $T_{B,2}=T_{can}+T_{cut}$. We can estimate a_{can} as Case 1 or Case 2 of Part I in the proof of Lemma 3.4 to conclude that

$$a_{can}(x,\xi)|\xi|^{2s}=C|\xi|^2.$$

Although $|\xi|>2$, a_{can} is not a symbol since C depends on $\cos\theta_0$ is not a symbol of 0 order as $|x|$ may be less one. This gives us the obstacle to use the result in the composition of pseudo-differential operators directly. However this can be remedied again by change of variables. The estimate we need to prove,

$$\left\|\langle D_x\rangle^{-(1+\rho)}(|x|_c^{1-2s}T_{can}h(x))\right\|_{L^2}\leq O((d_m)^{(1-1/s)})\|h\|_{H^{1-\rho}},$$

is equivalent to

$$\left\|\langle D_x\rangle^{-(1+\rho)}(|x|_c^{1-2s}T_{can}^{\rho-1}h(x))\right\|_{L^2}\leq O((d_m)^{(1-1/s)})\|h\|_{L^2}\tag{3.81}$$

where, by Plancherel theorem,

$$T_{can}^{\rho-1}\,h(x)=\int e^{ix\cdot\xi}\,a_{can}(x,\xi)|\xi|^{2s}\langle\xi\rangle^{\rho-1}\,h(\xi)d\xi.$$

Let

$$\begin{aligned}&\langle D_x\rangle^{-(1+\rho)}(|x|_c^{1-2s}T_{can}^{\rho-1}h(x))\\&=\iiint e^{i(x-y)\cdot\eta}\langle\eta\rangle^{-(1+\rho)}e^{iy\cdot\xi}|y|_c^{1-2s}a_{can}(y,\xi)|\xi|^{2s}\langle\xi\rangle^{\rho-1}h(\xi)d\xi dyd\eta\\&=\int e^{ix\cdot\xi}\,c(x,\xi)\,h(\xi)d\xi\\&\stackrel{\text{def}}{=}Sh(x)\end{aligned}$$

where

$$c(x,\xi)=|\xi|^{2s}\langle\xi\rangle^{\rho-1}\iint e^{i(x-y)\cdot(\eta-\xi)}\langle\eta\rangle^{-(1+\rho)}|y|_c^{1-2s}a_{can}(y,\xi)dyd\eta.$$

The proof of the L^2 boundedness, i.e., (3.81), is similarly to that of (3.54) except that the coefficient $O((d_m)^{(1-1/s)})$ in (3.81) and $|y|_c^{1-2s}$ in (3.82) were not seen before. Using the definition of $\chi_{B,2}$ and the decomposition

$$a_{can}(y,\xi)=\sum_{-j+l\geq 2,\,j\geq 0}\chi_{-j}(y)\chi_l(\xi)\beta_{can}(\theta)a^{tr}_{d_m}(y,\xi),$$

we have $c(x,\xi)=\sum c_{(-j,l)}(x,\xi)$ where

$$\begin{aligned}c_{(-j,l)}(x,\xi)=|\xi|^{2s}\langle\xi\rangle^{\rho-1}\iint e^{i(x-y)\cdot(\eta-\xi)}\langle\eta\rangle^{-(1+\rho)}|y|_c^{1-2s}\\\cdot\,\chi_{-j}(y)\chi_l(\xi)\beta_{can}(\theta)a^{tr}_{d_m}(y,\xi)dyd\eta.\end{aligned}\tag{3.82}$$

Thus we have corresponding decomposition

$$Sh(x)=\sum_{-j+l\geq 2,\,j\geq 0}S_{(-j,l)}h(x).$$

We will get L^2 boundedness of operator S by Coltar-Stein lemma and estimates

$$\|S_{(-j,l)}S^*_{(-k,m)}\|_{L^2\to L^2}\ ,\ \|S^*_{(-j,l)}S_{(-k,m)}\|_{L^2\to L^2}$$

for $-j+l\geq 2,\ j\geq 0$ and $-k+m\geq 2, k\geq 0$ where $S^*_{(-j,l)}$ is the adjoint of $S_{(-j,l)}$. It suffices to consider $\|S_{(-j,l)}S^*_{(-k,m)}\|$. We have

$$S_{(-j,l)}S^*_{(-k,m)}v(x)=\int K_{(-j,l),(-k,m)}(x,y)\,v(y)dy$$

where

$$\begin{aligned}&K_{(-j,l),(-k,m)}(x,y)\\&=\int e^{i(x-y)\cdot\xi}c_{(-j,l)}(x,\xi)\overline{c_{(-k,m)}(y,\xi)}d\xi.\end{aligned}$$

Since supp $\chi_l(\xi)\cap$ supp $\chi_m(\xi)=\emptyset$ when $|l-m|\geq 2$, we only have to consider $l=m-1, m$ or $m+1$. Without loss of generality, we assume $l=m+1$. Let

$$\lambda=2^j\ ,\ \xi=\lambda\tilde{\xi}\ ,\ x=\lambda^{-1}\tilde{x}\ ,\ y=\lambda^{-1}\tilde{y},$$

and $\eta=\lambda\tilde{\eta}$ in (3.82). Checking (3.79), we have $a^{tr}_{d_m}(y,\xi)=a^{tr}_{d_m}(\tilde{y},\tilde{\xi})$ and

$$\begin{aligned}c_{(-j,l)}(x,\xi)&=|\lambda\tilde{\xi}|^{2s}\langle\lambda\tilde{\xi}\rangle^{\rho-1}\iint e^{i(\tilde{x}-\tilde{y})\cdot(\tilde{\eta}-\tilde{\xi})}\langle\lambda\tilde{\eta}\rangle^{-(1+\rho)}|\tilde{y}/\lambda|^{1-2s}\\&\qquad\chi_0(\tilde{y})\chi_{-j+l}(\tilde{\xi})\beta_{can}(\theta)a^{tr}_{d_m}(\tilde{y},\tilde{\xi})d\tilde{y}d\tilde{\eta}\\&\stackrel{\text{def}}{=}c^{\lambda}_{(0,-j+l)}(\tilde{x},\tilde{\xi}).\end{aligned}\tag{3.83}$$

Similarly $c_{(-k,m)}(y,\xi) = c^{\lambda}_{(-k+j,-j+m)}(\tilde{y},\tilde{\xi})$. Note that here we have $-j+l \geq 2$, $-k+j \geq 1$ and $-j+m \geq 1$. The function $\beta_{can}(\theta)a^{tr}_{d_m}(\tilde{y},\tilde{\xi})$ in (3.83) nonvanishes for $(\tilde{y},\tilde{\xi}) \in \operatorname{supp}\chi_0(\tilde{y})\chi_{-j+l}(\tilde{\xi})$.

When $b^{tr}_{d_m} = b^{tr1}_{d_m}$, we calculate the function $\beta_{can}(\theta)a^{tr}_{d_m}(\tilde{y},\tilde{\xi})$ as Case 1 or Case 2 of the Part I of the proof of Lemma 3.4. The discrimination of Case 1 and Case 2, i.e., $\varepsilon^{-1}|y||\xi| \leq c_u^{-1}|y|(d_m)^{1/2s}$ or $\varepsilon^{-1}|y||\xi| > c_u^{-1}|y|(d_m)^{1/2s}$ is now

$$|\lambda\tilde{\xi}| \leq C(d_m)^{1/2s} \text{ or } |\lambda\tilde{\xi}| > C(d_m)^{1/2s} \tag{3.84}$$

in new variables. And $\beta_{can}(\theta)a^{tr}_{d_m}(\tilde{y},\tilde{\xi})$ equals to $C(\cos\theta_0)(d_m)^{(1-1/s)}|\lambda\tilde{\xi}|^{2-2s}$ if the former holds, equals to $C(\cos\theta_0)$ if the later holds. The function

$$|\tilde{y}/\lambda|_c^{1-2s} = \mathbf{c}(\tilde{y}/\lambda)|\tilde{y}/\lambda|^{1-2s} = \bar{\mathbf{c}}(\tilde{y})|\tilde{y}/\lambda|^{1-2s} \tag{3.85}$$

is bounded and its derivatives descend in order, see definition (3.3). Furthermore the integration part of $c^{\lambda}_{(0,-j+l)}(\tilde{x},\tilde{\xi})$ is a symbol for the compositions of pseudo-differential operators whose calculation can be done by (3.23). Thus $c^{\lambda}_{(0,-j+l)}(\tilde{x},\tilde{\xi})$ has principle symbol

$$C(\cos\theta_0)(d_m)^{(1-1/s)}|\tilde{x}/\lambda|^{1-2s}\chi_0(\tilde{x})\chi_{-j+l}(\tilde{\xi})\langle\lambda\tilde{\xi}\rangle^{-2}|\lambda\tilde{\xi}|^2 \tag{3.86}$$

if the former of (3.84) holds, principle symbol

$$C(\cos\theta_0)|\tilde{x}/\lambda|^{1-2s}\chi_0(\tilde{x})\chi_{-j+l}(\tilde{\xi})|\lambda\tilde{\xi}|^{2s}\langle\lambda\tilde{\xi}\rangle^{-2} \tag{3.87}$$

if the latter of (3.84) holds. By (3.85), the term $|\tilde{x}/\lambda|^{1-2s}$ in (3.86) and (3.87) is a symbol of order 0. And we can conclude that $c^{\lambda}_{(0,-j+l)}(\tilde{x},\tilde{\xi})$ is a symbol with principle symbol

$$C(\cos\theta_0)(d_m)^{(1-1/s)} \tag{3.88}$$

if the former of (3.84) holds, with principle symbol

$$C(\cos\theta_0)|\lambda\tilde{\xi}|^{2s}\langle\lambda\tilde{\xi}\rangle^{-2} \tag{3.89}$$

if the latter of (3.84) holds, here $|\lambda\tilde{\xi}|^{2s}\langle\lambda\tilde{\xi}\rangle^{-2} \leq C(d_m)^{(1-1/s)}$. The similar calculation works for $c^{\lambda}_{(-k+j,-j+m)}(\tilde{y},\tilde{\xi})$. From

$$\begin{aligned}
&K_{(-j,l),(-k,m)}(x,y)\\
&= \lambda^3 \int e^{i(\tilde{x}-\tilde{y})\cdot\tilde{\xi}} c^{\lambda}_{(0,-j+l)}(x,\xi)\overline{c^{\lambda}_{(-k+j,-j+m)}(y,\xi)}d\tilde{\xi}\\
&= \lambda^3 K^{\lambda}_{(0,-j+l),(-k+j,-j+m)}(\tilde{x},\tilde{y})
\end{aligned}$$

and

$$S_{(-j,l)}S^*_{(-k,m)}v(\lambda^{-1}\tilde{x}) = \int K^{\lambda}_{(0,-j+l),(-k+j,-j+m)}(\tilde{x},\tilde{y})\, v(\lambda^{-1}\tilde{y})d\tilde{y},$$

we know $\|S_{(-j,l)}S^*_{(-k,m)}\|_{L^2_x\to L^2_y} = \|S_{(0,-j+l)}S^*_{(-k+j,-j+m)}\|_{L^2_{\tilde{x}}\to L^2_{\tilde{y}}}$ where the later can be estimated as Lemma 2.1 whose value depends on $|j-k|$ and $|l-m|$ (see the paragraph after (3.57)) with the coefficient $(d_m)^{2(1-1/s)}$. We concludes (3.81). The function $a_{cut}(x,\xi)$ non-vanishes if the later of (3.84) holds. Then we need to prove

$$\left\| \langle D_x\rangle^{-(1+\rho)}(|x|_c^{1-2s}T_{cut}h(x))\right\|_{L^2} \leq O((d_m)^{(1-1/s)})\|h\|_{H^{1-\rho}}.$$

Similarly to the Case 2 of Part I of the proof of Lemma 3.4, we write $a_{cut} = a_g - a_l$ and $T_{cut} = T_g - T_l$. The operator T_l can be estimated by the same way as we did above to T_{can} using change of variables. To estimate T_g, we also need consider the function a_g in different cones Γ_j defined in (3.31) and refine each cone into region II and region III (no more region I here due to support of $\chi_{B,2}$). Then estimate a_g and the operators on different regions by using change of variables as before so that a_g in the new variables non-vanishes only on the set with magnitudes larger than 2. We skip the details to avoid the repeat.

When $b^{tr}_{d_m} = b^{tr2}_{d_m}$, we calculate the function $\beta_{can}(\theta)a^{tr}_{d_m}(\tilde{y}, \tilde{\xi})$ of (3.83) as Case 2 of the part I of the proof of Lemma 3.4 since the later of (3.84) holds. Due to the definition of $b^{tr2}_{d_m}$ is different with that of $b^{tr1}_{d_m}$, we have $\operatorname{supp}\beta_{cut}(\theta) \subset [0, \pi/2 - \varepsilon|x|^{-1}|\xi|^{-1}]$ compared to (3.30). This difference does not affect the result in view of $|x| \le c_u(d_m)^{-1/2s}$, by (3.73). For example the term $(d_m)^{(1-1/s)}|x|^{-2}\langle\xi\rangle^{\rho-1}$ in (3.26) is now $|x|^{-2s}\langle\xi\rangle^{\rho-1} = |x|^{2-2s}|x|^{-2}\langle\xi\rangle^{\rho-1} \le C(d_m)^{(1-1/s)}|x|^{-2}\langle\xi\rangle^{\rho-1}$. Since the discussion is essential the same as last paragraph, we can close the discussion of T_B.

Next we turn to the proof of

$$\left\| \langle D_x\rangle^{-(1+\rho)}(|x|_c^{1-2s} T_{C,2}h(x)) \right\|_{L^2} \le O((d_m)^{(1-1/s)})\|h\|_{H^{1-\rho}}. \tag{3.90}$$

Again we consider only $-1 < \rho \le 1$ for $\rho = -1$ is an easier case. Also the above estimate is equivalent to

$$\left\| \langle D_x\rangle^{-(1+\rho)}(|x|_c^{1-2s} T^{\rho-1}_{C,2}h(x)) \right\|_{L^2} \le O((d_m)^{(1-1/s)})\|h\|_{L^2} \tag{3.91}$$

where

$$T^{\rho-1}_{C,2}h(x) = \int e^{ix\cdot\xi} a_{C,2}(x,\xi)|\xi|^{2s}\langle\xi\rangle^{\rho-1}h(\xi)d\xi$$

with

$$\begin{aligned} a_{C,2}(x,\xi) &= \chi_{C,2}(x,\xi)\, a^{tr}_{d_m}(x,\xi) \\ &= \chi_{C,2}(x,\xi)\, |x|^{-2s}|\xi|^{-2s} \int\limits_{S^2} b^{tr}_{d_m}(\cos\theta)(e^{-i(x\cdot\omega)(\xi\cdot\omega)} - 1)d\Omega(\omega). \end{aligned}$$

Recall the definition of $\chi_{C,2}(x,\xi)$ and write it as

$$\begin{aligned} \chi_{C,2}(x,\xi) &= \sum_{j+k\le 1, j\le 0} \chi_j(x)\chi_k(\xi) = \sum_{j\le 0}[\sum_{k\le 1-j} \chi_j(x)\chi_k(\xi)] \\ &\stackrel{\text{def}}{=} \sum_{j\le 0} \chi_{C,2j}(x,\xi). \end{aligned}$$

We define

$$\sum_{k\le 0} \chi_1(x)\chi_k(\xi) \stackrel{\text{def}}{=} \chi_{cp}(x,\xi). \tag{3.92}$$

Then we have the decomposition

$$\begin{aligned} &a_{C,2}(x,\xi) = \chi_{C,2}(x,\xi)a^{tr}_{d_m}(x,\xi) = \sum_{j\le 0}\chi_{C,2j}(x,\xi)a^{tr}_{d_m}(x,\xi) \stackrel{\text{def}}{=} \sum_{j\le 0} a_{C,2j}(x,\xi), \\ &\langle D_x\rangle^{-(1+\rho)}(|x|_c^{1-2s}T^{\rho-1}_{C,2}h(x)) \\ &= \langle D_x\rangle^{-(1+\rho)}(|x|_c^{1-2s}\int e^{ix\cdot\xi}a_{C,2}(x,\xi)|\xi|^{2s}\langle\xi\rangle^{\rho-1}h(\xi)d\xi) \\ &= \iiint e^{i(x-y)\cdot\eta}\langle\eta\rangle^{-(1+\rho)}e^{iy\cdot\xi}|y|_c^{1-2s}a_{C,2}(y,\xi)|\xi|^{2s}\langle\xi\rangle^{\rho-1}h(\xi)d\xi dy d\eta \\ &= \int e^{ix\cdot\xi}\sum_{j\le 0}c_j(x,\xi)\,h(\xi)d\xi \\ &= \int K(x,\xi)h(\xi)d\xi \end{aligned} \tag{3.93}$$

where

$$c_j(x,\xi) = |\xi|^{2s}\langle\xi\rangle^{\rho-1}\iint e^{i(x-y)\cdot(\eta-\xi)}\langle\eta\rangle^{-(1+\rho)}|y|^{1-2s}a_{C,2j}(y,\xi)dy d\eta. \tag{3.94}$$

We will estimate the function $c_j(x,\xi)$ by change of variables and the argument in proof of the Part 3 of Lemma 3.4. Let

$$\lambda = 2^{-(j-1)},\ x = \lambda^{-1}\tilde{x},\ \xi = \lambda\tilde{\xi},\ y = \lambda^{-1}\tilde{y},\ \eta = \lambda\tilde{\eta}. \tag{3.95}$$

Then $\chi_{C,2j}(y,\xi) = \chi_{cp}(\tilde{y},\tilde{\xi})$ where χ_{cp} is given in (3.92). And $a_{C,2j}(y,\xi) = \chi_{cp}(\tilde{y},\tilde{\xi})a^{tr,\lambda}_{d_m}(\tilde{y},\tilde{\xi})$ where

$$a^{tr,\lambda}_{d_m}(\tilde{y},\tilde{\xi}) = |\tilde{y}|^{-2s}|\tilde{\xi}|^{-2s}\int\limits_{S^2} b^{tr,\lambda}_{d_m}(\cos\theta)(e^{-i(\tilde{y}\cdot\omega)(\tilde{\xi}\cdot\omega)}-1)d\Omega(\omega),$$

$$b^{tr,\lambda}_{d_m}(\cos\theta) = \begin{cases} b^{tr1,\lambda}_{d_m}(\cos\theta), & \text{if } |\tilde{y}| > c_u\lambda(d_m)^{-1/2s}, \\ b^{tr2,\lambda}_{d_m}(\cos\theta), & \text{if } |\tilde{y}| \le c_u\lambda(d_m)^{-1/2s}, \end{cases} \tag{3.96}$$

with

$$b^{tr1,\lambda}_{d_m}(\cos\theta) = \begin{cases} (\cos\theta)^{-(1+2s)}, & \text{if } \cos(\theta) \le c_u\lambda|\tilde{y}|^{-1}(d_m)^{-1/2s}, \\ 0, & \text{others}, \end{cases} \tag{3.97}$$

$$b^{tr2,\lambda}_{d_m}(\cos\theta) = (\cos\theta)^{-(1+2s)},\ 0 \le \theta < \pi/2. \tag{3.98}$$

Since $|\tilde{y}||\tilde{\xi}| = |y||\xi| < 128$ on the support of $\chi_{cp}(\tilde{y},\tilde{\xi})$, we can calculate $a^{tr,\lambda}_{d_m}(\tilde{y},\tilde{\xi})$ as the Case 1. of the part I of the proof of Lemma 3.4 by letting d_m be large enough. When $b^{tr,\lambda}_{d_m} = b^{tr1,\lambda}_{d_m}$, we have

$$a^{tr,\lambda}_{d_m}(\tilde{y},\tilde{\xi}) = C\begin{cases} (d_m)^{(1-1/s)}\lambda^{2-2s}|\tilde{\xi}|^{2-2s}, & \text{if } |\tilde{y}||\tilde{\xi}| \ge 1 \\ (d_m)^{1-1/s}\lambda^{2-2s}|\tilde{y}|^{-1}|\tilde{\xi}|^{1-2s}, & \text{if } |\tilde{y}||\tilde{\xi}| < 1. \end{cases} \tag{3.99}$$

When $b^{tr,\lambda}_{d_m} = b^{tr2,\lambda}_{d_m}$, we have

$$a^{tr,\lambda}_{d_m}(\tilde{y},\tilde{\xi}) = C\begin{cases} |\tilde{y}|^{2-2s}|\tilde{\xi}|^{2-2s}, & \text{if } |\tilde{y}||\tilde{\xi}| \ge 1 \\ |\tilde{y}|^{1-2s}|\tilde{\xi}|^{1-2s}, & \text{if } |\tilde{y}||\tilde{\xi}| < 1. \end{cases} \tag{3.100}$$

Note that $|\tilde{y}| \le c_u\lambda(d_m)^{-1/2s}$ implies $C|\tilde{y}|^{2-2s}|\tilde{\xi}|^{2-2s} \le C(d_m)^{1-1/s}\lambda^{2-2s}|\tilde{\xi}|^{2-2s}$ and $C|\tilde{y}|^{1-2s}|\tilde{\xi}|^{1-2s} \le C(d_m)^{1-1/s}\lambda^{2-2s}|\tilde{y}|^{-1}|\tilde{\xi}|^{1-2s}$. Applying (3.95) to $c_j(x,\xi)$ given in (3.94), using calculations of $a^{tr,\lambda}_{d_m}(\tilde{y},\tilde{\xi})$ above and the method used in the estimate of (3.66) in Part 3. of Lemma 3.4, we conclude that (3.94) is bounded by

$$\begin{cases} O((d_m)^{1-1/s})|\lambda^{-1}\tilde{x}|^{1-2s}\chi_{cp}(\tilde{x},\tilde{\xi}), & \text{if } |\tilde{x}||\tilde{\xi}| \ge 1, \\ O((d_m)^{1-1/s})|\lambda^{-1}\tilde{x}|^{-2s}\langle\lambda\tilde{\xi}\rangle^{-1}\chi_{cp}(\tilde{x},\tilde{\xi}), & \text{if } |\tilde{x}||\tilde{\xi}| < 1. \end{cases}$$

Equivalently

$$|c_j(x,\xi)| \le \begin{cases} O((d_m)^{1-1/s})|x|^{1-2s}\chi_{C,2j}(x,\xi), & \text{if } |x||\xi| \ge 1, \\ O((d_m)^{1-1/s})|x|^{-2s}\langle\xi\rangle^{-1}\chi_{C,2j}(x,\xi), & \text{if } |x||\xi| < 1. \end{cases}$$

From above and the definition of $\chi_{C,2j}$, we have the estimate for $K(x,\xi)$ of (3.93),

$$|K(x,\xi)| \le \begin{cases} O((d_m)^{1-1/s})|x|^{1-2s}\chi_{C,2}(x,\xi), & \text{if } |x||\xi| \ge 1, \\ O((d_m)^{1-1/s})|x|^{-2s}\langle\xi\rangle^{-1}\chi_{C,2}(x,\xi), & \text{if } |x||\xi| < 1. \end{cases} \tag{3.101}$$

It remains to check that $|K(x,\xi)|$ satisfies Schur test to conclude (3.91). When $|x||\xi| > 1$, we let $p(x) = |x|^{-2}$ and $q(\xi) = (1+|\xi|)^{-1}$. Then for any fixed $|x_0| < 8$,

$$\begin{aligned} \int |x_0|^{1-2s}\chi_{C,2}(x_0,\xi)\, q(\xi)d\xi &\le C\int\limits_{\{|\xi|<128|x_0|^{-1}\}} q(\xi)d\xi \\ &= C\int\limits_0^{128|x_0|^{-1}} \langle r\rangle^{-1}r^2dr \\ &\le C|x_0|^{-2} = Cp(x_0). \end{aligned} \tag{3.102}$$

For $|\xi_0| \le 16$,

$$\begin{aligned}\int |x|^{1-2s}\chi_{C,2}(x,\xi_0)\, p(x)dx &\le C\int \chi_{C,2}(x,\xi_0)\, p(x)dx \\ &= \int_0^8 r^{-2}r^2dr \le C \le Cq(\xi_0).\end{aligned} \tag{3.103}$$

For $|\xi_0| > 16$,

$$\begin{aligned}\int |x|^{1-2s}\chi_{C,2}(x,\xi_0)\, p(x)dx &\le C\int \chi_{C,2}(x,\xi_0)\, p(x)dx \\ &= C\int_0^{128|\xi_0|^{-1}} r^{-2}r^2dr \le C|\xi_0|^{-1} \le Cq(\xi_0).\end{aligned} \tag{3.104}$$

We note that in the estimates (3.102), (3.103) and (3.104), the factor $|x|^{1-2s} < 8^{1-2s}$ was not used. When $|x||\xi| < 1$, we let $p(x) = |x|^{-1-2s}$ and $q(\xi) = \langle\xi\rangle^{-1}$. Then

$$\begin{aligned}&\int_{\{|x_0||\xi|<1\}} |x_0|^{-2s}\langle\xi\rangle^{-1}\chi_{C,2}(x_0,\xi)\, q(\xi)d\xi \\ &= \int_{\{|\xi|<|x_0|^{-1}\}} |x_0|^{-2s}\langle\xi\rangle^{-1}q(\xi)d\xi \\ &\le C|x_0|^{-1-2s} = Cp(x_0) \\ &\int_{\{|x_0||\xi|<1\}} |x|^{-2s}\langle\xi_0\rangle^{-1}\chi_{C,2}(x,\xi_0)\, p(x)dx \\ &= \int_{|x|\le\{|\xi_0|^{-1} \text{ or } 8\}} \langle\xi_0\rangle^{-1}|x|^{-2s}|x|^{-1-2s}dx \\ &\le C(s)\langle\xi_0\rangle^{-1} = C(s)q(\xi_0).\end{aligned} \tag{3.105}$$

We remark that the coefficient $O((d_m)^{1-1/s})$ grows as s tends to $1/2$ due to last estimate above. □

Proof of Lemma 3.3. Small relative velocity, soft potential.

For $1/2 \le s < 1$, $\rho \in [-\frac{1}{2}-s, \frac{1}{2}+s]$ we wish to prove that

$$\langle Q^{tr}_{d_m,c}(f,f),h\rangle \le O((d_m)^{N(s)})\|f\|_{L^1}\|f\|_{H^{\frac{1}{2}+s+\rho}}\|h\|_{H^{\frac{1}{2}+s-\rho}} \tag{3.6}$$

where $N(s) = (1-s^\kappa)(1-1/s) < 0$ and small $\kappa > 0$ satisfies $2s^\kappa - 2s^{1+\kappa} - s > 0$.

By the same reduction as the proof of Lemma 3.2, (3.6) is implied by the estimate

$$\left\|\langle D_x\rangle^{-(\frac{1}{2}+s+\rho)}(|x|_c^{1-2s}T^{tr}_{d_m}h(x))\right\|_{L^2} \le O((d_m)^{N(s)})\|h\|_{H^{\frac{1}{2}+s-\rho}} \tag{3.106}$$

where $T^{tr}_{d_m}$ is defined in (3.78). The proof of above estimate is essential the same as that for the estimate (3.77). Below we explain how the factor $|x|_{1-2s}$ for $1/2 \le s < 1$ can be controlled by giving more regularity to f and h. Using (3.80), the operator $T^{tr}_{d_m}$ was split into $T_{B,2}$ and $T_{C,2}$ and the estimate (3.106) is the consequence following estimates,

$$\left\|\langle D_x\rangle^{-(\frac{1}{2}+s+\rho)}(|x|_c^{1-2s}T_{B,2}h(x))\right\|_{L^2} \le O((d_m)^{(1-1/s)})\|h\|_{H^{\frac{1}{2}+s-\rho}}, \tag{3.107}$$

$$\left\|\langle D_x\rangle^{-(\frac{1}{2}+s+\rho)}(|x|_c^{1-2s}T_{C,2}h(x))\right\|_{L^2} \le O((d_m)^{N(s)})\|h\|_{H^{\frac{1}{2}+s-\rho}}. \tag{3.108}$$

The estimate (3.107) has the same decay rate as before. Replacing the exponents $-(1+\rho), 1-\rho$ with $-(\frac{1}{2}+s+\rho), \frac{1}{2}+s-\rho$ respectively in the proof of the estimate about the operator $T_{B,2}$ in Lemma 3.2, we see that (3.86) and (3.87) become respectively

$$C(\cos\theta_0)(d_m)^{(1-1/s)}\,\lambda^{2s-1}|\tilde{x}|^{1-2s}\chi_0(\tilde{x})\chi_{-j+l}(\tilde{\xi})\langle\lambda\tilde{\xi}\rangle^{-1-2s}|\lambda\tilde{\xi}|^2,$$
$$C(\cos\theta_0)\,\lambda^{2s-1}|\tilde{x}|^{1-2s}\chi_0(\tilde{x})\chi_{-j+l}(\tilde{\xi})\langle\lambda\tilde{\xi}\rangle^{-(1+2s)}|\lambda\tilde{\xi}|^{2s}.$$

Here we can drop $|\tilde{x}|^{1-2s}$ since $s > 1/2$ and need to take care of λ. Use above estimates, the correspondences of (3.88) and (3.89) are respectively

$$C(\cos\theta_0)(d_m)^{(1-1/s)}\langle\tilde{\xi}\rangle^{1-2s},$$
$$C(\cos\theta_0)\langle\lambda\tilde{\xi}\rangle^{-1}\lambda^{2s-1} \le C(\cos\theta_0)\langle\tilde{\xi}\rangle^{-2+2s}\lambda^{2s-2} \le C(\cos\theta_0)(d_m)^{(1-1/s)}.$$

These first symbol is in our favor being negative order since $s \ge 1/2$. The remaining part of the proof for estimate (3.107) is the repeat of Lemma 3.2 for $T_{B,2}$ part using above results. Hence we turn to the discussion of $T_{C,2}$. Again we follows the argument in Lemma 3.2 for the proof about estimate $T_{C,2}$ by replacing the exponents $-(1+\rho), 1-\rho$ with $-(\frac{1}{2}+s+\rho), \frac{1}{2}+s-\rho$ respectively. It is straightforward to check that the correspondence of (3.101) is

$$|K(x,\xi)| \le \begin{cases} O((d_m)^{1-1/s})|x|^{1-2s}|\xi|^{1-2s}\chi_{C,2}(x,\xi), & \text{if } 1 \le |x||\xi|, \\ O((d_m)^{1-1/s})|x|^{-2s}\langle\xi\rangle^{-2s}\chi_{C,2}(x,\xi), & \text{if } |x||\xi| < 1. \end{cases} \tag{3.109}$$

Since $1-2s \le 0$ and $|x||\xi| \ge 1$, the first line of (3.109) satisfies Schur's test by (3.102), (3.103) and (3.104). The second line of (3.109) does not satisfy Schur test. Thus we need to sacrifice the decay rate here. Let $P(s) = 2s^\kappa - 2s^{1+\kappa}$ and note $-s < P(s) - 2s < 0$. We dominate the second line of (3.99), using $|\tilde{y}| > c_u\lambda(d_m)^{1-/2s}$, as

$$(d_m)^{1-1/s}|\lambda^{-1}\tilde{y}|^{-1}|\lambda\tilde{\xi}|^{1-2s} \le C(d_m)^{N(s)}|\lambda^{-1}\tilde{y}|^{P(s)-1}|\lambda\tilde{\xi}|^{1-2s}.$$

We also dominate the second line of (3.100), using $|\tilde{y}| \le c_u\lambda(d_m)^{-1/2s}$, as

$$|\tilde{y}|^{1-2s}|\tilde{\xi}|^{1-2s} \le C(d_m)^{N(s)}|\lambda^{-1}\tilde{y}|^{P(s)-1}|\lambda\tilde{\xi}|^{1-2s}.$$

Using these two results, the second line of (3.109) is replaced by

$$O((d_m)^{N(s)})|x|^{P(s)-2s}\langle\xi\rangle^{-2s}\chi_{C,2}(x,\xi)\ ,\ \text{if } |x||\xi| < 1$$

which is bounded by

$$O((d_m)^{N(s)})|x|^{-s}\langle\xi\rangle^{-2s}\chi_{C,2}(x,\xi) \tag{3.110}$$

since $-s < P(s) - 2s < 0$. Replacing $2s, 0 < s < 1/2$ in (3.105) by $s, 1/2 \le s < 1$, it is clear the bound (3.110) satisfies Schur test. □

Proof of Theorem 1.1. The proof can be done by exactly the same argument for proof of Theorem 1.2. The difference comes from $B^{tr}_{d_m}$ in (3.1) is now replaced by

$$B^H_{d_m} = \begin{cases} B^{H_s}_{d_m}(\mathcal{V},\theta) & \text{if } c_u\mathcal{V}^{-1}(d_m)^{-1/2s} \ge 1 \\ B^{H_l}_{d_m}(\mathcal{V},\theta) & \text{if } c_u\mathcal{V}^{-1}(d_m)^{-1/2s} < 1 \end{cases}$$

with $B^{H_s}_{d_m}$ defined in (1.6) and $B^{H_l}_{d_m}$ defined in (1.7). And therefore the operators $T^{tr}_{d_m}$ of (3.8) and (3.72) have to be replaced by

$$T^H_{d_m}h(x) = \int_{\omega\in S^2} |x|^{2s}(h(x-(x\cdot\omega)\omega) - h(x))b^H_{d_m}(\cos\theta)d\Omega(\omega) \tag{3.111}$$

with $\cos\theta = x\cdot\xi/|x||\xi|$ where

$$b^H_{d_m}(\cos\theta) = \begin{cases} b^{H_s}(\cos\theta) & \text{if } c_u|x|^{-1}(d_m)^{-1/2s} \ge 1 \\ b^{H_l}(\cos\theta) & \text{if } c_u|x|^{-1}(d_m)^{-1/2s} < 1. \end{cases} \qquad \square$$

4. Auxiliary lemmas

4.1. Collision kernel B_{d_m}

In this subsection, we calculate the formula of $B_{d_m}(\mathcal{V}, \theta)$. Recall that the inverse power law model means the intermolecular potential is given by

$$U(r) = c_{\overline{u}} r^{-(n-1)}, n > 2$$

where $c_{\overline{u}}$ is a constant. The collision kernel B_{d_m} is the one with inter-molecular potential $U(r)$ being cut-off at $r = d_m$. We define

$$s = (n-1)^{-1}.$$

The collision process for finite range potential is that the particle will convert the kinetic energy into potential energy, then at the critical point the process reverses if the particle penetrate the force field U_{d_m}, Fig. 1, otherwise it will bounce back like hard sphere. Thus two different cases of collision are under consideration.

Case 1.

The incoming particle will bounce back like hard sphere if kinetic energy or the projection of relative velocity to normal direction of potential is smaller than $U(d_m)$, i.e.,

$$\frac{1}{4} m(\mathcal{V} \cos\theta_e)^2 \le U(d_m) \tag{4.1}$$

where m is the mass of the particle. This gives (1.6) and (1.7), $\mathcal{V} d_m^2 \cos\theta_e \sin\theta_e = \mathcal{V} d_m^2 \cos\theta \sin\theta$, with $c_u = (4c_{\overline{u}} m^{-1})^{1/2}$.

Case 2.

The incoming particle penetrates the force field if

$$\frac{1}{4} m(\mathcal{V} \cos\theta_e)^2 > U(d_m). \tag{4.2}$$

In this case, the motion of the particle can be described by the system of equation

$$\begin{aligned} &\tfrac{1}{4} m(\dot{r}^2 + r^2\dot{\theta}^2) + U(r) = \tfrac{1}{4} m\mathcal{V}^2 \\ &r^2\dot{\theta} = -\mathcal{V} b \end{aligned} \tag{4.3}$$

which comes from conservation of energy and angular momentum respectively, Fig. 1. Here $\dot{r}, \dot{\theta}$ denote the derivatives of r, θ with respect to time. Solving above equations, we obtain

$$\begin{aligned} \theta &= \sin^{-1}\frac{b}{d_m} + b\int_{r_c}^{d_m} r^{-2}\Big(1 - \frac{4U(r)}{m\mathcal{V}^2} - \frac{b^2}{r^2}\Big)^{-1/2} dr \\ &:= \theta_e + \theta_p \end{aligned} \tag{4.4}$$

where r_c is the minimum solution of

$$1 - \frac{4U(r)}{m\mathcal{V}^2} - \frac{b^2}{r^2} = 0. \tag{4.5}$$

For the formula of the collision kernel

$$B(\mathcal{V}, \theta) = \mathcal{V} b \frac{db}{d\theta} d\theta d\phi,$$

we need to calculate $\frac{db}{d\theta}$ using (4.4). Cercignani [6] used (4.4) to get the estimate

$$b \simeq d_m[1 + s^{-1}(1 + K\mathcal{V}^2)^2 \cos^2\theta]^{\frac{-1}{2(n-1)}} \tag{4.6}$$

where $K = -m[2d_m(\partial U/\partial r)_{r=d_m}]^{-1}$. Then he used (4.6) to get

$$B(\theta, \mathcal{V}) \simeq d_m^2 \mathcal{V}(1+K\mathcal{V}^2)^2 \cos\theta \sin\theta \cdot [1+s^{-1}(1+K\mathcal{V}^2)^2\cos^2\theta]^{-(1+s)}. \tag{4.7}$$

However the hard sphere reflection (Case 1 above) was not included in his discussion. His formula (4.7) matches the second expression of (1.8) when $K\mathcal{V}^2$ is larger than 1. Also the condition $K\mathcal{V}^2 > 1$ is equivalent to the requirement $\cos\theta > c_u \mathcal{V}^{-1}(d_m)^{-1/2s}$ in (1.8). In fact the calculation for the finite range may follow the one for the infinite range. Let

$$x = \frac{b}{r},\ g = (\frac{m\mathcal{V}^2}{4c_{\overline{u}}})^{1/(n-1)} b,\ x_c = \frac{b}{r_c}$$

and note that x_c solves

$$1 - x^2 - (\frac{x}{g})^{n-1} = 0. \tag{4.8}$$

For large g, we want to estimate $dg/d\theta$ instead of $db/d\theta$. Rewrite

$$\begin{aligned}\theta_p &= \int_{\frac{b}{d_m}}^{x_c} \frac{dx}{\sqrt{1-x^2-(\frac{x}{g})^{n-1}}}\\ &= \int_{\frac{b}{d_m}}^{x_c} \frac{dx}{\sqrt{x_c^2+(\frac{x_c}{g})^{n-1}-x^2-(\frac{x}{g})^{n-1}}}\\ &= \int_{\frac{r_c}{d_m}}^{1} \frac{dy}{\sqrt{(1-y^2)[1+\frac{x_c^{(n-3)}}{g^{n-1}}(\frac{1-y^{n-1}}{1-y^2})]}}\\ &\approx \int_{\frac{r_c}{d}}^{1} \frac{dy}{\sqrt{1-y^2}}(1-\frac{x_c^{(n-3)}}{2g^{n-1}}(\frac{1-y^{n-1}}{1-y^2})) \text{ if } g \text{ is large}\end{aligned} \tag{4.9}$$

Using (4.8), (4.9), it is not hard to see that

$$\frac{\pi}{2} - \theta = O(g^{-(n-1)}) = O(g^{-1/s})$$

or

$$g = O((\frac{\pi}{2}-\theta)^{-1/(n-1)}) = O((\frac{\pi}{2}-\theta)^{-s}). \tag{4.10}$$

As the case of infinite rage potential, the second line of (1.8) follows by taking derivative to (4.10). The big O of (4.10) for finite range potential is uniform bounded above and below with respect to d_m. However there is a dichotomy for the derivative $dg/d\theta$ when we let d_m tends to infinity. First we note that

$$\frac{d\theta_e}{dg} = \frac{b}{d_m}(\frac{1}{g})\frac{1}{\sqrt{1-(\frac{b}{d_m})^2}} = \frac{\sin\theta_e}{g\cos\theta_e} \tag{4.11}$$

If $\lim_{d_m\to\infty} \frac{b}{d_m} = 0$ then $d\theta_e/dg$ could be smaller than the term $d\theta_p/dg = O(g^{-(1+1/s)})$ by asking sufficient large d_m. This means the coefficient c_{d_m} in the second line of (1.8) will tends to the constant, c_∞ of (1.11), as d_m tends to infinity.

On the other hand, if $\lim_{d_m\to\infty} \frac{b}{d_m} \neq 0$ and g is large. We see that $d\theta_e/dg$ is either of order g^{-1} or tends to ∞ (when $\cos\theta_e \approx c_u\mathcal{V}^{-1}(d_m)^{-1/s}$). In either case $d\theta_e/dg$ dominates the term $d\theta_p/dg = O(g^{-(1+1/s)})$. This means that c_{d_m} in the second line of (1.8) is much smaller than the c_{d_m} for the case $\lim_{d_m\to\infty}\frac{b}{d_m} = 0$. Heuristically, we can also observe that smallness of c_{d_m} is necessary for $\cos\theta$ close to $c_u\mathcal{V}^{-1}(d_m)^{-1/2s}$ so that the value of (1.8) at the critical angle $\cos\theta = c_u\mathcal{V}^{-1}(d_m)^{-1/2s}$ can match the value of (1.7) by the continuity of B_{d_m}.

Hence we conclude

Lemma 4.1. *When we consider the limit of B_{d_m} as d_m tends to ∞. There is a dichotomy to the coefficient c_{d_m} in the second line of* (1.8). *If* $\lim_{d_m\to\infty} \frac{b}{d_m} = 0$, *the coefficient c_{d_m} tends to a constant c_∞ in* (1.11). *If* $\lim_{d_m\to\infty} \frac{b}{d_m} \neq 0$, *the coefficient c_{d_m} is smaller than previous case and may tends to* 0.

4.2. Lemmas from harmonic analysis

We need the following Schur's test lemma ([14], Theorem 5.2). See also Stein's book [24] p. 284 for a special case and Sogge's book [21] Theorem 0.3.1 for the related Young's inequality.

Lemma 4.2 *(Schur test lemma). Let X, Y be two measurable spaces. Let T be an integral operator with the non-negative Schwartz kernel, i.e.*

$$Tf(x) = \int_Y K(x,y)f(y)dy.$$

If there exist functions $p(x) > 0$ and $q(x) > 0$ and numbers $\alpha,\ \beta > 0$ such that

$$\int_Y K(x,y)q(y)dy \le \alpha p(x)$$

for almost all x and

$$\int_X K(x,y)p(x)dx \le \beta q(y)$$

for almost all y. Then T is a continuous operator $L^2 \to L^2$ with the operator norm

$$\|T\|_{L^2\to L^2} \le \sqrt{\alpha\beta}$$

We also need the following Coltar-Stein lemma (see [24]).

Lemma 4.3 *(Coltar-Stein lemma). Assume a family of L^2 bounded operators $\{T_j\}_{j\in Z^n}$ and a sequence of positive constants $\{\gamma(j)\}_{j\in Z^n}$ satisfy*

$$\|T_i^* T_j\|_{L^2\to L^2} \le \{\gamma(i-j)\}^2\ ,\ \|T_i T_j^*\|_{L^2\to L^2} \le \{\gamma(i-j)\}^2$$

and

$$M = \sum_{j\in Z^n} \gamma(j) < \infty.$$

Then the operator $T = \sum_{j\in Z^n} T_j$ satisfies

$$\|T\|_{L^2\to L^2} \le M.$$

One of the most powerful tool in estimating the oscillatory integral

$$I_\Lambda(u,f) = \int_{\mathbb{R}^n} e^{i\Lambda f(y)} u(y)dy,$$

for large Λ is the following lemma of stationary phase asymptotics. There are several versions used widely, here we only record one of these which is from Theorem 7.7.5 of Hörmander [13]. We use the notation $D_j = -i\partial_j$.

Theorem 4.4 *(Stationary phase asymptotics). Let $K \subset \mathbb{R}^n$ be a compact set, X an open neighborhood of K and k a positive number. If $u \in C_c^{2k}(\mathbb{R}^n)$, $f \in C^{3k+1}(X)$ and $\operatorname{Im} f \ge 0$ in X, $\operatorname{Im} f(y_0) = 0$, $f'(y_0) = 0$, $\det f''(y_0) \neq 0$, $f' \neq 0$ in $K\backslash\{y_0\}$ then*

$$|I - e^{i\Lambda f(y_0)}(\det(\Lambda f''(y_0)/2\pi i))^{-1/2} \sum_{j<k} \Lambda^{-j} L_j u| \leq C\Lambda^{-k} \sum_{|\alpha|\leq 2k} \sup|D^\alpha u|\,,\ \Lambda > 0 \tag{4.12}$$

Here C is bounded when f stays in a bounded set in $C^{3k+1}(X)$ and $|y-y_0|/|f'(y)|$ has a uniform bound. With

$$g_{y_0}(y) = f(y) - f(y_0) - \langle f''(y_0)(y-y_0), y-y_0\rangle/2$$

which vanish of third order at x_0 we have

$$L_j u = \sum_{\nu-\mu=j} \sum_{2\nu\geq 3\mu} i^{-j} 2^{-\nu} \langle f''(y_0)^{-1} D, D\rangle^\nu (g_{y_0}^\mu u)(y_0)/\mu!\nu! \tag{4.13}$$

which is a differential operator of order $2j$ acting on u at y_0. The coefficients are rational homogeneous functions of degree $-j$ in $f''(y_0), \cdots, f^{2j+2}(y_0)$ with denominator $(\det f''(y_0))^{3j}$. In every term the total number of derivatives of u and f'' is at most $2j$.

Conflict of interest statement

There is no conflict of interest in this paper.

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