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# Global solution of 3D irrotational flow for gas dynamics in thermal nonequilibrium

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**Abstract**

We study the three-dimensional irrotational flow for gas dynamics in thermal nonequilibrium. The global existence and large time behavior of the classical solution to the Cauchy problem are established when the initial data are near the equilibrium state with an additional $L^1$-norm bound. We mention that the uniform bound on derivatives of the entropy is obtained by using the *a priori* decay-in-time estimate on the velocity.


*Keywords:* Gas dynamics in thermal nonequilibrium; Irrotational flow; Spectral analysis

## 1. Introduction

In this paper we study the three-dimensional vibrational nonequilibrium flow, where the motion of the gas is described by the following equations, cf. [11,19],

$$\begin{cases} \rho_t + \nabla\cdot(\rho u) = 0, \\ (\rho u)_t + \nabla\cdot(\rho u u^\top) + \nabla p = 0, \\ (\rho\mathcal{E})_t + \nabla\cdot(\rho\mathcal{E}u + pu) = 0, \\ (\rho q)_t + \nabla\cdot(\rho q u) = \rho\dfrac{Q-q}{\tau}. \end{cases} \tag{1.1}$$

Here $\rho, u = (u_1, u_2, u_3)^\top, p, \mathcal{E}, q, Q$ an $\tau$ are the gas density, velocity, pressure, specific total energy, specific vibrational energy, local equilibrium value of specific vibrational energy and local relaxation time, respectively. The first three equations of (1.1) are the conservation of mass, momentum and energy, while the last one describes the relaxation of the nonequilibrium vibrational mode to its local equilibrium value. The total energy $\mathcal{E}$ consists of internal energy and kinetic energy:

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$$\mathcal{E} = e + \frac{1}{2}|u|^2, \quad |u|^2 = \sum_{j=1}^{3} u_j^2,$$

where since we assume that the flow is everywhere in both instantaneous translational and rotational equilibrium, the internal energy $e$ is further divided into the equilibrium energy $e_1$ and the non-equilibrium energy $q$:

$$e = e_1 + q.$$

For gas flow with several thermal nonequilibrium modes, which is described by a hyperbolic system with several relaxation equations, we refer to [18].

The thermodynamic equations read as

$$de_1 = T_1 ds_1 - p dv, \quad dq = T_2 ds_2, \tag{1.2}$$

where $v = 1/\rho$ is the specific volume, $T_1, s_1$ are the equilibrium temperature and entropy while $T_2, s_2$ are the non-equilibrium temperature and entropy. Note that the vibrational energy $q$ is volume independent and for a nonequilibrium state it holds that $T_1 \neq T_2$. Then we arrive at the thermodynamics law for gas flow in vibrational nonequilibrium

$$de = T_1 ds + (T_2 - T_1) ds_2 - p dv,$$

where $s = s_1 + s_2$ is the total entropy. (1.2) yields that among $v, p, e_1, T_1$ and $s_1$ only two thermodynamic variables are independent, while among $q, T_2$ and $s_2$ any one variable determines the others. Particularly, both $Q$ and $\tau$ are given functions of $v$ and $e_1$, i.e.,

$$Q = Q(v, e_1), \quad \tau = \tau(v, e_1).$$

To be more specific, $Q$ is the local equilibrium value of $q$. $Q = q$ holds if and only if the gas is in equilibrium with local temperature $T_1 = T_2$. Let

$$q = \omega(T_2)$$

for some known increasing function $\omega$, which implies

$$Q = Q(v, e_1) = \omega(T_1). \tag{1.3}$$

We remark that when all relaxation processes within the gas take place infinitely rapidly as the internal structure of the molecules is negligible, that is $\tau \to 0$, it is necessary to have $q(t,x) = Q(t,x)$ in the equilibrium flow. On the other hand, when all relaxation processes take place infinitely slow, the model of frozen flow implies that $q(t,x) = q(0,x)$ as $\tau \to \infty$. In both limiting cases, the system (1.1) is reduced to the compressible Euler equations governing the motion of gas in local thermodynamic equilibrium, but with different equations of state, cf. [17].

Global existence and large time behavior of solutions for general hyperbolic systems with relaxation or other lower order dissipations under certain assumptions on the strong coupling of the inhomogeneous terms and the flux functions, e.g., the Shizuta-Kawashima condition [9], have been extensively studied, for instance, see [1,3,5,6,10,12–15] and references therein. However, the physical model (1.1) of gas dynamics in thermal nonequilibrium is a relaxation system of composite type. Precisely, the dissipation induced by the right-hand side of the fourth equation of (1.1) is too weak to have effect on all variables. This is due to the lack of coupling this term with certain part of the flux function, see (1.15) below. Compared with the results for dissipative systems mentioned above, the failure to satisfy the dissipative criterion makes it difficult to prove the global existence and investigating the behavior of the solution for (1.1). It should be noted that the global existence of smooth solution to compressible Euler equations with a lower order dissipation, frictional damping, was established in [7,16] with vacuum states where the system is degenerate. However, the mechanism for the global existence and decay of the solution is quite different from that for system (1.1).

In one-dimensional case, the Cauchy problem with smooth and small data was studied in [17] where the global existence and large time behavior of smooth solution in the pointwise sense are obtained while in the presence of physical boundaries, the initial boundary value problem was studied in [2]. We mention that in three space dimensions and the system is linearized around an equilibrium constant state, the pointwise description of the Green's functions reveals that compared with the one-dimensional flow, not only does the entropy wave not decay, but the velocity also contains a non-decaying part, which is strongly coupled with its decaying one, cf. [19]. Precisely, after extracting the entropy wave, which is represented by the non-decaying $\delta$-function $\delta(x)$ along the particle path, the resulting Green's function still contains a non-decaying part

$$\big(\mathrm{I}_{3\times 3} - \Delta^{-1}\nabla\nabla^\top\big)\delta(x), \tag{1.4}$$

corresponding to the "incompressible part" of the velocity $u$. (1.4) has no contradiction to the one-dimensional case, where the double Riesz transform $\Delta^{-1}\nabla\nabla^\top\delta(x)$ reduced to $\delta(x)$ such that the non-decaying term disappears.

Throughout this paper, we consider the irrotational flow, that is

$$\nabla \times u \equiv 0. \tag{1.5}$$

Precisely, (1.5) implies $u = \Delta^{-1}\nabla\nabla\cdot u$ by using the Hodge's decomposition

$$u = \Delta^{-1}\nabla\nabla\cdot u - \Delta^{-1}\nabla\times\nabla\times u. \tag{1.6}$$

For instance, (1.5) holds for spherically symmetric flow, i.e.,

$$(\rho, e_1, q)(t,x) = (\rho, e_1, q)(t,|x|), \quad u(x,t) = u(t,|x|)\frac{x}{|x|}; \tag{1.7}$$

denoted by $\varrho := (\arctan(x_2/x_1) + kx_3)$, $k\in\mathbb{R}$, (1.5) also holds for

$$(\rho, e_1, q)(t,x) = (\rho, e_1, q)(t,\varrho), \quad u(x,t) = u(t,\varrho)\nabla\varrho. \tag{1.8}$$

It is proved that the wave patterns of the three-dimensional irrotational flow are similar to the one-dimensional flow, based on the structure of the Green's function of the Cauchy problem for the linearized system, see Section 2.2 below.

System (1.1) is closed by appropriate equations of state. We take, cf. [2],

$$\begin{aligned} &e_1 = \frac{\alpha}{2}pv = \frac{\alpha R}{2}T_1, \quad Q = \frac{\alpha_f}{2}pv = \frac{\alpha_f R}{2}T_1, \quad q = \frac{\alpha_f R}{2}T_2,\\ &s_1 = R(\ln v + \frac{\alpha}{2}\ln e_1), \quad s_2 = \frac{\alpha_f R}{2}\ln q, \end{aligned} \tag{1.9}$$

by assuming $\alpha$ degrees of freedom adjust instantaneously and a further $\alpha_f$ degrees of freedom take longer to relax. $R > 0$ is the gas constant. The physical assumptions (1.9) satisfy the thermodynamics law (1.2), which also implies

$$\begin{aligned} &p = \frac{2}{\alpha}v^{-\gamma}\exp(\frac{2}{\alpha R}s_1) = \frac{2}{\alpha}v^{-\gamma}\exp(\frac{2}{\alpha R}s)(\frac{\alpha_f}{2}pv - \chi)^{-\frac{\alpha_f}{\alpha}},\\ &s_1 = s - \frac{\alpha_f R}{2}\ln(\frac{\alpha_f}{2}pv - \chi), \quad s_2 = \frac{\alpha_f R}{2}\ln(\frac{\alpha_f}{2}pv - \chi),\\ &T_1 = \frac{1}{R}pv, \quad T_2 = \frac{1}{R}(pv - \frac{2}{\alpha_f}\chi), \quad \gamma := \frac{2}{\alpha} + 1. \end{aligned}$$

Since we discuss the problem for some fixed relaxation parameter $\tau > 0$, for simplicity, we set $\tau = 1$ and write the system (1.1) as

$$\begin{cases} \rho_t + u\cdot\nabla\rho + \rho\nabla\cdot u = 0,\\ u_t + u\cdot\nabla u + \dfrac{1}{\rho}\nabla p = 0,\\ e_{1t} + u\cdot\nabla e_1 + \dfrac{p}{\rho}\nabla\cdot u = -\chi,\\ q_t + u\cdot\nabla q = \chi, \end{cases} \tag{1.10}$$

where and in the following,

$$\chi := Q - q = \frac{\alpha_f R}{2}(T_1 - T_2). \tag{1.11}$$

Using (1.2), it is equivalent to consider the system for $p, u, \chi$ and $s$:

$$\begin{cases} p_t + u\cdot\nabla p + \rho c_f^2 \nabla\cdot u = -p_{e_1}\chi, \\ u_t + u\cdot\nabla u + v\nabla p = 0, \\ \chi_t + u\cdot\nabla\chi + a\nabla\cdot u = -\zeta\chi, \\ s_t + u\cdot\nabla s = (\frac{1}{T_2} - \frac{1}{T_1})\chi, \end{cases} \tag{1.12}$$

where by (1.9) it holds that

$$\begin{aligned} &c_f^2 := p_\rho(\rho, s_1) = v^2(p p_{e_1} - p_v) = \gamma p v, \quad p_{e_1} = \frac{2}{\alpha v}, \\ &a := v(pQ_{e_1} - Q_v) = \omega'(T_1) v T_1 p_{e_1} = \frac{\alpha_f}{\alpha} p v, \quad \zeta := 1 + Q_{e_1} = 1 + \frac{\alpha_f}{\alpha}. \end{aligned} \tag{1.13}$$

We note that $c_f$ is the frozen speed of sound. The equilibrium speed of sound $c$ is given by $c^2 = p_\rho(\rho, s) = c_f^2/(1+b)$ where

$$b := \frac{p_{e_1} v}{(1+Q_{e_1})c_f^2/a - p_{e_1} v} = \frac{\gamma - 1}{c_f^2/a + 1}, \quad \frac{c_f^2}{a} = \frac{2+\alpha}{\alpha_f}. \tag{1.14}$$

We will use both (1.10) and (1.12) at our convenience. From (1.12) it is easy to see that the dissipation induced by the relaxation term has no effect on the entropy $s$. Precisely,

$$(\frac{1}{T_2} - \frac{1}{T_1})\chi = \frac{2}{\alpha_f R}\frac{\chi^2}{T_1 T_2} > 0 \tag{1.15}$$

implies that the entropy increases along the particle path.

In this paper, we are interested in the global existence and large time behavior of irrotational flow for gas dynamics in thermal nonequilibrium. We consider the initial value problem of the system (1.10) in $\mathbb{R}^3$ with irrotational initial data

$$(\rho, u, e_1, q)(0, x) = (\rho_0, u_0, e_{1,0}, q_0)(x), \quad \nabla \times u_0 = 0, \tag{1.16}$$

which is a small perturbation of an equilibrium state $(\bar\rho, \bar u, \bar e_1, \bar q)$, $\bar\rho, \bar e_1, \bar q > 0$ satisfying

$$\bar q = \bar Q, \quad \text{or} \quad \bar T_1 = \bar T_2. \tag{1.17}$$

Without loss of generality we take $\bar u = \mathbf{0}$. Here we use $\bar{\ }$ to denote the constant state and

$$\bar Q = \frac{\alpha_f}{\alpha}\bar e_1, \quad \bar T_2 = \frac{2}{\alpha_f R}\bar q, \quad \text{etc.}$$

Before stating the main result, we introduce some notations for later use. $A \lesssim B$ means that there is a generic constant $C > 0$ such that $A \le CB$. $\nabla^k$, $k \ge 1$ denotes all derivatives of order $k$ and for multi-index $\beta = (\beta_1, \beta_2, \beta_3)$,

$$\partial^\beta = \partial_{x_1}^{\beta_1}\partial_{x_2}^{\beta_2}\partial_{x_3}^{\beta_3}, \quad |\beta| = \sum_{j=1}^{3}\beta_j.$$

Then $\|\cdot\|_{H^k}$ denotes the norms in the Sobolev space $H^k(\mathbb{R}^3)$ and $\|\cdot\|_{L^p}$, $p \ge 1$ denotes the norms in $L^p(\mathbb{R}^3)$. For convenience, for $k = 0$ we use $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ to denote the inner product and norm in $L^2(\mathbb{R}^3)$ respectively.

**Theorem 1.1.** *Let $\bar\rho, \bar e_1, \bar q$ be positive constants such that (1.17) holds, the irrotational initial data $(\rho_0, u_0, e_{1,0}, q_0)$ be such that $\|\rho_0 - \bar\rho, u_0, e_{1,0} - \bar e_1, q_0 - \bar q\|_{H^3}$ is sufficiently small and $\|p_0 - \bar p, u_0, \chi_0\|_{L^1}$ is bounded. Then a unique global solution $(\rho, u, e_1, q)$ satisfying the irrotational condition $\nabla \times u = 0$ with $\rho, e_1, q > 0$ to the Cauchy problem of (1.10) and (1.16) exists and satisfies*

$$(\rho - \bar\rho, u, e_1 - \bar e_1, q - \bar q) \in C^0\big([0,\infty); H^3(\mathbb{R}^3)\big) \cap C^1\big([0,\infty); H^2(\mathbb{R}^3)\big). \tag{1.18}$$

*Moreover, it holds*

$$\|(\rho-\bar{\rho},u,e_1-\bar{e}_1,q-\bar{q})\|^2+\|\nabla(p,u,\chi,s)\|_{H^2}^2+\int_0^t\big(\|\chi\|_{H^3}^2+\|(\nabla p,\nabla\cdot u)\|_{H^2}^2\big)d\tau$$
$$\lesssim\|(\rho_0-\bar{\rho},u_0,e_{1,0}-\bar{e}_1,q_0-\bar{q})\|^2+\|\nabla(p_0,u_0,\chi_0)\|_{H^2}^2+\exp\{\|(p_0-\bar{p},u_0,\chi_0)\|_{L^1\cap H^3}\}\|\nabla s_0\|_{H^2}^2. \tag{1.19}$$

*And the solution $(p,u,\chi)$ has the following decay property*

$$\|\chi(t)\|_{H^3}+\|\nabla\big(p,u\big)(t)\|_{H^2}\lesssim(1+t)^{-\frac{5}{4}}\|(p_0-\bar{p},u_0,\chi_0)\|_{L^1\cap H^3},\quad\forall\, t\geq 0. \tag{1.20}$$

**Remark.** The above theorem says that if the flow is irrotational, then it exists globally. In particular, if the initial data are spherically symmetric, then there exists a globally defined spherically symmetric solution. This can be seen by a simple uniqueness argument and the invariance of the equations for the rotational coordinates transformation, $t'=t$, $x'=Sx$ for $S\in SO(3)$. Another example of irrotational flow is given by (1.8).

We give some remarks on the proof of Theorem 1.1. We prove the global existence and large time behavior of the solution to the Cauchy problem when the initial data are a small perturbation of an equilibrium constant state with an additional bound of the $L^1$-norm. If we study the spherically symmetric solution (1.7) on the exterior domain $|x|=r>r_0$ for some constant $r_0>0$. We can introduce the Lagrangian mass coordinate

$$\eta(r,\tau)=\int_1^r s^2\rho(t,s)ds,\quad \tau=t,$$

and the equations (1.12) are reduced to

$$\begin{cases} p_t+\dfrac{\gamma p}{v}(r^2u)_\eta=-\dfrac{2}{\alpha v}\chi,\\ u_t+r^2p_\eta=0,\\ \chi_t+\dfrac{\alpha_f}{\alpha}p(r^2u)_\eta=-(1+\dfrac{\alpha_f}{\alpha})\chi,\\ s_t=(\dfrac{1}{T_2}-\dfrac{1}{T_1})\chi. \end{cases} \tag{1.21}$$

Imposed with the boundary condition $u(r_0,t)=0$, the global existence and large time behavior of smooth solution for (1.21) on an exterior domain can be obtained following the argument in [2]. However, the entropy increase dictated by physics for any irreversible process makes it challenging to establish the global existence of irrotational Eulerian flow. Compared with the one-dimensional flow, cf. [17], using a standard energy method is not sufficient for proving the global *a priori* estimates, see (2.10). It is worth pointing out that we use the spectral method to obtain the large time behavior of the solution for the nonlinear system. Then we are able to show that the entropy increases but stays bounded. This framework has been applied in [4] to study the compressible Navier-Stokes equations.

## 2. Proof of Theorem 1.1

Theorem 1.1 follows from the standard continuity argument by the *a priori* estimates and the local existence result, which is standard for the system (1.1), e.g., see [8] and references therein. Thus it is sufficient to prove Proposition 2.1.

**Proposition 2.1** *(A priori estimate). Let $\bar{\rho},\bar{e}_1,\bar{q}$ be positive constants such that (1.17) holds and $(\rho_0-\bar{\rho},u_0,e_{1,0}-\bar{e}_1,q_0-\bar{q})\in H^3(\mathbb{R}^3)$. Suppose $(\rho,u,e_1,q)(t,x)$ is an irrotational solution of the Cauchy problem (1.10) and (1.16) in the time interval $[0,T]$, $T>0$, satisfying*

$$\begin{aligned}&(\rho-\bar{\rho},u,e_1-\bar{e}_1,q-\bar{q})\in C^0\big([0,T];H^3(\mathbb{R}^3)\big)\cap C^1\big([0,T];H^2(\mathbb{R}^3)\big),\\ &\quad\nabla p,\nabla\cdot u\in L^2\big([0,T];H^2(\mathbb{R}^3)\big),\quad \chi\in L^2\big([0,T];H^3(\mathbb{R}^3)\big).\end{aligned} \tag{2.1}$$

*There exists some constant $\delta>0$ sufficiently small, which is independent of $T$ such that if*

$$\sup_{0\le t\le T} \|\rho-\bar\rho, u, e_1-\bar e_1, q-\bar q\|_{H^3} \le \delta, \tag{2.2}$$

*and we further assume*

$$(p_0-\bar p, u_0, \chi_0) \in L^1(\mathbb{R}^3), \tag{2.3}$$

*then it holds*

$$\begin{aligned} &\|(\rho-\bar\rho, u, e_1-\bar e_1, q-\bar q)\|^2 + \|\nabla(p,u,\chi,s)\|_{H^2}^2 + \int_0^t \big(\|\chi\|_{H^3}^2 + \|(\nabla p, \nabla\cdot u)\|_{H^2}^2\big)d\tau \\ &\lesssim \|(\rho_0-\bar\rho, u_0, e_{1,0}-\bar e_1, q_0-\bar q)\|^2 + \|\nabla(p_0,u_0,\chi_0)\|_{H^2}^2 + \exp\{\|(p_0-\bar p, u_0,\chi_0)\|_{L^1\cap H^3}\}\|\nabla s_0\|_{H^2}^2. \end{aligned} \tag{2.4}$$

*Particularly, the solution $(p,u,\chi)$ has the following decay property*

$$\|\chi(t)\|_{H^3} + \|\nabla\big(p,u\big)(t)\|_{H^2} \lesssim (1+t)^{-\frac54}\|(p_0-\bar p, u_0, \chi_0)\|_{L^1\cap H^3}, \quad \forall\, t\in[0,T]. \tag{2.5}$$

**Proof.** Proposition 2.1 is a consequence of the energy estimates in Proposition 2.2 and the decay estimates of $(p,u,\chi)$ in Proposition 2.3.

Particularly, we use the *a priori* decay-in-time estimate on the velocity to obtain the uniform bound of the derivatives of entropy. Precisely, together with the decay property (2.36) of $\nabla u$ in Proposition 2.3, it follows from the energy estimate (2.10) of $\nabla s$ in Proposition 2.2 by using the Gronwall's inequality that

$$\begin{aligned} \|\nabla s\|_{H^2}^2 &\lesssim \exp\{\int_0^t \|\nabla(\nabla\cdot u)\|_{H^1}d\tau\}\big(\|\nabla s_0\|_{H^2}^2 + \delta\int_0^t \|\nabla\chi\|_{H^2}^2 d\tau\big) \\ &\lesssim \exp\{\|(p_0-\bar p, u_0,\chi_0)\|_{L^1\cap H^3}\}\big(\|\nabla s_0\|_{H^2}^2 + \delta\int_0^t \|\nabla\chi\|_{H^2}^2 d\tau\big). \end{aligned} \tag{2.6}$$

Integrate (2.9) in Proposition 2.2 over $[0,t]$, $t\le T$, under the smallness assumption (2.2), (2.4) follows directly from the summation of the result inequality and (2.6). □

## 2.1. Energy estimates

For later use, we review some Sobolev inequalities.

**Lemma 2.1.** *For $w\in H^k(\mathbb{R}^3)$, $\nabla w\in L^\infty(\mathbb{R}^3)$ and $v\in H^{k-1}(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$,*

$$\sum_{1\le|\beta|\le k} \|[\partial^\beta, w]v\| \le C_k\big(\|\nabla w\|_{L^\infty}\|\nabla^{k-1}v\| + \|v\|_{L^\infty}\|\nabla^k w\|\big).$$

*Particularly, for $w, v\in H^k(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$,*

$$\sum_{|\beta|\le k} \|\partial^\beta(wv)\| \le C_k\big(\|w\|_{L^\infty}\|\nabla^k v\| + \|v\|_{L^\infty}\|\nabla^k w\|\big),$$

*where*

$$\|w\|_{L^\infty} \lesssim \|\nabla w\|^{1/2}\|\nabla^2 w\|^{1/2} \lesssim \|\nabla w\|_{H^1}, \quad \forall\, w\in H^2(\mathbb{R}^3).$$

In this subsection, we will prove the following energy estimates.

**Proposition 2.2.** *Suppose that $(\rho,u,e_1,q)$ is an irrotational solution of the system (1.10) for $t\in[0,T]$, $T>0$. Assume all conditions of Proposition 2.1 hold. Under the assumption*

$$\sup_{0\le t\le T}\|\rho-\bar\rho,u,e_1-\bar e_1,q-\bar q\|_{H^3}\le\delta, \tag{2.7}$$

*where* $\delta>0$ *is sufficiently small, the following energy estimates hold.*

$$\frac{d}{dt}\big(\|\nabla(p,u)\|_{H^2}^2+\|\chi\|_{H^3}^2\big)+\|\chi\|_{H^3}^2+\|\nabla^2 p\|_{H^1}^2+\|\nabla(\nabla\cdot u)\|_{H^1}^2\lesssim\|\nabla(p,u)\|^2, \tag{2.8}$$

$$\frac{d}{dt}\big(\|(\rho-\bar\rho,u,e_1-\bar e_1,q-\bar q)\|^2+\|\nabla(p,u,\chi)\|_{H^2}^2\big)+\|\chi\|_{H^3}^2+\|\nabla p\|_{H^2}^2+\|\nabla\cdot u\|_{H^2}^2\le 0, \tag{2.9}$$

$$\frac{d}{dt}\|\nabla s\|_{H^2}^2\lesssim\|\nabla(\nabla\cdot u)\|_{H^1}\|\nabla s\|_{H^2}^2+\delta\|\nabla\chi\|_{H^2}^2. \tag{2.10}$$

**Proof.** Firstly, we study the basic energy estimates of (1.10). Multiplying $(1.10)_3$ by $\rho/e_1$ gives

$$\frac{d}{dt}(\rho\ln e_1)+\nabla\cdot(\rho u\ln e_1)+\frac{2}{\alpha}\rho\nabla\cdot u=-\frac{1}{e_1}\rho\chi,$$

where we used $(1.10)_1$ and (1.9). Together with $(1.10)_1$ and

$$\frac{d}{dt}(\rho e_1)+\nabla\cdot(\rho u e_1)+p\nabla\cdot u=-\rho\chi,$$

we can claim that

$$\begin{aligned}&\frac{d}{dt}\big(\bar q\rho(\frac{e_1}{\bar q}-\ln[(\frac{\alpha_f}{\alpha}\frac{e_1}{\bar q})^{\frac{\alpha}{\alpha_f}}]-\frac{\alpha}{\alpha_f})\big)\\&\quad+\nabla\cdot\big(\bar q\rho u(\frac{e_1}{\bar q}-\ln[(\frac{\alpha_f}{\alpha}\frac{e_1}{\bar q})^{\frac{\alpha}{\alpha_f}}]-\frac{\alpha}{\alpha_f})\big)+(p-\frac{2}{\alpha_f}\bar q\rho)\nabla\cdot u=(\frac{\bar q}{Q}-1)\rho\chi.\end{aligned} \tag{2.11}$$

Similarly, it follows from $(1.10)_1$ and $(1.10)_4$ respectively that

$$\frac{d}{dt}\big(\rho(\frac{\bar\rho}{\rho}-\ln\frac{\bar\rho}{\rho}-1)\big)-\nabla\cdot\big(\rho u(\ln\frac{\bar\rho}{\rho}+1)\big)+\rho\nabla\cdot u=0, \tag{2.12}$$

$$\frac{d}{dt}\big(\bar q\rho(\frac{q}{\bar q}-\ln\frac{q}{\bar q}-1)\big)+\nabla\cdot\big(\rho u(q-\bar q\ln\frac{q}{\bar q}-\bar q)\big)=(1-\frac{\bar q}{q})\rho\chi. \tag{2.13}$$

Finally, multiplying $(1.10)_2$ by $\rho u^\top$ yields that

$$\frac{1}{2}\frac{d}{dt}(\rho|u|^2)+\frac{1}{2}\nabla\cdot(\rho u|u|^2)+u\cdot\nabla p=0. \tag{2.14}$$

The summation of $(2.11)+2\bar q(2.12)/\alpha+(2.13)+(2.14)$ gives

$$\begin{aligned}&\frac{d}{dt}\Big(\frac{2\bar q}{\alpha_f}\rho(\frac{\bar\rho}{\rho}-\ln\frac{\bar\rho}{\rho}-1)+\frac{1}{2}\rho|u|^2+\bar q\rho(\frac{e_1}{\bar q}-\ln[(\frac{\alpha_f}{\alpha}\frac{e_1}{\bar q})^{\frac{\alpha}{\alpha_f}}]-\frac{\alpha}{\alpha_f}+\frac{q}{\bar q}-\ln\frac{q}{\bar q}-1)\Big)\\&\quad+\nabla\cdot\Big(\bar q\rho\big(\frac{e_1+q}{\bar q}-\ln[\frac{q}{\bar q}(\frac{\bar\rho}{\rho})^{\frac{2}{\alpha_f}}(\frac{\alpha_f}{\alpha}\frac{e_1}{\bar q})^{\frac{\alpha}{\alpha_f}}]-\frac{c_f^2}{a}-1\big)u+\frac{1}{2}\rho|u|^2u+pu\Big)\\&=\bar q(\frac{1}{Q}-\frac{1}{q})\rho\chi.\end{aligned} \tag{2.15}$$

Then under the smallness assumption (2.7), it follows from (2.15) that

$$\frac{d}{dt}\|(\rho-\bar\rho,u,e_1-\bar e_1,q-\bar q)\|^2+c_0\|\chi\|^2\le 0, \tag{2.16}$$

using

$$\bar q(\frac{1}{q}-\frac{1}{Q})\rho\chi\ge c_0\chi^2,$$

for some constant $c_0>0$ independent of $t$. Particularly, we multiply $(1.12)_3$ by $\chi$ and integrate the result equation over $\mathbb{R}^3$. Using the Cauchy-Schwarz inequality we have

$$\frac{d}{dt}\|\chi\|^2+\|\chi\|^2\lesssim\|\nabla\cdot u\|^2. \tag{2.17}$$

Then we consider the higher order energy estimates by using the thermodynamics properties of (1.12) as follows. Notice that the symmetrization procedure of (1.12) is motivated by choosing the positive definite matrix

$$\mathbb{S}=\begin{pmatrix} 1+b & 0 & -\frac{c_f^2}{a}b\rho \\ 0 & \gamma\rho p & 0 \\ -\frac{c_f^2}{a}b\rho & 0 & (\frac{c_f^2}{a}\rho)^2 b \end{pmatrix}, \tag{2.18}$$

where the constants $c_f^2/a, b>0$ are given in (1.14). For some multi-index $\beta$ with $1\le|\beta|\le 3$, using (2.18), applying $\partial^\beta$ to (1.12) gives

$$\begin{cases} (\partial^\beta p)_t+u\cdot\nabla\partial^\beta p=-\frac{2}{\alpha}\partial^\beta(\rho\chi)-\gamma\partial^\beta(p\nabla\cdot u)-[\partial^\beta,u\cdot\nabla]p, \\ (\partial^\beta u)_t+u\cdot\nabla\partial^\beta u=-\partial^\beta(v\nabla p)-[\partial^\beta,u\cdot\nabla]u, \\ (\partial^\beta p-\frac{c_f^2}{a}\rho\partial^\beta\chi)_t+u\cdot\nabla(\partial^\beta p-\frac{c_f^2}{a}\rho\partial^\beta\chi)-(\frac{c_f^2}{a}+1)\rho\partial^\beta\chi \\ =\frac{c_f^2}{a}\rho\nabla\cdot u\partial^\beta\chi-[\partial^\beta,u\cdot\nabla]p+\frac{c_f^2}{a}\rho[\partial^\beta,u\cdot\nabla]\chi+\gamma\rho[\partial^\beta,v](p\nabla\cdot u)-\frac{2}{\alpha}[\partial^\beta,\rho]\chi, \end{cases} \tag{2.19}$$

and

$$(\partial^\beta s)_t+u\cdot\nabla\partial^\beta s=\frac{2}{\alpha_f R}\partial^\beta(\frac{\chi^2}{T_1T_2})-[\partial^\beta,u\cdot\nabla]s. \tag{2.20}$$

We multiply (2.19) by $\big(\partial^\beta p, c_f^2\rho^2\partial^\beta u, b(\partial^\beta p-c_f^2/a\rho\partial^\beta\chi)\big)^\top$ to obtain that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\big(|\partial^\beta p|^2+\gamma\rho p|\partial^\beta u|^2+b|\partial^\beta p-\frac{c_f^2}{a}\rho\partial^\beta\chi|^2\big)+2\alpha\frac{c_f^2}{a}|\rho\partial^\beta\chi|^2 \\ &\quad+\frac{1}{2}\nabla\cdot\big(u\big(|\partial^\beta p|^2+\gamma\rho p|\partial^\beta u|^2+b|\partial^\beta p-\frac{c_f^2}{a}\rho\partial^\beta\chi|^2\big)+2\gamma p\partial^\beta u\partial^\beta p\big) \\ &=\frac{1}{2}\nabla\cdot u\big((1+b)|\partial^\beta p|^2-\rho p|\gamma\partial^\beta u|^2-b|\frac{c_f^2}{a}\rho\partial^\beta\chi|^2\big)+\gamma(\nabla p\cdot\partial^\beta u\partial^\beta p-\frac{\chi}{\alpha}|\rho\partial^\beta u|^2) \\ &\quad-\big((1+b)\partial^\beta p-b\frac{c_f^2}{a}\rho\partial^\beta\chi\big)([\partial^\beta,u\cdot\nabla]p+\frac{2}{\alpha}[\partial^\beta,\rho]\chi) \\ &\quad+b\rho(\partial^\beta p-\frac{c_f^2}{a}\rho\partial^\beta\chi)\big(\frac{c_f^2}{a}[\partial^\beta,u\cdot\nabla]\chi+\gamma[\partial^\beta,v](p\nabla\cdot u)\big) \\ &\quad-\gamma\partial^\beta p[\partial^\beta,p](\nabla\cdot u)-\gamma p\rho\partial^\beta u([\partial^\beta,u\cdot\nabla]u+[\partial^\beta,v]\nabla p). \end{aligned} \tag{2.21}$$

Under the assumption (2.2), by straightforward calculations we arrive that

$$\frac{d}{dt}\|\nabla(p,u,\chi)\|_{H^2}^2+\|\nabla\chi\|_{H^2}^2\lesssim\delta\|\nabla(p,u)\|_{H^2}^2, \tag{2.22}$$

where for instance we use the Sobolev inequalities in Lemma 2.1 to show that

$$\begin{aligned} \sum_{1\le|\beta|\le 3}\|[\partial^\beta,v](p\nabla\cdot u)\| &\lesssim\|\nabla v\|_{L^\infty}\big(\|p\|_{L^\infty}\|\nabla^2(\nabla\cdot u)\|+\|\nabla\cdot u\|_{L^\infty}\|\nabla^2 p\|\big)+\|(p\nabla\cdot u)\|_{L^\infty}\|\nabla^3 v\| \\ &\lesssim\delta(1+\delta)\|\nabla\cdot u\|_{H^2}, \end{aligned}$$

and

$$\sum_{1\le|\beta|\le 3}\|[\partial_x^\beta,\rho]\chi\|\lesssim\|\nabla\rho\|_{L^\infty}\|\nabla^2\chi\|+\|\chi\|_{L^\infty}\|\nabla^3\rho\|\lesssim\delta\|\nabla\chi\|_{H^1}.$$

The remained terms can be treated similarly.

Next, we apply $\partial^{\beta'}$, $|\beta'|\le 2$ to $(1.12)_3$ and multiply the obtained equation by $\partial^{\beta'}(\nabla\cdot u)$. Using $(1.12)_2$ it holds that

$$
\begin{aligned}
\frac{\alpha_f}{\alpha}pv|\partial^{\beta'}(\nabla\cdot u)|^2 &= \frac{\alpha_f}{\alpha}\partial^{\beta'}(\nabla\cdot u)\big(\partial^{\beta'}(pv\nabla\cdot u)-[\partial^{\beta'},pv](\nabla\cdot u)\big)\\
&= -\frac{d}{dt}\big(\partial^{\beta'}\chi\partial^{\beta'}(\nabla\cdot u)\big)+\nabla\cdot(\partial^{\beta'}\chi\partial^{\beta'}u_t)\\
&\quad -\partial^{\beta'}(u\cdot\nabla u+v\nabla p)\cdot\nabla\partial^{\beta'}\chi\\
&\quad -\big(\zeta\partial^{\beta'}\chi+\partial^{\beta'}(u\cdot\nabla\chi)+\frac{\alpha_f}{\alpha}[\partial^{\beta'},pv](\nabla\cdot u)\big)\partial^{\beta'}(\nabla\cdot u).
\end{aligned}
\tag{2.23}
$$

Similarly we use $(1.12)_1$ and $(1.12)_2$ to obtain that

$$
\begin{aligned}
v|\nabla\partial^{\beta'}p|^2 &= -\frac{d}{dt}(\partial^{\beta'}u\cdot\nabla\partial^{\beta'}p)+\nabla\cdot(\partial^{\beta'}u\partial^{\beta'}p_t)\\
&\quad +\partial^{\beta'}\big(u\cdot\nabla p+\gamma p\nabla\cdot u+\frac{2}{\alpha}\rho\chi\big)\partial^{\beta'}(\nabla\cdot u)-\big(\partial^{\beta'}(u\cdot\nabla u)+[\partial^{\beta'},v]\nabla p\big)\cdot\nabla\partial^{\beta'}p.
\end{aligned}
\tag{2.24}
$$

Since $\nabla\times u=0$, using (1.6) it is straightforward to verify that

$$
\|\nabla u(t,\cdot)\|_{H^k}^2=\|\nabla\cdot u(t,\cdot)\|_{H^k}^2.
$$

Then it follows from (2.23) by using the Cauchy-Scharwz inequality that

$$
\sum_{1\le|\beta'|\le 2}\frac{d}{dt}\langle\partial^{\beta'}\chi,\partial^{\beta'}(\nabla\cdot u)\rangle+\|\nabla(\nabla\cdot u)\|_{H^1}^2\lesssim C_\varepsilon\|\nabla\chi\|_{H^2}^2+(\delta+\varepsilon)\|\nabla^2 p\|_{H^1}^2,
\tag{2.25}
$$

where the constant $C_\varepsilon$ depends on the sufficiently small constant $\varepsilon>0$. Using (2.24) it also holds that

$$
\sum_{1\le|\beta'|\le 2}\frac{d}{dt}\langle\partial^{\beta'}u,\nabla\partial^{\beta'}p\rangle+\|\nabla^2 p\|_{H^1}^2\lesssim\|\nabla(\nabla u,\chi)\|_{H^1}^2.
\tag{2.26}
$$

A suitable summation of (2.22), (2.25) and (2.26) gives

$$
\frac{d}{dt}\|\nabla(p,u,\chi)\|_{H^2}^2+\|\nabla\chi\|_{H^2}^2+\|\nabla^2 p\|_{H^1}^2+\|\nabla(\nabla\cdot u)\|_{H^1}^2\lesssim\delta\|\nabla(p,u)\|^2,
\tag{2.27}
$$

together with (2.17), which leads to (2.8). Particularly, for $\beta'=0$ we have

$$
\frac{d}{dt}\big(\langle\chi,\nabla\cdot u\rangle,\langle u,\nabla p\rangle\big)+\|(\nabla p,\nabla\cdot u)\|^2\lesssim\|\chi\|_{H^1}.
\tag{2.28}
$$

Then (2.9) follows by using (2.16), (2.27) and (2.28).

Finally, multiplying (2.20) by $\partial^\beta s$ yields that

$$
\frac{1}{2}\frac{d}{dt}|\partial^\beta s|^2+\frac{1}{2}\nabla\cdot(u|\partial^\beta s|^2)=\frac{1}{2}\nabla\cdot u|\partial^\beta s|^2+\partial^\beta s\Big(\frac{2}{\alpha_f R}\partial^\beta\Big(\frac{\chi^2}{T_1T_2}\Big)-[\partial^\beta,u\cdot\nabla]s\Big).
$$

Using Lemma 2.1 again we can claim

$$
\begin{aligned}
\sum_{1\le|\beta|\le 3}\Big\|\partial^\beta\Big(\frac{\chi^2}{T_1T_2}\Big)\Big\| &\lesssim\|\chi^2\|_{L^\infty}\Big\|\nabla^3\Big(\frac{1}{T_1T_2}\Big)\Big\|+\Big\|\frac{1}{T_1T_2}\Big\|_{L^\infty}\|\nabla^3(\chi^2)\|\\
&\lesssim(1+\delta)\|\nabla\chi\|_{H^2}^2,\\
\sum_{1\le|\beta|\le 3}\|[\partial^\beta,u\cdot\nabla]s\| &\lesssim\|\nabla u\|_{L^\infty}\|\nabla^3 s\|+\|\nabla^3 u\|_{L^\infty}\|\nabla s\|_{L^\infty}\\
&\lesssim\|\nabla^2 u\|_{H^1}\|\nabla^2 s\|_{H^1}\lesssim\|\nabla(\nabla\cdot u)\|_{H^1}\|\nabla s\|_{H^2},
\end{aligned}
$$

which leads to (2.10). □

*2.2. Optimal decay estimates*

In this subsection we study the time-decay estimates of $(p, u, \chi)$ of the system (1.12). Derived from the spectral analysis of the linearized system, the large time decay estimate of $u$ in Proposition 2.3 plays a key role in obtaining the global *a priori* estimate of $\nabla s$, see (2.10).

Recall that the solution of (1.12) is a small perturbation around the equilibrium constant state $(\bar{p}, 0, 0, \bar{s})$. The linearized system of (1.12) around $(\bar{p}, 0, 0, \bar{s})$ can be decoupled into a system of $V = V(t, x) = (p - \bar{p}, u, \chi)^\top(t, x)$ as

$$V_t = \mathbb{A}V, \quad \mathbb{A} = -\begin{pmatrix} 0 & \gamma\bar{p}\nabla\cdot & 2/(\alpha\bar{v}) \\ \bar{v}\nabla & 0 & 0 \\ 0 & \bar{a}\nabla\cdot & \zeta \end{pmatrix} \in \mathbb{R}^{5\times 5}, \tag{2.29}$$

and the linearized entropy equation

$$s_t = 0. \tag{2.30}$$

Here $\bar{v}$ is the value of $v$ at the constant equilibrium state, $\bar{a} = \alpha_f\bar{p}\bar{v}/\alpha$. This subsection is devoted to studying the Cauchy problem of (2.29) with initial data

$$V_0 = V(0, x) = (p - \bar{p}, u, \chi)(0, x) = (p_0 - \bar{p}, u_0, \chi_0)(x), \tag{2.31}$$

satisfying the compatibility condition

$$\nabla \times u_0 = 0. \tag{2.32}$$

Then by Duhamel's principle, we have

$$\big(p - \bar{p}, u, \chi\big)^\top(t) = e^{t\mathbb{A}(x)}(p_0 - \bar{p}, u_0, \chi_0)^\top + \int_0^t e^{(t-\tau)\mathbb{A}(x)}\big(g_0, \mathbf{g}, g_4\big)^\top(\tau)d\tau, \quad \mathbf{g} = (g_1, g_2, g_3), \tag{2.33}$$

where the decay estimates on semigroup $e^{t\mathbb{A}(x)}$ will be given in Proposition 2.4 and the nonlinear terms $g_i$, $i = 0, \cdots, 4$,

$$\begin{cases} g_0 = -u\cdot\nabla p - \gamma(p - \bar{p})\nabla\cdot u - \dfrac{2}{\alpha}(\dfrac{1}{v} - \dfrac{1}{\bar{v}})\chi, \\ \mathbf{g} = -u\cdot\nabla u - (v - \bar{v})\nabla p, \\ g_4 = -u\cdot\nabla\chi - \dfrac{\alpha_f}{\alpha}(pv - \bar{p}\bar{v})\nabla\cdot u, \end{cases} \tag{2.34}$$

satisfy the compatibility condition (2.32).

**Proposition 2.3.** *Suppose that $(\rho, u, e_1, q)$ is an irrotational solution of the system (1.10) for $t \in [0, T]$, $T > 0$. Assume all conditions of Proposition 2.1 hold. If we further assume*

$$(p_0 - \bar{p}, u_0, \chi_0) \in L^1(\mathbb{R}^3), \tag{2.35}$$

*then the solution admits the following decay property*

$$\|\chi(t)\|_{H^3} + \|\nabla\big(p, u\big)(t)\|_{H^2} \lesssim (1+t)^{-\frac{5}{4}}\|(p_0 - \bar{p}, u_0, \chi_0)\|_{L^1\cap H^3}. \tag{2.36}$$

**Proof.** Firstly we study the *a priori* decay-in-time estimates of $(p, u)$ on the right-hand side of (2.8). Based on the decay properties of the linear system in Proposition 2.4, which will be proved later, it follows from (2.33) that

$$\begin{aligned} \|\nabla(p, u)\| &\lesssim (1+t)^{-\frac{5}{4}}\|(p_0 - \bar{p}, u_0, \chi_0)\|_{L^1\bigcap H^1} + \int_0^t (1+t-\tau)^{-\frac{5}{4}}\|(g_0, \mathbf{g})(\tau)\|_{L^1\bigcap H^1}d\tau \\ &\lesssim (1+t)^{-\frac{5}{4}}\|(p_0 - \bar{p}, u_0, \chi_0)\|_{L^1\bigcap H^1} + \delta\int_0^t (1+t-\tau)^{-\frac{5}{4}}\|\big(\chi, \nabla p, \nabla\cdot u\big)(\tau)\|_{H^1}d\tau, \end{aligned} \tag{2.37}$$

where it is straightforward to verify that

$$\|(g_0, \mathbf{g})(t)\|_{L^1} \lesssim \delta\|(\chi, \nabla p, \nabla u)(t)\|, \quad \|(g_0, \mathbf{g})(t)\|_{H^1} \lesssim \delta\|(\chi, \nabla p, \nabla u)(t)\|_{H^1}.$$

Set

$$\mathcal{M}(t) = \sup_{0\le\tau\le t} (1+\tau)^{\frac{5}{2}}\big(\|\chi(\tau)\|_{H^3}^2 + \|\nabla(p,u)(\tau)\|_{H^2}^2\big).$$

Adding $\|\nabla(p,u)\|^2$ to both sides of (2.8) gives

$$\frac{d}{dt}\big(\|\nabla(p,u)\|_{H^2}^2 + \|\chi\|_{H^3}^2\big) + \|\chi\|_{H^3}^2 + \|\nabla(p,u)\|_{H^2}^2 \lesssim \|\nabla(p,u)\|^2. \tag{2.38}$$

Then it follows from (2.38) by using the Gronwall's inequality and (2.37) that

$$\begin{aligned}\|\chi(t)\|_{H^3}^2 + \|\nabla(p,u)(t)\|_{H^2}^2 &\lesssim e^{-t}\big(\|\chi_0\|_{H^3}^2 + \|\nabla(p_0,u_0)\|_{H^2}^2\big)\\ &\quad + \int_0^t e^{-(t-\tau)}(1+\tau)^{-\frac{5}{2}}\|(p_0-\bar{p}, u_0, \chi_0)\|_{L^1\bigcap H^1}^2 d\tau\\ &\quad + \delta^2\mathcal{M}(t)\Big(\int_0^t (1+t-\tau)^{-\frac{5}{4}}(1+\tau)^{-\frac{5}{4}}d\tau\Big)^2\\ &\lesssim (1+t)^{-\frac{5}{2}}\big(\|(p_0-\bar{p}, u_0, \chi_0)\|_{L^1\bigcap H^3}^2 + \delta^2\mathcal{M}(t)\big).\end{aligned}$$

Since $\mathcal{M}(t)$ is non-decreasing and $\delta > 0$ is small enough, we can claim

$$\mathcal{M}(t) \lesssim \|(p_0-\bar{p}, u_0, \chi_0)\|_{L^1\bigcap H^3}^2, \quad \forall\, t\in[0,T], \tag{2.39}$$

which gives (2.36). □

It remains to prove the decay properties of the solution for the linearized system (2.29). In order to combine the constraint (2.32), we define the pseudo-differential operator $\mathcal{P}^r$, $r\in\mathbb{R}$, as

$$\mathcal{P}^r f = \mathcal{F}^{-1}\{|\xi|^r \hat{f}(\xi)\},$$

where $\mathcal{F}^{-1}\{\cdot\}$ denotes the inverse of the Fourier transform $\mathcal{F}\{\cdot\}$ with respect to $x$,

$$\mathcal{F}\{f\} = \hat{f}(\cdot,\xi) = \int_{\mathbb{R}^3} f(\cdot,x)e^{-\mathrm{i}x\cdot\xi}dx.$$

Then we introduce

$$w = \mathcal{P}^{-1}(\nabla\cdot u), \quad \text{i.e.} \quad \hat{w} = \frac{\mathrm{i}\xi}{|\xi|}\cdot\hat{u}. \tag{2.40}$$

By (1.6) it also holds that

$$u = -\mathcal{F}^{-1}\{|\xi|^{-2}\big(\mathrm{i}\xi(\mathrm{i}\xi\cdot\hat{u}) - \mathrm{i}\xi\times\mathrm{i}\xi\times\hat{u}\big)\} = -\mathcal{F}^{-1}\{|\xi|^{-1}(\mathrm{i}\xi\hat{w})\} = -\mathcal{P}^{-1}\{\nabla w\}. \tag{2.41}$$

Denoted by $U = U(t,x) = (\vartheta, w, \chi)^\top(t,x)$ where $\vartheta := p - \bar{p}$, the system (2.29), (2.31) and (2.32) can be rewritten as

$$U_t = \mathbb{B}U, \quad U(0,x) = U_0, \quad t\ge 0, \tag{2.42}$$

where $U_0 = (\vartheta, w, \chi)^\top(0,x)$ and $\mathbb{B}$ is given by

$$\mathbb{B} = -\begin{pmatrix} 0 & \gamma\bar{p}\mathcal{P} & 2/(\alpha\bar{v}) \\ -\bar{v}\mathcal{P} & 0 & 0 \\ 0 & \bar{a}\mathcal{P} & \zeta \end{pmatrix} \in \mathbb{R}^{3\times 3},$$

by noticing $\Delta\vartheta = -\mathcal{P}^2\vartheta$. Taking the Fourier transform of (2.42) we have

$$\hat{U}_t = \mathbb{E}\hat{U}, \quad \hat{U}(0,\xi) = \hat{U}_0, \quad t \geq 0,$$

where $\hat{U} = \hat{U}(t,\xi) = \big(\hat{\vartheta}, \hat{w}, \hat{\chi}\big)^\top(t,\xi)$ and

$$\mathbb{E} = \mathbb{E}(|\xi|) = -\begin{pmatrix} 0 & \gamma\bar{p}|\xi| & 2/(\alpha\bar{v}) \\ -\bar{v}|\xi| & 0 & 0 \\ 0 & \bar{a}|\xi| & \zeta \end{pmatrix}. \tag{2.43}$$

The characteristic equation of $\mathbb{E}$ is

$$\det(\lambda I - \mathbb{E}) = \lambda^3 + \zeta\lambda^2 + \bar{c}_f^2|\xi|^2\lambda + \zeta\bar{c}^2|\xi|^2,$$

which is the same as the one-dimensional flow. We point out that the matrix $\mathbb{E}$ admits three simple eigenvalues $\lambda_\pm(|\xi|)$ and $\lambda_d(|\xi|)$, which do not coincide on the complex plane except for a finite number of exceptional points. Obviously it holds that

$$\lambda_+ + \lambda_- + \lambda_d = -\zeta, \quad \lambda_+\lambda_- + \lambda_d(\lambda_+ + \lambda_-) = \bar{c}_f^2|\xi|^2, \quad \lambda_+\lambda_-\lambda_d = -\zeta\bar{c}^2|\xi|^2. \tag{2.44}$$

The functions $\lambda_\sigma = \lambda_\sigma(z)$, $\sigma = \pm, d$, are holomorphic in each simply connected domain containing no exceptional points while continuous at the exceptional point, which can be a branch point or a regular point with eigenvalue splitting, cf. [17,19].

**Lemma 2.2** *(Lemma 3.2. [19]). The functions $\lambda_\sigma(z)$, $\sigma = \pm, d$, are analytic at the origin and have at most a simple pole at the infinity. Precisely, for all $z \neq 0$ it holds that*

$$\mathrm{Re}\{\lambda_\sigma(z)\} < 0, \quad \sigma = \pm, d. \tag{2.45}$$

*Particularly, for $|z| \ll 1$,*

$$\lambda_\pm(z) = -\mu^{(r)}z^2 + \lambda_R(z^4) \pm \mathrm{i}z\big(\bar{c} + \lambda_I(z^2)\big), \quad \lambda_d(z) = -\zeta + 2\mu^{(r)}z^2 - 2\lambda_R(z^4), \tag{2.46}$$

*and for $|z| \to +\infty$,*

$$\lambda_\pm(z) = -\mu^{(1)} + \tilde{\lambda}_R(z^{-2}) \pm \mathrm{i}z\big(\bar{c}_f + \tilde{\lambda}_I(z^{-2})\big), \quad \lambda_d = -\mu^{(2)} - \tilde{\lambda}_R(z^{-2}), \tag{2.47}$$

*where*

$$\mu^{(r)} = \frac{\bar{c}_f^2 - \bar{c}^2}{2\zeta} = \frac{\alpha}{(\alpha + \alpha_f)^2}\bar{a}, \quad \mu^{(1)} = \frac{1}{2}(1 - \frac{\bar{c}^2}{\bar{c}_f^2})\zeta^2 = \frac{\alpha_f}{\alpha(2+\alpha)}, \quad \mu^{(2)} = \frac{\bar{c}^2}{\bar{c}_f^2}\zeta = 1 + \frac{\alpha_f}{2+\alpha},$$

*and $\lambda_R(z^4) = O(z^4)$, $\lambda_I(z^2) = O(z^2)$ are analytic of $z^2$ while $\tilde{\lambda}_R(z^{-2}) = O(z^{-2})$, $\tilde{\lambda}_I(z^{-2}) = O(z^{-2})$ are analytic of $z^{-2}$, which are all with real coefficients in their Taylor expansions.*

Corresponding to the eigenvalue $\lambda_\sigma$, $\sigma = \pm, d$, the right eigenvector $r_\sigma$ and the left eigenvector $l_\sigma$ are

$$r_\sigma = \big(\lambda_\sigma, \bar{v}|\xi|, -\alpha\bar{v}(\lambda_\sigma^2 + \bar{c}_f^2|\xi|^2)/2\big)^\top, \quad l_\sigma = \big(\bar{v}|\xi|^2/\lambda_\sigma, |\xi|, -2|\xi|^2/[\alpha\lambda_\sigma(\lambda_\sigma + \zeta)]\big).$$

Direct calculations by using (2.44) show that the eigenprojection

$$\mathrm{P}_\sigma = \mathrm{P}_\sigma(|\xi|) = [(l_\sigma^\top l_\sigma)^{-1} r_\sigma l_\sigma^\top](|\xi|), \quad \sigma = \pm, d,$$

corresponding to the eigenvalue $\lambda_\sigma$ of $\mathbb{E}$ is given by

$$\mathrm{P}_\sigma = \big(2\lambda_\sigma(\lambda_\sigma + \zeta) + \lambda_\sigma^2 + \bar{c}_f^2|\xi|^2\big)^{-1}\begin{pmatrix} \lambda_\sigma(\lambda_\sigma + \zeta) & \lambda_\sigma^2(\lambda_\sigma + \zeta)(\bar{v}|\xi|)^{-1} & -2\lambda_\sigma/(\alpha\bar{v}) \\ \bar{v}(\lambda_\sigma + \zeta)|\xi| & \lambda_\sigma(\lambda_\sigma + \zeta) & -2|\xi|/\alpha \\ -\bar{v}\bar{a}|\xi|^2 & -\lambda_\sigma\bar{a}|\xi| & \lambda_\sigma^2 + \bar{c}_f^2|\xi|^2 \end{pmatrix}. \tag{2.48}$$

The long time behavior of the Green's function $G = G(t,x) = e^{t\mathbb{B}(x)}$ depends on the expansions of the Fourier transform $\hat{G}(t,\xi)$ expressing as

$$\hat{G}(t,\xi) =: e^{t\mathbb{E}(\xi)} = e^{t\lambda_+(|\xi|)}\mathrm{P}_+(\xi) + e^{t\lambda_-|\xi|}\mathrm{P}_-(\xi) + e^{t\lambda_d|\xi|}\mathrm{P}_d(\xi), \tag{2.49}$$

in low frequency while the local behavior depends on those in high frequency. To derive the large time decay rate of the solution for (2.29), naturally we divide the integral in the inverse transform into three parts: over $|\xi| < \varepsilon,\ |\xi| > R$ and $\varepsilon \le |\xi| \le R$ for some small constant $\varepsilon$ and some large constant $R$.

Firstly, we study the asymptotic expansion of $\hat{G}(t,\xi)$ for small $|\xi|$. To achieve some cancelation, we pair the first two terms on the right-hand side of (2.49) to obtain that

$$\begin{aligned} |e^{t\lambda_+}\mathrm{P}_+ + e^{t\lambda_-}\mathrm{P}_-| &= |e^{t\mathrm{Re}\lambda_\pm}[\cos(t\mathrm{Im}\lambda_\pm)(\mathrm{P}_+ + \mathrm{P}_-) + \mathrm{i}\sin(t\mathrm{Im}\lambda_\pm)(\mathrm{P}_+ - \mathrm{P}_-)]| \\ &\lesssim e^{-\mu^{(r)}|\xi|^2 t}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ |\xi| & |\xi| & |\xi| \end{pmatrix}, \end{aligned}$$

by using $(2.46)_1$ and (2.48). And it follows from $(2.46)_2$ that

$$|e^{t\lambda_d}\mathrm{P}_d| \lesssim e^{-\zeta t}\begin{pmatrix} |\xi|^2 & |\xi| & 1 \\ |\xi|^3 & |\xi|^2 & |\xi| \\ |\xi|^2 & |\xi| & 1 \end{pmatrix}.$$

Then for some given $\varepsilon > 0$ sufficiently small we have

$$|\hat{G}(t,\xi)| \lesssim e^{-\mu^{(r)}|\xi|^2 t}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ |\xi| & |\xi| & |\xi| \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e^{-\zeta t} \end{pmatrix}, \quad |\xi| < \varepsilon. \tag{2.50}$$

Similarly, using (2.47), by direct calculations we can claim that for $|\xi| \to \infty$,

$$|\hat{G}(t,\xi)| \lesssim e^{-\mu^{(1)}t}\begin{pmatrix} 1 & 1 & |\xi|^{-1} \\ 1 & 1 & |\xi|^{-1} \\ 1 & 1 & |\xi|^{-1} \end{pmatrix} + e^{-\mu^{(2)}t}\begin{pmatrix} |\xi|^{-2} & |\xi|^{-3} & |\xi|^{-2} \\ |\xi|^{-1} & |\xi|^{-2} & |\xi|^{-1} \\ 1 & |\xi|^{-1} & 1 \end{pmatrix}.$$

Together with (2.45), it holds that

$$|\hat{G}(t,\xi)| \lesssim e^{-r_\varepsilon t}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad |\xi| \ge \varepsilon, \tag{2.51}$$

where the constant $r_\varepsilon > 0$ depends only on $\varepsilon$. Here for simplicity, $|\hat{G}|$ means the matrix corresponding to $\hat{G}$ with each element taken the absolute value.

With the help of the asymptotic analysis for the Green's function in Fourier space, we are able to establish the $L^2$-time decay rate of the global solution to the Cauchy problem for the linear system (2.29), (2.31)-(2.32). It should be noted that the decay rates obtained below are optimal.

**Proposition 2.4.** *Assume $(p_0 - \bar{p}, u_0, \chi_0) \in L^1(\mathbb{R}^3) \cap H^3(\mathbb{R}^3)$. Then $(p, u, \chi)$ solves the Cauchy problem (2.29), (2.31)-(2.32) for all $t > 0,\ x \in \mathbb{R}^3$. For $k \le 3$ it holds that*

$$\|\nabla^k(p - \bar{p}, u)(t)\| \lesssim (1+t)^{-\frac{3}{4}-\frac{k}{2}}\|(p_0 - \bar{p}, u_0, \chi_0)\|^2_{L^1\cap H^k}, \tag{2.52}$$

$$\|\nabla^k \chi(t)\| \lesssim (1+t)^{-\frac{5}{4}-\frac{k}{2}}\|(p_0 - \bar{p}, u_0, \chi_0)\|^2_{L^1\cap H^k}. \tag{2.53}$$

**Proof.** It is equivalent to consider the linear system (2.42). We use the pointwise estimates (2.50) and (2.51) on the Fourier transforms $\hat{G}(t,\xi)$ to derive time decay properties of solution $(p, u, \chi)$ for (2.29) as follows. For $k \ge 0$ we can claim

$$\|\nabla^k \vartheta(t,x)\|^2 = \int\limits_{|\xi|\le\varepsilon} |\xi|^{2k}|\hat{\vartheta}(t,\xi)|^2 d\xi + \int\limits_{|\xi|\ge\varepsilon} |\xi|^{2k}|\hat{\vartheta}(t,\xi)|^2 d\xi$$

$$\lesssim \int\limits_{|\xi|\le\varepsilon} e^{-\mu^{(r)}|\xi|^2 t}|\xi|^{2k}\left(|\hat{\vartheta}|^2+|\hat{w}|^2+|\hat{\chi}|^2\right)d\xi + e^{-r_\varepsilon t}\int\limits_{|\xi|\ge\varepsilon}|\xi|^{2k}\left(|\hat{\vartheta}|^2+|\hat{w}|^2+|\hat{\chi}|^2\right)d\xi$$
$$\lesssim (1+t)^{-\frac{3}{2}-k}\|(\vartheta_0,u_0,\chi_0)\|^2_{L^1\bigcap H^k},$$

by using the relation (2.41) between $u$ and $w$ such that

$$\||\xi|^k\hat{w}\| = \|\nabla^k u\|, \quad \|\hat{w}\|_{L^\infty}\le\|\hat{u}\|_{L^\infty}\le\|u\|_{L^1}.$$

Similarly, it holds that

$$\|\nabla^k u(t,x)\| \lesssim (1+t)^{-\frac{3}{4}-\frac{k}{2}}\|(\vartheta_0,u_0,\chi_0)\|_{L^1\bigcap H^k},$$
$$\|\nabla^k \chi(t,x)\| \lesssim (1+t)^{-\frac{5}{4}-\frac{k}{2}}\|(\vartheta_0,u_0,\chi_0)\|_{L^1\bigcap H^k},$$

which completed the proof of Proposition 2.4. □

## Conflict of interest statement

No conflict of interest.

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