Qualitative properties of positive singular solutions to nonlinear elliptic systems with critical exponent ☆

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Abstract

We studied the asymptotic behavior of local solutions for strongly coupled critical elliptic systems near an isolated singularity. For the dimension less than or equal to five we prove that any singular solution is asymptotic to a rotationally symmetric Fowler type solution. This result generalizes the celebrated work due to Caffarelli, Gidas and Spruck [1] who studied asymptotic proprieties to the classic Yamabe equation. In addition, we generalize similar results by Marques [12] for inhomogeneous context, that is, when the metric is not necessarily conformally flat.

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1. Introduction

Let g be a smooth Riemannian metric on the unit ball $B_1^n(0) \subset \mathbb{R}^n (n \geq 3)$. In this paper we derive asymptotic behavior for positive singular solutions to the following class of nonlinear elliptic system with critical exponent defined in the inhomogeneous context

$$-\Delta_g u_i + \sum_{j=1}^2 A_{ij}(x)u_j = \frac{n(n-2)}{4}|\mathcal{U}|^{\frac{4}{n-2}}u_i, \quad \text{for } i = 1, 2, \tag{1.1}$$

in the punctured ball $\Omega = B_1^n(0)\backslash\{0\}$. Here $\mathcal{U} = (u_1, u_2)$, $\Delta_g = \text{div}_g\nabla$ is the Laplace Beltrami operator of the metric g and A is a C^1 map from $B_1^n(0)$ to $M_2^s(\mathbb{R})$, the vector space of symmetrical 2×2 real matrices. We can write

$A = (A_{ij})_{i,j}$, where each A_{ij} is a C^1 real valued function. We say that $\mathcal{U} = (u_1, u_2)$ is a *positive* solution of (1.1) if each coordinate u_i is a positive function for $i = 1, 2$. Here $|\mathcal{U}|$ denotes the Euclidean norm of $\mathcal{U}$, that is, $|\mathcal{U}|^2 = u_1^2 + u_2^2$.

O. Druet and E. Hebey [7] have investigated and proved analytic stability for strongly coupled critical elliptic systems in the inhomogeneous context of a compact Riemannian manifold. This kind of coupled systems of nonlinear Schrödinger equations are part of important branches of mathematical physics, like in fiber-optic theory and in the behavior of deep water waves and freak waves in the ocean, (see [7] and [8] and the reference therein). The critical systems (1.1) are weakly coupled by the linear matrix A and strongly coupled by the Gross–Pitaevskii type nonlinearity in the right-hand side of (1.1). Moreover, system (1.1) can be seen as a natural generalization of the singular Yamabe problem for an arbitrary metric.

We also mention the recent work of Z. Chen and C.-S. Lin [6], where they have studied a different two-coupled elliptic system with critical exponent, although also is an extension of Yamabe-type equation in the strongly coupled regime.

We call a solution $\mathcal{U}$ of (1.1) singular at origin (non–removable singularity) if

$$\lim_{|x| \to 0} |\mathcal{U}(x)| = +\infty.$$

For easy reference, we introduce now the following hypotheses on the potential

(H1) The coefficient A_{ij} are nonpositive for any $i \neq j$, which is equivalent to say that $-A$ is *cooperative*.
(H2) In dimension $n = 5$, there exists a C^2-function f such that

$$A(x) = f(x)Id_2 + O(|x|)$$

near the origin, where Id_2 is the identity matrix of order 2×2.

Both hypotheses will be important to prove a removable singularity result, Theorem 1.5 (see on page 1578). While the hypotheses (H1) is used to prove a uniform spherical Harnack inequality (see Corollary 3.2) around the singularity, the hypotheses (H2) is used to get a good estimate of the Pohozaev integrals (1.5).

To understand our main result in this work we consider the Fowler solutions (or Delaunay-type solutions) which are radial solutions of

$$\Delta u + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad \mathbb{R}^n \backslash \{0\}, \tag{1.2}$$

which blow-up at zero. Here Δ denotes the Euclidean Laplacian. By Caffarelli, Gidas and Spruck in [1], it is well known that any solution to (1.2) with a non–removable singularity is rotationally invariant.

In order to derive asymptotic properties for singular positive solutions of (1.1) it is crucial in our argument to study qualitative properties of the limit system

$$\Delta u_i + \frac{n(n-2)}{4} |\mathcal{U}|^{\frac{4}{n-2}} u_i = 0 \quad \text{in} \quad \mathbb{R}^n \backslash \{0\}, \; i = 1, 2, \tag{1.3}$$

which is obtained after blowing up the system (1.1). First of all we prove that any solution of (1.3) is radially symmetric.

Theorem 1.1 (*Radial Symmetry*). *Let $n \geq 3$ and $\mathcal{U} \in C^2(\mathbb{R}^n \backslash \{0\})$ be a nonnegative singular solution of (1.3). Then $\mathcal{U}$ is radially symmetric about the origin.*

This theorem enables us to prove the following classification result.

Theorem 1.2 (*Classification*). *Let $n \geq 3$ and $\mathcal{U} \in C^2(\mathbb{R}^n \backslash \{0\})$ be a nonnegative singular solution of (1.3). Then there exists a Fowler solution u_0 and $\Lambda \in \mathbb{S}^1_+$ such that $\mathcal{U} = u_0 \Lambda$, where $\mathbb{S}^1_+$ consists of the elements (Λ_1, Λ_2) in $\mathbb{S}^1$, the unit circle in $\mathbb{R}^2$, which are such that $\Lambda_i \geq 0$ for $i = 1, 2$.*

We will call a solution of the limit system (1.3) as a *Fowler-type solution*.
Finally, our main result reads as follows

Theorem 1.3 (*Main Result*). *Assume* $3 \leq n \leq 5$ *and that the potential A satisfies the hypotheses* (*H1*) *and* (*H2*). *Let* $\mathcal{U} \in C^2(\Omega)$ *be a positive singular solution of* (1.1). *Then there exists* $\alpha > 0$ *and a Fowler-type solution* $\mathcal{U}_0$ *such that*

$$\mathcal{U}(x) = (1 + O(|x|^{\alpha}))\mathcal{U}_0(x) \quad as \quad x \to 0.$$

As a consequence of Theorem 1.3 we have that any positive solution $\mathcal{U}$ for the system

$$\Delta u_i + \frac{n(n-2)}{4}|\mathcal{U}|^{\frac{4}{n-2}}u_i = 0 \qquad in \quad \Omega, \, i = 1, 2,$$

either $\mathcal{U}$ can be smoothly extended to the origin or there is a Fowler-type solution $\mathcal{U}_0$ and $\alpha > 0$ such that

$$\mathcal{U}(x) = (1 + O(|x|^{\alpha}))\mathcal{U}_0(x) \quad as \quad x \to 0.$$

In [1], L. Caffarelli, B. Gidas and J. Spruck have proved that any positive solution to

$$\Delta u + \frac{n(n-2)}{4}u^{\frac{n+2}{n-2}} = 0 \ \ in \ \Omega$$

either can be smoothly extended to the origin or it is asymptotic to some Fowler solution u_0 when x goes to zero. Later, N. Korevaar, R. Mazzeo, F. Pacard and R. Schoen in [9] have improved this result by proving that

$$u(x) = (1 + O(|x|^{\alpha}))u_0(x) \ \ as \ x \to 0,$$

for some $\alpha > 0$. F. C. Marques in [12] extended this result to a more general setting, at least in low dimensions. More precisely, he has proved that Fowler solutions still serve as asymptotic model to the equation

$$\Delta_g u - \frac{n-2}{4(n-1)}R_g u + \frac{n(n-2)}{4}u^{\frac{n+2}{n-2}} = 0 \ \ in \ \Omega,$$

for $3 \leq n \leq 5$. C. C. Chen and C. S. Lin in a series of works, [2–4,10] have studied local singular solutions to the prescribed scalar curvature equation

$$\Delta u + K(x)u^{\frac{n+2}{n-2}} = 0,$$

where K is a positive C^1 function defined on a neighborhood around zero. They have studied conditions on the function K such that the Fowler solution still serve as asymptotic models. In the spirit of the previous works, the main motivation of this paper is the question that whether these asymptotic results could be extended to the system (1.1).

The first task to answer this question is to find an asymptotic model to the system. In [8], O. Druet, E. Hebey and J. Vetóis have proved that any nonnegative entire solution of the system

$$\Delta u_i + |\mathcal{U}|^{\frac{4}{n-2}}u_i = 0 \ \ in \ \ \mathbb{R}^n,$$

is of the form $U = u\Lambda$, where $\Lambda \in \mathbb{S}^1_+$ and u is an entire solution of (1.2). This conclusion motivated us to prove the classification result to the limit system, Theorem 1.2, which is the asymptotic model to the system (1.1).

Now let us briefly describe the strategy of the proof of the Theorem 1.3, which is inspired on the works [9] and [12]. The first step of our proof is to prove the Theorems 1.1 and 1.2. In Section 2.4, based on these theorems we can do a carefully analysis of the Jacobi fields to limit system (1.3) similar to the analysis done in [9]. A key result to the proof of the Theorem 1.3 is the following fundamental upper bound.

Theorem 1.4. *Suppose* $3 \leq n \leq 5$. *Assume that* $\mathcal{U} = (u_1, u_2)$ *is a positive solution of* (1.1) *in* $\Omega = B^n_1(0)\backslash\{0\}$. *Then there exists a constant* $c > 0$ *such that*

$$|\mathcal{U}(x)| \leq cd_g(x, 0)^{\frac{2-n}{2}}, \tag{1.4}$$

for $0 < d_g(x, 0) < 1/2$.

The difficulty here is the fact that we consider elliptic systems in a fully inhomogeneous medium breaking the various symmetries that these systems enjoy in the Euclidean setting. Inspired by the work of F. C. Marques [12], by

using the method of moving planes, we can overcome this difficulty at least for lower dimensions to prove this upper bound (1.4).

As a consequence of (1.4) we are able to define a Pohozaev-type invariant as

$$P(\mathcal{U}) = \lim_{r \to \infty} P(r, \mathcal{U}),$$

where

$$P(r, \mathcal{U}) = \int_{\partial B_r} \left(\frac{n-2}{2} \left\langle \mathcal{U}, \frac{\partial \mathcal{U}}{\partial \nu} \right\rangle - \frac{r}{2} |\nabla \mathcal{U}|^2 + r \left| \frac{\partial \mathcal{U}}{\partial \nu} \right|^2 + r \frac{(n-2)^2}{8} |\mathcal{U}|^{\frac{2n}{n-2}} \right) d\sigma. \tag{1.5}$$

In the light of the work of L. Caffarelli, B. Gidas and J. Spruck in [1], we prove a theorem on removable singularity. Precisely, we prove that this invariance is always non-negative and it is equal to zero if, and only if, the singularity is removable. Formally,

Theorem 1.5. *Assume $3 \leq n \leq 5$ and that the potential A satisfies the hypotheses (H1) and (H2). Let $\mathcal{U}$ be a positive solution to the system (1.1) in $B_1^n(0)\backslash\{0\}$. Then $P(\mathcal{U}) \leq 0$. Moreover, $P(\mathcal{U}) = 0$ if and only if the origin is a removable singularity.*

An important consequence of this result is the

Corollary 1.6. *Assume $3 \leq n \leq 5$ and that the potential A satisfies the hypotheses (H1) and (H2). Let $\mathcal{U}$ be a positive solution to the system (1.1) in $B_1^n(0)\backslash\{0\}$. If the origin is a non–removable singularity, then there exists $c > 0$ such that*

$$|\mathcal{U}(x)| \geq c d_g(x, 0)^{\frac{2-n}{2}} \tag{1.6}$$

for each i and $0 < d_g(x, 0) < 1/2$.

Remark 1.7. We can see that Theorems 1.1, 1.2 and 1.4 hold for systems with any number of equations by using easy modifications of our proofs.

The paper is organized as follows. In Section 2 we use the method of moving planes to prove that singular solutions of (1.3) are radially symmetric. Therefore system (1.3) is equivalent to a ODE system which is crucial in our argument to prove the classification result, Theorem 1.2. We also study growth properties of the Jacobi fields in order to obtain the asymptotic behavior of singular solutions of (1.1). In Section 3 we prove the upper bound, Theorem 1.4, Theorem 1.5 and the lower bound, Corollary 1.6 to solutions of (1.1). In Section 4 we conclude the proof of Theorem 1.3. Precisely, by using a delicate argument, originally due to Leon Simon, applied in [9] and [12], we apply the estimates (1.4) and (1.6) to prove that any solution of our system (1.1) is asymptotically to some Fowler-type solution.

2. Classification result for the limit system

Our main goal in this section is to study some properties of singular nonnegative solutions $\mathcal{U} = (u_1, u_2)$ to the following system

$$\Delta u_i + \frac{n(n-2)}{4} |\mathcal{U}|^{\frac{4}{n-2}} u_i = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}. \tag{2.1}$$

Similarly to the definition for the system (1.1) we say that a solution $\mathcal{U}$ of (2.1) is singular if $\lim_{|x| \to 0} |\mathcal{U}(x)| = +\infty$. It may happen that only one of the coordinates blows up at the singularity. The above system will be important in our analysis since its solutions will play a similar role to Fowler solutions in the singular Yamabe problem.

One of our main result in this section is the characterization of nonnegative singular solutions of the limit system (2.1). We show that any solution of this system is a multiple of a vector in the unit sphere by a classical Fowler solution, see Theorem 2.6 below.

2.1. Radial symmetry

The first step to reach our goal is to show that any solution of the system (2.1) is radially symmetric. To this end we will use the method of moving planes.

Theorem 2.1 (*Radial Symmetry*). *Let $n \geq 3$ and $\mathcal{U} = (u_1, u_2)$ be a nonnegative C^2 singular solution of the nonlinear elliptic system* (2.1). *Then $\mathcal{U}$ is radially symmetric about the origin.*

Proof. The proof will follow the argument in [1] and [6]. Without loss of generality we may assume that $\lim_{|x|\to 0} u_1(x) = +\infty$. Fix an arbitrary $z \neq 0$ in $\mathbb{R}^n$. Define the Kelvin transform

$$U_i(x) = \frac{1}{|x|^{n-2}} u_i \left(z + \frac{x}{|x|^2} \right), \quad \text{for } i = 1, 2.$$

We see that U_1 is singular at zero and $z_0 = -z/|z|^2$ and U_2 is singular at zero. Moreover, the Kelvin transform satisfies

$$\Delta U_i + \frac{n(n-2)}{4} |\mathbf{U}|^{\frac{4}{n-2}} U_i = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \{0, z_0\}. \tag{2.2}$$

It is easy to see that

$$U_i(x) = O(|x|^{2-n}) \quad \text{as } |x| \to \infty$$

and that each U_i has the following harmonic asymptotic expansion at infinity

$$
\begin{array}{rcl}
U_i(x) & = & a_{i0}|x|^{2-n} + a_{ij}x_j|x|^{-n} + O(|x|^{-n}), \\
\partial_{x_j} U_i & = & (2-n)a_{i0}x_j|x|^{-n} + O(|x|^{-n}), \\
\partial^2_{x_k x_j} U_i & = & O(|x|^{-n}),
\end{array}
$$

where $a_{i0} = u_i(z)$ and $a_{ij} = \partial_{y_j} u_i(z)$.

We will show that U_i are axisymmetric with respect to the axis going through 0 and z. Choose a reflection direction orthogonal to this axis and assume without loss of generality that it is equal to the positive x_n direction $(0, \ldots, 0, 1)$. For $\lambda > 0$ consider $\Sigma_\lambda := \{x \in \mathbb{R}^n : x_n > \lambda\}$ and the reflection

$$x = (x_1, \ldots, x_{n-1}, x_n) \in \Sigma_\lambda \mapsto x_\lambda = (x_1, \ldots, x_{n-1}, 2\lambda - x_n).$$

Using the asymptotic expansion at infinity for U_i and Lemma 2.3 in [1], we see that there exist positive constants $\overline{\lambda}$ and R such that for any $\lambda \geq \overline{\lambda}$ we have

$$U_i(x) < U_i(x_\lambda), \quad \text{for } x \in \Sigma_\lambda \text{ and } |x_\lambda| > R. \tag{2.3}$$

By Lemma 2.1 in [5] there exists a constant $C > 0$ such that

$$U_i(x) \geq C, \quad \text{for} \quad x \in \overline{B_R} \setminus \{0, z_0\}. \tag{2.4}$$

Since $U_i(x) \to 0$ as $|x| \to +\infty$, then by (2.3) and (2.4) we have that there exists $\lambda_0 > \overline{\lambda}$ such that when $|x| \geq 2\lambda_0 - R$ it holds $U_i(x) < C$. On the other hand if $x \in \Sigma_\lambda$ and $x_\lambda \in \overline{B}_R$ then $x \notin B_{2\lambda - R}$. Thus for every $\lambda \geq \lambda_0$ we have

$$U_i(x) \leq U_i(x_\lambda), \quad \text{for all } x \in \Sigma_\lambda \text{ and } x_\lambda \notin \{0, z_0\}. \tag{2.5}$$

Let

$$\lambda^* := \inf\{\tilde{\lambda} > 0; (2.5) \text{ holds for all } \lambda \geq \tilde{\lambda}\}.$$

It suffices to show that $\lambda^* = 0$. Indeed, this proves that U_i is axisymmetric with respect to the axis going through 0 and z and since z is arbitrary, each u_i is radially symmetric about the origin.

Suppose by contradiction that $\lambda^* > 0$. Then (2.5) holds for $\lambda = \lambda^*$. Since U_1 goes to infinity when x goes to z_0 we see that U_1 cannot be invariant by the reflection x_{λ^*}. Then by applying the maximum principle to $U_i(x_{\lambda^*}) - U_i(x)$ in (2.2), we conclude that

$$U_i(x) < U_i(x_{\lambda^*}), \text{ for } x \in \Sigma_{\lambda^*} \text{ and } x_{\lambda^*} \notin \{0, z_0\}. \tag{2.6}$$

Note that $0, z_0 \notin \partial\Sigma_{\lambda^*}$, since $\lambda^* > 0$. Then applying the Hopf boundary lemma for $x \in \partial\Sigma_{\lambda^*}$ we obtain

$$\partial_{x_n}(U_i(x_{\lambda^*}) - U_i(x)) = -2\partial_{x_n}U_i(x) > 0. \tag{2.7}$$

Now choose sequences $\lambda_j \nearrow \lambda^*$ and $x^j \in \Sigma_{\lambda_j}$ such that $U_1(x_{\lambda_j}^j) < U_1(x^j)$. By Lemma 2.4 in [1], we conclude that the sequence $|x^j|$ is bounded. Hence passing to a subsequence we may assume that $x_j \to \overline{x} \in \overline{\Sigma_{\lambda^*}}$ with $U_1(\overline{x}_{\lambda^*}) \leq U_1(\overline{x})$. By (2.6) we know that $\overline{x} \in \partial\Sigma_{\lambda^*}$ and then $\partial_{x_n}U_1(\overline{x}) \geq 0$, a contradiction with (2.7). Therefore, $\lambda^* = 0$. ∎

Since we proved the radial symmetry of the solutions we will use a change of variables to approach our problem using ODE analysis. This is the subject from the next subsection.

2.2. ODE analysis

If $\mathcal{U} = (u_1, u_2)$ is a solution of the system (2.1), then by Theorem 2.1 we have that $u_i(x)$ depends only on $|x|$. Let

$$v_i(t) := e^{-\delta t} u_i(e^{-t}\theta), \quad i = 1, 2,$$

where $t = -\ln r$, $\theta = x/|x|$ and $\delta = (n-2)/2$. Then, it is easy to see that $V = (v_1, v_2)$ satisfies the ordinary differential equation

$$v_i'' - \delta^2 v_i + \frac{n(n-2)}{4}|V|^{\frac{4}{n-2}}v_i = 0, \quad i = 1, 2 \tag{2.8}$$

with $t \in \mathbb{R}$, $v_i \geq 0$ and $v_i \in C^2(\mathbb{R})$. Setting $w_i := v_i'$, this ODE system transformed into a first order Hamiltonian system

$$\begin{cases} v_i' &= w_i \\ w_i' &= \delta^2 v_i - \dfrac{n(n-2)}{4}|V|^{\frac{4}{n-2}}v_i \end{cases}, \quad i = 1, 2.$$

Setting $W = (w_1, w_2)$, we can see that the Hamiltonian energy given by

$$H(V, W) = \frac{1}{2}\left(|W|^2 - \delta^2|V|^2 + \delta^2|V|^{\frac{2n}{n-2}}\right),$$

is constant along solutions of (2.8). It does not seem possible to integrate completely the ODE (2.8), as in the classical case. Besides the Hamiltonian energy above, it does not seem possible to find other quantities which are preserved along the trajectories of (2.8). This is the reason why we will rely on the following approach to prove the Theorem 1.2.

Since the Hamiltonian energy above is constant along any solution V of the ODE (2.8), $\Psi(t) := H(V(t), V'(t)) = K$, for some constant K.

In fact, by a direct computation, we easily deduce that

$$\Psi(t) = \frac{P(r, \mathcal{U})}{\sigma_{n-1}},$$

where σ_{n-1} is the volume of the unit sphere $\mathbb{S}^{n-1}$ and $P(r, \mathcal{U})$ is the Pohozaev integral defined in (1.5). As a consequence it is easy to see that V and $|V'|$ are bounded. Indeed, if we suppose that there exists a sequence $t_n \to +\infty$ such that $|V(t_n)| \to +\infty$, we have

$$\frac{1}{2}\left(\left(\frac{|V'(t_n)|}{|V(t_n)|}\right)^2 - \delta^2\right) + \frac{\delta^2}{2}|V(t_n)|^{\frac{4}{n-2}} = \frac{K}{|V(t_n)|^2}$$

which is a contradiction. Similarly, we conclude that $|V'|$ is bounded.

The first consequence of the radial symmetry of solutions is the fact that each coordinate u_i satisfies the following dichotomy: either it is strictly positive or it vanishes identically.

Lemma 2.2. *Let $\mathcal{U}$ be a nonnegative singular solution of the limit system* (2.1). *If there exists $y \in \mathbb{R}^n \backslash \{0\}$ such that $u_i(y) = 0$, then $u_i \equiv 0$.*

Proof. We already know that any solution $\mathcal{U}$ of the limit system (2.1) is radially symmetric. Suppose that there exist y and z in $\mathbb{R}^n \setminus \{0\}$, with $|z| > |y|$ and $u_i(z) > u_i(y) = 0$. By the maximum principle from Chen and Lin in [5, Lemma 2.1], we know that

$$u_i(x) \geq \inf_{\partial B_{|z|}} u_i = u_i(z) > 0,$$

for all $x \in B_{|z|} \backslash \{0\}$, which contradicts the fact that $u_i(y) = 0$. Hence, if there exists a point y such that $u_i(y) = 0$ then u_i must be zero for all z with $|z| \geq |y|$. However, we know that the corresponding function v_i is a solution for the ODE system (2.8) and by the uniqueness of solutions to the ODE system, it would be identically zero. ∎

In order to prove the classification result, we need the following lemma.

Lemma 2.3. *If v_i has a limit C_i at $+\infty$ or $-\infty$, then $C_i \leq \left(\frac{n-2}{n}\right)^{\frac{n-2}{4}}$. Moreover, if $\Psi(t) = K \geq 0$, then $C_i = 0$.*

Proof. Assume that $\lim_{t \to +\infty} v_1(t) = C_1 > 0$. For each coordinate, consider a sequence of translates $v_{i,k}(t) = v_i(t + k)$. By (2.8), up to passing to a subsequence, we may assume that $v_{1,k} \to C_1$ and $v_{2,k}^2 \to w \geq 0$ uniformly in compacts with respect to the C^2 topology. Thus

$$-\delta^2 C_1 + \frac{n(n-2)}{4}(C_1^2 + w)^{\frac{2}{n-2}} C_1 = 0,$$

which implies that $C_1 \leq \left(\frac{n-2}{n}\right)^{\frac{n-2}{4}}$ and w is also constant, C_2 for example. This implies that

$$K = -\frac{\delta^2}{2}|C|^2 + \frac{\delta^2}{2}|C|^{\frac{2n}{n-2}} \geq 0,$$

where $C = (C_1, C_2)$. Therefore, $C_i = 0$. ∎

Now, for each i define the auxiliary functions $f_i : \mathbb{R} \to \mathbb{R}$ by

$$f_i(t) := -\frac{1}{2}(v_i'(t))^2 + \frac{\delta^2}{2}v_i(t)^2 - \frac{\delta^2}{2}v_i(t)^{\frac{2n}{n-2}}.$$

By (2.8) it is easy to see that

$$f_i'(t) = \frac{\delta^2}{2}\left(|V|^{\frac{4}{n-2}}(t) - v_i(t)^{\frac{4}{n-2}}\right)v_i(t)v_i'(t).$$

Since $|V| \geq v_i$, then the monotonicity of f_i is exactly the same as the monotonicity of v_i. This monotonicity property is very important and it will be used in the sequel.

Lemma 2.4. *For all t in $\mathbb{R}$ we have that $v_i(t) < 1$.*

Proof. Suppose by contradiction that the result is not true. By Lemma 2.3 there exists a local maximum point $t_0 \in \mathbb{R}$ of v_1 such that $v_1(t_0) \geq 1$. This implies that

$$f_1(t_0) = \frac{\delta^2}{2}v_1^2(t_0) - \frac{\delta^2}{2}v_1(t_0)^{\frac{2n}{n-2}} \leq 0.$$

We claim that there exists $t_1 > t_0$ such that $v_1'(t) < 0$ for $t \in (t_0, t_1)$ and $v_1'(t_1) = 0$. Otherwise it would follow from Lemma 2.3 that $v_1 \downarrow C_1 \leq ((n-2)/n)^{\frac{n-2}{4}}$ and $f_1(t) \downarrow -C < 0$ as t goes to $+\infty$. Then we have that $v_1'(t) \to 0$ as t goes to $+\infty$ and so

$$0 > -C = \frac{\delta^2}{2}C_1^2 - \frac{\delta^2}{2}C_1^{\frac{2n}{n-2}} > 0,$$

which is a contradiction. Hence there exists such t_1 and it satisfies $v_1''(t_1) \geq 0$. On the other hand

$$f_1(t_1) = \frac{\delta^2}{2}v_1(t_1)^2 - \frac{\delta^2}{2}v_1(t_1)^{\frac{2n}{n-2}} < f_1(t_0) \leq 0,$$

namely $v_1(t_1) > 1$ which implies from (2.8) that $v_1''(t_1) < 0$, which is again a contradiction. Therefore, such t_0 can not exist. ∎

2.3. Classification result

In the beginning of this section, we mentioned that the solutions of our limit system play a similar role to the Fowler solutions in the singular Yamabe problem. We will summarize briefly basic properties about Fowler solutions.

Remember that a positive function u is a Fowler or Delaunay-type solution if it satisfies

$$\Delta u + \frac{n(n-2)}{4}u^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}. \tag{2.9}$$

It is well known by [1, Theorem 8.1] that u is rotationally invariant, and thus the equation can be reduced to an ODE. Since $\mathbb{R}^n \setminus \{0\}$ is conformally diffeomorphic to a cylinder, relative to the cylindrical metric this equation becomes

$$v'' - \frac{(n-2)^2}{4}v + \frac{n(n-2)}{4}v^{\frac{n+2}{n-2}} = 0. \tag{2.10}$$

The analysis of this equation can be done by converting into a system of first order equations

$$\begin{cases} v' = w \\ w' = \frac{(n-2)^2}{4}v - \frac{n(n-2)}{4}v^{\frac{n+2}{n-2}} \end{cases}$$

whose Hamiltonian energy given by

$$H(v, w) = w^2 - \frac{(n-2)^2}{4}v^2 + \frac{(n-2)^2}{4}v^{\frac{2n}{n-2}}$$

is constant along solutions of (2.10).

By examining the level curves of the energy, we see that all positive solutions lies in the bounded set $\{H < 0\} \cap \{v > 0\}$. The basic properties of these solutions are summarized in the next proposition whose proof can be found in [13].

Proposition 2.5. *For any $H_0 \in (-((n-2)/n)^{n/2}(n-2)/2, 0)$, there exists a unique bounded solution of (2.10) satisfying $H(v, v') = H_0$, $v'(0) = 0$ and $v''(0) > 0$. This solution is periodic and for all $t \in \mathbb{R}$ we have $v(t) \in (0, 1)$. This solution can be indexed by the parameter $\varepsilon = v_\varepsilon(0) \in (0, ((n-2)/n)^{(n-2)/4})$, which is the smaller of the two values v_ε assumes when $v_\varepsilon'(0) = 0$.*

In [8], Druet, Hebey and Vetóis have proved a characterization result for entire solutions in $\mathbb{R}^n$ for the limit system (2.1). They proved that any solution in $\mathbb{R}^n$ of the equation (2.1) is of the form $u\Lambda$, where u is an entire solution of the equation (2.10) and $\Lambda \in \mathbb{S}^1_+$. Inspired by their result, we may ask whether a similar description holds in the singular case. Indeed, we prove that every nonnegative singular solution of (2.1) is a Fowler solution, up to a vector in the unit sphere with nonnegative coordinates.

Theorem 2.6. *Suppose that $\mathcal{U}$ is a nonnegative singular solution for (2.1). Then there exists a Fowler solution u and $\Lambda \in \mathbb{S}^1_+$ such that $\mathcal{U} = u\Lambda$.*

Proof. We will prove the result in the case when v_1 and v_2 are positive solutions since otherwise the result follows directly from Lemma 2.2.

Let $V = (v_1, v_2)$ be a solution for (2.8) obtained from $\mathcal{U}$ after the change of variables of Fowler. We know that it satisfies

$$v_i'' - \delta^2 v_i + \frac{n(n-2)}{4}|V|^{\frac{4}{n-2}}v_i = 0, \quad t \in \mathbb{R}.$$

From this, we can note that

$$v_1'' v_2 = v_1 v_2''$$

which implies

$$v_1' v_2 - v_1 v_2' = c = constant.$$

Suppose that $c \neq 0$. Without loss of generality we can assume that $c > 0$. Since V is bounded, then by Lemma 2.4 we obtain

$$\left(\frac{v_1}{v_2}\right)' = \frac{c}{v_2^2} \geq c$$

which is a contradiction, since the quotient will be a monotone function and thus it would assume negative values. Therefore we conclude that

$$\left(\frac{v_1}{v_2}\right)' = 0$$

and so $v_1/v_2 = \eta$ is constant. It follows that

$$V = (v_1, v_2) = (\eta v_2, v_2) = (\eta, 1)v_2 = \sqrt{1 + \eta^2}v_2\left(\frac{\eta}{\sqrt{1 + \eta^2}}, \frac{1}{\sqrt{1 + \eta^2}}\right)$$

and a direct computation shows that $v_0 = \sqrt{1 + \eta^2}v_2$ is a Fowler solution. ∎

This result will be extremely important in the study of the Jacobi fields. The Theorem 2.6 will be very useful to simplify the analysis of the linearized system. Also it will allow us to use all the properties of Fowler solutions that we described previously.

Let us mention some important consequences of the Theorem 2.6. The first result concerns the singularity of the solutions.

Corollary 2.7. *Let $\mathcal{U} = (u_1, u_2)$ be a positive singular solution to the equation* (2.1). *Then* $\lim\limits_{|x|\to 0} u_i(x) = +\infty$, *for* $i = 1, 2$.

The second direct consequences is the following

Corollary 2.8. *If $\mathcal{U}$ is a nonnegative singular solution to the system* (2.1)*, then there exist positive constants c_1 and c_2 such that*

$$c_1|x|^{\frac{2-n}{2}} \leq |\mathcal{U}(x)| \leq c_2|x|^{\frac{2-n}{2}}.$$

To understand the third and last consequence, given a solution $\mathcal{U}$ of (2.1), it is an easy computation to see that

$$P(r, \mathcal{U}) = P(s, \mathcal{U}), \tag{2.11}$$

for any r and s, where $P(r, \mathcal{U})$ is defined in (1.5). Hence we can denote it by $P(\mathcal{U})$ which is called the Pohozaev invariant of the solution $\mathcal{U}$. Besides, note that by the classification result, there exist a Fowler solution u_0 and a unit vector $\Lambda = (\Lambda_1, \Lambda_2)$ with nonnegative coordinates, such that $\mathcal{U} = u_0\Lambda$. Consequently $P(\mathcal{U})$ coincides with the Pohozaev invariant of u_0 as defined in [1] and we obtain the following important result

Corollary 2.9. *Let $\mathcal{U} = (u_1, u_2)$ be a nonnegative solution of* (2.9). *Then $P(\mathcal{U}) \leq 0$. Moreover, $P(\mathcal{U}) = 0$ if and only if, $u_i \in C^2(\mathbb{R}^n)$, for all i.*

2.4. Jacobi fields for the limit system

In the study of the asymptotic behavior of the solution to the singular Yamabe problem, N. Korevaar et al. [9] and F. Marques [12] have used the growth properties of Jacobi fields as a tool to obtain properties of the singular solutions. In order to prove our main theorem, we need to do a similar analysis. In this case we are going to linearize the equation about a solution given by Theorem 2.6. In order to do this, let us first recall some details about the analysis in the simple case.

Let us briefly describe the analysis done in [9]. Consider the equation (2.9) which, using the cylindrical coordinates change is equivalent to

$$\partial_t^2 v + \Delta_{\mathbb{S}^{n-1}} v - \frac{(n-2)^2}{4} v + \frac{n(n-2)}{4} v^{\frac{n+2}{n-2}} = 0, \tag{2.12}$$

where $u(x) = |x|^{\frac{2-n}{2}} v(\frac{x}{|x|}, -\log|x|)$. The authors in [9] considered the linearization of the operator in (2.12) around a Fowler solution v_ε which is given by

$$L_\varepsilon = \partial_t^2 + \Delta_{\mathbb{S}^{n-1}} - \frac{(n-2)^2}{4} + \frac{n(n+2)}{4} v_\varepsilon^{\frac{4}{n-2}}. \tag{2.13}$$

Since the above operator has periodic coefficients, it could be studied using the classical Flocquet theory or also by separation of variables, see [15]. The elements in the kernel of the linearized operator L_ε, that is, the functions ψ satisfying $L_\varepsilon \psi = 0$, are called **Jacobi Fields**. The linearization (2.13) was studied using results due to R. Mazzeo, D. Pollack and K. Uhlenbeck in [14] based on the spectral decomposition of the Laplace operator $\Delta_{\mathbb{S}^{n-1}}$.

If $\{\lambda_k, \mathcal{X}_k(\theta)\}$ are the eigendata of $\Delta_{\mathbb{S}^{n-1}}$, using the convention that these eigenvalues are listed with multiplicity, we can write $\lambda_0 = 0, \lambda_1 = \cdots = \lambda_n = n - 1, \lambda_{n+1} = 2n$ and so on. Hence the linearized operator could be decomposed into many ordinary differential operators given by

$$L_{\varepsilon,k} = \partial_t^2 + \left(\frac{n(n+2)}{4} v_\varepsilon^{\frac{4}{n-2}} - \frac{(n-2)^2}{4} - \lambda_k \right). \tag{2.14}$$

With the decomposition of the Laplacian in terms of the eingenvalues, it is sufficient to consider solutions of the induced problems $L_{\varepsilon,k}(\psi_k) = 0$, where $\psi(t, \theta) = \sum_k \psi_k(t) \chi_k(\theta)$.

To the first eingenvalue $\lambda_0 = 0$, we consider the families of solutions to (2.12) given by

$$T \to v_\varepsilon(t + T) \quad \text{and} \quad \varepsilon \to v_\varepsilon(t).$$

One can obtain, by differentiating these families of solutions with respect to the parameters, the solutions of (2.14) for $k = 0$ given by

$$\psi_{\varepsilon,0}^+(t) = \frac{d}{dT} v_\varepsilon(t + T) = v_\varepsilon'(t + T), \quad \psi_{\varepsilon,0}^-(t) = \frac{d}{d\varepsilon} v_\varepsilon(t)$$

which are linearly independent Jacobi fields with periodic and linear growth, respectively.

Using a similar construction for $\lambda_k = n - 1$, they were able to build linearly independent solutions $\psi_{\varepsilon,k}^\pm$ that are exponentially increasing and decreasing. Finally when $k > n + 1$, we known that the term of zero order of the operator in (2.14) is negative, since $\lambda_k \geq 2n$ and $v_\varepsilon < 1$, which implies that $L_{\varepsilon,k}$ satisfies the maximum principle. This implies that the Jacobi fields ψ_k for $k > n + 1$ are exponentially increasing or decreasing.

Once we have listed the aforementioned properties, we can begin the study of the system (2.1). Consider the operator $H(V) = (H^1(V), H^2(V))$ in cylindrical coordinates, where

$$H^i(V) = \partial_t^2 v_i + \Delta_{\mathbb{S}^{n-1}} v_i - \frac{(n-2)^2}{4} v_i + \frac{n(n-2)}{4} |\mathcal{V}|^{\frac{4}{n-2}} v_i. \tag{2.15}$$

The linearization of this operator on a solution $V_0 = (v_{0,1}, v_{0,2})$ is given by $\mathcal{L}(\phi) = (\mathcal{L}^1(\phi), \mathcal{L}^2(\phi))$ where

$$\begin{aligned}
\mathcal{L}^i(\phi) &= \left. \frac{\partial}{\partial t} \right|_{t=0} H^i(V_0 + t\phi) = \partial_t^2 \phi_i + \Delta_{\mathbb{S}^{n-1}} \phi_i - \frac{(n-2)^2}{4} \phi_i \\
&+ \frac{n(n-2)}{4} \left(\frac{4}{n-2} |V_0|^{\frac{4}{n-2}-2} \langle V_0, \phi \rangle v_{0,i} + |V_0|^{\frac{4}{n-2}} \phi_i \right),
\end{aligned}$$

where $\phi = (\phi_1, \phi_2)$. However, by the Theorem 2.6 there exists a Fowler solution v_ε and $\Lambda = (\Lambda_1, \Lambda_2) \in \mathbb{S}^1$ with $\Lambda_i > 0$ such that $V_0 = v_\varepsilon \Lambda$. Using this, the operator $\mathcal{L}^i$ can be simplified and written as

$$\mathcal{L}^i_\varepsilon(\phi) = \partial_t^2 \phi_i + \Delta_{\mathbb{S}^{n-1}} \phi_i + n \Lambda_i \langle \Lambda, \phi \rangle v_\varepsilon^{\frac{4}{n-2}} + \frac{n-2}{4} \left(n v_\varepsilon^{\frac{4}{n-2}} + 2 - n \right) \phi_i. \tag{2.16}$$

Now we will study the **Jacobi fields** to the system (2.1), that is, solutions of the linear ODE system $\mathcal{L}_\varepsilon(\phi) = 0$.

Similarly to the case of a single equation, we consider the spectral decomposition of the operator $\Delta_{\mathbb{S}^{n-1}}$. Let $\phi = (\phi_1, \phi_2)$ be a solution of $\mathcal{L}_\varepsilon(\phi) = 0$ and consider $\phi_i(\theta, t) = \sum \phi_{ik}(t) \mathcal{X}_k(\theta)$. Then ϕ_{ik} satisfies

$$\mathcal{L}_{\varepsilon,k} \phi_{ik} + n \Lambda_i \langle \Lambda, (\phi_{1k}, \phi_{2k}) \rangle v_\varepsilon^{\frac{4}{n-2}} + \frac{n(n-2)}{4} v_\varepsilon^{\frac{4}{n-2}} \phi_{ik} = 0,$$

where $\mathcal{L}_{\varepsilon,k} = \partial_t^2 - (\lambda_k + \frac{(n-2)^2}{4})$.

In the one equation case we saw that the two linearly independent fields were obtained as derivatives of one-parameter families of solutions of (2.15). However, since we are dealing with an ODE linear system of second order with two equations, we expect to obtain four linearly independent Jacobi fields.

Consider the following one-parameter families of solutions to the system (2.1) in cylindrical coordinates

$$T \to v_\varepsilon(t + T)\Lambda \quad \text{and} \quad \varepsilon \to v_\varepsilon \Lambda.$$

The derivatives of these families provide us the solutions $\phi^1_{\varepsilon,0} = \psi^+_{\varepsilon,0} \Lambda$ and $\phi^2_{\varepsilon,0} = \psi^-_{\varepsilon,0} \Lambda$, where $\psi^\pm_{\varepsilon,0}$ are the previously described linearly independent Jacobi fields for one equation. Consequently $\phi^1_{\varepsilon,0}$ and $\phi^2_{\varepsilon,0}$ are also linearly independent and they are periodic and have linear growth.

Observe that there is another trivial one-parameter family of solutions to (2.1). In fact, let $\Lambda(\alpha) = (\cos\alpha, \sin\alpha)$ with $\Lambda(0) = \Lambda$ be a path in the sphere and consider the family

$$\alpha \to v_\varepsilon \Lambda(\alpha).$$

If we differentiate this family with respect to the parameter we get a third Jacobi field $\phi^3_{\varepsilon,0} = v_\varepsilon \overline{\Lambda}$ which is a periodic function, where $\overline{\Lambda} \neq 0$ is orthogonal to Λ. We automatically have that $\{\phi^1_{\varepsilon,0}, \phi^2_{\varepsilon,0}, \phi^3_{\varepsilon,0}\}$ is a linearly independent set.

Unfortunately, we are only able to construct the fourth Jacobi Field in an indirect way. First observe that v_ε satisfies $\overline{L}_{\varepsilon,0} v_\varepsilon = 0$, where

$$\overline{L}_{\varepsilon,0} := \partial_t^2 - \frac{(n-2)^2}{4} + \frac{n(n-2)}{4} v_\varepsilon^{\frac{4}{n-2}}. \tag{2.17}$$

Since this operator is linear we know that there exists a solution $\phi^-_{\varepsilon,0}$ to (2.17) which is linearly independent to $\phi^+_{\varepsilon,0} := v_\varepsilon$. Define $\phi^4_{\varepsilon,0} = \phi^-_{\varepsilon,0} \overline{\Lambda}$. It is easy to see that $\mathcal{L}_{\varepsilon,0}(\phi^4_{\varepsilon,0}) = 0$. Observe that $\overline{L}_{\varepsilon,0} < L_{\varepsilon,0}$, where $L_{\varepsilon,0}$ is given by (2.14). Using standard methods of comparison of solutions to ODE, we obtain that the growth of $\phi^-_{\varepsilon,0}$ is at most linear, see [9].

Therefore we constructed four linearly independent Jacobi fields to the system (2.16) with respect to the first eigenvalue.

Based on the above construction, for the higher eigenvalues λ_k the Jacobi fields is given by $\phi^1_k = \psi^+_{\varepsilon,k} \Lambda$, $\phi^2_k = \psi^-_{\varepsilon,k} \Lambda$, $\phi^3_k = \phi^+_{\varepsilon,k} \overline{\Lambda}$ and $\phi^4_k = \phi^-_{\varepsilon,k} \overline{\Lambda}$, where $\psi^\pm_{\varepsilon,k}$ are the original solutions constructed for one equation which we already know the growth, see [9], and $\phi^\pm_{\varepsilon,k}$ are solutions of

$$\overline{L}_{\varepsilon,k} = \partial_t^2 + \frac{n(n-2)}{4} v_\varepsilon^{\frac{4}{n-2}} - \frac{(n-2)^2}{4} - \lambda_k.$$

We know that the zero order term of the above operator is negative when $k > 0$ since $v_\varepsilon < 1$ and $\lambda_k \geq n - 1$ for such k. This implies that $\overline{L}_{\varepsilon,k}$ satisfies the maximum principle, so we are able to determine the growth of these fields, concluding our analysis.

We summarize the result in the following proposition.

Proposition 2.10. *Suppose that $\mathcal{L}_\varepsilon(\phi) = 0$ on $\mathbb{R}^+ \times \mathbb{S}^{n-1}$, where $\phi = (\phi_1, \phi_2)$. If $\phi_i(\theta, t) = \sum \phi_{ik}(t)\mathcal{X}_k(\theta)$, then for each $i \in \mathbb{N}$ we have that $\mathcal{L}^i_\varepsilon(\phi_{ik}) = 0$, where $\mathcal{L}^i_\varepsilon$ is given by* (2.16)*, and there are four linearly independent solution of this ODE. Moreover,*

(a) *There exist four linearly independent elements $\{\phi^1_{\varepsilon,0}, \phi^2_{\varepsilon,0}, \phi^3_{\varepsilon,0}, \phi^4_{\varepsilon,0}\}$ in the kernel of $\mathcal{L}^0_\varepsilon$ such that $\phi^1_{\varepsilon,0}$, $\phi^2_{\varepsilon,0}$ and $\phi^3_{\varepsilon,0}$ are periodic and the growth of $\phi^4_{\varepsilon,0}$ is at most linear.*
(b) *For $k > 0$ the functions ϕ_{ik} are exponentially increasing or decreasing.*

3. Coupled elliptic system in the punctured ball

Let g be a smooth Riemannian metric on the unit ball $B^n_1(0) \subset \mathbb{R}^n$ with $n \geq 3$. Consider a positive solution to the system

$$-\Delta_g u_i + \sum_{j=1}^2 A_{ij}(x)u_j = \frac{n(n-2)}{4}|\mathcal{U}|^{\frac{4}{n-2}}u_i \tag{3.1}$$

in the punctured ball $\Omega = B^n_1(0)\backslash\{0\}$. Here A is a C^1 map from the unit ball centered in the origin to the vector space of symmetrical 2×2 real matrices and $-A$ is cooperative, that is, the components in the nondiagonal A_{ij} of A, $i \neq j$, are nonpositive.

3.1. Upper and lower bounds near a singularity

In this section we will obtain upper and lower bounds for solutions to our system defined in the punctured ball. Since there exists a diffeomorphism between the half cylinder and the punctured ball it will be more convenient work in cylindrical coordinates. Explicitly, the diffeomorphism $\Phi : (\mathbb{R} \times S^{n-1}, g_{cyl} = dt^2 + d\theta^2) \rightarrow (\mathbb{R}^n\backslash\{0\}, \delta)$ is given by $\Phi(t, \theta) = e^{-t}\theta$ with inverse $\Phi^{-1}(x) = (-\log|x|, x|x|^{-1})$. One also verifies that $\Phi^*\delta = e^{-2t}g_{cyl}$.

Define $v_i(t, \theta) = |x|^{\frac{n-2}{2}}u_i(x)$ and $\hat{g} = e^{2t}\Phi^*g = (e^{\frac{n-2}{2}t})^{\frac{4}{n-2}}\Phi^*g$. Using that

$$L_{v^{\frac{4}{n-2}}g}(u) = v^{-\frac{n+2}{n-2}}L_g(vu),$$

where the linear operator $L_g = \Delta_g - \frac{n-2}{4(n-1)}R_g$ is the *conformal Laplacian*, we obtain that the system (3.1) is equivalent to

$$\mathcal{L}_{\hat{g}}(v_i) - \sum_{j=1}^2 B_{ij}v_j + \frac{n(n-2)}{4}|\mathcal{V}|^{\frac{4}{n-2}}v_i = 0, \tag{3.2}$$

where

$$\begin{aligned}\mathcal{L}_{\hat{g}}(v_i) &= \Delta_{\hat{g}}v_i - \frac{n-2}{4(n-1)}(R_{\hat{g}} - e^{-2t}R_{\Phi^*g})v_i, \\ B_{ij} &= e^{-2t}A_{ij} \circ \Phi \end{aligned} \tag{3.3}$$

and $\mathcal{V} = (v_1, v_2)$.

It is also useful to remember that in cylindrical coordinates we have

$$R_{\hat{g}} - e^{-2t}R_{\Phi^*g} = (n-2)(n-1) + 2(n-1)e^{-t}\frac{\partial_r\sqrt{|g|}}{\sqrt{|g|}} \circ \Phi. \tag{3.4}$$

The proof of the next result is strongly based on [12].

Theorem 3.1. *Suppose $3 \leq n \leq 5$. Assume that $\mathcal{U} = (u_1, u_2)$ is a positive solution of* (3.1) *in Ω. There exists a constant $c > 0$ such that*

$$|\mathcal{U}(x)| \leq cd_g(x, 0)^{\frac{2-n}{2}},$$

for $0 < d_g(x, 0) < 1/2$.

Proof. Given $x_0 \in \Omega$ with $d_g(x_0, 0) < 1/2$ and $s \in (0, 1/4)$ such that $\overline{B_s(x_0)} \subset \Omega$, for $x \in \overline{B_s(x_0)}$ define

$$f(x) = (s - d_g(x, x_0))^{\frac{n-2}{2}} |\mathcal{U}(x)|.$$

It suffices to show that there exists a positive constant C such that any such f satisfies $f(x) \leq C$ in $B_s(x_0)$. The result follows by taking $s = d_g(x_0, 0)/2$.

The proof is by contradiction. So assume there is no such constant C. Then we can find a sequence of points $x_{0,k}$ and positive numbers s_k such that if $x_{1,k}$ denotes the maximum point of the corresponding f_k, we have

$$f_k(x_{1,k}) = (s_k - d_g(x, x_{0,k}))^{\frac{n-2}{2}} |\mathcal{U}(x_{1,k})| \to \infty.$$

Since $s_k \in (0, 1/4)$, then $2^{n-2} f_k(x) < |\mathcal{U}(x)|$. This implies that $|\mathcal{U}(x_{1,k})| \to \infty$ and consequently $x_{1,k} \to 0$.

Let $\varepsilon_k = |\mathcal{U}(x_{1,k})|^{-\frac{2}{n-2}}$ and for $i \in \{1, 2\}$ define $\tilde{u}_{i,k}(y) = \varepsilon_k^{\frac{n-2}{2}} u_i(\exp_{x_{1,k}}(\varepsilon_k y))$. Note that $|\tilde{\mathcal{U}}_k(0)| = 1$, where $\tilde{\mathcal{U}}_k = (\tilde{u}_{1,k}, \tilde{u}_{2,k})$. Also note that the function $\tilde{u}_{i,k}$ is defined for all y such that $|y| \leq \varepsilon_k^{-1}(s_k - d_g(x_{0,k}, x_{1,k}))$. Moreover, if $d_g(x, x_{1,k}) \leq r_k := (s_k - d_g(x_{0,k}, x_{1,k}))/2$ then

$$|\mathcal{U}(x)| \leq 2^{\frac{n-2}{2}} |\mathcal{U}(x_{1,k})|.$$

This implies that $\tilde{u}_{i,k}(y) \leq 2^{\frac{n-2}{2}}$, for all y with $|y| \leq r_k \varepsilon_k^{-1} \to \infty$.

Now, if we define $(\tilde{g}_k)_{lm}(y) := g_{lm}(\varepsilon_k y)$, then we have

$$-\Delta_{\tilde{g}_k} \tilde{u}_{i,k}(y) + \varepsilon_k^2 \sum_{j=1}^{2} \tilde{A}_{ij}^k(y) \tilde{u}_{j,k}(y) = \frac{n(n-2)}{4} |\tilde{\mathcal{U}}_k(y)|^{\frac{4}{n-2}} \tilde{u}_{i,k}(y),$$

where $\tilde{A}_{ij}^k(y) := A_{ij}(\varepsilon_k y)$. Standard elliptic theory then implies that, after passing to a subsequence, that $\{\tilde{u}_{i,k}\}_k$ converge in C^2 norm on compact subsets of $\mathbb{R}^n$ to a positive solution $u_{i,0}$ to

$$\Delta u_{i0} + \frac{n(n-2)}{4} |\mathcal{U}_0|^{\frac{4}{n-2}} u_{i,0} = 0,$$

which satisfies $|\mathcal{U}_0(0)| = 1$ and $u_{i,0}(y) \leq 2^{\frac{n-2}{2}}$ for every $y \in \mathbb{R}^n$. Using a similar argument as in [12, Theorem 4] and a theorem due to Druet, Hebey and Vétois[8], we can suppose that $x_{1,k}$ is a local nondegenerate maximum point of $|\mathcal{U}|$, $|x_{1,k}| < 7/8$ and that $\tilde{u}_{i,k}$ is defined for $|y| \leq \frac{1}{16} \varepsilon_k^{-1}$, with a possible singularity at some point on the sphere of radius $|x_{1,k}| \varepsilon_k^{-1} \to \infty$. In this way, we get that

$$\mathcal{U}_0(y) = \left(\frac{1}{1 + \frac{1}{4}|y|^2} \right)^{\frac{n-2}{2}} \Lambda.$$

Let us introduce

$$v_{i,k}(t, \theta) = |y|^{\frac{n-2}{2}} \tilde{u}_{i,k}(y),$$

where $t = -\log|y|$ and $\theta = \frac{y}{|y|}$. These functions are defined for $t > \log(\frac{1}{16}\varepsilon_k^{-1})$, with a singularity at some point (t_k', θ_k'), $t_k' = \log(|x_{2,k}\varepsilon_k^{-1}|)$. Now define

$$\mathcal{V}_0(t) = |y|^{\frac{n-2}{2}} \mathcal{U}_0(y) = \left(e^t + \frac{1}{4} e^{-t} \right)^{\frac{2-n}{2}} \Lambda.$$

Since $\tilde{\mathcal{U}}_k \to \tilde{\mathcal{U}}_0$ in the C^2_{loc} topology, we know that given $R > 0$ the inequalities

$$|\mathcal{V}_k(t, \theta) - \mathcal{V}_0(t)| \leq R^{-1} e^{\frac{2-n}{2}t},$$
$$|\partial_t \mathcal{V}_k(t, \theta) - \mathcal{V}_0'(t)| \leq R^{-1} e^{\frac{2-n}{2}t}$$

are satisfied for $t > -\log R$ and sufficiently large k. In particular, for a small number $\delta > 0$, to be chosen later, we have $|\mathcal{V}_k(-\log(\delta \varepsilon_k^{-1}))| \geq c(\delta) > 0$ and also

$$\partial_t v_{j,k}(-\log 3n, \theta) > 0, \tag{3.5}$$

for all $\theta \in \mathbb{S}^{n-1}$ and for $R > 0$ large enough.

We will apply the moving plane method to $v_{i,k}$ on the region

$$\Gamma_k = [-\log(\delta\varepsilon_k^{-1}), \infty) \times \mathbb{S}^{n-1}$$

reflecting across the spheres $\{\lambda\} \times \mathbb{S}^{n-1}$.

To simplify the notation we will drop the subscript k.

Define $\Gamma_\lambda = [-\log(\delta\varepsilon^{-1}), \lambda]$, $\hat{g}_\lambda$ the pull-back of the metric $\hat{g}$ by the reflection across the sphere $\{\lambda\} \times \mathbb{S}^{n-1}$, $v = v_1 + v_2$, $v_{i,\lambda}(t, \theta) = v_i(2\lambda - t, \theta)$ and $v_\lambda(t) = v(2\lambda - t, \theta)$.

By (3.2) we can write

$$\mathbb{L}_{\hat{g}}(v_1 - v_{1,\lambda}, v_2 - v_{2,\lambda}) + \frac{n(n-2)}{4}\sum_i^2 b_{i\lambda}(v_i - v_{i,\lambda}) = Q_\lambda,$$

where

$$b_{i\lambda} = v(v_i + v_{i,\lambda})\frac{|\mathcal{V}|^{\frac{4}{n-2}} - |\mathcal{V}_\lambda|^{\frac{4}{n-2}}}{|\mathcal{V}|^2 - |\mathcal{V}_\lambda|^2} + |\mathcal{V}_\lambda|^{\frac{4}{n-2}} > 0,$$

$$Q_\lambda = (\mathcal{L}_{\hat{g}_\lambda} - \mathcal{L}_{\hat{g}})v_\lambda + \sum_{i,j=1}^2 (B_{ij}^\lambda - B_{ij})v_{j,\lambda},$$

$$\mathbb{L}_{\hat{g}}(w_1, w_2) = \mathcal{L}_{\hat{g}}(w_1 + w_2) - \sum_{i,j=1}^2 B_{ij}w_j \tag{3.6}$$

and $\mathcal{L}_{\hat{g}}$ is defined in (3.3).

Claim 1: There exists a constant $c_1 > 0$, not depending on δ, such that $|Q_\lambda(t, \theta)| \le q_\lambda(t) = c_1\varepsilon^2 e^{\frac{n-6}{2}t}e^{(2-n)\lambda}$.

The first term in the definition of Q_λ is estimated using (3.4), the estimates $\partial_r \log|g| = O(|x|)$, $v_\lambda(t) = O(e^{\frac{2-n}{2}(2\lambda-t)})$ and $\hat{g} = dt^2 + d\theta^2 + O(\varepsilon^2 e^{-2t})$. The second term is estimated using (3.3) and $v_\lambda(t) = O(e^{\frac{2-n}{2}(2\lambda-t)})$.

Claim 2: Suppose $3 \le n \le 5$ and let $\gamma > 0$ be a small number. Then there exists a family of functions $(h_{1\lambda}(t), h_{2\lambda}(t))$, defined on Γ_λ, satisfying the following properties

$$h_{i,\lambda}(\lambda) = 0;$$

$$h_{i,\lambda} \ge 0;$$

$$\mathbb{L}_{\hat{g}}(h_{1,\lambda}, h_{2,\lambda}) \ge Q_\lambda; \tag{3.7}$$

$$h_{i,\lambda} \le v_i - v_{i,\lambda}, \text{ if } \lambda \text{ is sufficiently large} \tag{3.8}$$

and

$$h_\lambda(-\log(\delta\varepsilon^{-1})) \le c_3 \max\left\{\varepsilon^{\frac{n-2}{2}}, \varepsilon^{\frac{6-n}{2}-\gamma}\right\}, \tag{3.9}$$

where $h_\lambda = h_{1\lambda} + h_{2\lambda}$, for some positive constant c which depends only on δ.

By (3.3) we obtain $B_{ij} = O(\varepsilon^2 e^{-2t})$. Thus, if f_1 and f_2 depend only on t, then

$$\mathbb{L}_{\hat{g}}(f_1, f_2) = f'' + O(\varepsilon^2 e^{-2t})f' - \left(\left(\frac{n-2}{2}\right)^2 + O(\varepsilon^2 e^{-2t})\right)f + \sum_{j=1}^2 O_j(\varepsilon^2 e^{-2t})f_j,$$

where $f = f_1 + f_2$. Given a small number $\gamma > 0$, let $\overline{L}$ be the linear operator

$$\overline{L}(f) = f'' + \gamma f' - \left(\left(\frac{n-2}{2}\right)^2 + \gamma\right)f.$$

Let $\gamma_1 = \frac{8-n}{2}\gamma > 0$ and $a(n) = \frac{1}{2(4-n)-\gamma_1}$. Define

$$h_{i,\lambda}(t) = \frac{a(n)}{2} c_1 \varepsilon^2 e^{(2-n)\lambda} e^{\frac{n-6}{2}t}(1 - e^{(4-n-\gamma_2)(t-\lambda)}),$$

where $\gamma_2 > 0$ is chosen so that the function $e^{(\frac{2-n}{2}-\gamma_2)t}$ is in the kernel of $\overline{L}$. Note that γ_1 and γ_2 are small.

Immediately we have $\overline{L}(h_\lambda) = q_\lambda$, $h_{i,\lambda}(\lambda) = 0$, $h_{i,\lambda} \geq 0$ and $h'_{i,\lambda} \leq 0$ in $(-\infty, \lambda]$. Since $t \geq -\log(\delta\varepsilon^{-1})$, then choosing $\delta^2 < \gamma$ we get

$$\mathbb{L}_{\hat{g}}(h_{1,\lambda}, h_{2,\lambda}) \geq \overline{L}(h_\lambda) = q_\lambda \geq |Q_\lambda|.$$

Using the same argument as in [12, Theorem 4] we get the estimates (3.8) and (3.9). This finishes the proof of the Claim 2.

Define

$$w_{i,\lambda} = v_i - v_{i,\lambda} - h_{i\lambda} \geq 0.$$

Note that $w_{i,\lambda}(\lambda, \theta) = 0$, for all $\theta \in \mathbb{S}^{n-1}$, and

$$\mathbb{L}_{\hat{g}}(w_{1,\lambda}, w_{2,\lambda}) + \frac{n(n-2)}{4}\sum_i^2 b_{i\lambda}w_{i,\lambda} = Q_\lambda - \mathbb{L}_{\hat{g}}(h_{1,\lambda}, h_{2,\lambda}) \leq 0. \tag{3.10}$$

We notice that by (3.3) and (3.6) we get, for some positive function D, that

$$\Delta_{\hat{g}}w_\lambda - Dw_\lambda \leq \mathbb{L}_{\hat{g}}(w_{1\lambda}, w_{2\lambda}). \tag{3.11}$$

Now we use (3.5), (3.7), (3.8), (3.10), (3.11), the Maximum Principle, the Hopf's Lemma and a similar argument as in the proof of [12, Theorem 4] to prove that

$$w_{\lambda_0}(-\log(\delta\varepsilon^{-1}), \theta_0) = 0,$$

for some $\lambda_0 > -\log(3n)$ and $\theta_0 \in \mathbb{S}^{n-1}$ and for $\varepsilon > 0$ small enough. Then, by definition of w_{λ_0} we get

$$0 < c(\delta) \leq v(-\log(\delta\varepsilon^{-1}), \theta_0) = (v_{\lambda_0} + h_{\lambda_0})(-\log(\delta\varepsilon^{-1}), \theta_0). \tag{3.12}$$

But, using that $v(t, \theta) = O(e^{\frac{2-n}{2}(2\lambda-t)})$ and (3.9) we obtain that the right hand side of (3.12) goes to zero when ε goes to zero, which is a contradiction. This completes the proof of the theorem. ∎

We notice that in the proof of the Theorem 3.1 the condition that $-A$ is cooperative is not needed. Besides that, this result holds for systems with any number of equations.

As a consequence of the upper bound we get the following *spherical Harnack inequality*.

Corollary 3.2. *Suppose* $\mathcal{U}$ *is a positive smooth solution of* (3.1) *in* Ω *with* $3 \leq n \leq 5$ *and that the potential A satisfies the hypotheses ([H1]). Then there exists a constant* $c_1 > 0$ *such that*

$$\max_{|x|=r} u_i \leq c_1 \min_{|x|=r} u_i$$

for every $0 < r < \frac{1}{4}$. *Moreover,* $|\nabla u_i| \leq c_1 |x|^{-1} u_i$ *and* $|\nabla^2 u_i| \leq c_1 |x|^{-2} u_i$.

Proof. Define $u_{i,r}(y) = r^{\frac{n-2}{2}} u_i(ry)$, for every $0 < r < 1/4$ and $|y| < r^{-1}$. Then the upper bound given by Theorem 3.1 implies that $u_{i,r}(y) \leq c|y|^{\frac{2-n}{2}}$, for $|y| < \frac{1}{2}r^{-1}$. In particular, if $\frac{1}{2} \leq |y| \leq \frac{3}{2}$, we have that $u_{i,r}(y) \leq 2^{\frac{n-2}{2}}c$. Moreover

$$\Delta_{g_r}u_{i,r}(y) - r^2\sum_{j=1}^2 A_{ij}(ry)u_{j,r}(y) + \frac{n(n-2)}{4}|\mathcal{U}_r(y)|^{\frac{4}{n-2}}u_{i,r}(y) = 0,$$

where $(g_r)_{ij}(y) = g_{ij}(ry)$, which implies that

$$\Delta_{g_r} u_{i,r}(y) - r^2 A_{ii}(ry) u_{i,r}(y) = r^2 A_{ij}(ry) u_{j,r}(y) - \frac{n(n-2)}{4} |\mathcal{U}_r(y)|^{\frac{4}{n-2}} u_{i,r}(y),$$

where $j \neq i$. Using that $-A$ is cooperative, the Harnack inequality for linear elliptic equations and standard elliptic theory imply that there exists $c_1 > 0$, not depending on r, such that

$$\max_{|x|=1} u_{i,r} \leq c_1 \min_{|x|=1} u_{i,r}$$

and $|\nabla u_{i,r}| + |\nabla^2 u_{i,r}| \leq c_1 u_{1,r}$ on the sphere of radius 1.

This finishes the proof of the corollary. ∎

3.2. Pohozaev invariant and removable singularities

In this section we will prove a Pohozaev Identity and define the Pohozaev invariant of a solution of (3.1). Also we will prove a removable singularity theorem. As a consequence we will derive a fundamental lower bound near the isolated singularity.

Given $\mathcal{U}$ a positive solution to the system (3.1), we define $P(r, \mathcal{U})$ as in (1.5). The following lemma gives the Pohozaev-type identity we are interested in. Its proof is a straightforward adaptation of the result in [11].

Lemma 3.3 (*Pohozaev Identity*). *Given $0 < s \leq r < 1$, we have*

$$P(r, \mathcal{U}) - P(s, \mathcal{U}) = -\sum_i \int_{B_r \backslash B_s} \left(x \cdot \nabla u_i + \frac{n-2}{2} u_i \right) \left((\Delta_g - \Delta) u_i - \sum_j A_{ij}(x) u_j \right).$$

In the case of the limit system we have seen in (2.11) that $P(r, \mathcal{U})$ does not depend on r, therefore it is an invariant of the solution $\mathcal{U}$.

Motivated by [12] we define the Pohozaev invariant of a solution $\mathcal{U}$ of (3.1) by using the Theorem 3.1. Since $g_{ij} = \delta_{ij} + O(|x|^2)$, then we get that

$$\left| \sum_i \left(x \cdot \nabla u_i + \frac{n-2}{2} u_i \right) \left((\Delta - \Delta_g) u_i - \sum_j A_{ij}(x) u_j \right) \right| \leq c |x|^{2-n}, \tag{3.13}$$

for some positive constant c which does not depend on x. The Pohozaev identity tells us that the limit

$$P(\mathcal{U}) := \lim_{r \to 0} P(r, \mathcal{U})$$

exists. The number $P(\mathcal{U})$ is called the *Pohozaev invariant* of the solution $\mathcal{U}$.

Once defined the Pohozaev invariant, our main result of this section reads as follows.

Theorem 3.4. *Let $\mathcal{U}$ be a positive solution to the system (1.1) in $B_1^n(0) \backslash \{0\}$ such that $3 \leq n \leq 5$ and the potential A satisfies the hypotheses (H1) and (H2). Then $P(\mathcal{U}) \leq 0$. Moreover, $P(\mathcal{U}) = 0$ if and only if each coordinate u_i is smooth on the origin.*

The strategy to reach the result is to assume that $P(\mathcal{U}) \geq 0$ and proof that the origin is a removable singularity and $P(\mathcal{U}) = 0$. In what follows let us denote by

$$u(x) = u_1(x) + u_2(x),$$

$\bar{u}$ the average of u over ∂B_r, that is,

$$\bar{u}(r) := \fint_{\partial B_r} u := \frac{1}{vol(\partial B_r)} \int_{\partial B_r} u$$

and define

$$w(t) = \overline{u}(r) r^{\frac{n-2}{2}},$$

where $t = -\log r$.

We have divided the proof into a sequence of lemmas.

Lemma 3.5. *Let $\mathcal{U}$ be a positive solution of* (1.1) *which satisfies $P(\mathcal{U}) \geq 0$. Then*

$$\liminf_{x \to 0} u(x) |x|^{\frac{n-2}{2}} = 0.$$

Proof. If this result is not true, without loss of generality we can suppose that there exist positive constants c_1 and c_2 such that

$$c_1 |x|^{\frac{2-n}{2}} \leq u_1(x) \leq c_2 |x|^{\frac{2-n}{2}}, \tag{3.14}$$

where the second inequality follows from Theorem 3.1. Choose any sequence $r_k \to 0$ and define

$$u_{i,k}(x) = r_k^{\frac{n-2}{2}} u_i(r_k x).$$

Then, using (3.14), we have

$$c_1 |x|^{\frac{2-n}{2}} \leq u_{1,k}(x) \leq c_2 |x|^{\frac{2-n}{2}}. \tag{3.15}$$

Moreover $u_{i,k}$ satisfies

$$-\Delta_{g_k} u_{i,k} + r_k^2 \sum_{j=1}^{2} A_{ij}(r_k x) u_{j,k} = \frac{n(n-2)}{4} |\mathcal{U}_k|^{\frac{4}{n-2}} u_{i,k} \quad \text{in} \quad B_{r_k^{-1}}(0) \setminus \{0\},$$

where $(g_k)_{lm}(x) = g_{lm}(r_k x)$. Elliptic theory then implies that there exists a subsequence, also denoted by $u_{i,k}$, which converges in compact subsets of $\mathbb{R}^n \setminus \{0\}$ to a solution $\mathcal{U}_0 = (u_{1,0}, u_{2,0})$ of the limit system

$$\Delta u_{i,0} + \frac{n(n-2)}{4} |\mathcal{U}_0|^{\frac{4}{n-2}} u_{i,0} = 0.$$

By (3.15) we get

$$u_{1,0}(x) \geq c_1 |x|^{\frac{2-n}{2}},$$

which implies that $\mathcal{U}_0$ is singular at the origin. However, by Theorem 2.6 and Corollary 2.9 we know that $P(\mathcal{U}_0) < 0$. This is a contradiction, because

$$P(\mathcal{U}_0) = P(1, \mathcal{U}_0) = \lim_{k \to \infty} P(1, \mathcal{U}_k) = \lim_{k \to \infty} P(r_k, \mathcal{U}) = P(\mathcal{U}) \geq 0. \quad \blacksquare$$

Lemma 3.6. *Assume that $\mathcal{U}$ is a positive solution of* (1.1) *and that the potential A satisfies the hypotheses* (H1) *and* (H2). *Suppose that there exists a sequence (t_k) of minimum points for w such that $\lim_{k \to \infty} w(t_k) = 0$, where $w = w_1 + w_2$, $w_i = r^{\frac{n-2}{2}} \overline{u}_i$ and $r = e^{-t}$. Along $|x| = r_k = e^{-t_k}$ there exist nonnegative constants c_1 and c_2 with $(c_1, c_2) \neq (0, 0)$, such that*

$$\begin{aligned} u_i(x) &= \overline{u}(r_k)(c_i + o(1)), \\ |\nabla u_i(x)| &= -\overline{u}'(r_k)(c_i + o(1)). \end{aligned} \tag{3.16}$$

Proof. Define $r_k = e^{-t_k}$ and $v_{i,k}(y) = r_k^{\frac{n-2}{2}} u_i(r_k y)$. Since $\overline{v}_{i,k}(1) = w_1(t_k) \leq w_1(t_k) + w_2(t_k) = w(t_k) \to 0$ we get from Harnack inequality that each coordinate $v_{i,k}$ converges uniformly in compact subsets of $\mathbb{R}^n \setminus \{0\}$ to 0. So, if we define

$$h_{i,k}(y) = v_k(p)^{-1} v_{i,k}(y),$$

where $v_k = \sum_i v_{i,k}$ and $p = (1, 0, \dots, 0)$, we have that

$$-\Delta_{g_k} h_{i,k} + r_k^2 \sum_{j=1}^{2} A_{ij}(r_k y) h_{j,k} = \frac{n(n-2)}{4} v_k(p)^{\frac{4}{n-2}} |H_k|^{\frac{4}{n-2}} h_{i,k},$$

where $(g_i)_{lm}(y) = g_{lm}(r_i y)$ and $H_k = (h_{1,k}, h_{2,k})$. By elliptic estimates we know that there exists a subsequence $h_{i,k}$ which converge in C_{loc}^2 to a nonnegative harmonic function h_i in $\mathbb{R}^n \backslash \{0\}$. Then

$$h_i(y) = a_i |y|^{2-n} + b_i,$$

and $a_1 + a_2 = b_1 + b_2 = 1/2$, since $h_1(p) + h_2(p) = 1$ and $\partial_r((h_1 + h_2)(r)r^{\frac{n-2}{2}}) = 0$ at $r = 1$. Note that if $|y| = 1$, then $h_i(y) = a_i + b_i$. This implies that $h_{i,k}(y) = a_i + b_i + o(1)$. From this, we will get the first equality in (3.16) if we prove that $a_i = b_i$. Analogously, we obtain the second equality.

Since

$$-\Delta_{g_k} v_{i,k} + r_k^2 \sum_{j} \tilde{A}_{ij} v_{jk} = \frac{n(n-2)}{4} |V_k|^{\frac{4}{n-2}} v_{ik},$$

where $\tilde{A}_{ij}(y) = A_{ij}(r_k y)$, then we get that

$$\int_{B_1 \backslash B_\varepsilon} (v_{i,k} \Delta_g v_{j,k} - v_{j,k} \Delta_g v_{i,k}) dv_g = r_k^2 \sum_{l=1}^{2} \int_{B_1 \backslash B_\varepsilon} (v_{j,k} \tilde{A}_{il} - v_{i,k} \tilde{A}_{jl}) v_{l,k} dv_g.$$

Integrating by parts, we obtain that

$$\int_{\partial B_1} (v_{i,k} \partial_r v_{j,k} - v_{j,k} \partial_r v_{i,k}) d\sigma_g = r_k^2 \sum_{l=1}^{2} \int_{B_1 \backslash B_\varepsilon} (v_{j,k} \tilde{A}_{il} - v_{i,k} \tilde{A}_{jl}) v_{l,k} dv_g$$
$$+ \int_{\partial B_\varepsilon} (v_{i,k} \partial_r v_{j,k} - v_{j,k} \partial_r v_{i,k}) d\sigma_g. \tag{3.17}$$

In order to analyse the last integral on the right-hand side of (3.17), define $\varphi_{i,k}^\varepsilon(z) = \varepsilon^{\frac{n-2}{2}} v_{i,k}(\varepsilon z)$. Thus

$$\int_{\partial B_\varepsilon} (v_{i,k} \partial_r u_{j,k} - u_{j,k} \partial_r u_{i,k}) d\sigma_g = \int_{\partial B_1} (\varphi_{i,k}^\varepsilon \partial_r \varphi_{j,k}^\varepsilon - \varphi_{j,k}^\varepsilon \partial_r \varphi_{j,k}^\varepsilon) d\sigma_g.$$

On the other hand

$$-\Delta_{g_{\varepsilon k}} \varphi_{i,k}^\varepsilon + (\varepsilon r_k)^2 \sum_{j=1}^{2} \tilde{A}_{ij}^\varepsilon \varphi_{j,k}^\varepsilon = c(n) |\varphi_k^\varepsilon|^{\frac{4}{n-2}} \varphi_{i,k}^\varepsilon$$

in $B_{(\varepsilon r_k)^{-1}}(0) \backslash \{0\}$ and by Theorem 3.1

$$|\varphi_k^\varepsilon(z)| \le C |z|^{\frac{2-n}{2}}.$$

Similarly to what we did at the beginning of this proof, after passing to a subsequence, $\varphi_{i,k}^\varepsilon$ converges in C^2 topology locally in compact subsets of $\mathbb{R}^n \backslash \{0\}$, to a positive solution of

$$\Delta u_i + \frac{n(n-2)}{4} |\mathcal{U}|^{\frac{4}{n-2}} u_i = 0$$

which using Theorem 2.6 as well Proposition 1.1 in [8], is of the form $u_0 \Lambda$, where $\Lambda = (\Lambda_1, \Lambda_2)$ is a vector in the unit sphere with nonnegative coordinates. Consequently, when ε goes to zero, we have

$$\lim_{\varepsilon \to 0} \int_{\partial B_1} (\varphi_{j,k}^\varepsilon \partial_r \varphi_{i,k}^\varepsilon - \varphi_{i,k}^\varepsilon \partial_r \varphi_{i,k}^\varepsilon) d\sigma_g = \int_{\partial B_1} (u_0 \partial_r u_0 \Lambda_i \Lambda_j - u_0 \partial_r u_0 \Lambda_j \Lambda_i) d\sigma_g = 0.$$

Thus by (3.17), we conclude that

$$\int\limits_{\partial B_1} (v_{i,k}\partial_r v_{j,k} - v_{j,k}\partial_r v_{i,k})d\sigma_g = r_k^2 \sum_{l=1}^{2} \int\limits_{B_1\backslash\{0\}} (v_{j,k}\tilde{A}_{il} - v_{i,k}\tilde{A}_{jl})v_{l,k}dv_g.$$

Since $|v_{i,k}| \leq c|x|^{\frac{2-n}{2}}$, then the integral in the right hand side above is convergent and it is uniformly bounded in k. This implies that

$$\int\limits_{\partial B_1} (h_{j,k}\partial_r h_{i,k} - h_{i,k}\partial_r h_{j,k})d\sigma_g = r_k^{4-n}u(r_k p)^{-2}\sum_{l=1}^{2} \int\limits_{B_1\backslash\{0\}} (v_{j,k}\tilde{A}_{il} - v_{i,k}\tilde{A}_{jl})v_{l,k}dv_g.$$

Since $\lim\limits_{k\to\infty} u(r_k p) = \infty$ and the integral in the right hand side is uniformly bounded and A satisfies the hypotheses (H2), we conclude that

$$\int\limits_{\partial B_1} (h_j\partial_r h_i - h_i\partial_r h_j)d\sigma_g = 0$$

and consequently $a_1 b_2 = a_2 b_1$. Since $a_1 + a_2 = b_1 + b_2 = 1/2$, then we conclude that $a_i = b_i$. This finishes the proof. ∎

Lemma 3.7. *Let $\mathcal{U}$ be a positive solution of* (1.1) *defined in the punctured ball. If*

$$\lim_{|x|\to 0} |x|^{\frac{n-2}{2}}u(x) = 0, \tag{3.18}$$

then $\mathcal{U}$ extends as a smooth solution to all unit balls B^n.

Proof. We begin by obtaining upper and lower bounds for the second derivatives of w in terms of w, where w is defined in Lemma 3.6. Indeed, observe that the upper bound in Theorem 3.1 implies that $w(t)$ is bounded. Then,

$$\overline{u}_r = \fint\limits_{\partial B_r} u_r$$

and since

$$w_t = -\overline{u}_r r^{\frac{n}{2}} - \frac{n-2}{2}w,$$

we also get that $|w_t|$ is bounded. Derivating again the function w we obtain

$$w_{tt} = \overline{u}_{rr}r^{\frac{n+2}{2}} + (n-1)\overline{u}_r r^{\frac{n}{2}} + \left(\frac{n-2}{2}\right)^2 w.$$

Similarly to [12, Theorem 6], we get

$$w_{tt} - \left(\frac{n-2}{2}\right)^2 w = r^{\frac{n+2}{2}} \fint\limits_{\partial B_r} \left((\Delta - \Delta_g)u + \sum_{i,j=1}^{2} A_{ij}(x)u_j - c(n)|\mathcal{U}|^{\frac{4}{n-2}}u \right).$$

Applying the spherical Harnack inequality obtained in Corollary 3.2 for each coordinate function we have that

$$C^{-1}\overline{u}^{\frac{n+2}{n-2}} \leq c(n) \fint\limits_{\partial B_r} |\mathcal{U}|^{\frac{4}{n-2}}u \leq C\overline{u}^{\frac{n+2}{n-2}}$$

and

$$\left| \fint\limits_{\partial B_r} \left((\Delta - \Delta_g)u + \sum_{i,j=1}^{2} A_{ij}(x)u_j \right) \right| \leq c\overline{u}.$$

With these estimates we obtain the following inequality

$$-c_1 w^{\frac{n+2}{n-2}} - c_3 e^{-2t} w \leq w_{tt} - \left(\frac{n-2}{2}\right)^2 w \leq -c_2 w^{\frac{n+2}{n-2}} + c_3 e^{-2t} w. \tag{3.19}$$

By hypothesis (3.18) we know that $\lim\limits_{t\to\infty} w(t) = 0$. The strategy is to show that $u \in L^p_{loc}(B_1^n(0))$ for some $p > 2n/(n-2)$ and by elliptic theory the function u extends smoothly across the origin. Consequently, each coordinate function u_i is smooth.

Note that the first inequality in (3.19) implies that there exists $\varepsilon_0 > 0$, such that if $w(t) \leq \varepsilon_0$ and t is sufficiently large, then $w_{tt}(t) > 0$. Since $\lim\limits_{t\to\infty} w(t) = 0$, there exists T_1 so that $w(t) < \varepsilon_0$ and $w_{tt} > 0$ for $t \geq T_1$. This implies that $w_t < 0$ for $t \geq T_1$.

By the first inequality in (3.19), given any positive number $0 < \delta < n - 2$, there exists T_0 sufficiently large such that

$$w_{tt} - \left(\frac{n-2}{2} - \delta\right)^2 w \geq (\delta(n-2) - \delta^2 - c_1 w^{\frac{4}{n-2}} - c_3 e^{-2t}) w \geq 0,$$

for $t \geq T_0$, which implies that

$$\left(w_t^2 - \left(\frac{n-2}{2} - \delta\right)^2 w^2\right)_t = 2w_t \left(w_{tt} - \left(\frac{n-2}{2} - \delta\right)^2 w\right) \leq 0,$$

for $t > T_2 = \max\{T_0, T_1\}$, and using that $\lim\limits_{t\to\infty} w_t(t) = 0$, we obtain

$$w_t^2 - \left(\frac{n-2}{2} - \delta\right)^2 w^2 \geq 0.$$

By integrating we get, for $t \geq T_2$, that

$$w(t) \leq w(T_0) e^{-\left(\frac{n-2}{2} - \delta\right)(t - T_0)}.$$

Equivalently, there exists $r_0(\delta) > 0$, so that

$$u(x) \leq c(\delta)|x|^{-\delta} \text{ for all } x \in B_{r_0}(0).$$

Since $\delta > 0$ is arbitrarily small, the estimate above implies that $u \in L^p_{loc}(B_1(0))$ for arbitrarily large p, which finishes our proof. ∎

Now we are ready to prove the main result of this section.

Proof of Theorem 3.4. Following the aforementioned strategy let us suppose that $P(\mathcal{U}) \geq 0$. By Lemma 3.7 the proof is completed by showing that $\lim\limits_{x\to0} u(x)|x|^{\frac{n-2}{2}} = 0$. Suppose by contradiction that this is false.

Since the Pohozaev invariant is nonnegative, it holds as a consequence of Lemma 3.5 that $\liminf\limits_{x\to0} u(x)|x|^{\frac{n-2}{2}} = 0$. Then we will assume that $\limsup\limits_{x\to0} u(x)|x|^{\frac{n-2}{2}} > 0$. Hence we can choose $\varepsilon_0 > 0$ sufficiently small so that we are able to construct sequences $\bar{t}_k \leq t_k \leq t_k^*$ satisfying $\lim\limits_{k\to\infty} \bar{t}_k = +\infty$, $w(\bar{t}_k) = w(t_k^*) = \varepsilon_0$, $w_t(t_k) = 0$ and $\lim\limits_{k\to\infty} w(t_k) = 0$, where w is defined in Lemma 3.6.

By a direct computation, using (3.16), we have that

$$P(r_k, \mathcal{U}) = \sigma_{n-1} \left(-|c|^2 \frac{\delta^2}{2} w^2(t_k) + \frac{\delta^2}{2} (c_1^2 + c_2^2)^{\frac{n}{n-2}} |W|^{\frac{2n}{n-2}}(t_k)\right)(1 + o(1)),$$

where $c = (c_1, c_2)$ comes from Lemma 3.6, so $|c| \neq 0$. Consequently

$$P(\mathcal{U}) = \lim\limits_{k\to\infty} P(r_k, \mathcal{U}) = 0.$$

Thus

$$w^2(t_k) \leq c|P(r_k, \mathcal{U})| \leq c(I_1 + I_2),$$

where

$$I_1 = \int\limits_{B_{r_k} \setminus B_{r_k^*}} |A(\mathcal{U})| dx,$$

$$I_2 = \int\limits_{B_{r_k^*}} |A(\mathcal{U})| dx$$

and $A(\mathcal{U}) = \sum_i \left(x \cdot \nabla u_i + \frac{n-2}{2} u_i \right) \left((\Delta_g - \Delta)(u_i) - \sum_j A_{ij}(x) u_j \right)$.

The rest of the proof follows exactly as the argument in [12, Theorem 6], by using the Lemma 3.7. ∎

As a consequence of the removable singularity theorem, we can now establish a fundamental lower bound.

Corollary 3.8. *Assume $3 \leq n \leq 5$ and that the potential A satisfies the hypotheses (H1) and (H2). Let $\mathcal{U}$ be a positive solution to the system (3.1) in $B_1^n(0) \setminus \{0\}$. If 0 is a non–removable singularity, then there exists $c > 0$ such that*

$$|\mathcal{U}|(x) \geq c d_g(x, 0)^{\frac{2-n}{2}}$$

for each i and $0 < d_g(x, 0) < \frac{1}{2}$.

Proof. Suppose by contradiction that this is not true. Then $\liminf\limits_{t \to \infty} w(t) = 0$, where $w(t) = r^{\frac{n-2}{2}} \overline{u}(r)$, $u = u_1 + u_2$ and $t = -\log r$. As 0 is a non–removable singularity, we also have $\limsup\limits_{t \to \infty} w(t) > 0$, otherwise we contradict the Lemma 3.7. Therefore there exists a sequence $t_k \to \infty$ such that $w'(t_k) = 0$ and $\lim\limits_{k \to \infty} w(t_k) = 0$. So, if $r_k = e^{-t_k}$ we can check that

$$P(r_k, \mathcal{U}) \leq \sigma_{n-1} \left(\frac{1}{2} \partial_t w(t_k)^2 - \frac{1}{2} \left(\frac{n-2}{2} \right)^2 w^2(t_k) + \frac{(n-2)^2}{8} |W|^{\frac{2n}{n-2}}(t_k) \right) (1 + o(1)),$$

where $|W|^2 = w_1^2 + w_2^2$, and $w_i(t) = r^{\frac{n-2}{2}} \overline{u}_i(r)$. But, in this case

$$P(\mathcal{U}) = \lim\limits_{i \to \infty} P(r_k, \mathcal{U}) = 0,$$

which is a contradiction. This finishes the proof. ∎

4. Convergence to a radial solution

Our main goal in this section is to prove that a local singular solution to our system is asymptotic to a radial Fowler-type solution, near the non–removable isolated singularity.

Theorem 4.1. *Suppose that $\mathcal{U}$ is a solution of the system (1.1) in the punctured ball $B_1^n(0) \setminus \{0\}$ and that the potential A satisfies the hypotheses (H1) and (H2). If there exist positive constants c_1 and c_2 such that*

$$c_1 |x|^{\frac{2-n}{2}} \leq |\mathcal{U}|(x) \leq c_2 |x|^{\frac{2-n}{2}} \tag{4.1}$$

then there exists a Fowler-type solution $\mathcal{U}_0 = u_0 \Lambda$ of (1.3), where u_0 is a Fowler solution such that

$$\mathcal{U}(x) = (1 + O(|x|^\alpha)) \mathcal{U}_0(x)$$

as $x \to 0$, for some $\alpha > 0$.

Proof. First we observe that (4.1) implies that the origin is a non–removable singularity. Thus, by Theorem 3.4 we get that $P(\mathcal{U}) < 0$.

Now, in cylindrical coordinates, consider $v_i(t, \theta) = |x|^{\frac{2-n}{2}} u_i(x)$, where $t = -\log |x|$ and $\theta = \frac{x}{|x|}$.

Let $\{\tau_k\}$ be a sequence of real numbers such that $\tau_k \to \infty$. Consider the translated sequence $v_{i,k}(t, \theta) = v_i(t + \tau_k, \theta)$ defined in $(-\tau_k, \infty) \times \mathbb{S}^{n-1}$. By (4.1) we get that

$$c_1 \leq |\mathcal{V}_k(t, \theta)| \leq c_2,$$

where $\mathcal{V}_k = (v_{1,k}, v_{2,k})$. Consequently, by standard elliptic estimates, we get the uniform boundedness of any derivative for $t > 0$. Since $v_{i,k}$ satisfies,

$$\mathcal{L}_{\hat{g}_k}(v_{i,k}) - \sum_j B_{ij} v_{j,k} + \frac{n(n-2)}{4} |\mathcal{V}_k|^{\frac{4}{n-2}} v_{i,k} = 0,$$

where $\mathcal{L}_{\hat{g}_k}$ and B_{ij} are given by (3.3) and $\hat{g}_k := dt^2 + d\theta^2 + O(e^{-2t}) \to dt^2 + d\theta^2$. Standard elliptic estimates imply that there exists a subsequence, also denoted by $v_{i,k}$, which converges in the C_{loc}^2 topology, to a positive solution of

$$\partial_t^2 v_{i,0} + \Delta_{\mathbb{S}^{n-1}} v_{i,0} - \frac{(n-2)^2}{4} v_{i,0} + \frac{n(n-2)}{4} |\mathcal{V}_0|^{\frac{4}{n-2}} v_{i,0} = 0,$$

defined in the whole cylinder. By the characterization result given by Theorem 2.6, such limit is a Fowler-type solution and we know that there exists a Fowler solution v_ε and a vector in the unit sphere with positive coordinates Λ such that $\mathcal{V}_\varepsilon(t) = \Lambda v_\varepsilon(t)$. Hence $\mathcal{V}_\varepsilon$ does not depend on θ, then we necessarily have that any angular derivative $\partial_\theta v_{i,k}$ converges uniformly to zero.

Besides, we claim that

$$\begin{aligned} v_{i,k}(t, \theta) &= \overline{v}_{i,k}(t)(1 + o(1)) \\ \nabla v_{i,k}(t, \theta) &= -\overline{v}'_{i,k}(t)(1 + o(1)), \end{aligned}$$

as $t \to \infty$. In fact, suppose that the first equality above is false. Then there exist $\overline{\varepsilon} > 0$ and sequences $\tau_k \to \infty$, $\theta_k \to \theta \in \mathbb{S}^{n-1}$ such that

$$\left| \frac{v_{i,k}(\tau_k, \theta_k)}{\overline{v}_{i,k}(\tau_k)} - 1 \right| \geq \overline{\varepsilon}$$

for some $i \geq 1$. This is a contradiction because, after passing to a subsequence, $\mathcal{V}_k$ converges to a rotationally symmetric Fowler-type solution $\mathcal{V}_0$. The second inequality follows from similar arguments.

In the cylindrical setting the Pohozaev integral $P(t, \mathcal{V}) = P(e^{-t}, \mathcal{U})$ becomes

$$P(t, \mathcal{V}) := \int_{t \times \mathbb{S}^{n-1}} \left(\frac{1}{2} |\partial_t \mathcal{V}|^2 - \frac{1}{2} |\nabla_\theta \mathcal{V}|^2 - \frac{(n-2)^2}{8} |\mathcal{V}|^2 + \frac{(n-2)^2}{8} |\mathcal{V}|^{\frac{2n}{n-2}} \right) d\sigma_1.$$

Hence

$$P(\mathcal{V}_\varepsilon) := P(0, \mathcal{V}_\varepsilon) = \lim_{k \to \infty} P(0, \mathcal{V}_k) = \lim_{k \to \infty} P(\tau_k, \mathcal{V}) = P(\mathcal{V}). \tag{4.2}$$

So we can conclude that the necksize ε of the limit function is independent of the sequence of numbers τ_k. Therefore, for each sequence $\tau_k \to \infty$ the correspondent sequence $\mathcal{V}_k$ converges to a function $\mathcal{V}_{\varepsilon,T}(t) = \Lambda v_\varepsilon(t + T)$, with $\Lambda \in \mathbb{S}_+^1$, for some $T \in \mathbb{R}$ which depends on the sequence τ_k.

We will show that there exists $T_0 \in \mathbb{R}$ such that $\mathcal{V}_k$ converges to $\mathcal{V}_{\varepsilon,T_0}$ for any sequence $\tau_k \to \infty$. The idea is to use a delicate rescaling argument due originally to Leon Simon. In order of that we will prove some claims using the Jacobi fields studied in subsection 2.4 as a tool.

Let T_ε be the period of $\mathcal{V}_\varepsilon$ and $A_\tau = \sup_{t \geq 0} |\partial_\theta \mathcal{V}_\tau|$, where $\mathcal{V}_\tau(t, \theta) = \mathcal{V}(t + \tau, \theta)$. Note that $A_\tau < \infty$, since $|\partial_\theta \mathcal{V}_\tau|$ converges uniformly to zero as $t \to \infty$.

Claim 1: For every $c > 0$, there exists a positive integer N such that, for any $\tau > 0$, either

(1) $A_\tau \le ce^{-2\tau}$ or
(2) A_τ is attained at some point in $I_N \times \mathbb{S}^{n-1}$, where $I_N = [0, NT_\varepsilon]$.

Suppose the Claim is not true.

Then there exist a constant $c > 0$ and sequences $\tau_k, s_k \to \infty, \theta_k \in \mathbb{S}^{n-1}$ such that $|\partial_\theta \mathcal{V}_\tau|(s_k, \theta_k) = A_{\tau_k}$ and $A_{\tau_k} > ce^{-2\tau_k}$ as $k \to \infty$. Then we can translate back further s_k and define $\tilde{v}_{i,k}(t, \theta) = v_{i,k}(t + s_k, \theta)$. Define $\varphi_{i,k} = A_{\tau_k}^{-1} \partial_\theta \tilde{v}_{i,k}$ and note that $|\boldsymbol{\varphi_k}| \le 1$, where $\boldsymbol{\varphi_k} = (\varphi_{1,k}, \varphi_{2,k})$. Now, we have

$$\mathcal{L}_{\hat{g}_k}(\tilde{v}_{i,k}) - \sum_j \tilde{B}_{ij} \tilde{v}_{j,k} + \frac{n(n-2)}{4} |\tilde{\mathcal{V}}_k|^{\frac{4}{n-2}} \tilde{v}_{i,k} = 0,$$

where the quantities with tilde are the originals replaced t by $t + \tau_k + s_k$. This implies that

$$L_{cyl}(\tilde{v}_{i,k}) + \frac{n(n-2)}{4} |\tilde{\mathcal{V}}_k|^{\frac{4}{n-2}} \tilde{v}_{i,k} = L_{cyl}(\tilde{v}_{i,k}) - \mathcal{L}_{\hat{g}_k}(\tilde{v}_{i,k}) + \sum_j \tilde{B}_{ij} \tilde{v}_{j,k}.$$

Taking derivative with respect to θ and multiplying by $A_{\tau_k}^{-1}$, we get

$$L_{cyl}(\varphi_{i,k}) + \frac{n(n-2)}{4} |\tilde{\mathcal{V}}_k|^{\frac{4}{n-2}} \varphi_{i,k} + n |\tilde{\mathcal{V}}_k|^{\frac{4}{n-2}-2} \tilde{v}_{ik} \langle \tilde{\mathcal{V}}_k, \boldsymbol{\varphi_k} \rangle$$
$$= L_{cyl}(\varphi_{i,k}) - \mathcal{L}_{\hat{g}_k}(\varphi_{i,k}) + \sum_j \tilde{B}_{ij} \varphi_{j,k}.$$

From (3.4) we have

$$\mathcal{L}_{\hat{g}_k}(\varphi_{i,k}) = \Delta_{\hat{g}_k} \varphi_{i,k} - \frac{(n-2)^2}{4} \varphi_{i,k} - \frac{n-2}{4} e^{-t} \partial_r \log |g_k| \circ \Phi(t, \theta) \varphi_{i,k}.$$

But using the fact $\hat{g}_k = dt^2 + d\theta^2 + O(e^{-2t})$ and the local expression of the laplacian in this metric, we find that

$$\Delta_{\hat{g}_k} \varphi_{i,k} = \partial_t^2 \varphi_{i,k} + \Delta_{\mathbb{S}^{n-1}} \varphi_{i,k} + O(e^{-2(t+\tau_k+s_k)}).$$

This implies that

$$L_{cyl}(\varphi_{i,k}) + \frac{n(n-2)}{4} |\tilde{\mathcal{V}}_k|^{\frac{4}{n-2}} \varphi_{i,k} + n |\tilde{\mathcal{V}}_k|^{\frac{4}{n-2}-1} \tilde{v}_{i,k} \langle \tilde{\mathcal{V}}_k, \boldsymbol{\varphi_k} \rangle = A_{\tau_k}^{-1} e^{-2(\tau_k+s_k)} O(e^{-2t}),$$

where $\boldsymbol{\varphi_k} = (\varphi_{1,k}, \varphi_{2,k})$.

Now we can use elliptic theory to extract a subsequence $\varphi_{i,k}$ which converges in compact subsets to a nontrivial and bounded Jacobi field $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$ which satisfies the following system

$$L_{cyl}(\varphi_i) + \frac{n(n-2)}{4} v_\varepsilon^{\frac{4}{n-2}} \varphi_i + n \Lambda_i \langle \Lambda, \boldsymbol{\varphi} \rangle v_\varepsilon^{\frac{4}{n-2}} = 0.$$

Since each coordinate function of the limit φ_i has no zero eingencomponent relative to Δ_θ, we get a contradiction because a Jacobi field with such property is necessarily unbounded. This proves the Claim 1.

Now suppose we have a sequence $v_{i,k}(t, \theta)$ converging to $\Lambda_i v_\varepsilon(t + T)$ as $k \to \infty$. Define

$$w_{i,k}(t, \theta) = v_{i,k}(t, \theta) - \Lambda_i v_\varepsilon(t + T).$$

Set $\mathcal{W}_k = (w_{1,k}, w_{2,k})$, $\eta_k = b \max_{I_N} |\mathcal{W}_k|$, $\overline{\eta}_k = \eta_k + e^{-(2-\delta)\tau_k}$ and $\varphi_{i,k} = \overline{\eta}_k^{-1} w_{i,k}$, where $\delta > 0$ is a small number and $b > 0$ is a fixed number to be chosen later. Note that $|(\varphi_{1,k}, \varphi_{2,k})| \le b^{-1}$ on I_N. Then

$$\mathcal{L}_{\hat{g}_k}(w_{i,k}) + \frac{n(n-2)}{4} \left(|\mathcal{V}_k|^{\frac{4}{n-2}} v_{i,k} - \Lambda_i v_\varepsilon^{\frac{n+2}{n-2}} \right) = E_{i,k}, \tag{4.3}$$

where $E_{i,k} = \sum_j B_{ij} v_{j,k} + \Lambda_i (L_{cyl} - \mathcal{L}_{\hat{g}_k}) v_\varepsilon$. First note that by (3.3) we get that $E_{i,k} = O(e^{-2(\tau_k+t)})$ when $t \to \infty$. Second, observe that

$$|\mathcal{V}_k|^{\frac{4}{n-2}} v_{i,k} - \Lambda_i v_\varepsilon^{\frac{n+2}{n-2}} = |\mathcal{V}_k|^{\frac{4}{n-2}} w_{i,k} + \Lambda_i v_\varepsilon \frac{|\mathcal{V}_k|^{\frac{4}{n-2}} - v_\varepsilon^{\frac{4}{n-2}}}{|\mathcal{V}_k|^2 - v_\varepsilon^2} \sum_j w_{j,k}(v_{j,k} + \Lambda_i v_\varepsilon).$$

Multiplying (4.3) by $\overline{\eta}^{-1}$ and taking the limit $k \to \infty$ we get

$$L_{cyl}(\varphi_i) + \frac{n(n-2)}{4} v_0^{\frac{4}{n-2}} \varphi_i + n\Lambda_i \langle \Lambda, \boldsymbol{\varphi} \rangle v_\varepsilon^{\frac{4}{n-2}} = 0,$$

on the whole cylinder, where $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$ is a Jacobi field.

Claim 2: The Jacobi field $\boldsymbol{\varphi}$ is bounded for $t \geq 0$.

To prove this claim we will use the analysis done in Section 2.4.

By the spectral decomposition for the Laplacian in the sphere, we know that it is possible to write the Jacobi field as

$$\boldsymbol{\varphi} = a_1 \phi_{\varepsilon,0}^1 + a_2 \phi_{\varepsilon,0}^2 + a_3 \phi_{\varepsilon,0}^3 + a_4 \phi_{\varepsilon,0}^4 + \tilde{\boldsymbol{\varphi}},$$

where $\phi_{\varepsilon,0}^i$ are the linearly independent Jacobi fields corresponding to the eingencomponent independent of θ and $\tilde{\boldsymbol{\varphi}}$ denotes the projection onto the orthogonal complement. We also know that the functions $\phi_{\varepsilon,0}^1$ and $\phi_{\varepsilon,0}^3$ are bounded and $\phi_{\varepsilon,0}^2$ and $\phi_{\varepsilon,0}^4$ are linearly growing.

Let us show that $\tilde{\boldsymbol{\varphi}}$ is bounded by proving that each $\partial_\theta \tilde{\varphi}_i = \partial_\theta \varphi_i$ is bounded for $t \geq 0$. In fact, the function $\partial_\theta \varphi_i$ is the limit of $\overline{\eta}_k^{-1} \partial_\theta v_{i,k}$ and we can suppose that $\partial_\theta \varphi_i$ is nontrivial, otherwise the result is immediate.

If the first item of Claim 1 happens then

$$\sup_{t \geq 0} \left(\overline{\eta}_k^{-1} |\partial_\theta v_{i,k}| \right) \leq \frac{ce^{-2\tau_k}}{\eta_k + e^{-(2-\delta)\tau_k}} \leq C.$$

While if the second item of Claim 1 is true then

$$\sup_{t \geq 0} \left(\overline{\eta}_k^{-1} |\partial_\theta v_{i,k}| \right) \leq \sup_{t \geq 0} \left(\overline{\eta}_k^{-1} |\partial_\theta \mathcal{V}_k| \right) = \sup_{t \in I_N} \left(\overline{\eta}_k^{-1} |\partial_\theta \mathcal{V}_k| \right) \leq C,$$

since the sequence $\overline{\eta}_k^{-1} |\partial_\theta \mathcal{V}_k|$ converges in the C_{loc}^2 topology. Therefore each $\tilde{\varphi}_i$ is bounded for $t \geq 0$, hence exponentially decaying.

To end the proof of the Claim 2 we need to show that $a_2 = a_4 = 0$. To see this note that the convergence $\varphi_{i,k} = \overline{\eta}_k^{-1} w_{i,k} \to \varphi_i$ implies that

$$\begin{aligned} \mathcal{V}_k &= \Lambda v_{\varepsilon,T} + \overline{\eta}_k \boldsymbol{\varphi} + o(\overline{\eta}_k) \\ &= \Lambda v_{\varepsilon,T} + \overline{\eta}_k (a_1 \phi_{\varepsilon,0}^1 + a_2 \phi_{\varepsilon,0}^2 + a_3 \phi_{\varepsilon,0}^3 + a_4 \phi_{\varepsilon,0}^4 + \tilde{\boldsymbol{\varphi}}) + o(\overline{\eta}_k), \end{aligned}$$

where $v_{\varepsilon,T}(t) = v_\varepsilon(t + T)$.

On the other hand by (3.13), (4.2) and the Pohozaev identity, Lemma 3.3, we have that

$$P(0, \mathcal{V}_k) = P(\tau_k, \mathcal{V}) = P(\mathcal{V}) + O(e^{-2\tau_k}) = P(v_{\varepsilon,T}) + O(e^{-2\tau_k}).$$

Since $\lim_{k \to \infty} (\overline{\eta}_k^{-1} e^{-2\tau_k}) = 0$, we would have a contradiction in case a_2 or a_4 is not zero. Thus each $\boldsymbol{\varphi}$ is bounded for $t \geq 0$.

Now we will show that there exists some T so that the difference between $\mathcal{V}$ and $\mathcal{V}_{\varepsilon,T} = \Lambda v_{\varepsilon,T}$ goes to zero as $t \to \infty$.

Since we do not know the correct translation parameter, define $\mathcal{V}_\tau(t, \theta) = \mathcal{V}(t + \tau, \theta)$ and $\mathcal{W}_\tau(t, \theta) = \mathcal{V}_\tau(t, \theta) - \Lambda v_\varepsilon(t)$. Let $C_1 > 0$ be a fixed constant and consider the interval I_N as before in the Claim 1. Set also $\eta(\tau) = b \max_{I_N} |\mathcal{W}_\tau|$ and $\overline{\eta}(\tau) = \eta(\tau) + e^{-(2-\delta)\tau}$, where $b > 0$ is a fixed constant to be chosen later. We observe that $\overline{\eta}(\tau) \to 0$ as $\tau \to \infty$. Let us prove the following claim.

Claim 3: If N, b and τ are sufficiently large and $\overline{\eta}$ is sufficiently small, then there exists s with $|s| \leq C_1 \overline{\eta}(\tau)$ so that $\overline{\eta}(\tau + NT_\varepsilon + s) \leq \frac{1}{2}\overline{\eta}(\tau)$.

Suppose the claim is not true. Then there exists some sequence $\tau_k \to \infty$ such that $\overline{\eta}(\tau_k) \to 0$ and for any s satisfying $|s| \le C_1 \overline{\eta}(\tau_k)$ we have that $\overline{\eta}(\tau_k + NT_\varepsilon + s) > \frac{1}{2}\overline{\eta}(\tau_k)$. Define $\varphi_{i,k} = \overline{\eta}(\tau_k)^{-1} w_{i,\tau_k}$, similarly to the previously claim. We can suppose that $\varphi_{i,k}$ converges in C^∞ on compact sets to a Jacobi field, which by Claim 2 it is bounded for $t \ge 0$. So we can write

$$\boldsymbol{\varphi} = a_1 \phi_{\varepsilon,0}^1 + a_3 \phi_{\varepsilon,0}^3 + \tilde{\boldsymbol{\varphi}}, \tag{4.4}$$

where $\tilde{\boldsymbol{\varphi}}$ has exponential decay. Note that $|\boldsymbol{\varphi}| \le b^{-1}$ on I_N, which implies that a_1 and a_3 are uniformly bounded, independently of the sequence τ_k. Moreover, since $\phi_{\varepsilon,0}^1 = v_\varepsilon' \Lambda$ and $\phi_{\varepsilon,0}^3 = v_\varepsilon \overline{\Lambda}$ then

$$|a_3 v_\varepsilon| \le |\langle \boldsymbol{\varphi}, \overline{\Lambda} \rangle| + |\langle \tilde{\boldsymbol{\varphi}}, \overline{\Lambda} \rangle| \le b^{-1} + |\tilde{\boldsymbol{\varphi}}| \tag{4.5}$$

on I_N. We know that $v_\varepsilon \ge \varepsilon$ and $\tilde{\boldsymbol{\varphi}}$ decreases exponentially with a fixed rate and so we can choose b and N sufficiently large such that $|a_3|$ is sufficiently small.

Set $s_k = -\overline{\eta}(\tau_k) a_1$ whose absolute value is less than $C_1 \overline{\eta}(\tau_k)$ if we choose C_1 sufficiently large. Hence for $t \in [0, 2NT_\varepsilon]$ we have

$$\begin{aligned}
\mathcal{W}_{\tau_k + s_k}(t, \theta) &= \mathcal{V}(t + \tau_k - \overline{\eta}(\tau_k) a_1, \theta) - \Lambda v_\varepsilon(t) \\
&= \mathcal{V}_{\tau_k}(t - \overline{\eta}(\tau_k) a_1, \theta) - \Lambda v_\varepsilon(t - \overline{\eta}(\tau_k) a_1) - \overline{\eta}(\tau_k) a_1 \Lambda \frac{v_\varepsilon(t - \overline{\eta}(\tau_k) a_1) - v_\varepsilon(t)}{-\overline{\eta}(\tau_k) a_1} \\
&= \overline{\eta}(\tau_k) \boldsymbol{\varphi}_k(t - \overline{\eta}(\tau_k) a_1, \theta) - \overline{\eta}(\tau_k) a_1 \phi_{\varepsilon,0}^1 + o(\overline{\eta}(\tau_k)) \\
&= \mathcal{W}_{\tau_k}(t, \theta) - \overline{\eta}(\tau_k) a_1 \phi_{\varepsilon,0}^1 + o(\overline{\eta}(\tau_k)),
\end{aligned}$$

where $\boldsymbol{\varphi}_k = (\varphi_{1,k}, \varphi_{2,k})$. Here we used the equality $\mathcal{W}_{\tau_k} = \overline{\eta}(\tau_k) \boldsymbol{\varphi}_k + o(\overline{\eta}(\tau_k))$ and $\boldsymbol{\varphi}_k(t - \overline{\eta}(\tau_k) a_1, \theta) - \boldsymbol{\varphi}_k(t, \theta)$ goes to zero as $\tau_k \to \infty$.

Consequently, by (4.4), for $t \in [0, 2NT_\varepsilon]$ we get that

$$\mathcal{W}_{\tau_k + s_k} = \overline{\eta}(\tau_k) \tilde{\boldsymbol{\varphi}} + \overline{\eta}(\tau_k) a_3 \phi_{\varepsilon,0}^3 + o(\overline{\eta}(\tau_k)),$$

which implies that

$$\max_{I_N} |\mathcal{W}_{\tau_k + s_k + NT_\varepsilon}| = \max_{[NT_\varepsilon, 2NT_\varepsilon]} |\mathcal{W}_{\tau_k + s_k}| \le \overline{\eta}(\tau_k) \max_{[NT_\varepsilon, 2NT_\varepsilon]} \left(|\tilde{\boldsymbol{\varphi}}| + |a_3 v_\varepsilon| \right) + o(\overline{\eta}(\tau_k)).$$

Since $\tilde{\boldsymbol{\varphi}}$ decreases exponentially with a fixed rate, by (4.5) we can choose N and $b > 0$ sufficiently large in a way that the last equality implies that

$$\max_{I_N} |\mathcal{W}_{\tau_k + s_k + NT_\varepsilon}| \le \frac{1}{4} \overline{\eta}(\tau_k).$$

On the other hand, note that

$$e^{-(2-\delta)(\tau_k + s_k + NT_\varepsilon)} \le e^{-(2-\delta)NT_\varepsilon} \overline{\eta}(\tau_k) \le \frac{1}{4} \overline{\eta}(\tau_k)$$

which implies that $\overline{\eta}(\tau + NT_\varepsilon + s) \le \frac{1}{2} \overline{\eta}(\tau)$, a contradiction. This ends the proof of the Claim 3.

Once the above claim is proved, using an iterative argument, we are ready to prove that there exists σ such that $w_{i,\sigma} \to 0$ as $t \to \infty$ for each coordinate. First choose τ_0 and N sufficiently large satisfying the Claim 3 and such that $C_1 \overline{\eta}(\tau_0) \le \frac{1}{2} NT_\varepsilon$. Let $s_0 = -\overline{\eta}(\tau_0) a_1$ be chosen as above. Thus we have $|s_0| \le C_1 \overline{\eta}(\tau_0) \le \frac{1}{2} NT_\varepsilon$. Define inductively three sequences as

$$\begin{aligned}
\sigma_k &= \tau_0 + \sum_{i=0}^{k-1} s_i \\
\tau_k &= \tau_{k-1} + s_{k-1} + NT_\varepsilon = \sigma_k + kNT_\varepsilon \\
s_k &= -\overline{\eta}(\tau_k) a_1.
\end{aligned}$$

By the Claim 3 we get by induction $\overline{\eta}(\tau_k) \le 2^{-k} \overline{\eta}(\tau_0)$ and $|s_k| \le 2^{-k-1} NT_\varepsilon$. Hence there exists the limit $\sigma = \lim \sigma_k \le \tau_0 + NT_\varepsilon$ and then $\tau_k \to \infty$ as $k \to \infty$.

We claim σ is the correct translation parameter.

In fact, choose k such that $t = kNT_\varepsilon + [t]$ with $[t] \in I_N$, and write

$$w_{i,\sigma}(t, \theta) = v_i(t + \sigma, \theta) - \Lambda_i v_\varepsilon(t)$$
$$= v_i(t + \sigma, \theta) - v_i(t + \sigma_k, \theta) + v_i(t + \sigma_k, \theta) - \Lambda_i v_\varepsilon(t).$$

Since $\partial_t v_i$ is uniform boundedness, then

$$v_i(t + \sigma, \theta) - v_i(t + \sigma_k, \theta) = \partial_t v_i(t_0) \sum_{j=k}^{\infty} s_j = O(2^{-k}),$$

for some t_0. Besides,

$$v_i(t + \sigma_k, \theta) - \Lambda_i v_\varepsilon(t) = v_i(\tau_k + [t], \theta) - \Lambda_i v_\varepsilon([t]) = w_{i, \tau_k}([t], \theta).$$

Thus,

$$\mathcal{W}_\sigma(t, \theta) = \mathcal{W}_{\tau_k}([t], \theta) + O(2^k).$$

Since $b \max_{I_N} |\mathcal{W}_{\tau_k}| = \eta(\tau_k) \leq \overline{\eta}(\tau_k) \leq 2^{-k} \overline{\eta}(\tau_0)$, we will have $|w_{i,\sigma}(t, \theta)| = O(2^{-k})$ or equivalently, using that $t = kNT_\varepsilon + [t]$, we have

$$|w_{i,\sigma}(t, \theta)| \leq C_1 e^{-\frac{\log 2}{NT_\varepsilon} t}$$

which finishes the proof of the theorem. ∎

As a direct consequence of the results proved in this section we have the following corollary.

Corollary 4.2. *Suppose that $\mathcal{U}$ is a solution of the system* (1.1) *in the punctured ball $B_1^n(0) \backslash \{0\}$, $3 \leq n \leq 5$ and that the potential A satisfies the hypotheses (H1) and (H2). Then there exists a Fowler-type solution $\mathcal{U}_0$ from* (1.3) *such that*

$$\mathcal{U}(x) = (1 + O(|x|^\alpha))\mathcal{U}_0(x)$$

as $x \to 0$, for some $\alpha > 0$.

Conflict of interest statement

The authors declare that there is no conflict of interest regarding the publication of this article.

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