Spreading in space–time periodic media governed by a monostable equation with free boundaries, Part 2: Spreading speed ☆

Weiwei Ding [a,b], Yihong Du [a], Xing Liang [c,*]

[a] *School of Science and Technology, University of New England, Armidale, NSW 2351, Australia*
[b] *Meiji Institute for Advanced Study of Mathematical Sciences, Meiji University, Tokyo 164-8525, Japan*
[c] *School of Mathematical Sciences and Wu Wen-Tsun Key Laboratory of Mathematics, University of Science and Technology of China, Hefei, Anhui, 230026, PR China*

Abstract

This is Part 2 of our work aimed at classifying the long-time behavior of the solution to a free boundary problem with monostable reaction term in space–time periodic media. In Part 1 (see [2]) we have established a theory on the existence and uniqueness of solutions to this free boundary problem with continuous initial functions, as well as a spreading-vanishing dichotomy. We are now able to develop the methods of Weinberger [15,16] and others [6–10] to prove the existence of asymptotic spreading speed when spreading happens, without knowing a priori the existence of the corresponding semi-wave solutions of the free boundary problem. This is a completely different approach from earlier works on the free boundary model, where the spreading speed is determined by firstly showing the existence of a corresponding semi-wave. Such a semi-wave appears difficult to obtain by the earlier approaches in the case of space–time periodic media considered in our work here.
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1. Introduction and main results

This is Part 2 of our work aimed at classifying the long-time dynamical behavior to a class of space–time periodic reaction–diffusion equations with free boundaries of the form

$$\begin{cases} u_t = du_{xx} + f(t, x, u), & g(t) < x < h(t), \ t > 0, \\ u(t, g(t)) = u(t, h(t)) = 0, & t > 0, \\ g'(t) = -\mu u_x(t, g(t)), & t > 0, \\ h'(t) = -\mu u_x(t, h(t)), & t > 0, \\ g(0) = g_0, \ h(0) = h_0, \ u(0, x) = u_0(x), & g_0 \le x \le h_0, \end{cases} \tag{1.1}$$

where $x = g(t)$ and $x = h(t)$ are the moving boundaries to be determined together with $u(t, x)$, and d, μ are given positive constants.

The initial function u_0 belongs to $\mathcal{H}(g_0, h_0)$ for some $g_0 < h_0$, where

$$\mathcal{H}(g_0, h_0) := \left\{ \phi \in C([g_0, h_0]) : \phi(g_0) = \phi(h_0) = 0, \ \phi(x) > 0 \text{ in } (g_0, h_0) \right\}.$$

The reaction term $f : \mathbb{R}^2 \times \mathbb{R}^+ \mapsto \mathbb{R}$ is continuous, of class $C^{\alpha/2, \alpha}(\mathbb{R}^2)$ in $(t, x) \in \mathbb{R}^2$ locally uniformly in $u \in \mathbb{R}^+$ (with $0 < \alpha < 1$), and of class C^1 in $u \in \mathbb{R}^+$ uniformly in $(t, x) \in \mathbb{R}^2$. The basic assumptions on f are:

$$f(t, x, 0) = 0 \quad \text{for all } (t, x) \in \mathbb{R}^2, \tag{1.2}$$

there exists $M > 0$ such that

$$f(t, x, u) \le 0 \text{ for all } (t, x) \in \mathbb{R}^2, \ u \ge M, \tag{1.3}$$

and f is ω-periodic in t and L-periodic in x for some positive constants ω and L, that is,

$$\begin{cases} f(t + \omega, x, u) = f(t, x, u) \\ f(t, x + L, u) = f(t, x, u) \end{cases} \text{ for all } (t, x) \in \mathbb{R}^2, \ u \ge 0. \tag{1.4}$$

In this work, we regard (1.1) as describing the spreading of a new or invasive species over a one-dimensional habitat, where $u(t, x)$ represents the population density of the species at location x and time t, the reaction term f measures the growth rate, the free boundaries $x = g(t)$ and $x = h(t)$ stand for the edges of the expanding population range, namely the spreading fronts. The Stefan conditions $g'(t) = -\mu u_x(t, g(t))$ and $h'(t) = -\mu u_x(t, h(t))$ may be interpreted as saying that the spreading front expands at a speed proportional to the population gradient at the front; a deduction of these conditions from ecological considerations can be found in [1]. When $f(t, x, u)$ is periodic with respect to x and t as described in (1.4), problem (1.1) represents spreading of the species in a heterogeneous environment that is periodic in both space and time.

In the special case that the function f does not depend on x and t, and is of logistic type, that is,

$$f(u) = u(a - bu) \text{ for some positive constants } a \text{ and } b,$$

such a problem was first studied in [4] for the spreading of a new or invasive species. It is proved that, when

$$u_0 \in C^2([g_0, h_0]), \ u_0(g_0) = u_0(h_0) = 0, \ u_0(x) > 0 \text{ in } (g_0, h_0),$$

there exists a unique solution (u, g, h) with $u(t, x) > 0$, $g'(t) < 0$ and $h'(t) > 0$ for all $t > 0$ and $g(t) < x < h(t)$, and a spreading-vanishing dichotomy holds, namely, there is a barrier R^* on the size of the population range, such that either

(i) **Spreading:** the population range breaks the barrier at some finite time (i.e., $h(t_0) - g(t_0) \ge R^*$ for some $t_0 \ge 0$), and then the free boundaries go to infinity as $t \to \infty$ (i.e., $\lim_{t \to \infty} h(t) = \infty$ and $\lim_{t \to \infty} g(t) = -\infty$), and the population spreads to the entire space and stabilizes at its positive steady state (i.e. $\lim_{t \to \infty} u(t, x) = a/b$ locally uniformly in $x \in \mathbb{R}$) or

(ii) **Vanishing:** the population range never breaks the barrier (i.e. $h(t) - g(t) < R^*$ for all $t > 0$), and the population vanishes (i.e. $\lim_{t \to \infty} u(t, x) = 0$).

Moreover, when spreading occurs, the asymptotic spreading speed can be determined, i.e.,

$$\lim_{t \to \infty} -g(t)/t = \lim_{t \to \infty} h(t)/t = c,$$

where c is the unique positive constant such that the problem

$$\begin{cases} dq_{xx} - cq_x + q(a - bq) = 0, \ q(x) > 0 \quad \text{for } x \in (0, \infty), \\ q(0) = 0, \quad \mu q_x(0) = c, \quad q(\infty) = 1 \end{cases}$$

has a (unique) solution q. Such a solution $q(x)$ is called a semi-wave with speed c.

These results have subsequently been extended to more general situations in several directions. But as we mentioned in the Introduction of Part 1 ([2]), in all the previous works on this problem, the spreading speed is determined by the corresponding semi-wave solution which, in our current space–time periodic case, appears difficult to establish by adapting the existing approaches.

In this paper we use a different approach to determine the spreading speed for the space–time periodic case of problem (1.1) with a monostable f. This approach is based on recent developments of Weinberger's ideas first appeared in [15], where the existence of spreading speed for the corresponding Cauchy problem is proved without knowing the existence of the corresponding traveling wave solutions. In [15], Weinberger established the existence of spreading speed for a scalar discrete-time recursion with a translation-invariant order-preserving monostable operator. Such a method was generalized in [10] to systems of discrete-time recursions, and then in [16] to scalar discrete-time recursions in spatially periodic habitats. The theory in [10,15] was further developed in [8] to the investigation of both discrete and continuous semiflows with a monostable structure, and then was extended to time-periodic semiflows in [7], to space-periodic semiflows in [9], and recently to space–time periodic semiflows in [6].

However, to adapt these ideas to treat our free boundary problem here, it is necessary to firstly extend the existence and uniqueness theory for (1.1) with C^2 initial functions (see [4]) to the case that the initial functions are merely continuous, which has not been considered before and requires new techniques. This step has now been carried out in Part 1 of this work. Moreover, in Part 1, we have also proved the continuous dependence of the solution to the initial function, and established some comparison principles and a spreading-vanishing dichotomy for (1.1).

With these preparations, we are now able to establish the existence of asymptotic spreading speed for (1.1), by further developing the techniques of Weinberger [15,16] and several other recent works [6–10]. To do this, we assume that the associated nonlinear term f admits a *monostable* structure, characterized by the following assumption (H).

Assumption (H):

(i) *The following problem*

$$\begin{cases} p_t = dp_{xx} + f(t, x, p) \ \text{ in } (t, x) \in \mathbb{R}^2, \\ p(t, x) \ \text{ is } \omega\text{-periodic in } t \text{ and } L\text{-periodic in } x, \end{cases} \tag{1.5}$$

admits a unique positive solution $p(t, x) \in C^{1,2}(\mathbb{R}^2)$;
(ii) *for any* $v_0 \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *with* $\inf_{x \in \mathbb{R}} v_0(x) > 0$, *there holds*

$$v(t + s, x; v_0) - p(t + s, x) \to 0 \ \text{ as } s \to \infty \tag{1.6}$$

uniformly in $(t, x) \in [0, \infty) \times \mathbb{R}$, *where* $v(t, x; v_0)$ *is the solution of the Cauchy problem*

$$\begin{cases} v_t = dv_{xx} + f(t, x, v), & x \in \mathbb{R}, \ t > 0, \\ v(0, x) = v_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.7}$$

By comparison principle, one easily sees that Assumption (H) (ii) is equivalent to the following: for any positive L-periodic $v_0 \in C(\mathbb{R})$, there holds

$$v(n\omega, x; v_0) - p(0, x) \to 0 \ \text{ as } n \to \infty \text{ uniformly in } x \in \mathbb{R},$$

where $v(t, x; v_0)$ is the solution of (1.7). Thus, our monostable assumption coincides with that given in [6] for space–time periodic semiflows. Under this condition, it is recently proved in [6] that, the Cauchy problem (1.7) has a rightward spreading speed $\bar{c}_+^*$ and a leftward spreading speed $\bar{c}_-^*$. More precisely, for any nonnegative non-null compactly supported initial datum v_0 with $v_0 \le p(0, x)$ for $x \in \mathbb{R}$, there holds

$$\begin{cases} \lim_{t\to\infty} \sup_{x\in[-c_2 t, c_1 t]} |v(t,x,v_0) - p(t,x)| = 0 & \text{when } -\bar{c}_-^* < -c_2 < c_1 < \bar{c}_+^*, \\ \lim_{t\to\infty} \sup_{x\in(-\infty, -c_2' t]\cup[c_1' t, +\infty)} v(t,x;v_0) = 0 & \text{when } c_2' > \bar{c}_-^* \text{ and } c_1' > \bar{c}_+^*. \end{cases}$$

In this current work, we show that under the same condition, whenever spreading occurs, the free boundary problem (1.1) also has a leftward and a rightward spreading speed. More precisely, we have the following theorem.

Theorem 1.1. *Suppose that* (1.2), (1.3), (1.4) *and* (H) *are satisfied. Then there exist constants* $c_{-,\mu}^* > 0$ *and* $c_{+,\mu}^* > 0$ *such that for any given* $u_0 \in \mathcal{H}(g_0, h_0)$ *with* $u_0(x) \le p(0,x)$ *in* $\mathbb{R}$ *such that* $\lim_{t\to\infty} h(t) = \lim_{t\to\infty} -g(t) = \infty$ *and*

$$\lim_{t\to\infty} |u(t,x) - p(t,x)| = 0 \text{ locally uniformly in } x \in \mathbb{R}, \tag{1.8}$$

the following conclusions hold:

$$\lim_{t\to\infty} \sup_{-c_2 t \le x \le c_1 t} |u(t,x) - p(t,x)| = 0 \text{ when } -c_{-,\mu}^* < -c_2 < c_1 < c_{+,\mu}^*, \tag{1.9}$$

and

$$\lim_{t\to\infty} \frac{g(t)}{t} = -c_{-,\mu}^*, \qquad \lim_{t\to\infty} \frac{h(t)}{t} = c_{+,\mu}^*. \tag{1.10}$$

Here (u, g, h) *is the solution to* (1.1) *with initial datum* u_0 *and* p *is the unique positive solution of* (1.5).

The above theorem indicates that $c_{+,\mu}^*$ (resp. $c_{-,\mu}^*$) is the rightward (resp. leftward) spreading speed for problem (1.1).

Remark 1.2. The restriction $u_0(x) \le p(0,x)$ in Theorem 1.1 is rather unnatural. We will show that it can be removed under mild additional assumptions on f near $u = p(t,x)$; see Section 2.1 below for details.

The proof of Theorem 1.1 is based on ideas in [6–9,16], but considerable technical changes are needed since the introduction of the free boundary here. As a consequence, the proof of Theorem 1.1 is rather involved.

We now give some examples of nonlinearities f for which the hypothesis (H) can be easily checked. The simplest example is the logistic nonlinearity

$$f(t,x,u) = u\big(a(t,x) - b(t,x)u\big) \tag{1.11}$$

where a, b are of class $C^{\alpha/2,\alpha}$ which are ω-periodic in t and L-periodic in x, and there are positive constants κ_1, κ_2 such that $\kappa_1 \le a(t,x) \le \kappa_2$ and $\kappa_1 \le b(t,x) \le \kappa_2$ for all $(t,x) \in \mathbb{R}^2$. It is well known that, with such a nonlinearity f, problem (1.5) admits a unique positive solution $p(t,x) \in C^{1,2}(\mathbb{R}^2)$, (1.6) holds and for any nonnegative bounded non-null initial function $v_0 \in C(\mathbb{R})$, there holds

$$v(t+s,x;v_0) - p(t+s,x) \to 0 \text{ as } s \to \infty \text{ locally uniformly in } (t,x) \in \mathbb{R}^2,$$

where $v(t,x;v_0)$ is the unique solution of the Cauchy problem (1.7). In fact, these existence, uniqueness and stability results hold for more general f satisfying (in addition to the basic assumptions (1.2), (1.3) and (1.4)),

$$\forall (t,x) \in \mathbb{R}^2, \text{ the function } u \mapsto f(t,x,u)/u \text{ is decreasing for } u > 0, \tag{1.12}$$

and the generalized principal eigenvalue of the linearized problem (at $u = 0$) is negative (see [11,13]).

An example satisfying (H) but not (1.12) is

$$f(t,x,u) = a(t,x)u^k(1-u) \text{ for some } k > 1, \tag{1.13}$$

where $a(t,x)$ is a positive function of class $C^{\alpha/2,\alpha}$, and is ω-periodic in t and L-periodic in x. It follows from [12, Proposition 1.7] that $p(t,x) \equiv 1$ is the unique positive solution for problem (1.5) with nonlinearity (1.13). A simple comparison argument involving a suitable ODE problem shows that (1.6) holds for such a nonlinearity.

Regarding sufficient conditions for spreading to happen for (1.1), when f is of type (1.11), the spreading-vanishing dichotomy proved in Part 1 shows that there exists a positive constant R (independent of u_0) such that $h_0 - g_0 \ge R$

implies spreading. When f is of type (1.13), sufficient conditions for spreading can be found in [14, Theorem 1.1 and Remark 2.4].

Finally, let us consider the behavior of the spreading speeds for problem (1.1) as μ increases to ∞. We have the following theorem.

Theorem 1.3. *Let $c^*_{\pm,\mu}$ be the spreading speeds obtained in Theorem 1.1. Then $c^*_{\pm,\mu}$ are nondecreasing in $\mu > 0$, and*

$$\lim_{\mu \to \infty} c^*_{-,\mu} = \bar{c}^*_- \quad and \quad \lim_{\mu \to \infty} c^*_{+,\mu} = \bar{c}^*_+,$$

where $\bar{c}^*_+$ (resp. $\bar{c}^*_-$) is the rightward (resp. leftward) spreading speed for problem (1.7).

The general strategy in proving Theorems 1.1 and 1.3 can be summarized as follows. We will first show the existence of rightward spreading speed for the following free boundary problem

$$\begin{cases} u_t = du_{xx} + f(t, x, u), & -\infty < x < h(t), \quad t > 0, \\ u(t, h(t)) = 0, \ h'(t) = -\mu u_x(t, h(t)), & t > 0, \\ h(0) = h_0, \quad u(0, x) = u_0(x), & -\infty < x \le h_0, \end{cases} \tag{1.14}$$

with initial data $u_0 \in \mathcal{H}_+(h_0)$, where

$$\mathcal{H}_+(h_0) := \left\{ \phi \in C\big((-\infty, h_0]\big) \cap L^\infty\big((-\infty, h_0]\big) : \phi(h_0) = 0, \ \phi(x) > 0 \text{ in } (-\infty, h_0) \right\}.$$

Then we will prove that this speed is indeed the rightward spreading speed for problem (1.1), and that it converges to the rightward spreading speed for the Cauchy problem (1.7) as $\mu \to \infty$. Similarly, to obtain the existence and convergence of the leftward spreading speed for (1.1), it suffices to prove these for the problem

$$\begin{cases} u_t = du_{xx} + f(t, x, u), & g(t) < x < \infty, \quad t > 0, \\ u(t, g(t)) = 0, \ g'(t) = -\mu u_x(t, g(t)), & t > 0, \\ g(0) = g_0, \quad u(0, x) = u_0(x), & g_0 \le x < \infty, \end{cases} \tag{1.15}$$

with initial data $u_0 \in \mathcal{H}_-(g_0)$, where

$$\mathcal{H}_-(g_0) := \left\{ \phi \in C\big([g_0, \infty)\big) \cap L^\infty\big([g_0, \infty)\big) : \phi(g_0) = 0, \ \phi(x) > 0 \text{ in } (g_0, \infty) \right\}.$$

Outline of the paper: The remaining part of this paper is organized as follows. In Section 2, we first give a proof for the statement in Remark 1.2 and then introduce some notations and state some common properties of the solutions to problems (1.1), (1.14) and (1.15). Section 3 is devoted to the proof for the existence of spreading speeds for problems (1.14) and (1.15). The proof of Theorem 1.1 is given in Section 4 and the proof of Theorem 1.3 is carried out in Section 5.

2. Preliminaries

In this section, we prove the statement in Remark 1.2, and then introduce some notations and basic facts to be used in the subsequent sections.

2.1. On the condition $u_0(x) \le p(0, x)$ in Theorem 1.1

The statement in Remark 1.2 clearly follows from the result below.

Proposition 2.1. *Suppose that f satisfies (1.2), (1.3), (1.4) and (H). Suppose further there exists $\varepsilon_0 > 0$ small such that for every $(t, x) \in \mathbb{R}^2$, we have $f(t, x, \cdot) \in C^2(I_{t,x})$ with $I_{t,x} := [(1 - \varepsilon_0) p(t, x), (1 + \varepsilon_0) p(t, x)]$, and*

$$\frac{f(t, x, u)}{u} \text{ is nonincreasing in } u, \ f_{uu}(t, x, u) \le 0 \text{ for } u \in I_{t,x}. \tag{2.1}$$

Let $u_0 \in \mathcal{H}(g_0, h_0)$ and (u, g, h) be the unique solution of (1.1). Then there exists $t_0 > 0$ such that

$$u(t_0, x) < p(0, x) \text{ for } x \in \mathbb{R}.$$

Let us note that the functions f given in (1.11) and (1.13) also satisfy (2.1).

To prove Proposition 2.1, we will use the following result on the corresponding Cauchy problem (1.7), which may have independent interest.

Proposition 2.2. *Suppose that f satisfies all the assumptions in Proposition 2.1. Let $v(t, x)$ be the unique solution of (1.7) with initial function $v_0 \in C(\mathbb{R})$ nonnegative and having compact support. Then there exist $t_0 > 0$ and $\delta_0 > 0$ such that*

$$v(t_0, x) \le p(0, x) - \delta_0 \text{ for } x \in \mathbb{R}. \tag{2.2}$$

Proof. Due to (2.1), for $\varepsilon \in (0, \varepsilon_0]$,

$$[(1 - \varepsilon)p]_t - d[(1 - \varepsilon)p]_{xx} \le f(t, x, (1 - \varepsilon)p) \text{ for } t, \ x \in \mathbb{R}.$$

It follows that

$$v_\varepsilon(t, x) := \max\{v(t, x), (1 - \varepsilon)p(t, x)\}$$

satisfies, in the weak sense,

$$(v_\varepsilon)_t - d(v_\varepsilon)_{xx} \le f(t, x, v_\varepsilon) \text{ for } t > 0, \ x \in \mathbb{R}.$$

For clarity, we divide the argument below into several steps.

Step 1. Define

$$w_\varepsilon(t, x) := v_\varepsilon(t, x) - p(t, x).$$

We show that for all large $t > 0$, say $t \ge T_0$,

$$(w_\varepsilon)_t - d(w_\varepsilon)_{xx} \le f_u(t, x, p(t, x))w_\varepsilon \text{ for all } x \in \mathbb{R}. \tag{2.3}$$

Clearly

$$(w_\varepsilon)_t - d(w_\varepsilon)_{xx} \le f(t, x, v_\varepsilon) - f(t, x, p).$$

By (1.6) and a simple comparison argument, for all large t, say $t \ge T_0 = T_0(\varepsilon)$, $v(t, x) \le (1 + \varepsilon)p(t, x)$ for $x \in \mathbb{R}$. It follows that

$$(1 - \varepsilon)p(t, x) \le v_\varepsilon(t, x) \le (1 + \varepsilon)p(t, x) \text{ for all } x \in \mathbb{R}, \ t \ge T_0.$$

Hence by the Taylor expansion and (2.1) we obtain, for $t \ge T_0$ and $x \in \mathbb{R}$,

$$
\begin{aligned}
& f(t, x, v_\varepsilon) - f(t, x, p) \\
&= f_u(t, x, p)(v_\varepsilon - p) + \frac{1}{2}f_{uu}(t, x, \theta_\varepsilon)(v_\varepsilon - p)^2 \\
&\le f_u(t, x, p)w_\varepsilon
\end{aligned}
$$

since

$$\theta_\varepsilon = \theta_\varepsilon(t, x) \in [(1 - \varepsilon)p(t, x), (1 + \varepsilon)p(t, x)].$$

This proves (2.3).

Step 2. Comparison via a linear equation.

In this step, we obtain an upper bound for w_ε by making use of (2.3) and the following eigenvalue problem

$$
\begin{cases}
\phi_t - d\phi_{xx} - f_u(t, x, p(t, x))\phi = \lambda\phi & \text{for } (t, x) \in \mathbb{R}^2, \\
\phi > 0 \text{ and is } \omega\text{-periodic in } t, \ L\text{-periodic in } x.
\end{cases}
$$

It is well known that this eigenvalue problem has an eigenpair $(\lambda, \phi) = (\lambda_1, \phi_1)$ (see [11]). So we have

$$(\phi_1)_t - d(\phi_1)_{xx} = a(t, x)\phi_1 \text{ for } t, \ x \in \mathbb{R},$$

where

$$a(t, x) := f_u(t, x, p(t, x)) + \lambda_1.$$

Set

$$V(t, x) := e^{\lambda_1 t} w_\varepsilon(t, x).$$

From (2.3) we obtain

$$V_t - dV_{xx} \le a(t, x)V \text{ for } t \ge T_0, \ x \in \mathbb{R}.$$

By our assumption on f, there exists $K > 0$ such that

$$f(t, x, v(t, x)) \le Kv(t, x) \text{ for all } t > 0, \ x \in \mathbb{R}.$$

Hence

$$v_t - dv_{xx} \le Kv, \ v(0, x) = v_0(x).$$

Since v_0 has compact support, this implies, by the heat kernel expression of v, that for every fixed $t > 0$,

$$v(t, x) \to 0 \text{ as } |x| \to \infty.$$

(See Lemma 2.2 in [5] for a simple proof.) Thus we can find $L_0 > 0$ large so that

$$v(T_0, x) \le \frac{1}{2}(1 - \varepsilon)p(T_0, x) \text{ for } |x| \ge L_0.$$

It follows that

$$V(T_0, x) = -e^{\lambda_1 T_0}\varepsilon p(T_0, x) \text{ for } |x| \ge L_0.$$

Therefore we can find $\delta > 0$ small such that

$$V(T_0, x) + \delta\phi_1(T_0, x) < 0 \text{ for } |x| \ge L_0.$$

We now define

$$W_0(x) := \max\{0, V(T_0, x) + \delta\phi_1(T_0, x)\}.$$

Clearly $W_0(x) = 0$ for $|x| \ge L_0$. Moreover,

$$\widetilde{V}(t, x) := V(t, x) + \delta\phi_1(t, x)$$

satisfies

$$\begin{cases} \widetilde{V}_t - d\widetilde{V}_{xx} \le a(t, x)\widetilde{V} & \text{for } t > T_0, \ x \in \mathbb{R}, \\ \widetilde{V}(T_0, x) \le W_0(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

Let $W(t, x)$ be the unique solution to

$$\begin{cases} W_t - dW_{xx} = a(t, x)W & \text{for } t > T_0, \ x \in \mathbb{R}, \\ W(T_0, x) = W_0(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

Then clearly

$$e^{\lambda_1 t}w_\varepsilon(t, x) + \delta\phi_1(t, x) = \widetilde{V}(t, x) \le W(t, x) \text{ for } t \ge T_0, \ x \in \mathbb{R}. \tag{2.4}$$

This is the estimate for w_ε we wanted to obtain in this step.

Step 3. We prove that

$$\lim_{t \to \infty} W(t, x) = 0 \text{ uniformly in } x \in \mathbb{R}.$$

If $W_0(x) \equiv 0$, then $W(t, x) \equiv 0$ and there is nothing left to prove. So assume that $W_0 \not\equiv 0$. Since $W_0 \in L^{\infty}(\mathbb{R})$, there exists $M_0 > 0$ such that

$$W_0(x) \le M_0 \phi_1(t, x) \text{ for } t, \ x \in \mathbb{R}.$$

It follows that $W(t, x) \le M_0 \phi_1(t, x)$ for all $t > T_0$ and $x \in \mathbb{R}$. Since $W_0(x)$ has compact support, and $a \in L^{\infty}(\mathbb{R}^2)$, as before we have

$$\lim_{|x| \to \infty} W(t, x) = 0 \text{ for any fixed } t > T_0.$$

Therefore, for each $t > T_0$, there exists $M(t) > 0$ and $x_t \in \mathbb{R}$ such that

$$W(t, x) \le M(t) \phi_1(t, x) \text{ for all } x \in \mathbb{R}, \ W(t, x_t) = M(t) \phi_1(t, x_t).$$

$M(t)$ must be nonincreasing in t, since if $T_0 < t_1 < t_2$, then from $W(t_1, x) \le M(t_1) \phi_1(t_1, x)$ and the comparison principle we deduce

$$W(t, x) \le M(t_1) \phi_1(t, x) \text{ for } t > t_1, \ x \in \mathbb{R}.$$

Hence $M(t_2) \le M(t_1)$. We may then define

$$M_{\infty} := \lim_{t \to \infty} M(t).$$

Clearly $M_{\infty} \ge 0$. If $M_{\infty} = 0$ then it follows immediately that $\lim_{t \to \infty} W(t, x) = 0$ uniformly in $x \in \mathbb{R}$, as required.

If $M_{\infty} > 0$, we are going to derive a contradiction. Choose an increasing sequence $\{t_n\}$ such that $\lim_{n \to \infty} t_n = \infty$, and denote $x_n := x_{t_n}$. So we have

$$W(t_n, x_n) = M(t_n) \phi_1(t_n, x_n).$$

Set

$$W_n(t, x) = W(t_n + t, x_n + x),$$

and write

$$t_n = k_n \omega + \widetilde{t}_n, \ x_n = l_n L + \widetilde{x}_n \text{ with } k_n, l_n \in \mathbb{N}, \ \widetilde{t}_n \in [0, \omega), \ \widetilde{x}_n \in [0, L).$$

Then

$$(W_n)_t - d(W_n)_{xx} = a(\widetilde{t}_n + t, \widetilde{x}_n + x) W_n.$$

By passing to a subsequence we may assume that

$$\lim_{n \to \infty} \widetilde{t}_n = \widetilde{t} \in [0, \omega], \ \lim_{n \to \infty} \widetilde{x}_n = \widetilde{x} \in [0, L].$$

By standard parabolic estimates and a diagonal argument, we may also assume (by passing to a further subsequence) that $W_n \to W_{\infty}$ in $C^{1,2}_{loc}(\mathbb{R} \times \mathbb{R})$. It follows that

$$(W_{\infty})_t - d(W_{\infty})_{xx} = a(\widetilde{t} + t, \widetilde{x} + x) W_{\infty} \text{ for } t, \ x \in \mathbb{R}. \tag{2.5}$$

Since

$$W_n(t, x) \le M(t_n + t) \phi_1(t_n + t, x_n + x) = M(t_n + t) \phi_1(\widetilde{t}_n + t, \widetilde{x}_n + x)$$

and

$$W_n(0, 0) = M(t_n) \phi_1(\widetilde{t}_n, \widetilde{x}_n),$$

we further have

$$W_\infty(t, x) \le M_\infty \phi_1(\widetilde{t} + t, \widetilde{x} + x), \ W_\infty(0, 0) = M_\infty \phi_1(\widetilde{t}, \widetilde{x}).$$

As $M_\infty \phi_1(\widetilde{t} + t, \widetilde{x} + x)$ also solves (2.5), the strong maximum principle infers that

$$W_\infty(t, x) \equiv M_\infty \phi_1(\widetilde{t} + t, \widetilde{x} + x).$$

We now fix $n_0 \in \mathbb{N}$ large such that

$$n_0 L > 2L_0,$$

and define, for $j, n \in \mathbb{N}$,

$$W^j(t, x) := W(t, x + jn_0 L), \ W_n^j(t, x) := W^j(t_n + t, x_n + x).$$

Then

$$W_t^j - dW_{xx}^j = a(t, x + jn_0 L)W^j = a(t, x)W^j \tag{2.6}$$

and

$$W_n(t, x) = W^j(t_n + t, x_n - jn_0 L + x) = W_n^j(t, x - jn_0 L).$$

Hence, after passing to the same subsequence (independent of j),

$$\lim_{n \to \infty} W_n^j(t, x - jn_0 L) = W_\infty(t, x) \equiv M_\infty \phi_1(\widetilde{t} + t, \widetilde{x} + x) \text{ in } C_{loc}^{1,2}(\mathbb{R} \times \mathbb{R}).$$

It follows that

$$\lim_{n \to \infty} W_n^j(t, x) = M_\infty \phi_1(\widetilde{t} + t, \widetilde{x} + x + jn_0 L) = M_\infty \phi_1(\widetilde{t} + t, \widetilde{x} + x),$$

and for each $k \in \mathbb{N}$,

$$\lim_{n \to \infty} \Sigma_{j=1}^k W_n^j(t, x) = k M_\infty \phi_1(\widetilde{t} + t, \widetilde{x} + x).$$

On the other hand, we note that due to the choice of n_0, $\{W^j(T_0, x)\}_{j \in \mathbb{N}}$ is a sequence of nonnegative functions with disjoint supporting sets, and for each $k \in \mathbb{N}$,

$$\Sigma_{j=1}^k W^j(T_0, x) \le M_0 \phi_1(T_0, x) \tag{2.7}$$

since

$$W^j(T_0, x) \le M_0 \phi_1(T_0, x + jn_0 L) = M_0 \phi_1(T_0, x) \text{ for } x \in [-L_0 - jn_0 L, L_0 - jn_0 L]$$

and $W^j(T_0, x) = 0$ for $x \notin [-L_0 - jn_0 L, L_0 - jn_0 L]$.

By (2.6) we see that $\Sigma_{j=1}^k W^j(t, x)$ and $M_0 \phi_1(t, x)$ satisfy the same linear differential equation, and so, in view of (2.7), we can apply the comparison principle to deduce

$$\Sigma_{j=1}^k W^j(t, x) \le M_0 \phi_1(t, x) \le M_0 \|\phi_1\|_\infty \text{ for } t > T_0, \ x \in \mathbb{R}.$$

It follows that

$$\Sigma_{j=1}^k W_n^j(t, x) \le M_0 \|\phi_1\|_\infty \text{ for } t > T_0 - t_n, \ x \in \mathbb{R}.$$

Letting $n \to \infty$ we thus obtain

$$k M_\infty \phi_1(\widetilde{t} + t, \widetilde{x} + x) \le M_0 \|\phi_1\|_\infty \text{ for } t, \ x \in \mathbb{R}.$$

Since $\sigma_0 := \min_{(t,x) \in \mathbb{R}^2} \phi_1(t, x) > 0$, the above inequality leads to a contradiction if we choose $k \in \mathbb{N}$ large enough. This proves our claim that $M_\infty = 0$. Thus we always have

$$\lim_{t \to \infty} W(t, x) = 0 \text{ uniformly for } x \in \mathbb{R}.$$

Step 4. Completion of the proof.

Since $\phi_1(t, x) \geq \sigma_0$, we have

$$V(t, x) + \delta\sigma_0 \leq \widetilde{V}(t, x) \leq W(t, x) \text{ for } t > T_0, \ x \in \mathbb{R}.$$

Choose $T_1 > T_0$ such that $W(t, x) \leq \frac{1}{2}\delta\sigma_0$ for $x \in \mathbb{R}$ and $t \geq T_1$. Then

$$V(t, x) \leq -\frac{1}{2}\delta\sigma_0 < 0 \text{ for } x \in \mathbb{R}, \ t \geq T_1.$$

Therefore, for $t \geq T_1$,

$$w_\varepsilon(t, x) = e^{-\lambda_1 t}V(t, x) \leq -\frac{1}{2}\delta\sigma_0 e^{-\lambda_1 t} < 0 \text{ for } x \in \mathbb{R}.$$

It follows that

$$v(t, x) \leq v_\varepsilon(t, x) = w_\varepsilon(t, x) + p(t, x) \leq p(t, x) - \frac{1}{2}\delta\sigma_0 e^{-\lambda_1 t} \text{ for } t \geq T_1, \ x \in \mathbb{R}.$$

Taking $k \in \mathbb{N}$ such that $t_0 := k\omega \geq T_1$, and denoting $\delta_0 := \frac{1}{2}\delta\sigma_0 e^{-\lambda_1 t_0}$, we then obtain

$$v(t_0, x) = v(k\omega, x) \leq p(k\omega, x) - \delta_0 = p(0, x) - \delta_0 \text{ for } x \in \mathbb{R}.$$

So (2.2) holds and the proof is complete. $\square$

Proof of Proposition 2.1. We use Proposition 2.2 with $v_0 = u_0$. By the comparison principle we have

$$u(t, x) \leq v(t, x) \text{ for } t > 0, \ x \in \mathbb{R}.$$

By (2.2) we thus have $u(t_0, x) \leq v(t_0, x) < p(0, x)$ for $x \in \mathbb{R}$. $\square$

2.2. Notations and basic facts

From now on, we always assume that

f satisfies (1.2), (1.3), (1.4) and (H).

For any $h_0 \in \mathbb{R}$ and any $u_0 \in \mathcal{H}_+(h_0)$, the unique solution of equation (1.14) with initial value $u_+(0, x) = u_0(x)$ in $(-\infty, h_0]$ is denoted by $\big(u_+(t, x; u_0), h_+(t; u_0)\big)$; for any $g_0 \in \mathbb{R}$ and any $u_0 \in \mathcal{H}_-(g_0)$, let $\big(u_-(t, x; u_0), g_-(t; u_0)\big)$ denote the unique solution of equation (1.15) with initial value $u_-(0, x) = u_0(x)$ in $[g_0, \infty)$; for any finite pair $g_0 < h_0$ and any $u_0 \in \mathcal{H}(g_0, h_0)$, we use $\big(u(t, x; u_0), g(t; u_0), h(t; u_0)\big)$ to denote the unique solution of equation (1.1) with initial value $u(0, x) = u_0(x)$ in $[g_0, h_0]$. Finally, for any $v_0 \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we let $v(t, x; v_0)$ denote the unique solution of the Cauchy problem (1.7) with initial function $v_0(x)$.

Let $p(t, x)$ be the unique positive solution for problem (1.5), and let $\mathcal{C}$ be the subset of $C(\mathbb{R})$ defined by

$$\mathcal{C} := \Big\{\varphi \in C(\mathbb{R}) : \text{ there exists } g_0 \in [-\infty, \infty) \text{ and } h_0 \in (-\infty, \infty] \text{ with } g_0 < h_0 \text{ such that}$$

$$0 < \varphi(x) \leq p(0, x) \text{ for } x \in (g_0, h_0), \text{ and } \varphi(x) = 0 \text{ for } x \in \mathbb{R} \setminus [g_0, h_0]\Big\}.$$

For the sake of convenience, for any given $\varphi \in \mathcal{C}$ with $g_0 \in [-\infty, \infty)$ and $h_0 \in (-\infty, \infty]$ such that $\varphi(x) > 0$ if and only if $g_0 < x < h_0$, we call g_0 the left supporting point of φ, and h_0 the right supporting point of φ.

We now define an operator U generated by the Poincaré map of the solution to problems (1.1), (1.14), (1.15) or (1.7), depending on the nature of $\varphi \in \mathcal{C}$ in the following way. Suppose that φ has left supporting point g_0 and right supporting point h_0.

- If $-\infty < g_0 < h_0 < \infty$, then

$$U[\varphi](x) := \begin{cases} u(\omega, x; \varphi), & \text{for } g(\omega; \varphi) \leq x \leq h(\omega; \varphi), \\ 0, & \text{for } x > h(\omega; \varphi) \text{ or } x < g(\omega; \varphi), \end{cases}$$

where (u, g, h) is the unique solution of (1.1) with initial function φ;

- if $-\infty = g_0 < h_0 < \infty$, then

$$U[\varphi](x) := \begin{cases} u_+(\omega, x; \varphi), & \text{for } x \le h_+(\omega; \varphi), \\ 0, & \text{for } x > h_+(\omega; \varphi), \end{cases}$$

  where (u_+, h_+) is the unique solution of (1.14) with initial function φ;
- if $-\infty < g_0 < h_0 = \infty$, then

$$U[\varphi](x) := \begin{cases} u_-(\omega, x; \varphi), & \text{for } x \ge g_-(\omega; \varphi), \\ 0, & \text{for } x < g_-(\omega; \varphi), \end{cases}$$

  where (u_-, g_-) is the unique solution of (1.15) with initial function φ;
- if $g_0 = -\infty$ and $h_0 = \infty$, then

$$U[\varphi](x) := v(\omega, x; \varphi) \text{ for all } x \in \mathbb{R},$$

  where v is the unique solution of (1.7) with initial function φ.

By the conclusions in Part 1 ([2]) and hypothesis (H) here, it is easy to check that U maps $\mathcal{C}$ into itself and has the following properties:

(A1) $U : \mathcal{C} \to \mathcal{C}$ is order-preserving in the sense that $U[\varphi_1](x) \ge U[\varphi_2](x)$ for $x \in \mathbb{R}$ whenever $\varphi_1(x) \ge \varphi_2(x)$ for $x \in \mathbb{R}$.

(A2) U is periodic with respect to $L\mathbb{Z}$ in the sense that $T_y\big[U[\varphi]\big] = U\big[T_y[\varphi]\big]$ for all $\varphi \in \mathcal{C}$ and $y \in L\mathbb{Z}$, where $T_y : \mathcal{C} \to \mathcal{C}$ is the translation operator defined by $T_y[\varphi] = \varphi[\cdot - y]$.

(A3) $U : \mathcal{C} \to \mathcal{C}$ is continuous in the sense that for any sequence $\varphi_n \in \mathcal{C}$ with left supporting points $g_n \in [-\infty, \infty)$ and right supporting points $h_n \in (-\infty, \infty]$, and any $\varphi \in \mathcal{C}$ with left supporting point $g \in [-\infty, \infty)$ and right supporting point $h \in (-\infty, \infty]$, if $\varphi_n(x)$ converges to $\varphi(x)$ locally uniformly for $x \in \mathbb{R}$ as $n \to \infty$, and g_n converges to g, h_n converges to h as $n \to \infty$, then $U[\varphi_n](x)$ converges to $U[\varphi](x)$ locally uniformly in $x \in \mathbb{R}$ as $n \to \infty$.

(A4) $U : \mathcal{C} \to \mathcal{C}$ is monostable in the sense that 0 and $p(0, x)$ are the only fixed points of U in $\mathcal{C}$. Moreover, if $w \in \mathcal{C}$ and $w(x) \ge \varepsilon$ for some $\varepsilon > 0$, then $\lim_{n\to\infty} U^n[w](x) = p(0, x)$ uniformly in $x \in \mathbb{R}$.

Moreover, we have the following comparison principle, which follows easily from the above properties and an induction argument.

Proposition 2.3. *Let U_1 and U_2 be two order-preserving operators defined on $\mathcal{C}$ as described above. Suppose that the sequence $(v_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ satisfies $v_{n+1}(x) \ge U_1[v_n](x)$ for $x \in \mathbb{R}$, and that the sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{C}$ satisfies $u_{n+1}(x) \le U_2[u_n](x)$ for $x \in \mathbb{R}$. Suppose also that $U_1[\varphi](x) \ge U_2[\varphi](x)$ for all $\varphi \in \mathcal{C}$, $x \in \mathbb{R}$ and that $v_0(x) \ge u_0(x)$ for $x \in \mathbb{R}$. Then $v_n(x) \ge u_n(x)$ for all $x \in \mathbb{R}$, $n \in \mathbb{N}$.*

3. Existence of spreading speed for problems (1.14) and (1.15)

In this section, we give a detailed proof for the fact that the free boundary problem (1.14) admits a spreading speed in the rightward direction. The existence of leftward spreading speed for problem (1.15) follows from the result for (1.14) with $f(t, x, u)$ replaced by $f(t, -x, u)$. Let us recall that f always satisfies (1.2), (1.3), (1.4) and (H).

First, we define a set $\mathcal{M}$ consisting of functions $\phi(\xi, x)$ in $C(\mathbb{R}^2)$ with the following properties:

(a) For each $\xi \in \mathbb{R}$, $\phi(\xi, x)$ is nonnegative and L-periodic in x;
(b) $\phi(\xi, x)$ is uniformly continuous in $(\xi, x) \in \mathbb{R}^2$;
(c) For each fixed x, $\phi(\xi, x)$ is nonincreasing in ξ;
(d) For any $\xi \in \mathbb{R}$, there exists a real number $H_0 = H_0(\xi)$ such that
$\phi(\xi + x, x) > 0$ for $x < H_0$ and $\phi(\xi + x, x) = 0$ for $x \ge H_0$;
(e) $0 < \phi(-\infty, x) < p(0, x)$ for all $x \in \mathbb{R}$.

(3.1)

For each $\alpha \in \mathbb{R}$ and $\beta \in (0, 1)$, let

$$\tau(\xi) := \beta \frac{\max\{\alpha - \xi, 0\}}{\max\{\alpha - \xi, 0\} + 1}.$$

Clearly the function $\tau(\xi) p(0, x)$ belongs to $\mathcal{M}$.

The above properties of $\phi \in \mathcal{M}$ imply the following one.

Lemma 3.1. *For any $\phi \in \mathcal{M}$, there exists $H_2 \geq H_1$ such that $\phi(\xi, \cdot) \equiv 0$ if $\xi \geq H_2$ and $\phi(\xi, x) > 0$ for all $x \in \mathbb{R}$ if $\xi < H_1$.*

Proof. By property (d) in (3.1), for any $\xi_0 \in \mathbb{R}$, there exists $H_0 = H_0(\xi_0)$ such that $\phi(\xi_0 + x, x) > 0$ if and only if $x < H_0$. By (c) we have

$$\phi(\xi_0 + H_0 - 2L, x) \geq \phi(\xi_0 + x, x) > 0 \text{ for } x \in [H_0 - 2L, H_0 - L].$$

By (a) this implies that $\phi(\xi_0 + H_0 - 2L, x) > 0$ for all $x \in \mathbb{R}$. Hence, in view of (c),

$$\phi(\xi, x) > 0 \text{ for } \xi < H_1 := \xi_0 + H_0 - 2L, \ x \in \mathbb{R}.$$

Similarly,

$$\phi(\xi_0 + H_0 + L, x) \leq \phi(\xi_0 + x, x) = 0 \text{ for } x \in [H_0, H_0 + L].$$

By (a) this implies that $\phi(\xi_0 + H_0 + L, x) = 0$ for all $x \in \mathbb{R}$. Hence, in view of (c),

$$\phi(\xi, x) = 0 \text{ for } \xi \geq H_2 := \xi_0 + H_0 + L, \ x \in \mathbb{R}.$$

The proof of Lemma 3.1 is thereby complete. $\quad\square$

For any $\phi \in \mathcal{M}$, and any fixed $\xi_0 \in \mathbb{R}$, by (3.1), the function $x \to \phi(\xi_0 + x, x)$ belongs to $\mathcal{C}$, with right supporting point $H_0 = H_0(\xi_0) \in \mathbb{R}$. Therefore $U[\phi(\xi_0 + \cdot, \cdot)](x)$ is well-defined, and

$$U[\phi(\xi_0 + \cdot, \cdot)](x) := \begin{cases} u_+(\omega, x; \phi(\xi_0 + \cdot, \cdot)), & \text{if } x \leq h_+(\omega; \phi(\xi_0 + \cdot, \cdot)), \\ 0, & \text{if } x > h_+(\omega; \phi(\xi_0 + \cdot, \cdot)). \end{cases}$$

For $(\xi, y) \in \mathbb{R}^2$, we now define the operator Q_+ on $\mathcal{M}$ by

$$Q_+[\phi](\xi, y) := U[\phi(\xi - y + \cdot, \cdot)](y) \quad \text{for any } \phi \in \mathcal{M}. \tag{3.2}$$

As will become clear below, we will make use of Q_+ and its iterations to determine the rightward spreading speed.

We now examine the properties of Q_+.

Lemma 3.2. *Q_+ maps $\mathcal{M}$ to $\mathcal{M}$, and Q_+ is order preserving in the sense that $Q_+[\phi_1] \geq Q_+[\phi_2]$ whenever $\phi_1 \geq \phi_2$ in $\mathcal{M}$.*

Proof. To prove that Q_+ maps $\mathcal{M}$ into itself, it is sufficient to prove that, for any $\phi \in \mathcal{M}$, $Q_+[\phi](\xi, y)$ has the properties stated in (3.1). In what follows, we divide the proof into five steps, and in each step, we show one property.

Step 1: We prove that $Q_+[\phi](\xi, y)$ is nonnegative and is L-periodic in y. The non-negativity is clear from the definition. It remains to show that it is L-periodic in y.

Since the operator U is periodic with respect to $L\mathbb{Z}$ in the sense of (A2), it is easy to check that

$$U[\phi(\cdot + \xi, \cdot)](y + L) = U[\phi(\cdot + L + \xi, \cdot + L)](y) \quad \text{for all } \xi \in \mathbb{R}, \ y \in \mathbb{R}.$$

This together with the L-periodicity of ϕ in the second variable implies that

$$\begin{aligned} Q_+[\phi](\xi, y + L) &= U[\phi(\cdot - y - L + \xi, \cdot)](y + L) \\ &= U[\phi(\cdot - y + \xi, \cdot + L)](y) \\ &= U[\phi(\cdot - y + \xi, \cdot)](y) \\ &= Q_+[\phi](\xi, y) \quad \text{for all } \xi \in \mathbb{R}, \ y \in \mathbb{R}. \end{aligned}$$

Thus, $Q_+[\phi](\xi, y)$ is L-periodic in y.

Step 2: We prove that $Q_+[\phi](\xi, y)$ is uniformly continuous in $(\xi, y) \in \mathbb{R}^2$. It suffices to show that $Q_+[\phi](\xi + y, y)$ is uniformly continuous in $(\xi, y) \in \mathbb{R}^2$. Note that

$$\left| Q_+[\phi](\xi_1 + y_1, y_1) - Q_+[\phi](\xi_2 + y_2, y_2) \right|$$
$$\leq \left| U[\phi(\xi_1 + \cdot, \cdot)](y_1) - U[\phi(\xi_1 + \cdot, \cdot)](y_2) \right| + \left| U[\phi(\xi_1 + \cdot, \cdot)](y_2) - U[\phi(\xi_2 + \cdot, \cdot)](y_2) \right|$$

for all $\xi_1, \xi_2, y_1, y_2 \in \mathbb{R}$.

We first claim that $\left\{ Q_+[\phi](\xi + y, y) : \xi \in \mathbb{R} \right\}$ is a family of equicontinuous functions of $y \in \mathbb{R}$. Indeed, for any fixed $\xi \in \mathbb{R}$, let $\left(u_+(t, y), h_+(t) \right)$ be the solution of equation (1.14) with initial value

$$u_+(0, y) = \phi(\xi + y, y) \text{ in } (-\infty, H_0(\xi)]. \tag{3.3}$$

It follows from [2, Remark 2.12] that the function $u_+(\omega, y)$ is Lipschitz continuous in $(-\infty, h_+(\omega)]$, and the Lipschitz constant depends only on ω and $\|\phi\|_{L^\infty(\mathbb{R}^2)}$ (and hence, it is independent of ξ). By the definition of Q_+, it is clear that $Q_+[\phi](\xi + y, y)$ is Lipschitz continuous in $y \in \mathbb{R}$ with the same Lipschitz constant. This in particular implies the equicontinuity of $\left\{ Q_+[\phi](\xi + y, y) : \xi \in \mathbb{R} \right\}$.

Therefore, for any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$\left| U[\phi(\xi_1 + \cdot, \cdot)](y_1) - U[\phi(\xi_1 + \cdot, \cdot)](y_2) \right| \leq \frac{\varepsilon}{2},$$

for all $|y_1 - y_2| \leq \delta_1$ and $\xi_1 \in \mathbb{R}$.

On the other hand, since $\phi(\xi, y)$ is uniformly continuous in $(\xi, y) \in \mathbb{R}^2$, it is easily seen that $|\phi(\xi + \xi_1, y) - \phi(\xi + \xi_2, y)|$ converges to 0 as $|\xi_1 - \xi_2| \to 0$, and this convergence is uniform with respect to $(\xi, y) \in \mathbb{R}^2$. Therefore, it follows from the (uniform) continuous dependence of the solution of the free boundary problem on its initial function that, for the above $\varepsilon > 0$, there exists $\delta_2 > 0$ such that

$$\left| U[\phi(\xi_1 + kL + \cdot, \cdot)](y_2) - U[\phi(\xi_2 + kL + \cdot, \cdot)](y_2) \right| \leq \frac{\varepsilon}{2}$$

for all $|\xi_1 - \xi_2| \leq \delta_2$, $y_2 \in [0, L]$ and $k \in \mathbb{Z}$. Furthermore, by the property (A2), we obtain

$$\left| U[\phi(\xi_1 + \cdot, \cdot)](y_2 + kL) - U[\phi(\xi_2 + \cdot, \cdot)](y_2 + kL) \right| \leq \frac{\varepsilon}{2}$$

for all $|\xi_1 - \xi_2| \leq \delta_2$, $y_2 \in [0, L]$ and $k \in \mathbb{Z}$. In particular, we have

$$\left| U[\phi(\xi_1 + \cdot, \cdot)](y_2) - U[\phi(\xi_2 + \cdot, \cdot)](y_2) \right| \leq \frac{\varepsilon}{2}$$

for all $|\xi_1 - \xi_2| \leq \delta_2$, $y_2 \in \mathbb{R}$.

Choosing $\delta = \min\{\delta_1, \delta_2\}$ and combining the above, we obtain

$$\left| Q_+[\phi](\xi_1 + y_1, y_1) - Q_+[\phi](\xi_2 + y_2, y_2) \right| \leq \varepsilon$$

for all $\xi_1, \xi_2, y_1, y_2 \in \mathbb{R}$ with $|y_1 - y_2| + |\xi_1 - \xi_2| \leq \delta$. This immediately gives the uniform continuity of $Q_+[\phi](\xi + y, y)$ in $(\xi, y) \in \mathbb{R}^2$. The proof of this step is thus complete.

Step 3: We prove that $Q_+[\phi](\xi, y)$ is nonincreasing in ξ. By (A1), we have

$$Q_+[\phi_1](\xi, y) \geq Q_+[\phi_2](\xi, y) \text{ for all } (\xi, y) \in \mathbb{R}^2,$$

whenever $\phi_1 \geq \phi_2$ in $\mathcal{M}$. This together with the property that $\phi(\xi, x)$ is nonincreasing in ξ implies that $Q_+[\phi](\xi, y)$ is also nonincreasing in ξ.

Step 4: We prove that for any $\xi \in \mathbb{R}$, there exists $H \in \mathbb{R}$ depending on ξ such that $Q_+[\phi](\xi + y, y) = 0$ if and only if $y \geq H$. As a matter of fact, for any given $\xi \in \mathbb{R}$, let $\left(u_+(t, y), h_+(t) \right)$ be the solution of equation (1.14) with initial value given as in (3.3). Set $H = h_+(\omega)$. Then by the definitions of Q_+ and U, it is easy to see that H is the desired critical number.

Step 5: We prove that the limit $Q_+[\phi](-\infty, y)$ exists and $0 < Q_+[\phi](-\infty, y) < p(0, y)$ for all $y \in \mathbb{R}$. To do this, we choose a sequence $(\xi_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ such that ξ_n is nonincreasing in n and that $\xi_n \to -\infty$ as $n \to \infty$. Due to the property (b) in (3.1), $\phi(-\infty, x)$ is continuous in x. Furthermore, since $\phi(\xi_n + x, x)$ is nondecreasing in n, and equi-continuous in x, $\phi(\xi_n + x, x)$ converges to $\phi(-\infty, x)$ locally uniformly in $x \in \mathbb{R}$ as $n \to \infty$. It then follows from the continuity of the operator U in the sense of (A3) that

$$U[\phi(\xi_n + \cdot, \cdot)](y) \to U[\phi(-\infty, \cdot)](y) \text{ as } n \to \infty \text{ locally uniformly in } y \in \mathbb{R},$$

that is,

$$Q_+[\phi](\xi_n + y, y) \to U[\phi(-\infty, \cdot)](y) \text{ as } n \to \infty \text{ locally uniformly in } y \in \mathbb{R}.$$

Since the nonincreasing sequence $(\xi_n)_{n \in \mathbb{N}}$ can be chosen arbitrarily, the limit $Q_+[\phi](-\infty, y)$ exists and

$$Q_+[\phi](-\infty, y) = U[\phi(-\infty, \cdot)](y).$$

Furthermore, by the property (e) in (3.1), it follows from the parabolic strong maximum principle that

$$0 < U[\phi(-\infty, \cdot)](y) < p(0, y) \text{ for all } y \in \mathbb{R}.$$

Thus, $Q_+[\phi](\xi, y)$ possesses the property stated in (e).

Therefore, the operator Q_+ maps $\mathcal{M}$ into itself. Lastly, the order-preserving property of Q_+ follows easily from that of U stated in (A1). The proof of Lemma 3.2 is thereby complete. $\square$

We now fix an arbitrary $\phi \in \mathcal{M}$ and, for any $c \in \mathbb{R}$, we define the sequence $\{(a_n^c, H_n^c)\}_{n \in \mathbb{N}}$ by the following recursions

$$a_n^c(\xi, x) = \max\left\{\phi(\xi, x), \, Q_+[a_{n-1}^c](\xi + c, x)\right\} \tag{3.4}$$

and

$$H_n^c(\xi) = \max\left\{H_0(\xi), \, h_+\left(\omega; a_{n-1}^c(\cdot + \xi + c, \cdot)\right)\right\} \tag{3.5}$$

where $a_0^c(\xi, x) = \phi(\xi, x)$ and $H_0(\xi)$ is the real number such that $\phi(\xi + x, x) = 0$ if and only if $x \geq H_0(\xi)$.

By Lemma 3.2, it is easily seen that for each $n \in \mathbb{N}$, $a_n^c \in \mathcal{M}$. Further properties are given below.

Lemma 3.3. *The following statements are valid:*

 (i) *For any fixed $n \in \mathbb{N}$, $0 \leq a_n^c(\xi, x) \leq p(0, x)$ for all $\xi \in \mathbb{R}$, $x \in \mathbb{R}$, and $a_n^c(\xi + x, x) = 0$ if and only if $x \geq H_n^c(\xi)$.*
 (ii) *The sequence $a_n^c(\xi, x)$ is nondecreasing in n, nonincreasing in ξ and c, L-periodic in x, and the sequence $H_n^c(\xi)$ is nondecreasing in n.*
 (iii) *For any fixed $n \in \mathbb{N}$, $a_n^c(\xi, x)$ is uniformly continuous in $(c, \xi, x) \in \mathbb{R}^3$ and $H_n^c(\xi)$ is uniformly continuous in $(c, \xi) \in \mathbb{R}^2$.*
 (iv) *$\left\{a_n^c(\xi + x, x) : \xi \in \mathbb{R}, \, c \in \mathbb{R}, \, n \in \mathbb{N}\right\}$ is a family of equicontinuous functions of $x \in \mathbb{R}$.*

Proof. We only give the proof for the statement (iv), since the others can be easily proved by an induction argument. For any fixed $\xi \in \mathbb{R}$ and $c \in \mathbb{R}$, let $\left(u_+(t, x), h_+(t)\right)$ be the solution of equation (1.14) with initial value

$$u_+(0, x) = \phi(x + \xi + c, x) \text{ in } (-\infty, H_0(\xi + c)].$$

By Step 2 in the proof of Lemma 3.2, $Q_+[\phi](x + \xi + c, x)$ is Lipschitz continuous in $x \in \mathbb{R}$ with the Lipschitz constant depending only on ω and $\|\phi\|_{L^\infty(\mathbb{R}^2)}$. In a similar way, one concludes that for any $n \in \mathbb{N}$, $Q_+[a_n^c](x + \xi + c, x)$ is Lipschitz continuous in $x \in \mathbb{R}$ with Lipschitz constant depending only on ω and $\|a_n^c\|_{L^\infty(\mathbb{R}^2)}$. Since the sequence $\{a_n^c\}_{n \in \mathbb{N}}$ is uniformly bounded, it follows that the family

$$\left\{Q_+[a_n^c](x + \xi + c, x) : \xi \in \mathbb{R}, \, c \in \mathbb{R}, \, n \in \mathbb{N}\right\}$$

is uniformly Lipschitz continuous in $x \in \mathbb{R}$. This, together with the uniform continuity of $\phi(\xi, x)$ in $(\xi, x) \in \mathbb{R}^2$, immediately implies the equicontinuity of the family $\left\{a_n^c(\xi + x, x) : \xi \in \mathbb{R}, \, c \in \mathbb{R}, \, n \in \mathbb{N}\right\}$. The proof of Lemma 3.3 is thereby complete. $\square$

Next, for any fixed $n \in \mathbb{N}$ and $c \in \mathbb{R}$, we consider the limits of $a_n^c(\xi, x)$ as $\xi \to \pm\infty$. By Lemma 3.1, for each fixed n and c, $a_n^c(\xi, \cdot) \equiv 0$ for all large ξ. Hence $a_n^c(+\infty, x) = 0$. The following lemma is concerned with

$$\alpha_n(x) = \alpha_n^c(x) := a_n^c(-\infty, x), \, x \in \mathbb{R}.$$

Lemma 3.4. *For each $n \in \mathbb{N}$, $\alpha_n(x)$ is L-periodic in x, nondecreasing in n, and $\alpha_n(x)$ converges to $p(0, x)$ as $n \to \infty$ uniformly in $x \in \mathbb{R}$.*

Proof. It is clear that $\alpha_n(x)$ is L-periodic in x and nondecreasing in n, since $a_n^c(\xi, x)$ has these properties. Next, we prove the convergence of $\alpha_n(x)$ as $n \to \infty$. Since the operator Q_+ is order preserving and is also translation invariant with respect to the variable ξ, one concludes by an induction argument that, for any $\phi \in \mathcal{M}$,

$$Q_+^n[\phi](\xi + nc, x) \le a_n^c(\xi, x) \le p(0, x) \quad \text{for all } \xi \in \mathbb{R}, \ x \in \mathbb{R}, \ n \in \mathbb{N}. \tag{3.6}$$

Now, for any fixed $n \in \mathbb{N}$, passing to the limit $\xi \to -\infty$ in the above inequality, we obtain

$$Q_+^n[\phi](-\infty, x) \le \alpha_n(x) \le p(0, x) \ \text{in } \mathbb{R}.$$

Furthermore, by the analysis in Step 5 of the proof of Lemma 3.2, there holds

$$Q_+^n[\phi](-\infty, x) = U^n[\phi(-\infty, \cdot)](x) \ \text{for each } n \in \mathbb{N}.$$

Since $\phi(-\infty, \cdot)$ is positive, L-periodic, it follows from the property (A4) that

$$\lim_{n \to \infty} U^n[\phi(-\infty, \cdot)](x) = p(0, x) \ \text{uniformly in } x \in \mathbb{R},$$

whence $\alpha_n(x)$ converges to $p(0, x)$ as $n \to \infty$ uniformly in $x \in \mathbb{R}$. The proof of Lemma 3.4 is thereby complete. $\square$

We now consider the limit function of $a_n^c(\xi, x)$ as $n \to \infty$.

Lemma 3.5. *The following statements are valid:*

(i) *For each fixed $c \in \mathbb{R}$ and $\xi \in \mathbb{R}$, there exist a function $a^c(\xi + \cdot, \cdot) \in C(\mathbb{R})$ and some $H^c(\xi) \in [H_0(\xi), +\infty]$ such that $\lim_{n \to \infty} H_n^c(\xi) = H^c(\xi)$ and*

$$\lim_{n \to \infty} a_n^c(\xi + x, x) = a^c(\xi + x, x) \ \text{locally uniformly in } x \in \mathbb{R}.$$

(ii) *For each fixed $c \in \mathbb{R}$ and $\xi \in \mathbb{R}$, if $H^c(\xi) < \infty$, then $a^c(\xi + x, x) = 0$ if and only if $x \ge H^c(\xi)$, and if $H^c(\xi) = \infty$, then $a^c(\xi + x, x) > 0$ for all $x \in \mathbb{R}$. Moreover,*

$$H^c(\xi + kL) = H^c(\xi) - kL \ \text{for all } k \in \mathbb{Z}. \tag{3.7}$$

(iii) *The function $a^c(\xi, x)$ is nonincreasing in $c \in \mathbb{R}$ and $\xi \in \mathbb{R}$, L-periodic in $x \in \mathbb{R}$, and $a^c(-\infty, x) \equiv p(0, x)$. Moreover, for any fixed $c \in \mathbb{R}$ and $\xi \in \mathbb{R}$,*

$$a^c(\xi + x, x) = \max\left\{\phi(\xi + x, x), \ U[a^c(\cdot + \xi + c, \cdot)](x)\right\} \ \text{for all } x \in \mathbb{R}. \tag{3.8}$$

Proof. (i) For any fixed $c \in \mathbb{R}$ and $\xi \in \mathbb{R}$, due to the monotonicity properties stated in Lemma 3.3 (ii), we may define

$$H^c(\xi) := \lim_{n \to \infty} H_n^c(\xi) \in [H_0(\xi), +\infty],$$

$$a^c(\xi, x) := \lim_{n \to \infty} a_n^c(\xi, x), \quad (\xi, x) \in \mathbb{R}^2.$$

Furthermore, by Lemma 3.3 (iv) and the Arzelà–Ascoli Theorem,

$$a_n^c(\xi + x, x) \to a^c(\xi + \cdot, \cdot) \ \text{as } n \to \infty \ \text{locally uniformly in } x \in \mathbb{R}.$$

This in particular implies that $a^c(\xi + \cdot, \cdot) \in C(\mathbb{R})$. We have thus proved all the conclusions in (i).

(ii) Fix $c \in \mathbb{R}$ and $\xi \in \mathbb{R}$. We first consider the case where $H^c(\xi) < \infty$. For any given $x < H^c(\xi)$, since $H_n^c(\xi)$ converges to $H^c(\xi)$ as $n \to \infty$, there exists some n_0 such that $x < H_n^c(\xi)$ for all $n \ge n_0$. Then by Lemma 3.3 (i), we have $a_n^c(\xi + x, x) > 0$ for all $n \ge n_0$. This together with the fact that $a_n^c(\xi + x, x)$ is nondecreasing in n implies that $a^c(\xi + x, x) > 0$.

On the other hand, for any given $x \geq H^c(\xi)$, since $H_n^c(\xi)$ is nondecreasing in n, there holds $x \geq H_n^c(\xi)$ for all $n \in \mathbb{N}$, whence $a_n^c(\xi + x, x) = 0$ by Lemma 3.3 (i) again. Therefore, $a^c(\xi + x, x) = 0$. Similarly, one concludes that if $H^c(\xi) = \infty$, then $a^c(\xi + x, x) > 0$ for all $x \in \mathbb{R}$.

We now show the equality (3.7). It suffices to prove that

$$H_n^c(\xi + kL) = H_n^c(\xi) - kL \text{ for all } k \in \mathbb{Z}, n \in \mathbb{N}.$$

By the definition of $H_n^c(\xi)$, $a_n^c(\xi + x, x) = 0$ if and only if $x \geq H_n^c(\xi)$. Since a_n^c is L-periodic in its second variable, it follows that

$$a_n^c(\xi + kL + x, x) = a_n^c(\xi + kL + x, kL + x),$$

and hence, $a_n^c(\xi + kL + x, x) = 0$ if and only if $x + kL \geq H_n^c(\xi)$. Thus, $H_n^c(\xi + kL) = H_n^c(\xi) - kL$ and (3.7) is proved.

(iii) Since for each fixed $n \in \mathbb{N}$, $a_n^c(\xi, x)$ is nonincreasing in $\xi \in \mathbb{R}$ and $c \in \mathbb{R}$, L-periodic in x, its limit $a^c(\xi, x)$ also possesses these properties. This in particular implies that the limits $a^c(\pm\infty, x)$ exist. Furthermore, letting $n \to \infty$ followed by sending $\xi \to -\infty$ in the first inequality of Lemma 3.3 (i), we obtain

$$0 \leq a^c(-\infty, x) \leq p(0, x) \text{ for } x \in \mathbb{R}.$$

On the other hand, since

$$a_n^c(-\infty, x) \leq a^c(-\infty, x) \text{ for all } n \in \mathbb{N}, x \in \mathbb{R},$$

and since $a_n^c(-\infty, x)$ converges to $p(0, x)$ uniformly in $x \in \mathbb{R}$ as $n \to \infty$ by Lemma 3.4, it follows that $a^c(-\infty, x) \equiv p(0, x)$. Finally, for any given $c \in \mathbb{R}$ and $\xi \in \mathbb{R}$, by (3.2) and (3.4), we have

$$a_{n+1}^c(\xi + x, x) = \max\left\{\phi(\xi + x, x), U[a_n^c(\cdot + \xi + c, \cdot)](x)\right\} \text{ for all } x \in \mathbb{R}, n \in \mathbb{N}.$$

Since $a_n^c(\xi + x, x)$ converges to $a^c(\xi + x, x)$ locally uniformly in $x \in \mathbb{R}$ and $H_n^c(\xi)$ converges to $H^c(\xi)$ as $n \to \infty$, it follows from the property (A3) that

$$U[a_n^c(\cdot + \xi + c, \cdot)](x) \to U[a^c(\cdot + \xi + c, \cdot)](x) \text{ locally uniformly in } x \in \mathbb{R} \text{ as } n \to \infty.$$

Then taking the limit $n \to \infty$ in the above equality, we arrive at (3.8). The proof of Lemma 3.5 is thereby complete. $\square$

By Lemma 3.3 (iv) and Lemma 3.5 (iii), it is easily seen that the limit $a^c(\infty, x)$ exists and it is continuous in $x \in \mathbb{R}$. The following two lemmas supply some crucial properties of $a^c(\infty, x)$, which are the key to obtain the spreading speed.

Lemma 3.6. *Either $a^c(\infty, x) \equiv 0$ or $a^c(\infty, x) \equiv p(0, x)$.*

Proof. Fix $c \in \mathbb{R}$. If there exists $\xi_0 \in \mathbb{R}$ such that $H^c(\xi_0) < \infty$, then it follows from Lemma 3.5 (ii) that $a^c(x + \xi_0, x) = 0$ if $x \geq H^c(\xi_0)$. This implies that, for any $x_0 \in \mathbb{R}$, there exists $k_0 \in \mathbb{N}$ large enough such that

$$a^c(\xi_0 + x_0 + kL, x_0 + kL) = 0 \text{ for all } k \geq k_0.$$

Since a^c is L-periodic in its second variable, it follows that

$$a^c(\xi_0 + x_0 + kL, x_0) = 0 \text{ for all } k \geq k_0.$$

Sending $k \to \infty$ yields $a^c(\infty, x_0) = 0$. Since x_0 is arbitrary, it follows that $a^c(\infty, \cdot) \equiv 0$.

Otherwise $H^c(\xi) = \infty$ for every $\xi \in \mathbb{R}$. Letting $\xi \to +\infty$ in (3.8), by the continuity property (A3), we obtain

$$a^c(\infty, x) = \max\left\{\phi(+\infty, x), U[a^c(\infty, \cdot)](x)\right\} = U[a^c(\infty, \cdot)](x) \text{ for } x \in \mathbb{R},$$

since $\phi(+\infty, x) = 0$. Thus $a^c(\infty, x)$ is an equilibrium of the operator U in $\mathcal{C}$. Due to the property (A4), it follows that either $a^c(\infty, x) \equiv p(0, x)$ or $a^c(\infty, x) \equiv 0$.

Consequently, either $a^c(\infty, x) \equiv p(0, x)$ and $H^c(\xi) \equiv \infty$, or $a^c(\infty, x) \equiv 0$. $\square$

Lemma 3.7. *Let H_2 be a real number such that $\phi(\xi, \cdot) \equiv 0$ for $\xi \geq H_2$ (whose existence is given in Lemma 3.1). Then $a^c(\infty, x) \equiv p(0, x)$ if and only if there is some $n_0 \in \mathbb{N}$ such that*

$$a_{n_0}^c(H_2, x) > \phi(-\infty, x) \ \text{for all } x \in \mathbb{R}.$$

Proof. If $a^c(\infty, x) = p(0, x)$, then by the monotonicity of $a^c(\xi, x)$ in ξ and $a^c(\xi, x) \leq p(0, x)$, we have $a^c(\xi, x) \equiv p(0, x)$ and hence

$$a^c(\xi + x, x) = p(0, x) \ \text{for all } \xi \in \mathbb{R}, \ x \in \mathbb{R}.$$

By choosing $\xi = H_2$, it follows from Lemma 3.5 (i) that $a_n^c(H_2 + x, x)$ converges to $p(0, x)$ locally uniformly in $x \in \mathbb{R}$ as $n \to \infty$. This together with the assumption that $p(0, x) > \phi(-\infty, x)$ for all $x \in \mathbb{R}$, implies that there is some $n_0 \in \mathbb{R}$ such that

$$a_{n_0}^c(H_2 + x, x) > \phi(-\infty, x) \ \text{for all } x \in [0, L].$$

Since $a_{n_0}^c(\xi, x)$ is nonincreasing in ξ, we obtain $a_{n_0}^c(H_2, x) > \phi(-\infty, x)$ for all $x \in [0, L]$. Furthermore, since both $a_{n_0}^c(H_2, x)$ and $\phi(-\infty, x)$ are L-periodic in x, it follows that $a_{n_0}^c(H_2, x) > \phi(-\infty, x)$ for all $x \in \mathbb{R}$.

Conversely, suppose that there is some $n_0 \in \mathbb{N}$ such that $a_{n_0}^c(H_2, x) > \phi(-\infty, x)$ for all $x \in \mathbb{R}$. Since both $a_{n_0}^c(\xi, x)$ and $\phi(-\infty, x)$ are L-periodic in x, and $a_{n_0}^c(\xi, x)$ is uniformly continuous in $(\xi, x) \in \mathbb{R}^2$, there exists a positive constant $\delta > 0$ such that

$$a_{n_0}^c(H_2 + \delta, x) > \phi(-\infty, x) \ \text{for all } x \in \mathbb{R}.$$

Furthermore, since $a_{n_0}^c(\xi, x)$ and $\phi(\xi, x)$ are nonincreasing in ξ and since $\phi(\xi, x) = 0$ for all $\xi \geq H_2$, it follows that

$$a_{n_0}^c(\delta + \xi, x) \geq \phi(\xi, x) \ \text{for all } (\xi, x) \in \mathbb{R}^2.$$

Now we claim that

$$a_{n_0+k}^c(\delta + \xi, x) \geq a_k^c(\xi, x) \quad \text{for all } (\xi, x) \in \mathbb{R}^2, \ k \in \mathbb{N}. \tag{3.9}$$

We already know that the above inequality holds for $k = 0$. Suppose that (3.9) holds for some integer $k = k_0 \geq 0$. Then

$$\begin{aligned}
a_{n_0+k_0+1}^c(\xi + \delta, x) &= \max\left\{\phi(\xi + \delta, x), \ Q_+[a_{n_0+k_0}^c](\xi + \delta + c, x)\right\} \\
&\geq Q_+[a_{n_0+k_0}^c](\xi + \delta + c, x) \\
&\geq Q_+[a_{k_0}^c](\xi + c, x) \ \text{for all } (\xi, x) \in \mathbb{R}^2.
\end{aligned}$$

We also have

$$a_{n_0+k_0+1}^c(\xi + \delta, x) \geq a_{n_0}^c(\xi + \delta, x) \geq \phi(\xi, x) \ \text{for } (\xi, x) \in \mathbb{R}^2.$$

Therefore, for $(\xi, x) \in \mathbb{R}^2$,

$$a_{n_0+k_0+1}^c(\xi + \delta, x) \geq \max\left\{\phi(\xi, x), \ Q_+[a_{k_0}^c](\xi + c, x)\right\} = a_{k_0+1}^c(\xi, x).$$

This implies that (3.9) holds for all $k \in \mathbb{N}$.

Passing to the limit $k \to \infty$ in (3.9) gives

$$a^c(\xi + \delta, x) \geq a^c(\xi, x) \ \text{for all } (\xi, x) \in \mathbb{R}^2.$$

This together with the fact that $a^c(\xi, x)$ is nonincreasing in ξ implies that $a^c(\xi, x)$ is independent of ξ. Furthermore, since $a^c(-\infty, x) \equiv p(0, x)$ by Lemma 3.5 (iii), it follows that $a^c(\infty, x) \equiv p(0, x)$. The proof of Lemma 3.7 is thereby complete. $\square$

Define

$$c_+ = \sup\left\{c \in \mathbb{R} : a^c(\infty, x) \equiv p(0, x)\right\}. \tag{3.10}$$

If there does not exist $c \in \mathbb{R}$ such that $a^c(\infty, x) \equiv p(0, x)$, then we define $c_+ = -\infty$.

Lemma 3.8. $c_+ > -\infty$, and $a^c(\infty, x) \equiv p(0, x)$ if $c < c_+$, $a^c(\infty, x) \equiv 0$ if $c \geq c_+$.

Proof. We first prove that $c_+ > -\infty$. It follows from the property (A4) that $U^n[\phi(-\infty, \cdot)](x)$ converges to $p(0, x)$ as $n \to \infty$ uniformly in $x \in \mathbb{R}$, and hence, for any small positive constant ε, there exists some $n_0 \in \mathbb{N}$ large enough such that

$$U^{n_0}[\phi(-\infty, \cdot)](x) - \varepsilon > \phi(-\infty, x) \text{ for all } x \in \mathbb{R}.$$

On the other hand, by (3.6), we have

$$a_{n_0}^c(H_2, x) \geq Q_+^{n_0}[\phi](H_2 + n_0 c, x) \text{ for all } c \in \mathbb{R}, \ x \in \mathbb{R},$$

with H_2 given in Lemma 3.1 such that $\phi(\xi, \cdot) \equiv 0$ for $\xi \geq H_2$. Furthermore, since $Q_+^{n_0}[\phi](-\infty, x) \equiv U^{n_0}[\phi(-\infty, \cdot)](x)$ (whose proof is the same as that in Step 5 of the proof of Lemma 3.2), there exists $c_0 < 0$ large negative such that

$$Q_+^{n_0}[\phi](H_2 + n_0 c_0, x) \geq U^{n_0}[\phi(-\infty, \cdot)](x) - \varepsilon \text{ for all } x \in \mathbb{R}.$$

Here we have implicitly used the fact that these two functions of x are L-periodic. Combining the above inequalities, we obtain

$$a_{n_0}^{c_0}(H_2, x) > \phi(-\infty, x) \text{ for all } x \in \mathbb{R}.$$

It then follows from Lemma 3.7 that $a^{c_0}(\infty, x) \equiv p(0, x)$, and hence, $c_+ \geq c_0 > -\infty$.

Next, since $a^c(\infty, x)$ is nonincreasing in c, it follows readily from Lemma 3.6 that $a^c(\infty, x) \equiv p(0, x)$ if $c < c_+$. By the definition of c_+, we clearly have $a^c(\infty, x) \equiv 0$ for $c > c_+$ when $c_+ < +\infty$. (We will prove later that $c_+ < +\infty$ always holds; see Proposition 3.13 below.)

To complete the proof, it remains to show that $a^{c_+}(\infty, x) \equiv 0$ when $c_+ < +\infty$. Assume by contradiction that $a^{c_+}(\infty, x) \equiv p(0, x)$. Then by Lemma 3.7 again, there exists some $n_1 \in \mathbb{N}$ such that

$$a_{n_1}^{c_+}(H_2, x) > \phi(-\infty, x) \text{ for all } x \in \mathbb{R}.$$

By the continuity of the function $a_{n_1}^c(H_2, x)$ with respect to c, and the fact that it is periodic in x, it follows that $a_{n_1}^c(H_2, x) > \phi(-\infty, x)$ for c in a neighborhood of c_+, whence $a^c(\infty, x) = p(0, x)$, which is in contradiction to the definition of c_+. The proof of Lemma 3.8 is now complete. $\quad\square$

The following lemma shows that c_+ is independent of the choice of the function ϕ.

Lemma 3.9. *Let* $\left\{\left(\widetilde{a}_n^c(\xi, x), \widetilde{H}_n^c(\xi)\right)\right\}_{n \in \mathbb{N}}$ *be the sequence obtained from the recursions* (3.4) *and* (3.5) *when* ϕ *is replaced by another function* $\widetilde{\phi} \in \mathcal{M}$ *with* $\widetilde{H}_0 := \widetilde{H}_0(\xi)$ *such that* $\widetilde{\phi}(\xi + x, x) = 0$ *if and only if* $x \geq \widetilde{H}_0$. *Then the limit* $\left\{\left(\widetilde{a}^c(\xi, x), \widetilde{H}^c(\xi)\right)\right\}$ *of* $\left\{\left(\widetilde{a}_n^c(\xi, x), \widetilde{H}_n^c(\xi)\right)\right\}$ *as* $n \to \infty$ *satisfies* $\widetilde{a}^c(\infty, x) = a^c(\infty, x)$ *and, for every* $\xi \in \mathbb{R}$, *either* $\widetilde{H}^c(\xi)$ *and* $H^c(\xi)$ *are both finite or they are both infinite.*

Proof. By the same proof as that in Lemma 3.4, we see that $\widetilde{a}_n^c(-\infty, x)$ converges to $p(0, x)$ as $n \to \infty$ uniformly in $x \in \mathbb{R}$. Since $\phi(-\infty, x) < p(0, x)$ and these functions are L-periodic in x, there exists $n_0 \in \mathbb{N}$ such that $\widetilde{a}_{n_0}^c(-\infty, x) > \phi(-\infty, x)$ for all $x \in \mathbb{R}$. This implies that

$$\widetilde{a}_{n_0}^c(\widehat{H}, x) > \phi(-\infty, x) \text{ for } x \in \mathbb{R} \text{ and all large negative } \widehat{H}.$$

We fix such a $\widehat{H}$ with the additional property that $\widehat{H} - H_2 \in L\mathbb{Z}$, where H_2 is given in Lemma 3.1 such that $\phi(\xi, \cdot) \equiv 0$ if $\xi \geq H_2$. Since $\widetilde{a}_{n_0}^c(\xi, x)$ and $\phi(\xi, x)$ are both nonincreasing in $\xi \in \mathbb{R}$, it follows that

$$\widetilde{a}_{n_0}^c(\xi + \widehat{H} - H_2, x) \geq \phi(\xi, x) \text{ for all } \xi \in \mathbb{R}, \ x \in \mathbb{R}.$$

Then by an induction argument similar to that used in the proof of Lemma 3.7, we obtain

$$\widetilde{a}_{n_0+k}^c(\xi + \widehat{H} - H_2, x) > a_k^c(\xi, x) \text{ for all } \xi \in \mathbb{R}, \ x \in \mathbb{R}, \ k \in \mathbb{N}.$$

Taking the limit as $k \to \infty$ in the above inequality yields that

$$\widetilde{a}^c(\xi + \widehat{H} - H_2, x) \geq a^c(\xi, x) \text{ for all } (\xi, x) \in \mathbb{R}^2,$$

whence

$$\widetilde{a}^c(\infty, x) \geq a^c(\infty, x) \text{ for all } x \in \mathbb{R}, \quad \text{and} \quad \widetilde{H}^c(\xi) \geq H^c(\xi - \widehat{H} + H_2) \text{ for all } \xi \in \mathbb{R}.$$

Furthermore, since $\widehat{H} - H_2 \in L\mathbb{Z}$, it follows from (3.7) that

$$\widetilde{H}^c(\xi) \geq H^c(\xi) + \widehat{H} - H_2 \text{ for all } \xi \in \mathbb{R}.$$

In a similar way, by reversing the roles of $\left(\widetilde{a}_n^c(\xi, x), \widetilde{H}_n^c(\xi)\right)$ and $\left(a_n^c(\xi, x), H_n^c(\xi)\right)$ in the above arguments, we obtain two real numbers $\bar{H}$ and $\widetilde{H}_2$ ($\widetilde{H}_2$ is a constant such that $\widetilde{\phi}(\xi, \cdot) \equiv 0$ if $\xi \geq \widetilde{H}_2$) with $\bar{H} - \widetilde{H}_2 \in L\mathbb{Z}$ such that

$$a^c(\xi + \bar{H} - \widetilde{H}_2, x) \geq \widetilde{a}^c(\xi, x) \text{ for all } (\xi, x) \in \mathbb{R}^2,$$

and hence

$$a^c(\infty, x) \geq \widetilde{a}^c(\infty, x) \text{ for all } x \in \mathbb{R}, \quad \text{and} \quad H^c(\xi) \geq \widetilde{H}^c(\xi) + \bar{H} - \widetilde{H}_2 \text{ for all } \xi \in \mathbb{R}.$$

The proof of Lemma 3.9 is now complete. $\quad\square$

Summarizing the above results we immediately obtain

Proposition 3.10. *Let c_+ be given in* (3.10). *Then $c_+ > -\infty$ and is independent of the choice of $\phi \in \mathcal{M}$ in the recursion* (3.4) *leading to $a^c(\infty, x)$.*

Notice that if ξ is replaced by $x + \xi - (n+1)c$ in (3.4), then

$$\left.\begin{array}{l} a_{n+1}^c\big(x + \xi - (n+1)c, x\big) \\[4pt] = \max\left\{\phi\big(x + \xi - (n+1)c, x\big), Q_+\big[a_n^c\big]\big(x + \xi - nc, x\big)\right\} \\[4pt] \geq Q_+\big[a_n^c\big]\big(x + \xi - nc, x\big) \\[4pt] = U\big[a_n^c(\cdot + \xi - nc, \cdot)\big](x) \text{ for all } (\xi, x) \in \mathbb{R}^2. \end{array}\right\} \tag{3.11}$$

We will make use of this observation to prove that c_+ is the rightward spreading speed for the recursion

$$u_{n+1} = U[u_n], \ u_0 \in \mathcal{C}, \ n = 0, 1, 2, \dots.$$

Proposition 3.11. *Let c_+ be given in* (3.10). *Suppose $u_0 \in \mathcal{C}$ has left supporting point $g_0 = -\infty$ and right supporting point $h_0 < \infty$, and*

$$\liminf_{x \to -\infty} \big(p(0, x) - u_0(x)\big) > 0. \tag{3.12}$$

If for every $C \in \mathbb{R}$,

$$\lim_{n \to \infty} \big|U^n[u_0](x) - p(0, x)\big| = 0 \text{ uniformly in } x \in (-\infty, C], \tag{3.13}$$

then $c_+ > 0$ and

$$\lim_{n \to \infty} \sup_{x \geq c_1 n} U^n[u_0](x) = 0 \quad \text{for any } c_1 > c_+, \tag{3.14}$$

$$\lim_{n \to \infty} \sup_{x \leq c_2 n} \big|U^n[u_0](x) - p(0, x)\big| = 0 \quad \text{for any } c_2 < c_+. \tag{3.15}$$

Moreover,

$$\liminf_{n \to \infty} \frac{h_+\big(n\omega; u_0\big)}{n} \geq c_+. \tag{3.16}$$

Proof. Without loss of generality, we assume that the right supporting point of u_0 is $h_0 = 0$. By replacing u_0 by $U[u_0]$ if necessary, we can also assume without loss of generality that, u_0 satisfies the assumptions (3.12), (3.13) and that

$$u_0(x) < p(0, x)(1 - \varepsilon) \text{ for all } x \in \mathbb{R} \text{ for some } \varepsilon > 0.$$

Then we choose some continuous function $l : \mathbb{R} \to [0, 1 - \varepsilon/2)$ such that $l(x)$ is strictly decreasing in $x \in \mathbb{R}$, that $l(-\infty) = 1 - \varepsilon/2, l(0) = 1 - \varepsilon, l(1) = 0$, and that

$$l(x)p(0, x) \geq u_0(x) \text{ for all } x \in \mathbb{R}. \tag{3.17}$$

Set

$$\phi(\xi, x) := l(\xi)p(0, x) \text{ for } (\xi, x) \in \mathbb{R}^2.$$

It is easy to see that $\phi \in \mathcal{M}$. In what follows, we will make use of the recursions (3.4) and (3.5) starting from ϕ to prove all the conclusions.

We first show $c_+ \in (0, +\infty]$ (we will show in Proposition 3.13 that $c_+ < +\infty$). Let $\{(a_n^0(\xi, x), H_n^0(\xi))\}_{n \in \mathbb{N}}$ be the sequence obtained from the recursions (3.4) and (3.5) with $c = 0$. By choosing $\xi = 0$ in (3.11), we have

$$a_{n+1}^0(x, x) \geq U[a_n^0(\cdot, \cdot)](x) \text{ for all } x \in \mathbb{R}.$$

Then the comparison principle Proposition 2.3 together with (3.17) implies that

$$a_n^0(x, x) \geq U^n[u_0](x) \text{ for all } x \in \mathbb{R}.$$

Furthermore, due to the assumption (3.13), we see that

$$\lim_{n \to \infty} a_n^0(x, x) = p(0, x) \text{ locally uniformly in } x \in \mathbb{R},$$

whence $a^0(x, x) \equiv p(0, x)$ by Lemma 3.5 (i). Since the functions $a^0(\xi, x)$ and $p(0, x)$ are both L-periodic in $x \in \mathbb{R}$, we obtain

$$a^0(\infty, x) = \lim_{k \to \infty} a^0(x + kL, x) = \lim_{k \to \infty} a^0(x + kL, x + kL) = p(0, x).$$

It then follows from Lemma 3.8 that $c_+ > 0$.

Next, we prove the convergence property stated in (3.14). If $c_+ = +\infty$ then there is nothing to prove. So we assume $c_+ < +\infty$.

Let $\{(a_n^{c_+}(\xi, x), H_n^{c_+}(\xi))\}_{n \in \mathbb{N}}$ be the sequence obtained from the recursions (3.4) and (3.5) with $c = c_+$. By choosing $\xi = 0$ and $c = c_+$ in (3.11), we have

$$a_{n+1}^{c_+}(x - (n + 1)c_+, x) \geq U[a_n^{c_+}(\cdot - nc_+, \cdot)](x) \text{ for all } x \in \mathbb{R}, n \in \mathbb{N}.$$

It then follows from the comparison principle Proposition 2.3 and (3.17) that

$$a_n^{c_+}(x - nc_+, x) \geq U^n[u_0](x) \text{ for all } x \in \mathbb{R}.$$

Furthermore, for any $c_1 > c_+$, suppose $\{x_k\} \subset [c_1 n, \infty)$ satisfies

$$\lim_{k \to \infty} U^n[u_0](x_k) = \sup_{x \geq c_1 n} U^n[u_0](x).$$

Then since $a_n^{c_+}(\xi, x)$ is nonincreasing in $\xi \in \mathbb{R}$ and nondecreasing in $n \in \mathbb{N}$, we have

$$
\begin{aligned}
\sup_{y \in \mathbb{R}} a^{c_+}(nc_1 - nc_+, y) &\geq a^{c_+}(nc_1 - nc_+, x_k) \\
&\geq a^{c_+}(x_k - nc_+, x_k) \\
&\geq a_n^{c_+}(x_k - nc_+, x_k) \\
&\geq U^n[u_0](x_k).
\end{aligned}
$$

Letting $k \to \infty$, we obtain

$$\sup_{y \in \mathbb{R}} a^{c_+}(nc_1 - nc_+, y) \geq \sup_{x \geq c_1 n} U^n[u_0](x) \geq 0 \text{ for all } n \in \mathbb{N}. \tag{3.18}$$

Since $a^{c_+}(\infty, x) \equiv 0$ by Lemma 3.8 and since $a^{c_+}(nc_1 - nc_+, x)$ converges to $a^{c_+}(\infty, x)$ as $n \to \infty$ uniformly in $x \in \mathbb{R}$, (3.14) follows by letting $n \to \infty$ in (3.18).

We now prove (3.15) and (3.16). Fix $c_2 < c_+$ and let $\{(a_n^{c_2}(\xi, x), H_n^{c_2}(\xi))\}_{n \in \mathbb{N}}$ be the sequence obtained from the recursions (3.4) and (3.5) with $c = c_2$. We claim that there exists $n_0 \geq 0$ such that

$$a_{n+1}^{c_2}(x + \xi - c_2, x) = U[a_n^{c_2}(\cdot + \xi, \cdot)](x) \text{ for all } x \in \mathbb{R}, \xi \in \mathbb{R}, n \geq n_0. \tag{3.19}$$

Since $c_2 < c_+$, it follows from Lemma 3.8 that $a^{c_2}(\infty, x) \equiv p(0, x)$. By Lemma 3.7, there exists some $n_0 \geq 0$ such that $a_{n_0}^{c_2}(H_2, x) > \phi(-\infty, x)$ for all $x \in \mathbb{R}$, where H_2 is a real number such that $\phi(\xi, x) = 0$ for all $\xi \geq H_2$ (with our choice of ϕ, we may take $H_2 = 1$). Then we easily see $a_{n_0}^{c_2}(\xi, x) > \phi(\xi, x)$ for all $\xi \leq H_2$, $x \in \mathbb{R}$, and hence

$$a_n^{c_2}(\xi, x) \geq a_{n_0}^{c_2}(\xi, x) > \phi(\xi, x) \text{ for all } \xi \in \mathbb{R}, x \in \mathbb{R} \text{ and } n \geq n_0.$$

Thus, by the definition of $a_n^{c_2}$, we immediately obtain (3.19).

By our choice of ϕ, we can prove by an induction argument that

$$a_n^{c_2}(x, x) \leq \max_{0 \leq k \leq n} U^k[(1 - \varepsilon/2)p(0, \cdot)](x) \text{ for all } x \in \mathbb{R}, n \in \mathbb{N}.$$

Since $U^k[(1 - \varepsilon/2)p(0, \cdot)](x) < p(0, x)$ for all $x \in \mathbb{R}$, $k \in \mathbb{N}$ by the strong parabolic maximum principle, and these two functions are L-periodic in x, there exists $\varepsilon_n > 0$ such that

$$\max_{0 \leq k \leq n} U^k[(1 - \varepsilon/2)p(0, \cdot)](x) \leq p(0, x) - \varepsilon_n \text{ for } x \in \mathbb{R} \text{ and } n \in \mathbb{N}.$$

It follows that

$$a_{n_0}^{c_2}(x, x) \leq p(0, x) - \varepsilon_{n_0} \text{ for all } x \in \mathbb{R}.$$

Hence, by the assumption (3.13) and the fact that $a_{n_0}^{c_2}(x, x) = 0$ for all $x \geq H_{n_0}^{c_2}(0)$, we have

$$U^{n_1}[u_0](x) \geq a_{n_0}^{c_2}(x, x) \text{ for all } x \in \mathbb{R} \text{ and large integer } n_1.$$

This together with (3.19) and the comparison principle Proposition 2.3 implies that

$$U^{n_1+n}[u_0](x) \geq U^n[a_{n_0}^{c_2}(\cdot, \cdot)](x) = a_{n+n_0}^{c_2}(x - nc_2, x) \text{ for all } x \in \mathbb{R}, n \in \mathbb{N},$$

which yields

$$U^n[u_0](x) \geq a_{n-n_1+n_0}^{c_2}(x - (n - n_1)c_2, x) \text{ for all } x \in \mathbb{R}, n \geq n_1. \tag{3.20}$$

Since $a_n^{c_2}(\xi, x)$ is nonincreasing in $\xi \in \mathbb{R}$, it then follows that

$$\inf_{x \leq c_2 n} U^n[u_0](x) \geq \inf_{y \in \mathbb{R}} a_{n-n_1+n_0}^{c_2}(n_1 c_2, y) \text{ for } n \geq n_1 + 1.$$

Since $a^{c_2}(\infty, x) \equiv p(0, x)$ by Lemma 3.8, we have $a^{c_2}(\xi, x) \equiv a^{c_2}(\infty, x)$ by the monotonicity of $a^{c_2}(\xi, x)$ in ξ. Therefore, letting $n \to \infty$ in the above inequality, we deduce (3.15).

Moreover, it follows from (3.20) that

$$U^n[u_0](x) \geq a_{n-n_1+n_0}^{c_2}(x - m_n L, x) \text{ for all } x \in \mathbb{R}, n \geq n_1,$$

where m_n is the positive integer such that

$$m_n L \leq (n - n_1)c_2 < (m_n + 1)L.$$

It follows that

$$h_+(n\omega; u_0) \geq H_{n-n_1+n_0}^{c_2}(-m_n L) = H_{n-n_1+n_0}^{c_2}(0) + m_n L \text{ for all } n \geq n_1.$$

Thus,

$$\frac{h_+(n\omega; u_0)}{n} \geq \frac{H_{n-n_1+n_0}^{c_2}(0)}{n} + \frac{m_n L}{n}.$$

Since $H_{n-n_1+n_0}^{c_2}(0) > 0$ (which follows from the monotonicity of $H_n^{c_2}(0)$ in n and our choice of ϕ), passing to the limit $n \to \infty$ yields

$$\liminf_{n \to \infty} \frac{h_+(n\omega; u_0)}{n} \geq c_2.$$

Since $c_2 < c_+$ is arbitrary, this implies (3.16). The proof of Proposition 3.11 is thereby complete. □

Remark 3.12. It is easy to find sufficient conditions for (3.13) to hold. For example, if the nonlinearity f is of type (1.11), then (3.13) holds for any $u_0 \in \mathcal{C}$ with left supporting point $g_0 = -\infty$, right supporting point $h_0 < \infty$ and $\liminf_{x \to -\infty} u_0(x) > 0$. Indeed, in this case, the spreading-vanishing dichotomy in [2, Theorem 1.2] infers that

$$\lim_{t \to \infty} h_+(t; u_0) = +\infty \text{ and } \lim_{t \to \infty} u_+(t, x; u_0) = p(t, x) \text{ locally uniformly in } x \in \mathbb{R}.$$

This in particular implies that, for any $C \in \mathbb{R}$, $U^n[u_0](x)$ converges to $p(0, x)$ uniformly in $x \in [C - L, C]$ as $n \to \infty$. By the spatial L-periodicity assumption in (1.4),

$$U^n[u_0](x - kL) = U^n[u_0(\cdot - kL)](x) \text{ for } x \in \mathbb{R}, \; k \in \mathbb{Z}, \; n \in \mathbb{N}.$$

By replacing u_0 with some $\widetilde{u}_0 \in \mathcal{C}$ with left supporting point $\widetilde{g}_0 = -\infty$ such that $\widetilde{u}_0(x)$ is nonincreasing in x, $\liminf_{x \to -\infty} \widetilde{u}_0(x) > 0$ and $\widetilde{u}_0(x) \leq u_0(x)$ for $x \in \mathbb{R}$ if necessary, one can assume without loss of generality that, $u_0(x)$ is nonincreasing in $x \in \mathbb{R}$. It then follows from the order-preserving property of U in (A1) that

$$U^n[u_0](x - kL) \geq U^n[u_0](x) \text{ for } x \in [C - L, C], \; k \in \mathbb{N}, \; n \in \mathbb{N}.$$

Therefore, $U^n[u_0](x)$ converges to $p(0, x)$ uniformly in $x \leq C$.

Proposition 3.13. *Let c_+ be given in (3.10). Then $c_+ < +\infty$.*

Proof. By the assumptions on f, there exists $K > 0$ such that

$$f(t, x, u) \leq Ku \text{ for } (t, x) \in \mathbb{R}^2, u \in [0, M]; \; f(t, x, u) \leq 0 \text{ for } u \geq M, \; (t, x) \in \mathbb{R}^2.$$

It follows that

$$f(t, x, u) \leq F(u) := \frac{K}{M} u(2M - u) \text{ for } (t, x) \in \mathbb{R}^2, \; u \in [0, 2M].$$

By [4], there exist $c^* > 0$ and $\Phi(\xi) \in C^2([0, \infty))$ satisfying

$$\begin{cases} d\Phi_{xx} - c^*\Phi_x + F(\Phi) = 0, \; 0 < \Phi(x) < 2M & \text{for } x \in (0, \infty), \\ \Phi(0) = 0, \quad \mu\Phi'(0) = c^*, \quad \Phi(\infty) = 2M. \end{cases}$$

Let

$$\Psi(t, x) := \Phi(c^*t - x + C), \quad H(t) := c^*t + C$$

with $C > 0$ to be determined later. Then

$$\begin{cases} \Psi_t = d\Psi_{xx} + F(\Psi) \geq d\Psi_{xx} + f(t, x, \Psi) & \text{for } x < H(t), \; t > 0, \\ \Psi(t, H(t)) = 0, \; H'(t) = -\mu\Psi_x(t, c^*t + C) & \text{for } t > 0, \\ \Psi(0, x) = \Phi(-x + C) & \text{for } x \leq H(0) = C. \end{cases}$$

Since $f(t, x, u) \leq 0$ for $u \geq M$, a simple comparison argument shows that $p(0, x) \leq M$ for all $x \in \mathbb{R}$. Let u_0 be chosen as in Proposition 3.11. Then

$$u_0(x) < M \text{ for } x < h_0; \; u_0(x) = 0 \text{ for } x \geq h_0.$$

Therefore we can fix $C > 0$ large enough so that

$$\Psi(0, x) = \Phi(-x + C) > M > u_0(x) \text{ for } x \leq h_0.$$

The comparison principle then yields

$$h_+(t; u_0) \leq H(t) = c^* t + C, \ u_+(t, x; u_0) \leq \Psi(t, x) \text{ for } x \leq h_+(t; u_0), \ t > 0.$$

It then follows that

$$\limsup_{t \to \infty} \frac{h_+(t; u_0)}{t} \leq c^*.$$

In particular,

$$\limsup_{n \to \infty} \frac{h_+(n\omega; u_0)}{n} \leq c^* \omega.$$

In view of (3.16), we deduce $c_+ \leq c^* \omega < +\infty$. $\quad \square$

We are now ready to prove the existence of spreading speed for the problem (1.14).

Theorem 3.14. *Let $c_+^* = c_+/\omega$ where c_+ is given in (3.10). If there exists $u_0 \in \mathcal{H}_+(h_0)$ such that $u_0(x) \leq p(0, x)$ for all $x \in (-\infty, h_0]$ and u_0 satisfies (3.12) and (3.13), then*

$$\lim_{t \to \infty} \sup_{x \leq ct} \left| u_+(t, x; u_0) - p(t, x) \right| = 0 \quad \text{for any } c < c_+^*, \tag{3.21}$$

and

$$\lim_{t \to \infty} \frac{h_+(t; u_0)}{t} = c_+^*. \tag{3.22}$$

Proof. In what follows, for any $t \geq 0$, we extend the function $u_+(t, x; u_0)$ to the whole real line $\mathbb{R}$ by defining $u_+(t, x; u_0) = 0$ for all $x > h_+(t; u_0)$. By a slight abuse of notation, we still use $u_+(t, x; u_0)$ to denote the extended function.

For each $n \geq 1$, we set

$$h_n := h_+(n\omega; u_0) \quad \text{and} \quad u_n(x) := u_+(n\omega, x; u_0) = U^n[u_0](x) \text{ for all } x \in \mathbb{R}.$$

Clearly

$$u_+(t + n\omega, x; u_0) = u_+(t, x; u_n) \text{ and } h_+(t + n\omega; u_0) = h_+(t; u_n) \tag{3.23}$$

for all $n \in \mathbb{N}$, $x \in \mathbb{R}$ and $t \geq 0$. Furthermore, by the assumption (3.13), we find that, as $n \to \infty$, $h_n \to \infty$ and for any given $C \in \mathbb{R}$, $u_n(x)$ converges to $p(0, x)$ uniformly in $x \in (-\infty, C]$.

Now we prove (3.21). It suffices to show that, for any $c < c_+^*$ and any $C \in \mathbb{R}$,

$$\lim_{t \to \infty} \left| u_+(t, x + ct; u_0) - p(t, x + ct) \right| = 0 \text{ uniformly in } x \leq C. \tag{3.24}$$

Without loss of generality, we can assume that $u_0(x)$ is nonincreasing in $x \in (-\infty, h_0]$. Indeed, we could first prove (3.24) with u_0 replaced by some nonincreasing $\widetilde{u}_0 \in \mathcal{H}_+(h_0)$ satisfying (3.12) and (3.13) such that $\widetilde{u}_0 \geq u_0$ in $(-\infty, h_0]$. Then since (3.13) implies $U^{n_0}[u_0](x) \geq \widetilde{u}_0(x)$ in $\mathbb{R}$ for some large integer n_0, the comparison principle [2, Proposition 2.14] gives

$$\begin{aligned}
u_+(t, x + ct; \widetilde{u}_0) &\leq u_+(t, x + ct; U^{n_0}[u_0]) \\
&= u_+(t + n_0\omega, x + ct; u_0) \\
&\leq p(t + n_0\omega, x + ct) \\
&= p(t, x + ct) \text{ for all } t > 0, \ x \in \mathbb{R}.
\end{aligned}$$

This together with (3.24) holding for $\widetilde{u}_0$ implies that, for any $C \in \mathbb{R}$,

$$\lim_{t \to \infty} \left| u_+(t + n_0\omega, x + ct; u_0) - p(t, x + ct) \right| = 0 \text{ uniformly in } x \leq C,$$

that is,

$$\lim_{t\to\infty} \left| u_+\big(t, x + ct - cn_0\omega; u_0\big) - p(t, x + ct - cn_0\omega)\right| = 0 \text{ uniformly in } x \le C.$$

Thus, (3.24) holds for the original u_0.

With the assumption that u_0 nonincreasing in $x \in (-\infty, h_0]$, it follows from similar analysis as that used in Remark 3.12 that, to prove (3.24), it suffices to show that

$$\lim_{t\to\infty} \left| u_+\big(t, x + ct; u_0\big) - p(t, x + ct)\right| = 0 \text{ locally uniformly in } x \in \mathbb{R}.$$

Thus, by (3.23), to complete the proof of (3.24), we only need to show that for any $c < c_+^*$,

$$\lim_{n\to\infty} \left| u_+(t, x + ct + cn\omega; u_n) - p(t, x + ct + cn\omega)\right| = 0 \tag{3.25}$$

locally uniformly in $x \in \mathbb{R}$, and uniformly in $t \in [0, \omega]$.

Write $cn\omega = x_n + x_n'$ with $x_n \in [0, L]$ and $x_n' \in L\mathbb{Z}$. Due to the spatial L-periodicity assumption in (1.4) and the L-periodicity of the function $p(t, x)$ in x,

$$\begin{cases} u_+(t, x + ct + cn\omega; u_n) = u_+(t, x + ct + x_n; u_n(\cdot + x_n')), \\ \quad p(t, x + ct + cn\omega) = p(t, x + ct + x_n) \end{cases}$$

for any $n \in \mathbb{N}$, $t \in [0, \omega]$, $x \in \mathbb{R}$. Notice that $x_n' \le c\omega n$ for all $n \in \mathbb{N}$. Since $c < c_+^*$, and by (3.15),

$$\lim_{n\to\infty} \sup_{x \le c_1\omega n} \left| u_n(x) - p(0, x)\right| = 0 \text{ for any } c_1 \in (c, c_+^*),$$

we easily see that

$$\lim_{n\to\infty} \left| u_n(x + x_n') - p(0, x)\right| = 0 \text{ locally uniformly in } x \in \mathbb{R}.$$

Furthermore, by (3.16), we have $h(n\omega) - c\omega n \to \infty$ as $n \to \infty$, which clearly implies $h(n\omega) - x_n' \to \infty$ as $n \to \infty$. It then follows from the continuous dependence property stated in [2, Proposition 2.13] that

$$\lim_{n\to\infty} \left| u_+\big(t, x; u_n(\cdot + x_n')\big) - p(t, x)\right| = 0$$

locally uniformly in $x \in \mathbb{R}$ and uniformly in $t \in [0, \omega]$, which implies (3.25). The proof of (3.21) is thus complete.

Next we prove (3.22). To this end, we first prove the following conclusion:

$$\lim_{t\to\infty} \sup_{x \ge c't} u_+(t, x; u_0) = 0 \quad \text{for any } c' > c_+^*. \tag{3.26}$$

As above, we can assume without loss of generality that $u_0(x)$ is nonincreasing in $x \in (-\infty, h_0]$. For any given $c' > c_+^*$, write $c'n\omega = y_n + y_n'$ with $y_n \in [0, L]$ and $y_n' \in L\mathbb{Z}$. As in the analysis leading to (3.25), to prove (3.26), it is sufficient to show that

$$\lim_{n\to\infty} u_+(t, x; u_n(\cdot + y_n')) = 0 \tag{3.27}$$

locally uniformly in $x \in \mathbb{R}$ and uniformly in $t \in [0, \omega]$.

Since $c' > c_+^*$, it follows from (3.14) that

$$\lim_{n\to\infty} u_n(x + y_n') = 0 \text{ locally uniformly in } x \in \mathbb{R}.$$

Denote

$$\widetilde{u}_n(x) := u_n(x + y_n') \quad \text{and} \quad \widetilde{h}_n = h_n - y_n'.$$

It is easily seen that $\widetilde{u}_n \in \mathcal{H}_+(\widetilde{h}_n)$, and $\lim_{n\to\infty} \widetilde{u}_n(x) = 0$ locally uniformly in $x \in \mathbb{R}$. By the comparison principle we have

$$0 \le u_+\big(t, x; \widetilde{u}_n\big) \le v(t, x; \widetilde{u}_n) \text{ for } t > 0, \ x \in \mathbb{R},$$

where $v(t, x; \widetilde{u}_n)$ is the solution of the corresponding Cauchy problem, which converges to 0 as $n \to \infty$ uniformly for (t, x) over any bounded set of $[0, +\infty) \times \mathbb{R}$. It then follows that

$$\lim_{n \to \infty} u_+(t, x; \widetilde{u}_n) = 0 \text{ locally uniformly in } x \in \mathbb{R} \text{ and uniformly in } t \in [0, \omega],$$

that is, (3.27) holds. The proof of (3.26) is finished.

We are now ready to give the proof for (3.22). We first show that

$$\liminf_{t \to \infty} \frac{h_+(t; u_0)}{t} \geq c_+^*. \tag{3.28}$$

Assume by contradiction that $\liminf_{t \to \infty} h_+(t; u_0)/t < c_+^*$. Then there would exist some real number $\delta_1 > 0$ and a sequence $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ such that $t_n \to \infty$ as $n \to \infty$, and that $h_+(t_n; u_0) \leq (c_+^* - \delta_1) t_n$ for all $n \in \mathbb{N}$. It follows from (3.21) that

$$\liminf_{n \to \infty} \left| u_+(t_n, h_+(t_n); u_0) - p(t_n, h_+(t_n)) \right| = 0,$$

which is in contradiction with the fact that $u_+(t, h_+(t); u_0) \equiv 0$. Therefore (3.28) holds.

To complete the proof, it remains to show

$$\limsup_{t \to \infty} \frac{h_+(t; u_0)}{t} \leq c_+^*. \tag{3.29}$$

Suppose, to the contrary, that $\limsup_{t \to \infty} h_+(t; u_0)/t > c_+^*$. Then we can find a real number $\delta_2 > 0$ such that $\limsup_{t \to \infty} h_+(t; u_0)/t \geq c_+^* + \delta_2$. Thus, there would exist a sequence $\{\tau_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ such that $\tau_n \to \infty$ as $n \to \infty$, and that

$$h_+'(\tau_n; u_0) \geq c_+^* + \frac{\delta_2}{4}, \qquad h_+(\tau_n; u_0) \geq \left(c_+^* + \frac{\delta_2}{4} \right) \tau_n \text{ for all } n \in \mathbb{N} \tag{3.30}$$

Since $h_+'(t; u_0) = -\mu \partial_x u_+(t, h_+(t); u_0)$ for all $t \in \mathbb{R}$, it follows from the first inequality of (3.30) that

$$\partial_x u_+(\tau_n, h_+(\tau_n); u_0) \leq -\frac{c_+^* + \delta_2/4}{\mu} \text{ for all } n \in \mathbb{N}.$$

Furthermore, by standard parabolic theory, the function $\partial_{xx} u_+(\tau_n, x; u_0)$ is bounded in $(-\infty, h_+(\tau_n; u_0)]$ uniformly in n. This implies that there exist two positive constants C_1 and C_2 independent of n such that

$$u_+(\tau_n, h_+(\tau_n) - C_1; u_0) \geq C_2 \quad \text{for all } n \in \mathbb{N}.$$

However, due to the second inequality of (3.30), it follows from (3.26) that

$$\limsup_{n \to \infty} u_+(\tau_n, h_+(\tau_n) - C_1; u_0) = 0,$$

which is a contradiction. Therefore, (3.29) holds, and (3.22) is proved. $\quad\square$

Theorem 3.14 indicates that c_+^* is the spreading speed for the problem (1.14) in the rightward direction. In a similar way, we can consider the existence of spreading speed in the leftward direction for the problem (1.15). As a matter of fact, if we define

$$\widetilde{f}(t, x, u) = f(t, -x, u),$$

then (1.15) reduces to a problem of the form (1.14) with reaction function $\widetilde{f}$. Therefore a parallel theory holds.

For clarity, we state the corresponding results precisely below. We define

$$Q_-[\phi](\xi, y) := U[\phi(\cdot + \xi - y, \cdot)](y) \text{ for } \phi \in \widetilde{\mathcal{M}},$$

where

$$\widetilde{\mathcal{M}} := \left\{ \phi \in C(\mathbb{R}^2) : \widetilde{\phi}(\xi, x) := \phi(-\xi, -x) \text{ belongs to } \mathcal{M} \right\}.$$

Clearly, for any fixed $\xi_0 \in \mathbb{R}$, $U[\phi(\xi_0 + \cdot, \cdot)](y)$ is well-defined, and

$$U[\phi(\xi_0 + \cdot, \cdot)](y) := \begin{cases} u_-(\omega, y; \phi(\xi_0 + \cdot, \cdot)), & \text{if } y \geq g_-(\omega; \phi(\xi_0 + \cdot, \cdot)), \\ 0, & \text{if } y < g_-(\omega; \phi(\xi_0 + \cdot, \cdot)). \end{cases}$$

Now, for any given $\phi \in \widetilde{\mathcal{M}}$ and $c \in \mathbb{R}$, we define the sequence $\left\{ (b_n^c, G_n^c) \right\}_{n \in \mathbb{N}}$ by the following recursions

$$b_n^c(\xi, x) = \max \left\{ \phi(\xi, x), \, Q_-[b_{n-1}^c](\xi - c, x) \right\} \quad \text{with} \quad b_0^c(\xi, x) = \phi(\xi, x),$$

and

$$G_n^c(\xi) = \min \left\{ G_0(\xi), \, g_-\left(\omega; b_{n-1}^c(\cdot + \xi - c, \cdot)\right) \right\},$$

where $G_0(\xi)$ is the real number such that $\phi(\xi + x, x) = 0$ if and only if $\xi \leq G_0(\xi)$.

Lemma 3.15. *The limits $G^c(\xi) = \lim_{n \to \infty} G_n^c(\xi)$ and $b^c(\xi, x) = \lim_{n \to \infty} b_n^c(\xi, x)$ exist. Moreover, either $b^c(-\infty, x) \equiv p(0, x)$ or $b^c(-\infty, x) \equiv 0$.*

Let c_- be defined by

$$c_- = \sup \left\{ c \in \mathbb{R} : b^c(-\infty, x) \equiv p(0, x) \right\}. \tag{3.31}$$

Then we have the following two results.

Proposition 3.16. *Let c_- be given in (3.31). Then $c_- \in (0, \infty)$ and is independent of the choice of $\phi \in \widetilde{\mathcal{M}}$ in the definition of the recursion. Suppose $u_0 \in \mathcal{C}$ has left supporting point $g_0 > -\infty$, right supporting point $h_0 = \infty$ and*

$$\liminf_{x \to \infty} \left(p(0, x) - u_0(x) \right) > 0. \tag{3.32}$$

If for every $C \in \mathbb{R}$,

$$\lim_{n \to \infty} \left| U^n[u_0](x) - p(0, x) \right| = 0 \quad \text{uniformly in } x \in [C, \infty), \tag{3.33}$$

then

$$\lim_{n \to \infty} \sup_{x \leq -c_1 n} U^n[u_0](x) = 0 \quad \text{for any } c_1 > c_-,$$

and

$$\lim_{n \to \infty} \sup_{x \geq -c_2 n} \left| U^n[u_0](x) - p(0, x) \right| = 0 \quad \text{for any } c_2 < c_-.$$

Theorem 3.17. *Let $c_-^* = c_-/\omega$ where $c_- \in (0, \infty)$ is given in (3.31). If there exists $u_0 \in \mathcal{H}_-(g_0)$ such that $u_0(x) \leq p(0, x)$ for all $x \in [g_0, \infty)$ and u_0 satisfies the assumptions (3.32) and (3.33), then*

$$\lim_{t \to \infty} \sup_{x \geq -ct} \left| u_-(t, x; u_0) - p(t, x) \right| = 0 \quad \text{for any } c < c_-^*,$$

and

$$\lim_{t \to \infty} \frac{g_-(t; u_0)}{t} = -c_-^*.$$

4. Proof of Theorem 1.1

In this section, we will complete the proof of Theorem 1.1 by showing that the rightward and leftward spreading speeds of (1.1) are the same as the spreading speeds determined by (1.14) and (1.15), respectively. With the preparations in the previous section, we are now able to adapt the approximation technique of Weinberger [15] to our situation here, similar in spirit to [9, Theorem 3.3] where the Cauchy problem was considered.

Let $\eta : \mathbb{R} \to \mathbb{R}$ be a smooth, nonincreasing function such that $\eta(s) > 0$ for $s < 1$, and

$$\eta(s) = \begin{cases} 1, & \text{if } s \leq \frac{1}{2}, \\ 0, & \text{if } s \geq 1. \end{cases}$$

For any real number $B > 0$, we define the map U_B on $\mathcal{C}$ by

$$U_B[\varphi](x) := U\left[\eta\left(\frac{|\cdot - x|}{B}\right)\varphi(\cdot)\right](x) \quad \text{for all } \varphi \in \mathcal{C},$$

where $U : \mathcal{C} \to \mathcal{C}$ is the operator defined in Section 2.

Thus, for any given $\varphi \in \mathcal{C}$ with left supporting point g_0 and right supporting point h_0,

- if $x \notin (g_0 - B, h_0 + B)$, then $\eta(|\cdot - x|/B)\varphi(\cdot) \equiv 0$ and hence $U_B[\varphi](x) = 0$;
- if $x \in (g_0 - B, h_0 + B)$, then $U_B[\varphi](x)$ is equal to $u_{B,x}(\omega, x)$, where $u = u_{B,x}(t, y)$, and its left supporting point $g = g_{B,x}(t)$, right supporting point $h = h_{B,x}(t)$ form a triplet $(u_{B,x}, g_{B,x}, h_{B,x})$ that solves the following free boundary problem[1]

$$\begin{cases} u_t(t, y) = d u_{yy}(t, y) + f(t, y, u), & g(t) < y < h(t), \quad t > 0, \\ u(t, g(t)) = u(t, h(t)) = 0, & t > 0, \\ g'(t) = -\mu u_y(t, g(t)), \quad h'(t) = -\mu u_y(t, h(t)), & t > 0, \\ g(0) = \max\{g_0, x - B\}, \quad h(0) = \min\{h_0, x + B\}, \\ u(0, y) = \eta\left(\frac{|y - x|}{B}\right)\varphi(y), & g(0) \leq y \leq h(0). \end{cases} \tag{4.1}$$

Lemma 4.1. *The operator U_B possesses the following properties.*

(i) For any $\varphi \in \mathcal{C}$ and $x \in \mathbb{R}$, $U_B[\varphi](x)$ only depends on the values of $\varphi(y)$ for $y \in [x - B, x + B]$.

(ii) For any $\varphi \in \mathcal{C}$, $U_B[\varphi](x)$ is nondecreasing in B and converges to $U[\varphi](x)$ locally uniformly in $x \in \mathbb{R}$ as $B \to \infty$.

(iii) U_B maps $\mathcal{C}$ into itself, and U_B has the properties stated in (A1)–(A3).

Proof. (i) This statement follows directly from the definition of U_B.

(ii) Let φ be a given function in $\mathcal{C}$ with left supporting point g_0 and right supporting point h_0. We first prove the monotonicity of $U_B[\varphi](x)$ in B. It suffices to show that for any $B_2 \geq B_1 > 0$ and any $x \in (g_0 - B_1, h_0 + B_1)$, there holds $U_{B_1}[\varphi](x) \leq U_{B_2}[\varphi](x)$. Indeed, for any such x,

$$\max\{g_0, x - B_2\} \leq \max\{g_0, x - B_1\}, \quad \min\{h_0, x + B_2\} \geq \min\{h_0, x + B_1\},$$

and due to the monotonicity of $\eta(s)$ in $s \in \mathbb{R}^+$,

$$\eta\left(\frac{|y - x|}{B_1}\right)\varphi(y) \leq \eta\left(\frac{|y - x|}{B_2}\right)\varphi(y) \quad \text{for all } y \in \mathbb{R}.$$

Applying the comparison principle [2, Proposition 2.10] to (4.1), we obtain

$$g_{B_1,x}(t) \geq g_{B_2,x}(t), \quad h_{B_1,x}(t) \leq h_{B_2,x}(t) \text{ for } t > 0,$$

and

$$u_{B_1,x}(t, y) \leq u_{B_2,x}(t, y) \text{ for } t > 0, \ y \in [g_{B_1,x}(t), h_{B_1,x}(t)].$$

---

[1] As before, for any $t > 0$, $u_{B,x}(t, y)$ is extended to $y \in \mathbb{R}$ by defining $u_{B,x}(t, y) = 0$ for $y > h_{B,x}(t)$ or $y < g_{B,x}(t)$.

In particular,

$$u_{B_1,x}(\omega, y) \leq u_{B_2,x}(\omega, y) \text{ for } g_{B_1,x}(\omega) \leq y \leq h_{B_1,x}(\omega).$$

This clearly implies that $U_{B_1}[\varphi](x) \leq U_{B_2}[\varphi](x)$.

We now show the convergence of $U_B[\varphi]$ as $B \to \infty$. For any given bounded subset $S \subset \mathbb{R}$, when B is sufficiently large, clearly $S \subset (g_0 - B, h_0 + B)$. Moreover, as $B \to \infty$,

$$\max\{g_0, x - B\} \to g_0, \quad \min\{h_0, x + B\} \to h_0 \quad \text{uniformly in } x \in S,$$

and

$$\eta\Big(\frac{|y - x|}{B}\Big)\varphi(y) \to \varphi(y) \quad \text{locally uniformly in } y \in \mathbb{R} \text{ and uniformly in } x \in S.$$

It then follows from the continuity of the operator U in (A3) that

$$U\Big[\eta\Big(\frac{|\cdot - x|}{B}\Big)\varphi(\cdot)\Big](x) \to U[\varphi](x) \text{ as } B \to \infty \text{ uniformly in } x \in S.$$

That is, $U_B[\varphi](x)$ converges to $U[\varphi](x)$ locally uniformly for $x \in \mathbb{R}$ as $B \to \infty$.

(iii) We only prove that U_B maps $\mathcal{C}$ into itself, since the other properties can be easily checked. It follows easily from the definition of U_B and the properties of U stated in (A1) and (A3) that $U_B[\varphi] \in C(\mathbb{R})$, and that $0 \leq U_B[\varphi](x) \leq p(0, x)$ for all $x \in \mathbb{R}$.

We now show that, for any $\varphi \in \mathcal{C}$ with left supporting point $g_0 = -\infty$ and right supporting point $h_0 < \infty$, $U_B[\varphi]$ has the same type of supporting points. Set

$$h_1 = \sup\big\{x \in \mathbb{R} : U_B[\varphi](x) > 0\big\}.$$

By the continuity of $U_B[\varphi](x)$, we have $U_B[\varphi](h_1) = 0$. If $x_0 \leq h_0$, then the right supporting point of $u_{B,x_0}(0, x)$ is $\min\{x_0 + B, h_0\} \geq x_0$, and its left supporting point is $x_0 - B$. So $h_{B,x_0}(\omega) > x_0$, $g_{B,x_0}(\omega) < x_0 - B$. Hence

$$U_B[\varphi](x_0) = u_{B,x_0}(\omega, x_0) > 0,$$

which implies $h_1 > h_0$. For any $x_0 \geq h_0 + B$, we have $u_{B,x_0}(0, x) \equiv 0$ and hence $U_B[\phi](x_0) = 0$. It follows that $h_1 \leq h_0 + B$. Summarizing the above, we have

$$h_0 < h_1 \leq h_0 + B, \quad U_B[\varphi](x) > 0 \text{ for all } x \leq h_0.$$

We show next that $U_B[\varphi](x_0) > 0$ for all $h_0 < x_0 < h_1$. Indeed, for any such x_0, by the definition of h_1, there exists $x_1 \in (x_0, h_1)$ such that $U_B[\varphi](x_1) > 0$. That is, $u_{B,x_1}(\omega, x_1) > 0$ and so $h_{B,x_1}(\omega) > x_1 > x_0$. Furthermore, since $x_1 < h_1 \leq h_0 + B$, the left supporting point of $u_{B,x_1}(0, x)$ is $x_1 - B < h_0$. It follows that $g_{B,x_1}(\omega) < h_0 < x_0$. Thus

$$g_{B,x_1}(\omega) < x_0 < h_{B,x_1}(\omega).$$

On the other hand, since the function $\eta(s)$ is nonincreasing in $s \in \mathbb{R}^+$, it follows that

$$\eta\Big(\frac{|x_0 - y|}{B}\Big)\varphi(y) \geq \eta\Big(\frac{|x_1 - y|}{B}\Big)\varphi(y) \text{ for } y \in [x_1 - B, h_0].$$

Then applying the comparison principle [2, Proposition 2.10] to (4.1), we obtain

$$u_{B,x_0}(\omega, y) \geq u_{B,x_1}(\omega, y) \text{ for } g_{B,x_1}(\omega) < y < h_{B,x_1}(\omega).$$

This in particular implies that $u_{B,x_0}(\omega, x_0) \geq u_{B,x_1}(\omega, x_0) > 0$, that is, $U_B[\varphi](x_0) > 0$. Thus the left supporting point of $U_B[\varphi]$ is $-\infty$, and its right supporting point is h_1.

The analysis for $\varphi \in \mathcal{C}$ with other types of supporting points is similar. The proof of Lemma 4.1 is thereby complete. $\square$

In our analysis below, we need a fixed positive L-periodic function $w \in C(\mathbb{R})$ such that $0 < w(x) < p(0, x)$ for all $x \in \mathbb{R}$. We fix a small positive constant ε such that $p(0, x) - \varepsilon > w(x) \geq \varepsilon$ for all $x \in \mathbb{R}$. Our first result involving this function $w(x)$ is the lemma below.

Lemma 4.2. *There exists $B_1 > 0$ and $N_1 \in \mathbb{N}$ such that*

$$U_B^n[w](x) \geq p(0, x) - \varepsilon \text{ for all } x \in \mathbb{R}, \ B \geq B_1, \ n \geq N_1.$$

Proof. It follows from the property (A4) that $U^n[w](x)$ converges to $p(0, x)$ as $n \to \infty$ uniformly in $x \in \mathbb{R}$. This implies that there exists some $N_0 \in \mathbb{N}$ such that

$$U^n[w](x) \geq p(0, x) - \varepsilon/2 \text{ for all } x \in \mathbb{R}, \ n \geq N_0.$$

By Lemma 4.1 (ii), we find some large $B_1 > 0$ such that

$$U_B^{N_0+k}[w](x) \geq p(0, x) - \varepsilon > w(x) \text{ for all } x \in [0, L], \ B \geq B_1, \ k = 0, 1, \cdots, N_0 - 1.$$

For every $n \geq N_0$, there exists $m \in \mathbb{N}$ and $k \in \{0, 1, \cdots, N_0 - 1\}$ such that $n = mN_0 + k$, whence it follows from the order-preserving property of U_B that

$$U_B^n[w](x) = U_B^{mN_0+k}[w](x) \geq U_B^{(m-1)N_0}[w](x) \text{ for all } x \in [0, L], \ B \geq B_1.$$

For $n \geq N_1 := 2N_0$, we have $m \geq 2$ and thus

$$U_B^{(m-1)N_0}[w](x) \geq U_B^{N_0}[w](x) \geq p(0, x) - \varepsilon \text{ for all } x \in [0, L], \ B \geq B_1,$$

where we have used $U_B^{N_0}[w](x) > w(x)$ to obtain the first inequality. We thus obtain

$$U_B^n[w](x) \geq p(0, x) - \varepsilon \text{ for all } x \in [0, L], \ B \geq B_1, \ n \geq N_1.$$

Since U_B has property (A2), and $w(x)$ is L-periodic, we see that $U_B^n[w](x)$ is L-periodic in x for all $n \in \mathbb{N}$. Thus the above inequality holds for all $x \in \mathbb{R}$, and the proof of Lemma 4.2 is complete. □

Let B_1 and N_1 be given by Lemma 4.2. Fix $B \geq B_1$ and let $\{w_B^n\}_{n \in \mathbb{N}}$ be determined by the following recursion

$$w_B^n(x) = \max\left\{w(x), U_B[w_B^{n-1}](x)\right\}, \quad w_B^0(x) = w(x).$$

It is easy to see that for each fixed $n \in \mathbb{N}$, $w_B^n(x)$ is positive, continuous and L-periodic in x, and it is nondecreasing in n. Moreover, since $w_B^k(x) \geq w(x)$ for all $k \in \mathbb{N}$, we have, by the proof of Lemma 4.2,

$$w_B^n(x) \geq U_B^n[w](x) > w(x) \text{ for } n \geq N_1, \ x \in \mathbb{R}. \tag{4.2}$$

Then necessarily

$$w_B^n(x) = U_B[w_B^{n-1}](x) \text{ for all } x \in \mathbb{R}, \ n \geq N_1. \tag{4.3}$$

Now, for the same fixed $B \geq B_1$ we define

$$Q_{+,B}[\phi](\xi, x) := U_B[\phi(\cdot + \xi - x, \cdot)](x) \text{ for } \phi \in \mathcal{M}.$$

Then $Q_{+,B}$ is a map from $\mathcal{M}$ to $\mathcal{M}$.[2]

We next fix a function $\phi_0(\xi, x)$ in $\mathcal{M}$ such that $\phi_0(\xi, x) \equiv w(x)$ for all $\xi \leq -1$, and $\phi_0(\xi, x) \equiv 0$ for $\xi \geq 0$. Then for any $0 < c < c_+$, we define

$$\widetilde{a}_{n+1}^c(\xi, x) = \max\left\{\phi_0(\xi, x), \ Q_{+,B}[\widetilde{a}_n^c](\xi + c, x)\right\} \quad \text{with} \quad \widetilde{a}_0^c(\xi, x) = \phi_0(\xi, x). \tag{4.4}$$

It is easily checked that $\widetilde{a}_n^c(\xi, x)$ is nondecreasing in n, nonincreasing in ξ and c, and L-periodic in x. Moreover, we have the following conclusions.

---

[2] For $\phi \in \mathcal{M}$, since U_B has the properties stated in (A1)–(A3), similar analysis to that in Lemma 3.2 indicates that $Q_{+,B}[\phi](\xi, x)$ possesses the properties (a)–(c) and (e). To prove (d), one may use the same arguments as those used in the proof of Lemma 4.1 to conclude that, for every $\xi \in \mathbb{R}$, $Q_{+,B}[\phi](\xi + x, x) = 0$ if and only if $x \geq \widetilde{H}(\xi)$ where $\widetilde{H}(\xi) = \sup\{x \in \mathbb{R} : U_B[\phi(\cdot + \xi, \cdot)](x) > 0\}$.

Lemma 4.3. (i) *For every $n \in \mathbb{N}$ and $x \in \mathbb{R}$,*

$$\widetilde{a}_n^c(\xi, x) = \begin{cases} w_B^n(x), & \text{if } \xi \leq -n(B+c) - 1, \\ 0, & \text{if } \xi \geq n(B-c). \end{cases} \tag{4.5}$$

(ii) *There is some $N_2 \in \mathbb{N}$, $B_2 > 0$ such that*

$$\widetilde{a}_{n+1}^c(\xi, x) = Q_{+,B}[\widetilde{a}_n^c](\xi + c, x) \text{ for all } x \in \mathbb{R}, \xi \in \mathbb{R}, B \geq B_2, n \geq N_2. \tag{4.6}$$

Proof. We prove (4.5) by an induction argument. By our choice of ϕ_0, (4.5) trivially holds in the case of $n = 0$. Now suppose that (4.5) holds for some $n = n_0$.

By definition,

$$Q_{+,B}[\widetilde{a}_{n_0}^c](\xi + c, x) = U_B[\widetilde{a}_{n_0}^c(\cdot + \xi + c - x, \cdot)](x) \text{ for all } \xi \in \mathbb{R}, x \in \mathbb{R}.$$

By Lemma 4.1 (i), $Q_{+,B}[\widetilde{a}_{n_0}^c](\xi + c, x)$ only depends on the values of $\widetilde{a}_{n_0}^c(y + \xi + c - x, y)$ with $|x - y| \leq B$. For any $\xi \leq -(n_0 + 1)(B + c) - 1$, $x \in \mathbb{R}$ and $y \in \mathbb{R}$ with $|y - x| \leq B$, we have $y + \xi + c - x \leq -n_0(B + c) - 1$, whence by the induction assumption $\widetilde{a}_{n_0}^c(y + \xi + c - x, y) = w_B^{n_0}(y)$, and so $Q_{+,B}[\widetilde{a}_{n_0}^c](\xi + c, x) = U_B[w_B^{n_0}](x)$. This together with the fact $\phi_0(\xi, x) \equiv w(x)$ for such ξ gives

$$\widetilde{a}_{n_0+1}^c(\xi, x) = \max\{w(x), U_B[w_B^{n_0}](x)\} = w_B^{n_0+1}(x).$$

Similarly, one concludes that $\widetilde{a}_{n_0+1}^c(\xi, x) \equiv 0$ for $\xi \geq (n_0 + 1)(B - c)$. Thus (4.5) also holds for $n = n_0 + 1$. The induction principle then concludes that (4.5) holds for all $n \in \mathbb{N}$.

Next, we prove (4.6). Since $c < c_+$, from Lemma 3.5 (i) and Lemma 3.8 we see that, for any fixed $\xi \in \mathbb{R}$, $a_n^c(\xi + x, x)$ converges to $p(0, x)$ locally uniformly in $x \in \mathbb{R}$ as $n \to \infty$, where $\{a_n^c(\xi, x)\}_{n \in \mathbb{N}}$ is the sequence obtained from the recursion (3.4) with $a_0^c = \phi_0$. Furthermore, by Lemma 3.7 and the monotonicity of $a_n^c(\xi, x)$ in n, $c < c_+$ implies the existence of $N_2 \in \mathbb{N}$ such that $a_{n+1}^c(0, x) > \phi_0(-\infty, x) = w(x)$ for all $x \in \mathbb{R}$, $n \geq N_2$. Then Lemma 4.1 (ii) implies that there is some $B_2 > 0$ such that

$$\widetilde{a}_{n+1}^c(0, x) \geq \widetilde{a}_{N_2}^c(0, x) > w(x) \text{ for all } x \in [0, L], n \geq N_2, B \geq B_2.$$

Since both $\widetilde{a}_{n+1}^c(0, x)$ and $w(x)$ are L-periodic in x, the above inequality holds for all $x \in \mathbb{R}$.

Since $\widetilde{a}_{n+1}(\xi, x)$ is nonincreasing in ξ, and since $\phi_0(\xi, x) \leq \phi_0(-\infty, x) \equiv w(x)$, we obtain

$$\widetilde{a}_{n+1}^c(\xi, x) \geq \widetilde{a}_{n+1}^c(0, x) > \phi_0(\xi, x) \text{ for all } x \in \mathbb{R}, \xi \leq 0, n \geq N_2, B \geq B_2.$$

In view of (4.4), this implies that (4.6) holds for $\xi \leq 0$. Since $\phi_0(\xi, x) \equiv 0$ for $\xi > 0$, we see that (4.6) also holds for $\xi > 0$. The proof of Lemma 4.3 is thereby complete. □

Correspondingly, we define

$$Q_{-,B}[\widetilde{\phi}](\xi, x) := U_B[\widetilde{\phi}(\cdot + \xi - x, \cdot)](x) \text{ for } \widetilde{\phi} \in \widetilde{\mathcal{M}},$$

and choose a function $\widetilde{\phi}_0(\xi, x) \in \widetilde{\mathcal{M}}$ such that $\widetilde{\phi}_0(\xi, x) \equiv w(x)$ for $\xi \geq 1$ and $\widetilde{\phi}_0(\xi, x) \equiv 0$ for $\xi \leq 0$. For any $0 < c' < c_-$, we define

$$\widetilde{b}_{n+1}^{c'}(\xi, x) = \max\left\{\widetilde{\phi}_0(\xi, x), Q_{-,B}[\widetilde{b}_n^{c'}](\xi - c', x)\right\} \quad \text{with} \quad \widetilde{b}_0^{c'}(\xi, x) = \widetilde{\phi}_0(\xi, x).$$

Then $\widetilde{b}_n^{c'}(\xi, x)$ is nondecreasing in n and ξ, nonincreasing in c', and L-periodic in x. Moreover, we have

Lemma 4.4. (i) *For every $n \in \mathbb{N}$ and $x \in \mathbb{R}$,*

$$\widetilde{b}_n^{c'}(\xi, x) = \begin{cases} w_B^n(x), & \text{if } \xi \geq n(B+c') + 1, \\ 0, & \text{if } \xi \leq -n(B-c'). \end{cases}$$

(ii) *There is $N_3 \in \mathbb{N}$, $B_3 > 0$ such that*

$$\widetilde{b}_{n+1}^{c'}(\xi, x) = Q_{-,B}[\widetilde{b}_n^{c'}](\xi - c', x) \text{ for all } x \in \mathbb{R}, \xi \in \mathbb{R}, B \geq B_3, n \geq N_3.$$

For fixed $c \in (0, c_+)$ and $c' \in (0, c_-)$, let B_1, B_2, B_3 and N_1, N_2, N_3 be given by Lemmas 4.2, 4.3 and 4.4. Then fix some $B > \max\{B_1, B_2, B_3\}$, some $m > \max\{N_1, N_2, N_3\}$, and choose constants A and A' such that

$$A \geq \frac{1 + m(B + c) + 2B}{c}, \quad A' \geq \frac{1 + m(B + c') + 2B}{c'}.$$

We now define, for every $n \in \mathbb{N}$,

$$e_n(x) = \begin{cases} \widetilde{a}_m^c(x - (n + A)c, x), & \text{if } x \geq 0, \\ \widetilde{b}_m^{c'}(x + (n + A')c', x), & \text{if } x \leq 0. \end{cases}$$

By Lemma 4.3 (i) and Lemma 4.4 (i), it is easy to check that for each $n \in \mathbb{N}$, $e_n \in \mathcal{C}$ and that

$$e_n(x) = \begin{cases} w_B^m(x), & \text{if } x \in [-l_{m,n} + 1, \widetilde{l}_{m,n} - 1], \\ 0, & \text{if } x \notin [-l_{m,n} - 2m(B + c'), \widetilde{l}_{m,n} + 2m(B - c)], \end{cases} \tag{4.7}$$

where

$$l_{m,n} := (n + A')c' - m(B + c'), \quad \widetilde{l}_{m,n} := (n + A)c - m(B + c).$$

Furthermore, the sequence $\{e_n\}_{n \in \mathbb{N}}$ has the following key property.

Lemma 4.5. $e_{n+1}(x) \leq U_B[e_n](x)$ for all $x \in \mathbb{R}$, $n \in \mathbb{N}$.

Proof. We follow similar lines as the proof of [9, Lemma 3.10]. For the sake of completeness, we include the details here.

For each $n \in \mathbb{N}$, if $x \in [-l_{m,n} + 1 + B, \widetilde{l}_{m,n} - 1 - B]$, then for any $y \in \mathbb{R}$ with $|y - x| \leq B$, we have $y \in [-l_{m,n} + 1, \widetilde{l}_{m,n} - 1]$, and hence $U_B[e_n](x) = U_B[w_B^m](x)$ by (4.7). It then follows from (4.3) and (4.7) that

$$U_B[e_n](x) = w_B^{m+1}(x), \quad w_B^m(x) = e_{n+1}(x).$$

By the monotonicity of $w_B^k(x)$ in k, we have $w_B^{m+1}(x) \geq w_B^m(x)$ and hence

$$U_B[e_n](x) \geq e_{n+1}(x).$$

Now suppose that $x > \widetilde{l}_{m,n} - 1 - B$. Then for any $y \in \mathbb{R}$ with $|y - x| \leq B$, we have $y > \widetilde{l}_{m,n} - 1 - 2B$. By the choice of A, we have $\widetilde{l}_{m,n} - 1 - 2B > nc > 0$, and so $y > 0$. Then by the definition of e_n and the monotonicity of $\widetilde{a}_m^c(\xi, x)$ in m, we have

$$e_n(y) = \widetilde{a}_m^c(y - (n + A)c, y) \geq \widetilde{a}_{m-1}^c(y - (n + A)c, y),$$

and hence,

$$U_B[e_n](x) \geq U_B[\widetilde{a}_{m-1}^c(\cdot - (n + A)c, \cdot)](x).$$

It then follows from Lemma 4.3 (ii) (by choosing $\xi = x - (n + A)c - c$ and $n = m - 1$) and the definition of e_{n+1} that

$$U_B[e_n](x) \geq U_B[\widetilde{a}_{m-1}^c(\cdot - (n + A)c, \cdot)](x) = \widetilde{a}_m^c(x - (n + A)c - c, x) = e_{n+1}(x).$$

Similarly, one can prove that $U_B[e_n](x) \geq e_{n+1}(x)$ if $x < -l_{m,n} + 1 + B$. The proof of Lemma 4.5 is thereby complete. $\square$

We are now ready to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. For any $g_0 < h_0$ and $u_0 \in \mathcal{H}(g_0, h_0)$ satisfying the assumptions in Theorem 1.1, we first extend u_0 to the whole real line by defining $u_0(x) = 0$ for $x \notin [g_0, h_0]$. After the extension, clearly $u_0 \in \mathcal{C}$.

For any $n \in \mathbb{N}$, set $u_n(x) = U^n[u_0](x)$. To complete the proof of Theorem 1.1, it is sufficient to prove that

$$\lim_{n \to \infty} \sup_{-c_2 n \leq x \leq c_1 n} |u_n(x) - p(0, x)| = 0 \quad \text{for any } c_1 \in (0, c_+), \, c_2 \in (0, c_-), \tag{4.8}$$

and

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}\setminus[-c_2'n,c_1'n]} u_n(x) = 0 \quad \text{for any } c_1' > c_+ \text{ and } c_2' > c_-. \tag{4.9}$$

Indeed, once (4.8) and (4.9) are obtained, similar analysis to that used in the proof of Theorem 3.14 would imply (1.9) and (1.10).

We prove (4.9) first. Choose some nondecreasing function $\widetilde{u}_0 \in \mathcal{C}$ with left supporting point $g_0 = -\infty$ and right supporting point $h_0 < \infty$ such that $\widetilde{u}_0(x) \geq u_0(x)$ for all $x \in \mathbb{R}$ and that $\widetilde{u}_0$ satisfies (3.12). Since the operator U is order-preserving in the sense of (A1) and since $u_n(x)$ converges to $p(0,x)$ as $n \to \infty$ locally uniformly in $x \in \mathbb{R}$ by our assumption (1.8), it follows that $U^n[\widetilde{u}_0]$ also converges to $p(0,x)$ as $n \to \infty$ locally uniformly in $x \in \mathbb{R}$. This together with the monotonicity of $\widetilde{u}_0$ implies that $U^n[\widetilde{u}_0]$ satisfies (3.13). Proposition 3.11 then infers

$$\lim_{n\to\infty} \sup_{x\geq c_1'n} U^n[\widetilde{u}_0](x) = 0 \text{ for any } c_1' > c_+,$$

whence $\lim_{n\to\infty} \sup_{x\geq c_1'n} u_n(x) = 0$ by (A1) again. Similarly, by applying Proposition 3.16, one concludes that

$$\lim_{n\to\infty} \sup_{x\leq -c_2'n} u_n(x) = 0 \text{ for any } c_2' > c_-.$$

Thus (4.9) holds.

Next, we prove (4.8). For any given $c_1 \in (0,c_+)$ and $c_2 \in (0,c_-)$, we fix some $c \in (c_1,c_+)$ and $c' \in (c_2,c_-)$. For any $\varepsilon > 0$ satisfying $p(0,x) - \varepsilon \geq w(x)$, let $B \in \mathbb{R}^+$ and $m \in \mathbb{N}$ large enough such that the conclusions in Lemmas 4.2–4.5 are all valid with $n = m$. Since $\lim_{n\to\infty} |u_n(x) - p(0,x)| = 0$ locally uniformly in $x \in \mathbb{R}$ by (1.8), and since e_0 is compactly supported in $\mathbb{R}$, there is $l \in \mathbb{N}$ such that $u_l(x) \geq e_0(x)$ for all $x \in \mathbb{R}$. Thus using $U[\varphi] \geq U_B[\varphi]$ for all $\varphi \in \mathcal{C}$, by Proposition 2.3 and Lemma 4.5, we have

$$u_{l+n}(x) \geq e_n(x) \text{ for all } x \in \mathbb{R}, \ n \in \mathbb{N}.$$

We may now apply (4.7) to obtain

$$u_{l+n}(x) \geq w_B^m(x) \text{ for } x \in [-l_{m,n}+1, \widetilde{l}_{m,n}-1],$$

where

$$l_{m,n} := (n+A')c' - m(B+c'), \ \widetilde{l}_{m,n} := (n+A)c - m(B+c).$$

By Lemma 4.2 and our choice of m and B, we have

$$U_B^m[w](x) \geq p(0,x) - \varepsilon \text{ for all } x \in \mathbb{R}.$$

Since $c_1 < c < c_+$ and $c_2 < c' < c_-$, there exists $n_1 = n_1(c,c_1,c',c_2)$ such that for any $n \geq n_1$,

$$[-(l+n)c_2, (l+n)c_1] \subset [-l_{m,n}+1, l_{m,n}-1].$$

Then, for $n \geq n_1$ and $x \in [-(l+n)c_2, (l+n)c_1]$, we have

$$u_{l+n}(x) \geq w_B^m(x) \geq U_B^m[w](x) \geq p(0,x) - \varepsilon.$$

It follows that

$$\limsup_{n\to\infty} \sup_{-c_2n\leq x\leq c_1n} u_n(x) \geq p(0,x) - \varepsilon.$$

Since ε is arbitrary and $u_n(x) \leq p(0,x)$, we thus obtain (4.8). The proof of Theorem 1.1 is thereby complete. $\quad\square$

5. Proof of Theorem 1.3

In this section we prove that the spreading speeds for the free boundary problem (1.1) converge to those for the corresponding Cauchy problem (1.7) as $\mu \to \infty$. By Theorem 1.1, it suffices to show the convergence of the spreading speed for problem (1.14) in the rightward direction and the same for problem (1.15) in the leftward direction, as $\mu \to \infty$. We only consider the former, since the latter follows from the former by a simple change of variables.

Throughout this section, to indicate the dependence on μ, for any $u_0 \in \mathcal{H}_+(h_0)$, we denote the unique solution of (1.14) by $\left(u_{+,\mu}(t, x; u_0), h_{+,\mu}(t; u_0)\right)$; and we rewrite U, Q_+, $a_n^c(\xi, x)$, H_n^c, $a^c(\xi, x)$, H^c, c_+ and c_+^* in Section 3 by U_μ, $Q_{+,\mu}$, $a_{n,\mu}^c(\xi, x)$, $H_{n,\mu}^c$, $a_\mu^c(\xi, x)$, H_μ^c, $c_{+,\mu}$ and $c_{+,\mu}^*$, respectively.

Before starting the proof, let us recall some existing results on the spreading speeds of the Cauchy problem (1.7). Let

$$\bar{Q}_+[\phi](\xi, x) := \bar{U}[\phi(\cdot + \xi - x, \cdot)](x) \quad \text{for } \phi \in \mathcal{M},$$

where $\mathcal{M}$ is given in the beginning of Section 3, and $\bar{U}$ is the Poincaré map for the Cauchy problem (1.7), that is,

$$\bar{U}[\psi](x) = v(\omega, x; \psi) \quad \text{for } \psi \in C(\mathbb{R}). \tag{5.1}$$

It is easily checked that $\bar{Q}_+$ maps $\mathcal{M}$ into $C(\mathbb{R}^2)$ and for any $\phi \in \mathcal{M}$, $\bar{Q}_+[\phi](\xi, x)$ has the properties (a)–(c) and (e). We fix a number $h_0 \in \mathbb{R}$ and a function $\phi \in \mathcal{M}$ such that $\phi(\xi, x) \equiv 0$ if and only if $\xi \geq h_0$. For any $c \in \mathbb{R}$, we define the sequence $\{\bar{a}_n^c\}_{n \in \mathbb{N}}$ by the following recursion

$$\bar{a}_{n+1}^c(\xi, x) = \max\left\{\phi(\xi, x), \ \bar{Q}_+[\bar{a}_n^c](\xi + c, x)\right\}, \qquad \bar{a}_0^c(\xi, x) = \phi(\xi, x). \tag{5.2}$$

It follows from the analysis in [16, Section 3] that

$$\bar{a}^c(\xi, x) := \lim_{n \to \infty} \bar{a}_n^c(\xi, x) \text{ exists pointwisely in } \mathbb{R}^2$$

and that

$$\text{either } \bar{a}^c(\infty, x) \equiv 0 \text{ or } \bar{a}^c(\infty, x) \equiv p(0, x).$$

Set

$$\bar{c}_+ = \sup\left\{c \in \mathbb{R} : \bar{a}^c(\infty, x) \equiv p(0, x)\right\}. \tag{5.3}$$

Then, [16, Lemma 3.2] implies that

$$c < \bar{c}_+ \text{ if and only if } \bar{a}_{n_0}^c(h_0, x) > \phi(-\infty, x) \text{ for some } n_0 \in \mathbb{N} \text{ and all } x \in \mathbb{R}. \tag{5.4}$$

Moreover, the following lemma is an easy application of [6, Theorem 2.2].

Lemma 5.1. *Let $\bar{c}_+^* = \bar{c}_+/\omega$. Then $\bar{c}_+^*$ is the rightward spreading speed for problem* (1.7).

The next result is the key of this section.

Lemma 5.2. *Let $c_{+,\mu}$ and $\bar{c}_+$ be given in* (3.10) *and* (5.3), *respectively. Then $c_{+,\mu}$ is nondecreasing in $\mu > 0$ and* $\lim_{\mu \to \infty} c_{+,\mu} = \bar{c}_+$.

Proof. Let $h_0 \in \mathbb{R}$ be a fixed number and $\phi \in \mathcal{M}$ such that $\phi(\xi, x) \equiv 0$ if and only if $\xi \geq h_0$. For any real number $c \in \mathbb{R}$ and any $\mu > 0$, let the sequence $\left\{\left(a_{n,\mu}^c(\xi, x), H_{n,\mu}^c(\xi)\right)\right\}_{n \in \mathbb{N}}$ be obtained from the recursions (3.4) and (3.5), and $\{\bar{a}_n^c(\xi, x)\}_{n \in \mathbb{N}}$ be obtained from (5.2).

We first claim that, for any $n \in \mathbb{N}$ and any bounded subsets $K_1 \subset \mathbb{R}$ and $K_2 \subset \mathbb{R}$, $a_{n,\mu}^c(\xi + x, x)$ is Lipschitz continuous in $x \in K_2$ uniformly in $\xi \in K_1$ and uniformly in μ for all large positive μ, and there holds

$$\lim_{\mu \to \infty} H_{n,\mu}^c(\xi) = \infty \text{ uniformly in } \xi \in K_1, \tag{5.5}$$

$$\lim_{\mu \to \infty} a_{n,\mu}^c(\xi + x, x) = \bar{a}_n^c(\xi + x, x) \text{ uniformly in } x \in K_2, \ \xi \in K_1. \tag{5.6}$$

In the case of $n = 1$, according to the definitions of $H_{n,\mu}^c$ and $a_{n,\mu}^c$, we have

$$H_{1,\mu}^c(\xi) = \max\left\{H_0(\xi), h_{+,\mu}(\omega; \phi(\cdot + \xi + c, \cdot))\right\},$$

and

$$a_{1,\mu}^c(\xi + x, x) = \max\left\{\phi(\xi + x, x), U_\mu[\phi(\cdot + \xi + c, \cdot)](x)\right\},$$

where U_μ is the operator defined in Section 2.2. By [3, Theorem 5.4],[3] we obtain

$$h_{+,\mu}(\omega; \phi(\cdot + \xi + c, \cdot)) \to \infty \text{ as } \mu \to \infty \text{ uniformly in } \xi \in K_1,$$

and

$$U_\mu[\phi(\cdot + \xi + c, \cdot)](x) \to \bar{U}[\phi(\cdot + \xi + c, \cdot)](x) \text{ as } \mu \to \infty \text{ in } C^{1+\alpha}(K_2)$$

uniformly in $\xi \in K_1$, where $\bar{U}$ is given in (5.1). It then follows that (5.5) and (5.6) hold for $n = 1$, and $\partial_x U_\mu[\phi(\cdot + \xi + c, \cdot)](x)$ is uniformly bounded in $x \in K_2$, $\xi \in K_1$ for all large μ, say $\mu \geq \mu_0$. Thus, $a_{1,\mu}^c(\xi + x, x)$ is Lipschitz continuous in $x \in K_2$ uniformly in $\xi \in K_1$ and $\mu \geq \mu_0$. This proves our claim in the case of $n = 1$.

Now, suppose that our claim is valid for some $n = n_0 \in \mathbb{N}$; we want to prove that it still holds for $n = n_0 + 1$. Since $a_{n_0,\mu}^c(\xi + x, x)$ is Lipschitz continuous in $x \in K_2$ uniformly in $\xi \in K_1$ and $\mu \geq \mu_0$, by obvious modifications of [3, Theorem 5.4], we have

$$\lim_{\mu\to\infty} h_{+,\mu}(\omega; a_{n_0,\mu}^c(\cdot + \xi + c, \cdot)) = \infty \text{ uniformly in } \xi \in K_1,$$

and that

$$U_\mu[a_{n_0,\mu}^c(\cdot + \xi + c, \cdot)](x) \to \bar{U}[\bar{a}_{n_0}^c(\cdot + \xi + c, \cdot)](x) \text{ as } \mu \to \infty \text{ in } C^{1+\alpha}(K_2)$$

uniformly in $\xi \in K_1$. Then the same reasoning as in the case of $n = 1$ shows that our claim also holds for $n = n_0 + 1$.

Next, we prove that $c_{+,\mu_1} \lessapprox c_{+,\mu_2}$ whenever $0 < \mu_1 \leq \mu_2$. Indeed, it follows from the comparison principle [2, Proposition 2.14] that for any $\tilde{\phi} \in \mathcal{M}$,

$$U_{\mu_1}[\tilde{\phi}(\cdot + \xi - x, \cdot)](x) \leq U_{\mu_2}[\tilde{\phi}(\cdot + \xi - x, \cdot)](x) \text{ for all } \xi \in \mathbb{R}, x \in \mathbb{R}.$$

Since U_{μ_1} and U_{μ_2} are order-preserving operators, it follows from an induction argument that

$$a_{n,\mu_1}^c(\xi, x) \leq a_{n,\mu_2}^c(\xi, x) \text{ for all } \xi \in \mathbb{R}, x \in \mathbb{R}. \tag{5.7}$$

Passing to the limits $n \to \infty$ and $\xi \to \infty$, we obtain

$$a_{\mu_1}^c(\infty, x) \leq a_{\mu_2}^c(\infty, x) \text{ for all } x \in \mathbb{R}.$$

It follows immediately that

$$c_{+,\mu_1} \leq c_{+,\mu_2}.$$

Moreover, using (5.6) and (5.7), we also obtain

$$a_{n,\mu}^c(\xi, x) \leq \bar{a}_n^c(\xi, x) \text{ for all } \xi \in \mathbb{R}, x \in \mathbb{R}, \mu > 0.$$

By passing to the limit $n \to \infty$ and $\xi \to \infty$ it follows that $c_{+,\mu} \leq \bar{c}_+$. Thus, the limit $\lim_{\mu\to\infty} c_{+,\mu}$ exists and $\lim_{\mu\to\infty} c_{+,\mu} \leq \bar{c}_+$.

To end the proof, we need to show that $\lim_{\mu\to\infty} c_{+,\mu} = \bar{c}_+$. Assume by contradiction that $\lim_{\mu\to\infty} c_{+,\mu} < \bar{c}_+$. Then there exists some $c' \in \mathbb{R}$ such that $c_{+,\mu} < c' < \bar{c}_+$ for all $\mu > 0$. It follows from (5.4) that there is some $n_0 > 0$ such that

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[3] We remark that [3, Theorem 5.4] is concerned with the convergence (as $\mu \to \infty$) of the weak solutions in the sense of [3, Definition 2.1] in high space dimensions. By some slight modifications of the proof in [3, Theorem 5.4], we can conclude that such a convergence result is still valid for classical solutions to problem (1.1) with Lipschitz continuous initial data in $\mathcal{H}(g_0, h_0)$ and for classical solutions to problem (1.14) with Lipschitz continuous initial data in $\mathcal{H}_+(h_0)$.

$$\bar{a}_{n_0}^{c'}(h_0, x) > \phi(-\infty, x) \ \text{ for all } \ x \in \mathbb{R}.$$

Since the functions $a_{n_0,\mu}^{c'}(h_0, x)$ and $\bar{a}_{n_0}^{c'}(h_0, x)$ are L-periodic in $x \in \mathbb{R}$, it follows from (5.6) that

$$a_{n_0,\mu}^{c'}(h_0, x) \to \bar{a}_{n_0}^{c'}(h_0, x) \ \text{ as } \ \mu \to \infty \ \text{ uniformly in } \ x \in \mathbb{R}.$$

We then find some $\mu_1 > 0$ sufficiently large such that

$$a_{n_0,\mu_1}^{c'}(h_0, x) > \phi(-\infty, x) \ \text{ for all } \ x \in \mathbb{R}.$$

It follows from Lemma 3.7 that $c' < c_{+,\mu_1}$, which is in contradiction with the assumption that $c_{+,\mu} < c'$ for all $\mu > 0$. The proof of Lemma 5.2 is thereby complete. □

Proof of Theorem 1.3. Let $c_{+,\mu}^*$ be the rightward spreading speed for the free boundary problem (1.1). It then follows from Theorem 3.14 that $c_{+,\mu}^* = c_{+,\mu}/\omega$. By Lemmas 5.1 and 5.2, $c_{+,\mu}^*$ is nondecreasing in $\mu > 0$ and $\lim_{\mu\to\infty} c_{+,\mu}^* = \bar{c}_+^*$.

Correspondingly, we conclude that $c_{-,\mu}^*$ is nondecreasing in $\mu > 0$ and $\lim_{\mu\to\infty} c_{-,\mu}^* = \bar{c}_-^*$, where $c_{-,\mu}^*$ is the leftward spreading speed for problem (1.1) and $\bar{c}_-^*$ is the leftward spreading speed for problem (1.7). The proof of Theorem 1.3 is now complete. □

Conflict of interest statement

There is no conflict of interest.

Acknowledgement

References

[1] G. Bunting, Y. Du, K. Krakowski, Spreading speed revisited: analysis of a free boundary model, in: Special Issue Dedicated to H. Matano, Netw. Heterog. Media 7 (2012) 583–603.
[2] W. Ding, Y. Du, X. Liang, Spreading in space–time periodic media governed by a monostable equation with free boundaries, Part 1: Continuous initial functions, J. Differ. Equ. 262 (2017) 4988–5021.
[3] Y. Du, Z. Guo, The Stefan problem for the Fisher–KPP equation, J. Differ. Equ. 253 (2012) 996–1035.
[4] Y. Du, Z. Lin, Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary, SIAM J. Math. Anal. 42 (2010) 377–405.
[5] Y. Du, H. Matano, Convergence and sharp thresholds for propagation in nonlinear diffusion problems, J. Eur. Math. Soc. 12 (2010) 279–312.
[6] J. Fang, X. Yu, X.-Q. Zhao, Traveling waves and spreading speeds for time–space periodic monotone systems, J. Funct. Anal. 272 (2017) 4222–4262.
[7] X. Liang, Y. Yi, X.-Q. Zhao, Spreading speeds and traveling waves for periodic evolution systems, J. Differ. Equ. 231 (2006) 57–77.
[8] X. Liang, X.-Q. Zhao, Asymptotic speeds of spread and traveling waves for monotone semiflows with applications, Commun. Pure Appl. Math. 60 (2007) 1–40.
[9] X. Liang, X.-Q. Zhao, Spreading speeds and traveling waves for abstract monostable evolution systems, J. Funct. Anal. 259 (2010) 857–903.
[10] R. Lui, Biological growth and spread modeled by systems of recursions, I. Mathematical theory, Math. Biosci. 93 (1989) 269–295.
[11] G. Nadin, The principal eigenvalue of a space–time periodic parabolic operator, Ann. Mat. Pura Appl. 188 (2009) 269–295.
[12] G. Nadin, Traveling fronts in space–time periodic media, J. Math. Pures Appl. 92 (2009) 232–262.
[13] G. Nadin, Existence and uniqueness of the solutions of a space–time periodic reaction–diffusion equation, J. Differ. Equ. 249 (2010) 1288–1304.
[14] N. Sun, Asymptotic behavior of solutions of a degenerate Fisher–KPP equation with free boundaries, Nonlinear Anal., Real World Appl. 24 (2015) 98–107.
[15] H.F. Weinberger, Long-time behavior of a class of biological models, SIAM J. Math. Anal. 13 (1982) 353–396.
[16] H.F. Weinberger, On spreading speeds and traveling waves for growth and migration models in a periodic habitat, J. Math. Biol. 45 (2002) 511–548.