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Recurrence for the wind-tree model

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Abstract

In this paper, we give a geometric criterion ensuring the recurrence of the vertical flow on $\mathbb{Z}^d$-covers of compact translation surfaces ($d \geq 2$). We prove that the linear flow in the wind-tree model is recurrent for every pair of parameters and almost every direction.
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1. Introduction

Very little is known on the dynamics of the linear flows on non-compact translation surfaces. Some results exist for classes of examples. "Periodic" translation surfaces form a natural class actively studied. In this paper, we consider a translation surface $\hat{X}$ which is a ramified cover over a compact translation surface X, the covering group being $\mathbb{Z}^d$ ($d \geq 1$).[1] Let Σ be the finite set of branched points. Since the intersection form is non-degenerate between $H_1(X, \Sigma, \mathbb{Z})$ and $H_1(X \setminus \Sigma, \mathbb{Z})$, every cover is defined by a d-tuple of independent elements $\Gamma = (\gamma_1, \dots, \gamma_d)$ in the group of relative homology $H_1(X, \Sigma, \mathbb{Z})$. The d-tuple Γ is called the *cocycle* defining the covering $\hat{X}$. The holonomy of an element of $H_1(X, \Sigma, \mathbb{Z})$ is $\int_\gamma \omega$ where ω is the holomorphic 1-form defining the translation surface X. A necessary condition for recurrence is the so called *no drift condition*

$$hol(\gamma_i) = 0, \text{ for } i = 1 \dots d.$$

The Lebesgue measure is invariant by the linear flow on $\hat{X}$, it is an infinite measure. For $d = 1$ under the no drift condition, recurrence of the linear flow is a consequence of general principles: ergodicity of the flow on X implies

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[1] The reader interested only in the wind-tree model may ignore everything concerning ramification and, in particular, assume that the cover is defined by a sublattice in the *absolute* homology.

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recurrence on $\hat{X}$. This is not true in dimension $d \geq 2$. For translation surfaces, the first counter example is due to Delecroix [3].

In this paper, we give a geometric criterion ensuring recurrence for the linear flow on a $\mathbb{Z}^2$-cover of a compact translation surface (see sections 3 and 4). We apply this criterion to periodic versions of the wind-tree model introduced by Ehrenfest in 1912 ([5]). The model is the following: a point moves in the plane and collides with rectangular scatterers with the usual law of reflexion. The scatterers are identical rectangular obstacles located periodically along a square lattice on the plane, one obstacle centered at each point of $\mathbb{Z}^2$. The scatterers are rectangles of size (a, b), with $0 < a < 1$, $0 < b < 1$. We call the subset of the plane obtained by removing the obstacles the billiard table $T(a, b)$. Polygonal billiard is one of the main motivation to develop the theory of translation surfaces. Thus, it is important to understand the dynamics in this situation. The phase space splits into a family of invariant surfaces since the angles at the boundary of $T(a, b)$ are integer multiples of $\pi/2$. We prove

Theorem 1. *For every $(a, b) \in (0, 1) \times (0, 1)$, the billiard flow in the table $T_{a,b}$ is recurrent for almost every direction θ.*

Using the Katok–Zemliakov's construction, we replace the billiard flow in $T(a, b)$ by the linear flow on a non-compact translation surface $X^\infty_{a,b}$. The surface $X^\infty_{a,b}$ is a cover of a translation surface $X_{a,b}$ of genus 5 (see [4] for details). That's why we can apply the geometric criterion proven in section 4. Theorem 1 is a generalization of a result in [12]. Our result is optimal for two different reasons. For all rational parameters (a, b) there exists a set of positive Hausdorff dimension of non-recurrent directions on $X^\infty_{a,b}$ (see [3]). Moreover ergodicity is false by a result of Frączek and Ulcigrai (see [10]).

1.1. Outline of the paper

In section 3, we prove a general criterion for recurrence for linear flows on $\mathbb{Z}^2$-covers of compact translation surfaces. In section 4, we derive a geometric criterion for recurrence. In section 5, we check this criterion for the wind-tree model for generic parameters. This relies on a careful analysis of the existence of "good" cylinders. A crucial fact is that the surface $X_{a,b}$ is a cover of an L-shaped surface $L_{a,b}$. In section 6, we prove that the result is in fact true for every parameter. A key point is McMullen's classification of $\mathrm{SL}(2, \mathbb{R})$ invariant measures in the stratum $\mathcal{H}(2)$ (see [15]).

2. Background

For general references on translation surfaces we refer the reader to the survey of A. Zorich [18] or the course of M. Viana [17].

A *translation surface* is a surface which can be obtained by edge-to-edge gluing of polygons in the plane using translations only. Such a surface is endowed with a flat metric (the one inherited from $\mathbb{R}^2$) and with a choice of a canonical direction. There is a one to one correspondence between translation surfaces and compact Riemann surfaces equipped with a non-zero holomorphic 1-form. There is a canonical vertical direction in each translation surface and we refer to the flow in this direction as the *vertical flow*.

A *cylinder* on a translation surface is a maximal open annulus filled by homotopic simple closed geodesics. The direction of a cylinder is the direction of these geodesics. A cylinder is isometric to the product of an open interval and a circle. The core curve of a cylinder $\mathcal{C}$, denoted by $m(\mathcal{C})$, is the geodesic projecting to the middle of the interval. By convention, we puncture all the zeroes and the marked points. In particular, a maximal flat cylinder never has zeroes nor marked points inside it.

The moduli space of translation surfaces of genus g is stratified according to the degrees of zeros of the corresponding 1-forms. If $\alpha = (\alpha_1, ..., \alpha_s)$ is a partition of the even number $2g - 2$, $\mathcal{H}(\alpha)$ denotes the stratum consisting of 1-forms with zeros of degrees α_1, ..., α_s, on a Riemann surface of genus g. We denote by $\mathcal{H}^{(1)}(\alpha) \subset \mathcal{H}(\alpha)$ the codimension 1 subspace which consists of area 1 translation surfaces.

There is a natural action of $\mathrm{SL}(2, \mathbb{R})$ on strata $\mathcal{H}(\alpha)$ coming from the linear action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathbb{R}^2$. The *Teichmüller geodesic flow* on $\mathcal{H}_g$ is the action of the diagonal matrices $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$. We denote by $\mathcal{M}_g$ the moduli

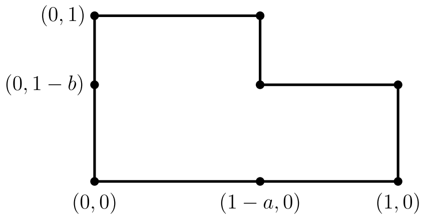

Fig. 1. The surface $L_{a,b}$: opposite sides are identified.

space $\mathcal{M}_g$ of closed compact Riemann surfaces of genus g. The image of the orbits $(g_t \cdot (X, \omega))_t$ in $\mathcal{M}_g$ are geodesic with respect to the Teichmüller metric. Each stratum $\mathcal{H}_g(\alpha)$ carries a natural *Lebesgue measure*, invariant under the action of $\mathrm{SL}(2, \mathbb{R})$. Moreover, this action preserves the area and hence $\mathcal{H}^{(1)}(\alpha)$. H. Masur [13] and independently W. Veech [16] proved that on each component of a normalized stratum $\mathcal{H}^{(1)}(\alpha)$ the total mass of the Lebesgue measure is finite and the geodesic flow acts ergodically with respect to this measure. Another important one parameter flow on $\mathcal{H}(\alpha)$ is the *horocycle flow* given by the action of $h_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$.

Stabilizers for the action of $\mathrm{SL}(2, \mathbb{R})$ in the stratum $\mathcal{H}(\alpha)$, called *Veech groups*, are discrete subgroups of $\mathrm{SL}(2, \mathbb{R})$. In exceptional cases Veech groups are lattices (i.e. finite-covolume subgroups) in $\mathrm{SL}(2, \mathbb{R})$ (see [16]), though they are never cocompact. Closed compact translation surfaces with a lattice Veech group are exactly those whose $\mathrm{SL}(2, \mathbb{R})$-orbit is closed in the corresponding stratum. They are called *Veech surfaces*. Their orbits project to *Teichmüller curves* in the moduli space $\mathcal{M}_g$.

The stratum $\mathcal{H}(2)$ is connected and is the best understood in higher genus. It was proven that the Teichmüller curves are generated by L-shaped surfaces of the form $\mathrm{L}(a, b)$ (see Fig. 1).

In his fundamental work, C. McMullen [15] proved a complete classification theorem for $\mathrm{SL}(2, \mathbb{R})$-invariant measures and closed invariant sets in the stratum $\mathcal{H}(2)$. The only $\mathrm{SL}(2, \mathbb{R})$-invariant irreducible closed subsets of $\mathcal{H}(2)$ are the Teichmüller curves and the whole stratum. The only ergodic $\mathrm{SL}(2, \mathbb{R})$-invariant probability measures are the Haar measure carried by Teichmüller curves and the Lebesgue measure on the stratum. Following McMullen, these measures will be called *Euclidean measures*.

Let $g \geq 2$. The *Hodge bundle* E_g is the real vector bundle of dimension $2g$ over $\mathcal{M}_g$ where the fiber over $X \in \mathcal{M}_g$ is the real cohomology $H^1(X; \mathbb{R})$. Each fiber $H^1(X; \mathbb{R})$ has a natural lattice $H^1(X; \mathbb{Z})$ which allows identification of nearby fibers and definition of the Gauss–Manin (flat) connection. Since branched $\mathbb{Z}^d$-covers are defined by relative cycles, we will also consider the *extended Hodge bundle* with fiber $H_1(X, \Sigma, \mathbb{R})$. The holonomy along the Teichmüller geodesic flow provides a cocycle called the *Kontsevich–Zorich cocycle*. Given a Teichmüller geodesic starting from a translation surface X and $\gamma \in H_1(X, \Sigma, \mathbb{Z})$ we denote by $G_t(\gamma) \in H_1(g_t(X), \Sigma, \mathbb{Z})$ the value of the Kontsevich–Zorich cocycle after time t. When $\Gamma = (\gamma_1, \ldots, \gamma_d)$ is a vector with coordinates in $H_1(X, \Sigma, \mathbb{Z})$, $G_t(\Gamma)$ is the vector $(G_t(\gamma_1), \ldots, G_t(\gamma_d))$. In the sequel, we will only work in local coordinates, thus homology (resp. cohomology) can be locally identified. Given a simply connected small open set U in a stratum, the Kontsevich–Zorich cocycle tells us how a cycle has been modified when a Teichmüller geodesics comes back to U.

Given X a compact translation surface, consider $\hat{X}$ a $\mathbb{Z}^d$-cover of X ramified over a finite set of points Σ. An isomorphism class of $\mathbb{Z}^d$-cover is defined by the kernel of an homomorphism ϕ from $\pi_1(X \setminus \Sigma)$ to $\mathbb{Z}^d$. As $\mathbb{Z}^d$ is an abelian group, the morphism ϕ factors through homology. It induces a morphism

$$\psi\ : H_1(X \setminus \Sigma, \mathbb{Z}) \to \mathbb{Z}^d.$$

As the intersection form $\iota : H_1(X \setminus \Sigma, \mathbb{Z}) \times H_1(X, \Sigma, \mathbb{Z}) \to \mathbb{Z}$ is non-degenerate, by Poincaré duality, there exists $\Gamma = (\gamma_1, \cdots, \gamma_d) \in H_1(X, \Sigma, \mathbb{Z})$ such that for every $\gamma \in H_1(X \setminus \Sigma, \mathbb{Z})$,

$$\psi(\gamma) = (\iota(\gamma, \gamma_1), \cdots \iota(\gamma, \gamma_d)).$$

Since $\hat{X}$ is a $\mathbb{Z}^d$-cover of X, $\Gamma = (\gamma_1, \cdots, \gamma_d)$ is a family of independent cycles, thus Γ is a sublattice in $H_1(X, \Sigma, \mathbb{Z})$. In the sequel, we will denote by X_Γ the cover of X associated to Γ.

A necessary condition for the vertical flow on X_Γ to be recurrent (see [11] for a proof) is

$$\Re(hol(\gamma_1)) = \cdots = \Re(hol(\gamma_d)) = 0.$$

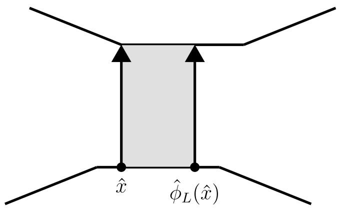

Fig. 2. A C-recurrent rectangle in the vertical direction. The horizontal sides are identified together.

This is called the no drift condition for $\Re(\omega)$ where ω is the holomorphic 1-form defining X. A necessary condition to get recurrence of the linear flow in two (or more) directions on X_Γ is

$$hol(\gamma_1) = \cdots = hol(\gamma_d) = 0.$$

This is called the no drift condition.

3. Recurrence criterion

Definition 1. Let X be a compact translation surface, Γ a cocycle, X_Γ the $\mathbb{Z}^d$-cover of X associated to Γ and $\hat{\phi}_t$ the vertical flow on X_Γ. Given a positive number C we say that X_Γ is C-recurrent if there is an embedded rectangle $R = I \times [0, L)$ in X of measure larger than $1/C$ with $L > 1/C$ such that if $x \in R$ then for every preimage $\hat{x} \in X_\Gamma$ we have:

- $\hat{x} \in X_\Gamma$ and $\hat{\phi}_L(\hat{x})$ belong to the same horizontal leaf
- the distance d_H along the horizontal leaf between $\hat{x}$ and $\hat{\phi}_L(\hat{x})$ satisfies

$$d_H(\hat{x}, \hat{\phi}_L(\hat{x})) < C.$$

Remark 1. An example of a C-recurrent X_Γ is depicted in Fig. 2.

To avoid overloading the definition, only one constant C is introduced. Nevertheless, the hypothesis on the area of R, its height and the distance between $\hat{x}$ and $\hat{\phi}_L(\hat{x})$ are independent.

Proposition 1. *Let X be a compact translation surface of area 1, Γ a cocycle and $C > 0$. Assume that there exists a sequence of real numbers (t_n) tending to infinity such that $g_{t_n}(X)_{\Gamma_n}$ is C-recurrent for every n where $\Gamma_n = G_{t_n}(\Gamma)$. Then if the flow ϕ_t is ergodic on X, the vertical flow $\hat{\phi}_t$ is recurrent on X_Γ.*

Proof. We denote by $\tilde{R}_n$ the rectangle which is C-recurrent for Γ_n on $g_{t_n}(X)$ and by R_n its preimage by g_{t_n}. Teichmüller flow in backward direction contracts horizontals and expands verticals. Thus, the height L_n of R_n, i.e. the length L_n of the vertical side of R_n, is at least e^{t_n}/C and its width is at most e^{-t_n}. Therefore, if $x \in X$ belongs to R_n, we have $d_H(\hat{x}, \hat{\phi}_{L_n}(\hat{x})) < Ce^{-t_n}$. Thus the vertical trajectory of a point which belongs to R_n for infinitely many n is recurrent.

Lemma 1. *Almost every point $x \in X$ belongs to R_n for infinitely many n.*

Proof. The width of the rectangle R_n is at most e^{-t_n}/C. We consider a subsequence of real numbers (still denoted by (t_n)) such that $\sum_{n=0}^{\infty} e^{-t_n}$ is finite.

Denote by

$$\Omega = \{x \in X \ \text{ such that, for infinitely many } n,\ x \in R_n\}.$$

We have $\lambda(\Omega) \geq 1/C$ since $\lambda(R_n) = \lambda(R) > C$. Let us prove that Ω is ϕ_t invariant mod 0 for every $t > 0$. This will prove that $\lambda(\Omega) = 1$. For $t > 0$ fixed, the t-top of R_n is the set $A_n = I \times [e^{t_n}L_n - 2t, e^{t_n}L_n)$ where L_n is the length

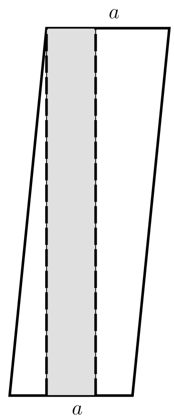

Fig. 3. A cylinder in the deformed surface and the associated C-recurrent rectangle.

of $\tilde{R}_n$. This set is defined for n large enough. As $\sum_{n=0}^{\infty} e^{-t_n}$ is finite, by Borel–Cantelli Lemma, almost every $x \in X$ belongs to a finite number of t-top of the rectangles R_n. Take $x \in \Omega$ and in this set of full measure. Consequently, for infinitely many n, x belongs to $R_n \setminus A_n$. Thus $\phi_t(x)$ belongs to infinitely many rectangles R_n which means that $\phi_t(x) \in \Omega$. This ends the proof of the lemma. □

The proof of Proposition 1 is now complete. □

Remark 2. This proposition is an avatar of Masur's criterion for unique ergodicity of the vertical flow on a compact translation surface [14].

4. Geometric criterion for recurrence

Through this section, $\mathcal{L}$ will be a closed g_t invariant locus and $\mathcal{B}$ a flat continuous linear subbundle of the extended Hodge bundle over $\mathcal{L}$.

Remark 3. Since $\mathcal{L}$ is g_t-invariant and $\mathcal{B}$ is flat, $\mathcal{B}$ is invariant under the Kontsevich–Zorich cocycle.

Definition 2. Fix $X \in \mathcal{L}$. A cylinder $\mathcal{C} \subset X$ is said to be $\mathcal{B}$-good if $i(m(\mathcal{C}), \gamma) = 0$ for all $\gamma \in \mathcal{B}_X$.

It immediately follows from Definition 2 that if a cylinder $\mathcal{C}$ on a flat surface X from the family $\mathcal{L}$ is $\mathcal{B}$-good, then for any $\Gamma \subset \mathcal{B}_X \cap H_1(X, \Sigma, \mathbb{Z})$ the pullback to the (branched) cover X_Γ is a union of disjoint cylinders isometric to $\mathcal{C}$. (Recall that by convention the cylinder may not contain any branching points of the cover.)

Now we give a strong relation between the existence of good cylinders and the C-recurrence property.

Lemma 2. *Let X in $\mathcal{L}$ with a vertical $\mathcal{B}$-good cylinder. There exists a neighborhood $U \subset \mathcal{L}$ of X and a $C > 0$ such that every surface $Y \in U$ is C-recurrent for every Γ with coordinates in $\mathcal{B}_Y \cap H_1(Y, \Sigma, \mathbb{Z})$.*

Proof. Assume that X has area 1 and contains a $\mathcal{B}$-good vertical cylinder of area at least $2/C$ and width at most $2/C$. Cylinders are stable under small perturbations in the strata of abelian differentials. Thus, in a neighborhood of X, there is a metric cylinder whose core curve is homologous to $m(\mathcal{C})$ and direction close to be vertical. In a nearby direction, this cylinder contains a rectangle which takes up an arbitrary large proportion of the cylinder (see Fig. 3).

We fix U a neighborhood of X small enough so that this cylinder contains a rectangle whose sides are horizontal and vertical, area is at least $1/C$ and height L at least $1/C$. Let $Y \in U$ and Γ with coordinates in $\mathcal{B}_Y$. The previous part of the argument provides on Y a cylinder $\mathcal{C}(Y)$ and a rectangle $\mathcal{R}(Y)$. Note that $\mathcal{C}(Y)$ is $\mathcal{B}$-good, so the lift of the cylinder $\mathcal{C}(Y)$ in the cover Y_Γ is a union of cylinders isometric to $\mathcal{C}(Y)$. Moreover, the width w of the rectangle $\mathcal{R}(Y)$ satisfies $w \le C$ since $L > 1/C$. Thus, if $\hat{x} \in \mathcal{R}(Y)$, then $d(\hat{x}, \hat{\phi}_L(\hat{x})) < C$. □

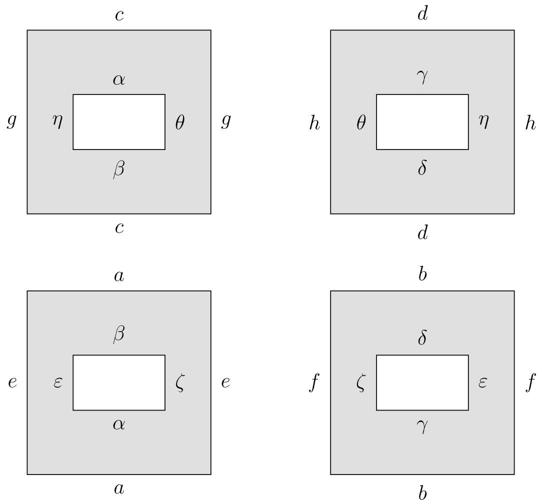

Fig. 4. The genus 5 surface obtained from the wind-tree model. Identifications are indicated by Greek and Latin letters.

Proposition 2. *Let X in $\mathcal{L}$ with a vertical $\mathcal{B}$-good cylinder. Let $Y \in \mathcal{L}$ and Γ be a d-tuple of elements in $\mathcal{B}_Y \cap H_1(Y, \Sigma, \mathbb{Z})$. If the positive g_t-orbit of Y accumulates on X then the vertical flow is recurrent on Y_Γ.*

Proof. Denote by (t_n) the subsequence such that $g_{t_n}(Y)$ tends to X and let $\Gamma_n = G_{t_n}(\Gamma)$. By Masur's criterion, the vertical flow on Y is ergodic. Since $g_{t_n}(Y)$ tends to X and $\mathcal{B}$ is invariant by the Kontsevich–Zorich cocycle, by Lemma 2, for n large enough, $g_{t_n}(Y)$ is C-recurrent and then by Proposition 1 the vertical flow is recurrent on Y_Γ. □

5. Recurrence for the wind-tree model: almost everywhere statement

In this section, we apply our method to the wind-tree model. We check the geometric criterion for the $SL(2, \mathbb{R})$-invariant locus responsible for the dynamics of the wind-tree model.

5.1. *Summary of results on the wind-tree model*

We recall here results from [4]. The billiard flow is described by the linear flow on a non-compact translation surface $X^\infty_{a,b}$. The surface $X^\infty_{a,b}$ is a $\mathbb{Z}^2$-cover of a genus 5 surface $X_{a,b}$ which is itself a non-ramified cover of degree 4 of an L-shaped surface $L_{a,b}$ in the stratum $\mathcal{H}(2)$. This covering construction can be done for every surface in $\mathcal{H}(2)$. We denote by $\mathcal{L}$ the locus of these covers which is isomorphic to $\mathcal{H}(2)$.

The Klein group K acts on $X_{a,b}$ by translations (see Fig. 4). This action induces a splitting of $H_1(X_{a,b}, \mathbb{Z})$ which is $SL(2, \mathbb{R})$ invariant.

Denote by τ_h, τ_v and $\tau_h\tau_v$ the non-trivial elements of K. τ_h (resp. τ_v) permutes the fundamental domains horizontally (resp. vertically).

We have:

$$H_1(X_{a,b}, \mathbb{R}) = E^{++} \oplus (E^{+-} \oplus E^{-+}) \oplus E^{--}$$

where E^{++} is the vector space invariant by τ_h and τ_v, $E^{+-}(\mathbb{Z})$ the vector space invariant by τ_h and anti-invariant by τ_v, etc. This decomposition is $SL(2, \mathbb{R})$ equivariant. The coordinates of the $\mathbb{Z}^2$ cocycle defining $X^\infty_{a,b}$ belong to $E^{+-} \oplus E^{-+}$. The invariant vector space by $\tau_h\tau_v$ is $E^{++} \oplus E^{--}$. The quotient surface $X_{a,b}/\tau_h\tau_v$ is a hyperelliptic surface (it belongs to the hyperelliptic locus of the non-hyperelliptic component of the stratum $\mathcal{H}(2, 2)$).

On the surface $L_{a,b}$ two Weierstrass points are distinguished by the cocycle defining $X^\infty_{a,b}$ and are denoted by E and F (see Fig. 5). In geometric terms, E is uniquely defined by the condition that the horizontal trajectory through the Weierstrass point E hits the singularity; F is uniquely defined by the condition that the vertical trajectory through the Weierstrass point F hits the singularity.

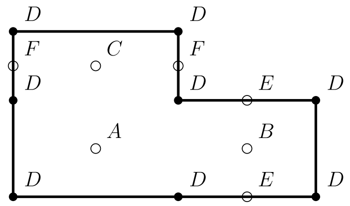

Fig. 5. Weierstrass points on $L_{a,b}$.

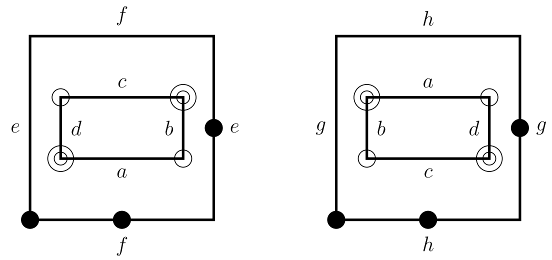

Fig. 6. Weierstrass points on the surface $X_{a,b}/\tau_h\tau_v$. The identification are indicated by the letters a, b, c, d, e, f, g, h. The singularities are Weierstrass points.

5.2. Good cylinders in the wind-tree model

We now give a refined version of Lemma 10 of [12] in the adequate language for our purpose.

Lemma 3. *The lift to $X_{a,b}$ of a cylinder in $L_{a,b}$ whose core curve contains E and F is the union of two cylinders which are homologous. The homology class of the core curve of each cylinder belongs to E^{++}.*

Proof. Since E and F are Weierstrass points on $L_{a,b}$, every trajectory from E to F closes up in $L_{a,b}$. The length of the closed curve γ is twice the length of the segment EF. In [12] Lemma 10 a symmetry argument shows that γ lifts in $X^{\infty}_{a,b}$ to the core of a cylinder whose length is twice the length of γ. The proof is explained in the language of billiards. The symmetry argument shows that in the billiard table $T_{a,b}$, the trajectory is symmetric with respect to a lattice point. It contains two preimages of E (and F) with opposite vector. We now translate this argument in $X_{a,b}$. Let $\hat{\gamma}$ be a preimage of γ in $X_{a,b}$ containing $\hat{E}$ a preimage of E. The previous argument means that $\hat{\gamma}$ contains $\hat{E}$ and $\tau_h\tau_v(\hat{E})$. This implies that the homology class of $\hat{\gamma}$ is $\tau_h\tau_v$-invariant. Thus, it belongs to $E^{++} \oplus E^{--}$. The same is true for the other preimage of γ denoted by $\hat{\gamma'}$. The vector space $E^{++} \oplus E^{--}$ is identified with $H_1(X_{a,b}/\tau_h\tau_v, \mathbb{R})$ (it is the $\tau_h\tau_v$-invariant part of $H_1(X_{a,b}, \mathbb{R})$). Denote by $\hat{\gamma}$ and $\hat{\gamma'}$ the projections of $\hat{\gamma}$ and $\hat{\gamma'}$ in $X_{a,b}/\tau_h\tau_v$.

We now prove that $\hat{\gamma}$ and $\hat{\gamma'}$ are equal in $H_1(X_{a,b}/\tau_h\tau_v, \mathbb{R})$. A simple calculation shows that the Weierstrass points in $X_{a,b}/\tau_h\tau_v$ are the preimages of the Weierstrass points A, B, C, D in $L_{a,b}$ (see Fig. 6).

In $L_{a,b}$ the curve γ contains the two Weierstrass points E and F thus it does not contain any other Weierstrass point. Therefore $\hat{\gamma}$ and $\hat{\gamma'}$ are not fixed by the hyperelliptic involution ι in $X_{a,b}/\tau_h\tau_v$. Thus, we have $\iota(\hat{\gamma}) = -\hat{\gamma'}$. This implies that $\hat{\gamma}$ and $\hat{\gamma'}$ are homologous in $X_{a,b}/\tau_h\tau_v$. Consequently $\hat{\gamma}$ and $\hat{\gamma'}$ are homologous in $X_{a,b}$. We also have

$$\hat{\gamma'} = \tau_h(\hat{\gamma}) = \tau_v(\hat{\gamma}).$$

This yields that the homology class of $\hat{\gamma}$ is τ_h and τ_v invariant thus it belongs to E^{++}. □

Corollary 1. *Any lift to $X_{a,b}$ of a cylinder in $L_{a,b}$ whose core curve contains E and F is a $E^{+-} \oplus E^{-+} \oplus E^{--}$ good cylinder.*

Proof. Lemma 3 shows that the core curve of such a cylinder belongs to E^{++} which is the symplectic orthocomplement of $E^{+-} \oplus E^{-+} \oplus E^{--}$. This means that this cylinder is $E^{+-} \oplus E^{-+} \oplus E^{--}$ good on $X_{a,b}$. □

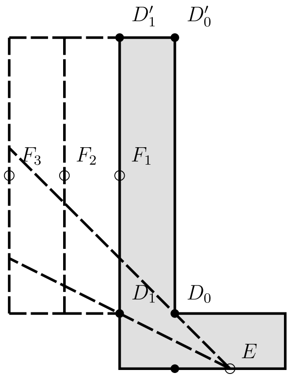

Fig. 7. Orbit from E to F.

5.3. Almost everywhere statement

We now apply the geometric criterion to prove an intermediate statement. We prove recurrence of the cocycle for almost every surface in $\mathcal{L}$ with respect to any $\mathrm{SL}(2,\mathbb{R})$ ergodic invariant probability measure.

Proposition 3. *Let $\mathcal{C}$ be a connected closed invariant subspace of $\mathcal{L}$, μ be its Euclidean measure and let X be in $\mathcal{C}$. Let U be a neighborhood in $\mathcal{C}$ of X. Denote by Γ a cocycle with values in $E^{+-} \oplus E^{+-} \oplus E^{--}$. For μ almost every $Y \in U$ the vertical flow is recurrent in Y_Γ.*

Remark 4. This applies to the wind-tree model since the cocycle defining it belongs to $E^{+-} \oplus E^{+-}$.

We need the following lemma.

Lemma 4. *On every L-shaped surface there is a cylinder with Weierstrass points E and F in its core curve.*

Proof. We take coordinates on the L-shaped surface as in Fig. 7. We denote by $\mathcal{R}$ the rectangle $D_0D_1D_1'D_0'$. We unfold this rectangle along the vertical side containing D_1 and F_1. We obtain points $F_1, \ldots, F_n$ with the same y-coordinate in the complex plane. In the horizontal strip ending at the vertical segment D_0D_0', there is no singularity except on the horizontal boundaries. We now consider the cone bounded by the lines ED_0 and ED_1. As the slope of ED_0 is larger than the slope of ED_1, every point in $\mathcal{R}$ has a preimage in the strip contained in the cone. Thus there is a segment joining E to each point of $\mathcal{R}$ and thus to F. □

Proof of Proposition 3. Fix μ a $\mathrm{SL}(2,\mathbb{R})$ ergodic invariant probability measure on $\mathcal{L}$ and Y a generic surface for μ. By McMullen's classification, the support of every ergodic measure in $\mathcal{H}(2)$ contains a L-shaped surface. Thus the support of μ contains a surface $X_{a,b}$ for some $(a,b) \in (0,1)^2$. By Corollary 1 and Lemma 4, the surface $X_{a,b}$ contains a $E^{+-} \oplus E^{-+} \oplus E^{--}$ good cylinder. As Y is a generic point, its orbit under the geodesic flow accumulates to $X_{a,b}$. Thus by Proposition 2, the vertical flow is recurrent on Y. □

6. Everywhere statement

The result of this section is an easy consequence of the work of Eskin, Mirzakhani and Mohammadi (see [8] and [9]). It can also be deduced from [2]. We give a direct proof in this elementary situation.

First fix some notations. Our convention for subgroups and elements in $\mathrm{SL}(2,\mathbb{R})$ is the following: the rotation of angle θ is denoted by r_θ, the subgroup P is the group of upper triangular matrices K is the orthogonal group $\mathrm{SO}(2,\mathbb{R})$. We recall that the geodesic flow is the one parameter flow $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, $t \in \mathbb{R}$, the horocycle flow is the one parameter flow $h_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$, $s \in \mathbb{R}$.

Let $\mathcal{L}$ be the locus defined in § 5.1, let X be a translation surface belonging to $\mathcal{L}$ and Ω be a set of positive measure of the unit circle. We denote by ν the normalized Lebesgue measure on Ω . X considered as a subset of K . X. We first prove some useful lemmas.

Lemma 5. *Every limit point ν_∞ of the family of probability measure*

$$\frac{1}{T}\int_0^T g_t \nu$$

is a probability measure which is P invariant.

Proof. Every limit of $g_t \nu$ is a probability measure in $\mathcal{L}$. This is a direct consequence of the results of Eskin–Masur [6] and Athreya [1]. No mass can escape to infinity.

Claim 1. *Any limit μ of $g_t \nu$ is h_s invariant.*

The proof of the claim is an adaptation of Eskin–Marklof–Morris Lemma 7.3 (see [7]). We fix a sequence t_i such that $g_{t_i}\nu$ tends to μ. Fix $s \in \mathbb{R}$, a matrix calculation proves that there exists a sequence (θ_j) tending to zero such that $g_{t_i} r_{\theta_j} g_{t_i}^{-1}$ tends to h_s. Let us prove that

$$g_{t_i} r_{\theta_j} g_{t_i}^{-1} g_{t_i}\nu - g_{t_i}\nu$$

tends to zero as j tends to infinity. Passing to the limit this will prove that μ is h_s invariant. We use the fact that ν is absolutely continuous with respect to Lebesgue measure on the circle. This means that there is a non-negative measurable function ϕ such that

$$d\nu = \phi d\theta \text{ and } \int_{\mathbb{S}^1} \phi d\theta = 1.$$

Let f be a bounded continuous function on $\mathcal{L}$ and let

$$\Delta_j = \int_{\mathcal{L}} f(M) g_{t_i} r_{\theta_j} g_{t_i}^{-1} g_{t_i}\nu - \int_{\mathcal{L}} f(M) g_{t_i}\nu = \int_{\mathcal{L}} f(M) g_{t_i}(r_{\theta_j} d\nu - d\nu) = \int_{\mathcal{L}} f(g_{t_i}^{-1} M)(\phi \circ r_{\theta_j} - \phi) d\theta.$$

Thus

$$|\Delta_j| \leq ||f||_\infty ||\phi \circ r_{\theta_j} - \phi||_1$$

where $|| \, ||_\infty$ is the sup norm on bounded continuous functions on $\mathcal{L}$ and $|| \, ||_1$ is L^1 norm in the unit circle with respect to Lebesgue measure. For every $\phi \in L^1$, $||\phi \circ r_{\theta_j} - \phi||_1$ tends to zero as j tends to infinity. Consequently (Δ_j) tends to zero as j tends to infinity which proves the claim.

Now ν_∞ is obtained as a Cesaro mean. Thus it is g_t invariant. Moreover, by the claim, it is a convex combination of h_s invariant measures. Since the set of h_s invariant measures is a convex set, ν_∞ is h_s invariant. This means that it is P invariant. □

Consequently the support Σ of ν_∞ is P invariant.

Lemma 6. *The set $K\Sigma$ is* SL(2, $\mathbb{R}$) *invariant. The set Σ contains a Teichmüller curve.*

Proof. A direct calculation shows that the set $K\Sigma$ is SL(2, $\mathbb{R}$) invariant since the set Σ is P invariant. By McMullen classification, $K\Sigma$ contains a Teichmüller curve $\mathcal{M}$. Thus $\Sigma \cap \mathcal{M}$ is a closed P invariant set and contained in the homogeneous space $\mathcal{M}$. Thus, by Ratner's theorem, $\Sigma \cap \mathcal{M} = \mathcal{M}$. □

We now prove that:

Lemma 7. *For every surface X in the locus $\mathcal{L}$, for almost every θ, the set of limit points of $g_t r_\theta X$ contains a Teichmüller curve.*

Proof. Assume by contraction that there is a set of positive Lebesgue measure Ω_1 contained in the circle which does not satisfy the conclusion of the lemma. For every Teichmüller curve $\mathcal{M}$, fix a point $x_{\mathcal{M}}$ with dense g_t-orbit in $\mathcal{M}$. By hypothesis, this point $x_{\mathcal{M}}$ does not belong to the omega-limit set of $g_t r_\theta X$ for θ in Ω_1 (otherwise this limit set would contain a Teichmüller curve since it is a closed set).

For each k, denote by $V_k(x_{\mathcal{M}})$ a neighborhood of $x_{\mathcal{M}}$ of diameter $1/k$. For every $\theta \in \Omega_1$ there is k and a t_θ such that

$$V_k(x_{\mathcal{M}}) \cap \{g_t r_\theta X, t \geq t_\theta\} = \emptyset.$$

Consequently, there exists $\kappa(\mathcal{M})$ and a set of positive measure $\Omega_2 \subset \Omega_1$ such that, for every $\theta \in \Omega_2$, there is a t_θ with

$$V_{\kappa(\mathcal{M})}(x_{\mathcal{M}}) \cap \{g_t r_\theta X, t \geq t_\theta\} = \emptyset.$$

Let U be the open subset of $\mathcal{L}$ defined by

$$U = \bigcup_{\mathcal{M}} V_{\kappa(\mathcal{M})}(x_{\mathcal{M}}).$$

For every $\theta \in \Omega_2$, $g_t r_\theta X$ does not enter U for t large enough. Restricting Ω_2 further, there is a set of positive measure $\Omega_3 \subset \Omega_2$ and a T such that for $t \geq T$, $g_t r_\theta X$ does not enter U.

We perform the same construction as in Lemma 5 replacing Ω by Ω_3. We fix a limiting measure ν'_∞ obtained by this process. By construction of ν'_∞, $\nu'_\infty U = 0$. This is a contradiction with Lemma 6 since U intersects every Teichmüller curve. $\square$

We now complete the proof of Theorem 1. Let $X = X_{a,b}$ be a surface obtained by the wind-tree construction for some parameters (a, b). By Lemma 7, for almost every θ, the limit points of $g_t r_\theta X$ contains a Teichmüller curve. By McMullen classification of Teichmüller curves in genus 2 every Teichmüller curve is generated by a L-shaped polygon L. Thus L is a limit point of $g_t r_\theta X$. Moreover L contains a $E^{+-} \oplus E^{-+} \oplus E^{--}$ good cylinder by Corollary 1. As it is explained in § 5.1 the flow in $X^\infty_{a,b}$ is defined by a cocycle over $X_{a,b}$ with coordinates in $E^{+-} \oplus E^{-+}$. Therefore by Proposition 2, the flow on $X^\infty_{a,b}$ is recurrent for almost every θ. This ends the proof of Theorem 1.

Conflict of interest statement

No conflict of interest.

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