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# On decay of periodic entropy solutions to a scalar conservation law ☆

E.Yu. Panov

*Novgorod State University, 41, B. St-Peterburgskaya, 173003 Veliky Novgorod, Russia*

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**Abstract**

We establish a necessary and sufficient condition for decay of periodic entropy solutions to a multidimensional conservation law with merely continuous flux vector.


**Résumé**

Nous considérons les lois de conservation [hyperboliques] en plusieurs dimensions d'espace avec la fonction de flux seulement continue. Nous établissons une condition nécessaire et suffisante pour la décroissance des solutions entropiques périodiques de ce problème.


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*Keywords:* Conservation laws; Periodic entropy solutions; Decay property; $H$-measures

## 1. Introduction

In the half-space $\Pi = \mathbb{R}_+ \times \mathbb{R}^n$, $\mathbb{R}_+ = (0, +\infty)$, we consider a first order multidimensional conservation law

$$u_t + \operatorname{div}_x \varphi(u) = 0, \tag{1.1}$$

where the flux vector $\varphi(u)$ is supposed to be only continuous: $\varphi(u) = (\varphi_1(u), \dots, \varphi_n(u)) \in C(\mathbb{R}, \mathbb{R}^n)$. Recall the notion of entropy solution of (1.1) in the sense of S.N. Kruzhkov [4].

**Definition 1.** A bounded measurable function $u = u(t, x) \in L^\infty(\Pi)$ is called an entropy solution (e.s. for short) of (1.1) if for all $k \in \mathbb{R}$

$$|u-k|_t + \operatorname{div}_x\big[\operatorname{sign}(u-k)\big(\varphi(u) - \varphi(k)\big)\big] \leqslant 0 \tag{1.2}$$

in the sense of distributions on $\Pi$ (in $\mathcal{D}'(\Pi)$).

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*E-mail address:* Eugeny.Panov@novsu.ru.

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Condition (1.2) means that for all non-negative test functions $f = f(t,x) \in C_0^1(\Pi)$

$$\int_\Pi \big[|u-k|f_t + \operatorname{sign}(u-k)\big(\varphi(u)-\varphi(k)\big)\cdot \nabla_x f\big]\,dt\,dx \geqslant 0$$

(here "·" denotes the inner product in $\mathbb{R}^n$).

As was shown in [13] (see also [14]), an e.s. $u(t,x)$ always admits a strong trace $u_0 = u_0(x) \in L^\infty(\mathbb{R}^n)$ on the initial hyperspace $t=0$ in the sense of relation

$$\operatorname*{ess\,lim}_{t\to 0} u(t,\cdot) = u_0 \quad \text{in } L^1_{loc}\big(\mathbb{R}^n\big), \tag{1.3}$$

that is, $u(t,x)$ is an e.s. to the Cauchy problem for Eq. (1.1) with initial data

$$u(0,x) = u_0(x). \tag{1.4}$$

**Remark 1.1.** It was also established in [13, Corollary 7.1] that, after possible correction on a set of null measure, an e.s. $u(t,x)$ is continuous on $\mathbb{R}_+$ as a map $t \mapsto u(t,\cdot)$ of $\mathbb{R}_+$ into $L^1_{loc}(\mathbb{R}^n)$.

When the flux vector is Lipschitz continuous, the existence and uniqueness of e.s. to the problem (1.1), (1.4) are well-known (see [4]). In the case under consideration when the flux functions are merely continuous, the effect of infinite speed of propagation for initial perturbations appears, which leads even to the nonuniqueness of e.s. to problem (1.1), (1.4) if $n > 1$ (see examples in [5,6]).

But, if initial function is periodic (at least in $n-1$ independent directions), the uniqueness holds: an e.s. of (1.1), (1.4) is unique and space-periodic, see the proof in [11,12]. In the present paper we assume that the requirement of space-periodicity holds: $u(t,x+e_i) = u(t,x)$ for almost all $(t,x) \in \Pi$ and all $i = 1,\dots,n$, where $\{e_i\}_{i=1}^n$ is a basis of periods in $\mathbb{R}^n$. We assume that this basis is fixed. Then, without loss of generality, we may suppose that $\{e_i\}_{i=1}^n$ is the canonical basis. We denote by $P = [0,1)^n$ the corresponding fundamental parallelepiped (cube) (which can be identified with a torus).

As was established by G.-Q. Chen and H. Frid [1], under the conditions $\varphi(u) \in C^2(\mathbb{R},\mathbb{R}^n)$ and

$$\forall (\tau,\xi) \in \mathbb{R}^{n+1},\ (\tau,\xi) \neq 0, \quad \operatorname{meas}\big\{u \in \mathbb{R} \bigm| \tau + \varphi'(u)\cdot\xi = 0\big\} = 0, \tag{1.5}$$

the following decay result holds for space-periodic e.s. $u(t,x)$ of (1.1), (1.4):

$$\operatorname*{ess\,lim}_{t\to\infty} u(t,\cdot) = \text{const} = \frac{1}{|P|}\int_P u_0(x)\,dx \quad \text{in } L^1(P). \tag{1.6}$$

Here $|P|$ denotes the Lebesgue measure of $P$ (in the case under consideration $P$ is a unite cube and, therefore, $|P| = 1$).

**Definition 2.** We will say that Eq. (1.1) satisfies *the decay property* if (1.6) holds for every periodic e.s.

In the present paper we propose the following necessary and sufficient condition for the decay property (by $\mathbb{Z}$ we denote the set of integers)

$$\forall (\tau,\xi) \in \mathbb{R}\times\mathbb{Z}^n,\ (\tau,\xi) \neq 0, \quad \text{the function } u \mapsto \tau u + \varphi(u)\cdot\xi \text{ is not constant on non-empty intervals,} \tag{1.7}$$

in the general case of only continuous flux vector $\varphi(u)$. Obviously, this condition is equivalent to the requirement that $\forall \xi \in \mathbb{Z}^n$, $\xi \neq 0$, *the functions* $u \mapsto \varphi(u)\cdot\xi$ *are not affine on non-empty intervals*.

Thus, our main result is the following theorem.

**Theorem 1.1.** *Eq.* (1.1) *satisfies the decay property if and only if condition* (1.7) *holds.*

**Remark 1.2.** In the case of arbitrary basis of periods $\{e_i\}_{i=1}^n$ one can make the linear change of variables $x = x(y) = \sum_{i=1}^n y_i e_i$. Then, as is easily verified, if $u(t,x)$ is a periodic entropy solution of (1.1) then the function $v(t,y) = u(t,x(y))$ is an entropy solution of the equation

$$v_t + \operatorname{div}_y \tilde{\varphi}(v) = 0,$$

where $\tilde{\varphi}_j(v) = \varphi(v) \cdot e'_j$, $\{e'_j\}_{j=1}^n$ being the dual basis:

$$e'_j \cdot e_i = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Clearly, $v(t, y)$ is a $y$-periodic function with the canonical basis of periods. By Theorem 1.1 the necessary and sufficient condition for $v(t, y)$ to satisfy the decay property is condition (1.7):

$$\forall \eta \in \mathbb{Z}^n, \ \eta \neq 0, \quad \text{the function } v \mapsto \tilde{\varphi}(v) \cdot \eta \text{ is not affine on non-empty intervals.}$$

Observe that $\tilde{\varphi}(v) \cdot \eta = \varphi(v) \cdot \xi$, where $\xi = \sum_{j=1}^n \eta_j e'_j \in L'$ and $L' = \{\sum_{j=1}^n \eta_j e'_j \mid \eta_j \in \mathbb{Z}\}$ is the dual lattice to the lattice of periods $L = \{\sum_{i=1}^n \xi_i e_i \mid \xi_i \in \mathbb{Z}\}$. It is obvious that the decay property for $v(t, y)$ holds if and only if this property holds for original solution $u(t, x)$.

Therefore, in the case of arbitrary basis of periods, Theorem 1.1 remains valid after replacement of the lattice $\mathbb{Z}^n$ in (1.7) by $L'$:

$$\forall \xi \in L', \ \xi \neq 0, \quad \text{the function } u \mapsto \varphi(u) \cdot \xi \text{ is not affine on non-empty intervals.}$$

By Remark 1.2 in the case when the basis of periods is not fixed and may depend on a solution, the statement of Theorem 1.1 remains valid after replacement of condition (1.7) by the following stronger one:

$$\forall (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n, \ (\tau, \xi) \neq 0, \quad \text{the function } u \mapsto \tau u + \varphi(u) \cdot \xi \text{ is not constant on non-empty intervals.} \tag{1.8}$$

Obviously, condition (1.8) is strictly weaker than (1.5) even in the case of smooth flux $\varphi(u)$.

## 2. Preliminaries

To prove Theorem 1.1, we use, as in [1], the strong pre-compactness property for the self-similar scaling sequence $u(kt, kx)$, $k \in \mathbb{N}$. This pre-compactness property will be obtained under condition (1.7) with the help of localization principles for $H$-measures with "continuous indexes", introduced in [8]. The strong pre-compactness property for arbitrary sequences of e.s. of (1.1) under exact non-degeneracy condition (1.8) was derived in [9] (see also [10,15] for the case of general flux vector $\varphi = \varphi(t, x, u)$). In the present paper we take into account the periodicity condition, which allows to refine the localization principle.

First, we recall the original concept of $H$-measure introduced by L. Tartar [17] and P. Gerárd [3]. Let $F(u)(\xi)$, $\xi \in \mathbb{R}^N$, be the Fourier transform of a function $u(x) \in L^2(\mathbb{R}^N)$, $S = S^{N-1} = \{\xi \in \mathbb{R}^N \mid |\xi| = 1\}$ be the unit sphere in $\mathbb{R}^N$. Denote by $u \to \bar{u}$, $u \in \mathbb{C}$ the complex conjugation.

Let $\Omega$ be an open domain in $\mathbb{R}^N$, $l \in \mathbb{N}$, and let $U_k(x) = (U_k^1(x), \dots, U_k^l(x)) \in L^2_{loc}(\Omega, \mathbb{R}^l)$ be a sequence of vector-functions weakly convergent to the zero vector.

**Proposition 2.1.** *(See Theorem 1.1 in [17].) There exist a family of complex Borel measures $\mu = \{\mu^{ij}\}_{i,j=1}^l$ in $\Omega \times S$ and a subsequence $U_r(x) = U_k(x)$, $k = k_r$, such that*

$$\langle \mu^{ij}, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi) \rangle = \lim_{r \to \infty} \int_{\mathbb{R}^N} F(\Phi_1 U_r^i)(\xi) \overline{F(\Phi_2 U_r^j)(\xi)} \psi\left(\frac{\xi}{|\xi|}\right) d\xi \tag{2.1}$$

*for all $\Phi_1(x), \Phi_2(x) \in C_0(\Omega)$ and $\psi(\xi) \in C(S)$.*

The family $\mu = \{\mu^{ij}\}_{i,j=1}^l$ is called the $H$-measure corresponding to $U_r(x)$.

**Remark 2.1.** In the case when the sequence $U_k(x)$ is bounded in $L^\infty(\Omega)$ it follows from (2.1) and the Plancherel identity that $\operatorname{pr}_x |\mu^{pq}| \leqslant C \operatorname{meas}$, and that (2.1) remains valid for all $\Phi_1(x), \Phi_2(x) \in L^2(\Omega)$, cf. [15, Remark 2(a)]. Here we denote by $|\mu|$ the variation of measure $\mu$ (it is a non-negative measure), and by meas the Lebesgue measure on $\Omega$.

We need also the concept of measure-valued functions (Young measures). Let $\Omega \subset \mathbb{R}^N$ be an open domain. Recall (see [2,16]) that a measure-valued function on $\Omega$ is a weakly measurable map $x \mapsto \nu_x$ of $\Omega$ into the space $\mathrm{Prob}_0(\mathbb{R})$ of probability Borel measures with compact support in $\mathbb{R}$.

The weak measurability of $\nu_x$ means that for each continuous function $g(\lambda)$ the function $x \to \langle \nu_x, g(\lambda)\rangle = \int g(\lambda)\,d\nu_x(\lambda)$ is measurable on $\Omega$.

Measure-valued functions of the kind $\nu_x(\lambda) = \delta(\lambda - u(x))$, where $u(x) \in L^\infty(\Omega)$ and $\delta(\lambda - u^*)$ is the Dirac measure at $u^* \in \mathbb{R}$, are called *regular*. We identify these measure-valued functions and the corresponding functions $u(x)$, so that there is a natural embedding of $L^\infty(\Omega)$ into the set $\mathrm{MV}(\Omega)$ of measure-valued functions on $\Omega$.

Measure-valued functions naturally arise as weak limits of bounded sequences in $L^\infty(\Pi)$ in the sense of the following theorem by L. Tartar [16].

**Theorem 2.1.** *Let $u_m(x) \in L^\infty(\Omega)$, $m \in \mathbb{N}$, be a bounded sequence. Then there exist a subsequence (we keep the notation $u_m(x)$ for this subsequence) and a measure-valued function $\nu_x \in \mathrm{MV}(\Omega)$ such that*

$$\forall g(\lambda) \in C(\mathbb{R}) \quad g(u_m) \underset{m\to\infty}{\rightharpoonup} \langle \nu_x, g(\lambda)\rangle \quad \textit{weakly-}* \textit{ in } L^\infty(\Omega). \tag{2.2}$$

*Besides, $\nu_x$ is regular, i.e., $\nu_x(\lambda) = \delta(\lambda - u(x))$ if and only if $u_m(x) \to_{m\to\infty} u(x)$ in $L^1_{loc}(\Omega)$ (strongly).*

In [8] the new concept of $H$-measures with "continuous indexes" was introduced, corresponding to sequences of measure-valued functions. We describe this concept in the particular case of "usual" sequences in $L^\infty(\Omega)$. Let $u_m(x)$ be a bounded sequence in $L^\infty(\Omega)$. Passing to a subsequence if necessary, we can suppose that this sequence converges to a measure-valued function $\nu_x \in \mathrm{MV}(\Omega)$ in the sense of relation (2.2). We introduce the measures $\gamma_x^m(\lambda) = \delta(\lambda - u_m(x)) - \nu_x(\lambda)$ and the corresponding distribution functions $U_m(x,p) = \gamma_x^m((p,+\infty))$, $u_0(x,p) = \nu_x((p,+\infty))$ on $\Omega \times \mathbb{R}$. Observe that $U_m(x,p), u_0(x,p) \in L^\infty(\Omega)$ for all $p \in \mathbb{R}$, see [8, Lemma 2]. We define the set

$$E = E(\nu_x) = \Big\{ p_0 \in \mathbb{R} \;\Big|\; u_0(x,p) \underset{p\to p_0}{\to} u_0(x,p_0) \text{ in } L^1_{loc}(\Omega) \Big\}.$$

As was shown in [8, Lemma 4], the complement $\mathbb{R} \setminus E$ is at most countable and if $p \in E$ then $U_m(x,p) \rightharpoonup_{m\to\infty} 0$ weakly-$*$ in $L^\infty(\Omega)$.

The next result, similar to Proposition 2.1, has been established in [8, Theorem 3], [10, Proposition 2, Lemma 2].

**Proposition 2.2.**

(1) *There exist a family of locally finite complex Borel measures $\{\mu^{pq}\}_{p,q\in E}$ in $\Omega \times S$ and a subsequence $U_r(x,p) = U_{m_r}(x,p)$ such that for all $\Phi_1(x), \Phi_2(x) \in C_0(\Omega)$ and $\psi(\xi) \in C(S)$*

$$\big\langle \mu^{pq}, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\big\rangle = \lim_{r\to\infty} \int_{\mathbb{R}^N} F\big(\Phi_1 U_r(\cdot,p)\big)(\xi)\overline{F\big(\Phi_2 U_r(\cdot,q)\big)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right) d\xi; \tag{2.3}$$

(2) *The correspondence $(p,q) \to \mu^{pq}$ is a continuous map from $E \times E$ into the space $\mathrm{M}_{loc}(\Omega \times S)$ of locally finite Borel measures on $\Omega \times S$ (with the standard locally convex topology);*

(3) *For any $p_1, \dots, p_l \in E$ the matrix $\{\mu^{p_i p_j}\}_{i,j=1}^l$ is Hermitian and positive semidefinite, that is, for all $\zeta_1, \dots, \zeta_l \in \mathbb{C}$ the measure*

$$\sum_{i,j=1}^{l} \mu^{p_i p_j} \zeta_i \overline{\zeta_j} \geqslant 0.$$

Notice that assertion (3) readily follows from relation (2.3).

We call the family of measures $\{\mu^{pq}\}_{p,q\in E}$ the $H$-measure corresponding to the subsequence $u_r(x) = u_{m_r}(x)$.

As was demonstrated in [8], the $H$-measure $\mu^{pq} = 0$ for all $p, q \in E$ if and only if the subsequence $u_r(x)$ converges as $r \to \infty$ strongly (in $L^1_{loc}(\Omega)$). Observe also that assertion (3) in Proposition 2.2 implies that measures $\mu^{pp} \geqslant 0$ for all $p \in E$, and that

$$\left|\mu^{pq}(A)\right| \leqslant \sqrt{\mu^{pp}(A)\mu^{qq}(A)} \tag{2.4}$$

for any Borel set $A \subset \Omega \times S$ and all $p,q \in E$. Indeed, this directly follows from the fact that the matrix $\begin{pmatrix} \mu^{pp}(A) & \mu^{pq}(A) \\ \mu^{qp}(A) & \mu^{qq}(A) \end{pmatrix}$ is Hermitian and positive semidefinite.

## 3. Main results

We fix a periodic e.s. $u = u(t,x)$ of (1.1). Without loss of generality, we may assume that $u(t,\cdot) \in C(\mathbb{R}_+, L^1(P))$ (see Remark 1.1 above).

**Lemma 3.1.** *Let $s(u)$ be a Lipschitz function, $v(t,x) = s(u(t,x))$, and*

$$v(t,x) = \sum_{\kappa \in \mathbb{Z}^n} a_\kappa(t) e^{2\pi i \kappa \cdot x}$$

*be the Fourier series of $v(t,\cdot)$ in $L^2(P)$, so that $a_\kappa(t) = \int_P e^{-2\pi i \kappa \cdot x} v(t,x)\,dx$. Then this series converges to $v(t,\cdot)$ in $L^2(P)$ uniformly with respect to $t$, that is, for each $\varepsilon > 0$ there exists a value $N \in \mathbb{N}$ such that*

$$\sum_{|\kappa|>N} \left|a_\kappa(t)\right|^2 < \varepsilon^2 \quad \forall t > 0. \tag{3.1}$$

**Proof.** Let $u_0(x) \in L^\infty(\mathbb{R}^n)$ be a strong trace of $u(t,x)$ on the initial hyper-plane $t = 0$ (recall that its existence follows from the results of [13,14]). Obviously, $u_0(x)$ is a periodic function. Since for each $h \in \mathbb{R}^n$ $u(t,x+h)$ is a periodic e.s. of the Cauchy problem for (1.1) with initial data $u_0(x+h)$ then for all $t > 0$

$$\begin{aligned}\int_P \left|v(t,x+h) - v(t,x)\right|^2 dx &\leqslant 2L^2 \|u\|_\infty \int_P \left|u(t,x+h) - u(t,x)\right| dx \\ &\leqslant 2L^2 \|u\|_\infty \int_P \left|u_0(x+h) - u_0(x)\right| dx \end{aligned} \tag{3.2}$$

by the $L^1$-contraction property, see for example [7, Corollary 3.3]. Here $L$ is a Lipschitz constant of $s(u)$, i.e., $|s(u_2) - s(u_1)| \leqslant L|u_2 - u_1|$ for every $u_1, u_2 \in \mathbb{R}$.

In view of (3.2), the set of functions $F = \{v(t,\cdot) \mid t > 0\}$ is precompact in $L^2(P)$. By Hausdorff's compactness criterion there exists a finite $\varepsilon/2$-net $\{g_k(x)\}_{k=1}^m$ for $F$ in $L^2(P)$. Let $b_{\kappa,k} = \int_P e^{-2\pi i \kappa \cdot x} g_k(x)\,dx$, $\kappa \in \mathbb{Z}^n$, be Fourier coefficients of $g_k(x)$. Observe that

$$\sum_{\kappa \in \mathbb{Z}^n} |b_{\kappa,k}|^2 = \|g_k\|^2_{L^2(P)} < +\infty.$$

Therefore, there exists an integer $N$ such that

$$\sum_{|\kappa|>N} |b_{\kappa,k}|^2 < \varepsilon^2/4 \tag{3.3}$$

for all $k = 1,\dots,m$. Since $\{g_k(x)\}_{k=1}^m$ is an $\varepsilon/2$-net for $F$ then for each $t > 0$ one can find such $k \in \{1,\dots,m\}$ that

$$\sum_{\kappa \in \mathbb{Z}^n} \left|a_\kappa(t) - b_{\kappa,k}\right|^2 = \left\|v(t,\cdot) - g_k\right\|^2_{L^2(P)} < \varepsilon^2/4. \tag{3.4}$$

In view of (3.3), (3.4) and Minkowski inequality we find

$$\left(\sum_{|\kappa|>N} \left|a_\kappa(t)\right|^2\right)^{1/2} \leqslant \left(\sum_{|\kappa|>N} \left|a_\kappa(t) - b_{\kappa,k}\right|^2\right)^{1/2} + \left(\sum_{|\kappa|>N} |b_{\kappa,k}|^2\right)^{1/2} < \varepsilon,$$

and (3.1) follows. □

Let $v = v(t,x) \in L^\infty(\Pi)$ be the function introduced in Lemma 3.1. We consider the sequence $v_k = v(kt,kx)$, $k \in \mathbb{N}$, and the Tartar's $H$-measure $\hat{\mu}$ corresponding to the scalar sequence $v_r - v^*$, where $v_r = v_{k_r}(t,x)$ is a subsequence of $v_k$, and $v^* = v^*(t,x)$ is a weak-$*$ limit of $v_r$ as $r \to \infty$ in $L^\infty(\Pi)$.

**Lemma 3.2.**

(i) *The function $v^*(t,x) = v^*(t)$ does not depend on $x$;*
(ii) $\operatorname{supp}\hat{\mu} \subset \Pi \times S_0$, *where*

$$S_0 = \bigl\{\hat{\xi}/|\hat{\xi}| \in S \bigm| \hat{\xi} = (\tau,\xi) \neq 0,\ \tau \in \mathbb{R},\ \xi \in \mathbb{Z}^n\bigr\}.$$

**Proof.** For $m \in \mathbb{N}$ we introduce the sets

$$S_m = \bigl\{\hat{\xi}/|\hat{\xi}| \in S \bigm| \hat{\xi} = (\tau,\xi) \neq 0,\ \tau \in \mathbb{R},\ \xi \in \mathbb{Z}^n,\ |\xi| \leqslant m\bigr\}.$$

It is clear that $S_m$ is a closed subset of the sphere $S$ (it is the union of the finite set of circles $\{(p, q\xi/|\xi|) | p^2 + q^2 = 1\}$, where $\xi \in \mathbb{Z}^n$, $0 < |\xi| \leqslant m$), and $S_0 = \bigcup_{m=1}^\infty S_m$. Let

$$v(t,x) = s\bigl(u(t,x)\bigr) = \sum_{\kappa \in \mathbb{Z}^n} a_\kappa(t) e^{2\pi i \kappa \cdot x}$$

be the Fourier series for $v(t,\cdot)$ in $L^2(P)$. Then

$$v_r(t,x) = v(k_r t, k_r x) = \sum_{\kappa \in \mathbb{Z}^n} a_\kappa(k_r t) e^{2\pi i k_r \kappa \cdot x}. \tag{3.5}$$

It follows from (3.5) that the function $v^*(t,x)$ does not actually depend on $x$: $v^*(t,x) = v^*(t)$, and $v^*(t)$ is the weak-$*$ limit of the sequence $a_0(k_r t)$, $r \in \mathbb{N}$, in $L^\infty(\mathbb{R}_+)$. Thus, statement (i) is proved.

We denote $b_{0,r} = a_0(k_r t) - v^*(t)$; $b_{\kappa,r} = a_\kappa(k_r t)$, where $\kappa \in \mathbb{Z}^n$, $\kappa \neq 0$. Let $\alpha(t) \in C_0(\mathbb{R}_+)$, and $\beta(x) \in L^2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ be such that its Fourier transform is a continuous compactly supported function:

$$\tilde{\beta}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} \beta(x)\,dx \in C_0\bigl(\mathbb{R}^n\bigr). \tag{3.6}$$

We take $R = \max_{\xi \in \operatorname{supp}\tilde{\beta}} |\xi|$. Let $\Phi(t,x) = \alpha(t)\beta(x)$. By (3.5) we find that

$$\bigl(v_r(t,x) - v^*(t)\bigr)\Phi(t,x) = \sum_{\kappa \in \mathbb{Z}^n} b_{\kappa,r}(t)\alpha(t) e^{2\pi i k_r \kappa \cdot x} \beta(x). \tag{3.7}$$

Observe that the Fourier transform of $e^{2\pi i k_r \kappa \cdot x}\beta(x)$ in $\mathbb{R}^n$ coincides with $\tilde{\beta}(\xi - k_r\kappa)$. Since for $k_r > 2R$ supports of these functions do not intersect, then for such $r$ the series

$$\sum_{\kappa \in \mathbb{Z}^n} b_{\kappa,r}(t)\alpha(t)\tilde{\beta}(\xi - k_r\kappa) \tag{3.8}$$

is orthogonal in $L^2(\mathbb{R}^n)$ for each $t > 0$. Besides, by the Plancherel equality $\|\tilde{\beta}(\xi - k_r\kappa)\|_{L^2(\mathbb{R}^n)} = \|\tilde{\beta}\|_2 = \|\beta\|_2$, and

$$\begin{aligned}\sum_{\kappa \in \mathbb{Z}^n} \bigl|b_{\kappa,r}(t)\alpha(t)\bigr|^2 \bigl\|\tilde{\beta}(\xi - k_r\kappa)\bigr\|^2_{L^2(\mathbb{R}^n)} &= \bigl|\alpha(t)\bigr|^2 \|\beta\|_2^2 \sum_{\kappa \in \mathbb{Z}^n} \bigl|b_{\kappa,r}(t)\bigr|^2 \\ &= \bigl|\alpha(t)\bigr|^2 \|\beta\|_2^2 \cdot \bigl\|v(k_r t,\cdot) - v^*(t)\bigr\|^2_{L^2(P)} < +\infty.\end{aligned}$$

Therefore, orthogonal series (3.8) converges in $L^2(\mathbb{R}^n)$ for each $t > 0$. Moreover, by Lemma 3.1

$$\sum_{\kappa \in \mathbb{Z}^n,\, |\kappa| > N} \bigl|b_{\kappa,r}(t)\bigr|^2 = \sum_{\kappa \in \mathbb{Z}^n,\, |\kappa| > N} \bigl|a_\kappa(k_r t)\bigr|^2 \underset{N \to \infty}{\to} 0$$

uniformly with respect to $t > 0$. Hence, series (3.8) converges in $L^2(\mathbb{R}^n)$ uniformly with respect to $t$. Since the Fourier transformation is an isomorphism on $L^2(\mathbb{R}^n)$, we conclude that series (3.7) also converges in $L^2(\mathbb{R}^n)$ (not only in $L^2(P)$) uniformly with respect to $t$. Since $\alpha(t) \in C_0(\mathbb{R})$, this implies that (3.7) converges in $L^2(\Pi)$, and

$$F\big((v_r - v)\Phi\big)(\hat{\xi}) = \sum_{\kappa\in\mathbb{Z}^n} F^t(\alpha b_{\kappa,r})(\tau)\tilde{\beta}(\xi - k_r\kappa), \quad \hat{\xi} = (\tau, \xi), \tag{3.9}$$

where $F^t(h)(\tau) = \int_{\mathbb{R}} e^{-2\pi i\tau t} h(t)\,dt$ denotes the Fourier transform over the time variable (we extend functions $h(t) \in L^2(\mathbb{R}_+)$ on the whole line $\mathbb{R}$, setting $h(t) = 0$ for $t < 0$). It follows from (3.9) that for $k_r > 2R$

$$\begin{aligned}
&\int\limits_{\mathbb{R}^{n+1}} \big|F\big(\Phi\big(v_r - v^*\big)\big)(\hat{\xi})\big|^2 \psi\big(\hat{\xi}/|\hat{\xi}|\big)\,d\hat{\xi} \\
&\quad = \sum_{\kappa\in\mathbb{Z}^n} \int\limits_{\mathbb{R}^{n+1}} \big|F^t(\alpha b_{\kappa,r})(\tau)\big|^2 \big|\tilde{\beta}(\xi - k_r\kappa)\big|^2 \psi\big(\hat{\xi}/|\hat{\xi}|\big)\,d\hat{\xi},
\end{aligned} \tag{3.10}$$

where $\psi(\hat{\xi}) \in C(S)$ is arbitrary. Now we fix $\varepsilon > 0$. Recall that $b_{\kappa,r} = a_\kappa(k_r t)$ for $\kappa \neq 0$, and by Lemma 3.1 there exists $m \in \mathbb{N}$ such that

$$\begin{aligned}
&\sum_{\kappa\in\mathbb{Z}^n,\,|\kappa|>m} \int\limits_{\mathbb{R}^{n+1}} \big|F^t(\alpha b_{\kappa,r})(\tau)\big|^2 \big|\tilde{\beta}(\xi - k_r\kappa)\big|^2\,d\hat{\xi} \\
&\quad = \sum_{\kappa\in\mathbb{Z}^n,\,|\kappa|>m} \int\limits_{\Pi} \big|\alpha(t)a_\kappa(k_r t)\big|^2 \big|\beta(x)\big|^2\,dt\,dx \\
&\quad \leqslant \|\Phi\|_2^2 \cdot \sup_{t>0} \sum_{\kappa\in\mathbb{Z}^n,\,|\kappa|>m} \big|a_\kappa(t)\big|^2 < \varepsilon.
\end{aligned} \tag{3.11}$$

Now we suppose that $\|\psi\|_\infty \leqslant 1$ and $\psi(\hat{\xi}) = 0$ on the set $S_m$. By (3.11)

$$\sum_{\kappa\in\mathbb{Z}^n,\,|\kappa|>m} \int\limits_{\mathbb{R}^{n+1}} \big|F^t(\alpha b_{\kappa,r})(\tau)\big|^2 \big|\tilde{\beta}(\xi - k_r\kappa)\big|^2 \big|\psi\big(\hat{\xi}/|\hat{\xi}|\big)\big|\,d\hat{\xi} \leqslant \varepsilon. \tag{3.12}$$

Since continuous function $\psi(\hat{\xi})$ is uniformly continuous on the compact $S$ then we can find such $\delta > 0$ that $|\psi(\hat{\xi}_1) - \psi(\hat{\xi}_2)| < \varepsilon$ whenever $\hat{\xi}_1, \hat{\xi}_2 \in S$, $|\hat{\xi}_1 - \hat{\xi}_2| < \delta$. Suppose that $\kappa \neq 0$, $\tilde{\beta}(\xi - k_r\kappa) \neq 0$. Then $|\xi - k_r\kappa| \leqslant R$. For a fixed $\tau \in \mathbb{R}$ we denote $\hat{\xi} = (\tau, \xi)$, $\hat{\eta} = (\tau, k_r\kappa)$. As is easy to compute,

$$\left|\frac{\hat{\xi}}{|\hat{\xi}|} - \frac{\hat{\eta}}{|\hat{\eta}|}\right| \leqslant \frac{2|\hat{\xi} - \hat{\eta}|}{|\hat{\eta}|} = \frac{2|\xi - k_r\kappa|}{|\hat{\eta}|} \leqslant 2R/|\hat{\eta}|. \tag{3.13}$$

Observe that for each nonzero $\kappa \in \mathbb{Z}^n$, $|\hat{\eta}| \geqslant k_r$. Then, by (3.13) we see that for all $r \in \mathbb{N}$ such that $k_r > 2R/\delta$ and all $\kappa \in \mathbb{Z}^n$, $0 < |\kappa| \leqslant m$,

$$\big|\psi\big(\hat{\xi}/|\hat{\xi}|\big)\big| = \big|\psi\big(\hat{\xi}/|\hat{\xi}|\big) - \psi\big(\hat{\eta}/|\hat{\eta}|\big)\big| < \varepsilon. \tag{3.14}$$

We use here that $\hat{\eta}/|\hat{\eta}| \in S_m$ and, therefore, $\psi(\hat{\eta}/|\hat{\eta}|) = 0$. In view of (3.14), for all $k_r > 2R/\delta$

$$\begin{aligned}
&\sum_{\kappa\in\mathbb{Z}^n,\,0<|\kappa|\leqslant m} \int\limits_{\mathbb{R}^{n+1}} \big|F^t(\alpha b_{\kappa,r})(\tau)\big|^2 \big|\tilde{\beta}(\xi - k_r\kappa)\big|^2 \big|\psi\big(\hat{\xi}/|\hat{\xi}|\big)\big|\,d\hat{\xi} \\
&\quad \leqslant \varepsilon \sum_{\kappa\in\mathbb{Z}^n,\,0<|\kappa|\leqslant m} \int\limits_{\mathbb{R}^{n+1}} \big|F^t(\alpha b_{\kappa,r})(\tau)\big|^2 \big|\tilde{\beta}(\xi - k_r\kappa)\big|^2\,d\hat{\xi} \\
&\quad \leqslant \varepsilon\|\beta\|_2^2 \sum_{\kappa\in\mathbb{Z}^n} \int\limits_{\mathbb{R}} \big|\alpha(t)b_{\kappa,r}(t)\big|^2\,dt \leqslant \varepsilon\|\Phi\|_2^2 \sup_{t>0} \sum_{\kappa\in\mathbb{Z}^n} \big|b_{\kappa,r}(t)\big|^2 \\
&\quad = \varepsilon\|\Phi\|_2^2 \sup_{t>0} \big\|v(k_r t,\cdot) - v^*(t)\big\|_{L^2(P)}^2 \leqslant C\varepsilon\|\Phi\|_2^2,
\end{aligned} \tag{3.15}$$

where $C = 4\|v\|_\infty^2$. Further, it follows from (3.13) with $\hat\eta = (\tau, 0)$ that for $|\xi| \leqslant R$ and $|\tau| > R_1 = 2R/\delta$

$$\left|\psi\left(\hat\xi/|\hat\xi|\right)\right| = \left|\psi\left(\hat\xi/|\hat\xi|\right) - \psi\left(\tau/|\tau|, 0\right)\right| < \varepsilon.$$

Therefore,

$$\int\limits_{\mathbb{R}^{n+1}} \theta\left(|\tau| - R_1\right)\left|F^t(\alpha b_{0,r})(\tau)\right|^2\left|\tilde\beta(\xi)\right|^2\left|\psi\left(\hat\xi/|\hat\xi|\right)\right| d\hat\xi \leqslant C\varepsilon\|\Phi\|_2^2. \tag{3.16}$$

Here $\theta(r) = \begin{cases} 1, & r>0, \\ 0, & r\leqslant 0\end{cases}$ is the Heaviside function.

For $|\tau| \leqslant R_1$ we are reasoning in the following way. Since $\alpha(t)b_{0,r}(t) = \alpha(t)(a_{0,r}(t) - v^*(t)) \rightharpoonup 0$ as $r \to \infty$, and $\|\alpha b_{0,r}\|_1 \leqslant C_1 = 2\|v\|_\infty\|\alpha\|_1$, the Fourier transform $F^t(\alpha b_{0,r})(\tau) \to_{r\to\infty} 0$ for all $\tau \in \mathbb{R}$ and uniformly bounded: $|F^t(\alpha b_{0,r})(\tau)| \leqslant C_1$. By Lebesgue dominated convergence theorem

$$\int\limits_{\mathbb{R}} \theta\left(R_1 - |\tau|\right)\left|F^t(\alpha b_{0,r})(\tau)\right|^2 d\tau \underset{r\to\infty}{\to} 0.$$

Therefore (recall that $\|\psi\|_\infty \leqslant 1$),

$$\begin{aligned}&\int\limits_{\mathbb{R}^{n+1}} \theta\left(R_1 - |\tau|\right)\left|F^t(\alpha b_{0,r})(\tau)\right|^2\left|\tilde\beta(\xi)\right|^2\left|\psi\left(\hat\xi/|\hat\xi|\right)\right| d\hat\xi \\ &\quad\leqslant \|\beta\|_2 \int\limits_{\mathbb{R}} \theta\left(R_1 - |\tau|\right)\left|F^t(\alpha b_{0,r})(\tau)\right|^2 d\tau \underset{r\to\infty}{\to} 0.\end{aligned} \tag{3.17}$$

In view of (3.16), (3.17) we find

$$\varlimsup_{r\to\infty} \int\limits_{\mathbb{R}^{n+1}} \left|F^t(\alpha b_{0,r})(\tau)\right|^2\left|\tilde\beta(\xi)\right|^2\left|\psi\left(\hat\xi/|\hat\xi|\right)\right| d\hat\xi \leqslant C\varepsilon\|\Phi\|_2^2. \tag{3.18}$$

Using (3.10), (3.12), (3.15) and (3.18), we arrive at the relation

$$\varlimsup_{r\to\infty} \int\limits_{\mathbb{R}^{n+1}} \left|F\left(\Phi\left(v_r - v^*\right)\right)(\hat\xi)\right|^2\left|\psi\left(\hat\xi/|\hat\xi|\right)\right| d\hat\xi \leqslant C_2\varepsilon, \tag{3.19}$$

where $C_2$ is a constant independent on $\psi$ and $m$. By the definition of $H$-measure and Remark 2.1

$$\begin{aligned}&\lim_{r\to\infty} \int\limits_{\mathbb{R}^{n+1}} \left|F\left(\Phi\left(v_r - v^*\right)\right)(\hat\xi)\right|^2\left|\psi\left(\hat\xi/|\hat\xi|\right)\right| d\hat\xi \\ &\quad= \left\langle\hat\mu, \left|\Phi(t,x)\right|^2\left|\psi(\hat\xi)\right|\right\rangle = \int\limits_{\Pi\times(S\setminus S_m)} \left|\Phi(t,x)\right|^2\left|\psi(\hat\xi)\right| d\hat\mu(t,x,\hat\xi),\end{aligned}$$

and (3.19) implies that

$$\int\limits_{\Pi\times(S\setminus S_m)} \left|\Phi(t,x)\right|^2\psi(\hat\xi)\, d\hat\mu(t,x,\hat\xi) \leqslant C_2\varepsilon$$

for all $\psi(\hat\xi) \in C_0((S \setminus S_m))$ such that $0 \leqslant \psi(\hat\xi) \leqslant 1$. Therefore, we can claim that

$$\int\limits_{\Pi\times(S\setminus S_m)} \left|\Phi(t,x)\right|^2 d\hat\mu(t,x,\hat\xi) \leqslant C_2\varepsilon,$$

and since $S \setminus S_0 \subset S \setminus S_m$, we obtain the relation

$$\int\limits_{\Pi\times(S\setminus S_0)} \left|\Phi(t,x)\right|^2 d\hat{\mu}(t,x,\hat{\xi}) \leqslant C_2\varepsilon,$$

which holds for arbitrary positive $\varepsilon$. Therefore,

$$\int\limits_{\Pi\times(S\setminus S_0)} \left|\Phi(t,x)\right|^2 d\hat{\mu}(t,x,\hat{\xi}) = 0. \tag{3.20}$$

Since for every point $(t_0,x_0)\in\Pi$ one can find functions $\alpha(t)$, $\beta(x)$ with the prescribed above properties in such a way that $\Phi(t,x)=\alpha(t)\beta(x)\neq 0$ in a neighborhood of $(t_0,x_0)$, we derive from (3.20) the desired inclusion $\operatorname{supp}\hat{\mu}\subset \Pi\times S_0$. $\square$

We consider the $H$-measure $\{\mu^{pq}\}_{p,q\in E}$ corresponding to a subsequence $u_r=u_{k_r}(t,x)$ of the sequence $u_k(t,x)=u(kt,kx)$, $k\in\mathbb{N}$, defined in accordance with Proposition 2.2.

**Theorem 3.1.** *For every $p,q\in E$, $\operatorname{supp}\mu^{pq}\subset\Pi\times S_0$.*

**Proof.** Let $\nu_{t,x}$ be a weak measure valued limit of the sequence $u_r$. We introduce measures

$$\gamma^r_{t,x}(\lambda)=\delta\bigl(\lambda-u_r(t,x)\bigr)-\nu_{t,x}(\lambda),$$

and set $U_r(t,x,p)=\gamma_{t,x}((p,+\infty))$. Let $s(u)\in C^1(\mathbb{R})$ be such that its derivative $s'(u)$ is compactly supported, and $v_r(t,x)=s(u_r(t,x))$, $r\in\mathbb{N}$. Then $v_r \rightharpoonup v^*(t)=\int s(\lambda)\,d\nu_{t,x}(\lambda)$ as $r\to\infty$ weakly-$*$ in $L^\infty(\Pi)$ (by Lemma 3.2(i), the limit function $v^*(t)$ does not depend on $x$). Integrating by parts, we find that

$$v_r(t,x)-v^*(t)=\int s(\lambda)\,d\gamma^r_{t,x}(\lambda)=\int s'(\lambda)U_r(t,x,\lambda)\,d\lambda. \tag{3.21}$$

Let $\Phi(t,x)\in C_0(\Pi)$, $\psi(\hat{\xi})\in C(S)$. Then, in view of (3.21), we find

$$\begin{aligned}&\int\limits_{\mathbb{R}^{n+1}} \left|F\bigl(\Phi\bigl(v_r-v^*\bigr)\bigr)(\hat{\xi})\right|^2\psi\bigl(\hat{\xi}/|\hat{\xi}|\bigr)\,d\hat{\xi}\\ &\quad=\int\int s'(p)s'(q)\Biggl(\int\limits_{\mathbb{R}^{n+1}} F\bigl(\Phi U_r(\cdot,p)\bigr)(\hat{\xi})\overline{F\bigl(\Phi U_r(\cdot,q)\bigr)(\hat{\xi})}\psi\bigl(\hat{\xi}/|\hat{\xi}|\bigr)\,d\hat{\xi}\Biggr)\,dp\,dq.\end{aligned} \tag{3.22}$$

By the definition of $H$-measure, for each $p,q\in E$

$$\lim_{r\to\infty}\int\limits_{\mathbb{R}^{n+1}} F\bigl(\Phi U_r(\cdot,p)\bigr)(\hat{\xi})\overline{F\bigl(\Phi U_r(\cdot,q)\bigr)(\hat{\xi})}\psi\bigl(\hat{\xi}/|\hat{\xi}|\bigr)\,d\hat{\xi}=\bigl\langle\mu^{pq},\left|\Phi(t,x)\right|^2\psi(\hat{\xi})\bigr\rangle.$$

Using Lebesgue dominated convergence theorem, we can pass to the limit as $r\to\infty$ in equality (3.22) and arrive at

$$\begin{aligned}\bigl\langle\hat{\mu},\left|\Phi(t,x)\right|^2\psi(\hat{\xi})\bigr\rangle&=\lim_{r\to\infty}\int\limits_{\mathbb{R}^{n+1}}\left|F\bigl(\Phi(v_r-v)\bigr)(\hat{\xi})\right|^2\psi\bigl(\hat{\xi}/|\hat{\xi}|\bigr)\,d\hat{\xi}\\ &=\iint s'(p)s'(q)\bigl\langle\mu^{pq},\left|\Phi(t,x)\right|^2\psi(\hat{\xi})\bigr\rangle\,dp\,dq,\end{aligned} \tag{3.23}$$

where $\hat{\mu}=\hat{\mu}(t,x,\hat{\xi})$ is the Tartar's $H$-measure, corresponding to the scalar sequence $v_r-v^*$. Clearly, the equality

$$\bigl\langle\hat{\mu},\left|\Phi(t,x)\right|^2\psi(\hat{\xi})\bigr\rangle=\iint s'(p)s'(q)\bigl\langle\mu^{pq},\left|\Phi(t,x)\right|^2\psi(\hat{\xi})\bigr\rangle\,dp\,dq$$

remains valid for every Borel function $\psi(\hat{\xi})$. Taking $\psi(\hat{\xi})$ being the indicator function of the set $S\setminus S_0$ and using Lemma 3.2, we obtain the relation

$$\iint s'(p)s'(q)\langle\mu^{pq}, |\Phi(t,x)|^2\psi(\hat{\xi})\rangle\,dp\,dq = 0. \tag{3.24}$$

Now we take in (3.24) $s'(p) = l\omega(l(p-p_0))$, where $p_0 \in E$, $l \in \mathbb{N}$, and $\omega(y) \in C_0((0,1))$ be a non-negative function such that $\int \omega(y)dy = 1$. Since the $H$-measure $\mu^{pq}$ is strongly continuous with respect to $(p,q)$ at point $(p_0,p_0)$, we derive from (3.24) in the limit as $l \to \infty$ that

$$\begin{aligned}&\langle\mu^{p_0p_0}, |\Phi(t,x)|^2\psi(\hat{\xi})\rangle\\ &\quad= \lim_{l\to\infty} l^2 \iint \omega\big(l(p-p_0)\big)\omega\big(l(q-p_0)\big)\langle\mu^{pq}, |\Phi(t,x)|^2\psi(\hat{\xi})\rangle\,dp\,dq = 0.\end{aligned}$$

Since $\Phi(t,x) \in C_0(\Pi)$ is arbitrary, we conclude that $\mu^{p_0p_0}(\Pi \times (S \setminus S_0)) = 0$ (remark that $\mu^{p_0p_0} \geqslant 0$). Hence, for every $p = p_0 \in E$, $\operatorname{supp}\mu^{pp} \subset \Pi \times S_0$. Finally, as directly follows from (2.4), for $p,q \in E$ $\operatorname{supp}\mu^{pq} \subset \operatorname{supp}\mu^{pp} \subset \Pi \times S_0$. The proof is complete. □

Let us define the minimal linear subspace $L = L(p) \subset \mathbb{R}^{n+1}$ such that $\operatorname{supp}\mu^{pp} \subset \Pi \times L$. Since $u_r(t,x)$ is a bounded sequence of e.s. of Eq. (1.1) from the results of [9] (see Lemma 2 with $q = p_0$ and the proof of Theorem 4) it follows the localization principle:

**Theorem 3.2.** *There exists $\delta > 0$ such that the function $u \mapsto \tau u + \xi \cdot \varphi(u)$ is constant on the interval $(p-\delta, p+\delta)$ for all $\hat{\xi} = (\tau,\xi) \in L$.*

Now we are ready to prove our main Theorem 1.1.

**Proof of Theorem 1.1.** We fix $p \in E$ and assume that $\mu^{pp} \neq 0$. Then the space $L = L(p)$ is not trivial: $\dim L > 0$. By Theorem 3.1 there exists a nonzero vector $\hat{\xi} = (\tau,\xi) \in (\mathbb{R} \times \mathbb{Z}^n) \cap L$. Then, by Theorem 3.2 the function $u \mapsto \tau u + \xi \cdot \varphi(u)$ is constant on some interval $(p-\delta, p+\delta)$, which contradicts to condition (1.7). Hence $\mu^{pp} = 0$ for all $p \in E$. In view of (2.4) this implies that the $H$-measure $\mu^{pq} \equiv 0$. Therefore the sequence $u_r(t,x)$ converges as $r \to \infty$ to a function $u^*(t,x) \in L^\infty(\Pi)$ strongly, in $L^1_{loc}(\Pi)$. By Lemma 3.2(i) the limit function does not depend on $x$: $u^*(t,x) = u^*(t)$. Passing to the limit as $r \to \infty$ in equalities $(u_r)_t + \operatorname{div}_x \varphi(u_r) = 0$ in $\mathcal{D}'(\Pi)$ we derive, in view of the strong convergence $u_r \to u^*$, $\varphi(u_r) \to \varphi(u^*)$ as $r \to \infty$, the relation $(u^*)_t + \operatorname{div}_x \varphi(u^*) = 0$ in $\mathcal{D}'(\Pi)$. Since $u^* = u^*(t)$ we find $(u^*)' = 0$ in $\mathcal{D}'(\mathbb{R}_+)$, which yields $u^* = \text{const}$. The relation $u_r(t,x) \to_{r\to\infty} u^*$ in $L^1_{loc}(\Pi)$ implies that (after possible extraction of a subsequence) for a.e. $t > 0$, $u_r(t,x) \to_{r\to\infty} u^*$ in $L^1_{loc}(\mathbb{R}^n)$. By the periodicity, this reads

$$\int_P |u(k_rt, k_rx) - u^*|\,dx \underset{r\to\infty}{\to} 0.$$

Making the change of variables $y = k_rx$ and using the space periodicity of $u$, we find that for a.e. $t > 0$

$$\int_P |u(k_rt, y) - u^*|\,dy = \int_P |u(k_rt, k_rx) - u^*|\,dx \underset{r\to\infty}{\to} 0. \tag{3.25}$$

We fix such $t = t_0 > 0$. Then for a.e. $t > k_rt_0$

$$\int_P |u(t,y) - u^*|\,dy \leqslant \int_P |u(k_rt_0, y) - u^*|\,dy, \tag{3.26}$$

by the $L^1(P)$-contraction property. In view of (3.25) it follows from (3.26) that $\operatorname{ess\,lim}_{t\to\infty} u(t,x) = u^*$ in $L^1(P)$. Finally, by the conservation of "mass" (see [11]), for a.e. $t > 0$

$$\int_P u(t,x)\,dx = \int_P u_0(x)\,dx,$$

where $u_0(x)$ is a strong trace of $u(t,x)$ on the initial hyper-space $t = 0$. Passing in this relation to the limit as $t \to \infty$, we obtain that

$$u^* = \frac{1}{|P|}\int_P u_0(x)\,dx = \int_P u_0(x)\,dx.$$

Hence,

$$\operatorname*{ess\,lim}_{t\to\infty} u(t,x) = \int_P u_0(x)\,dx \quad \text{in } L^1(P),$$

and decay property (1.6) holds.

Conversely, assume that Eq. (1.1) satisfies the decay property. Let us demonstrate that it satisfies condition (1.7). Assuming the contrary, we can find the segment $[a,b]$, $a<b$, and a nonzero point $(\tau,\xi)\in\mathbb{R}\times\mathbb{Z}^n$ such that the function $u\mapsto \tau u+\xi\cdot\varphi(u)$ is constant on the segment $[a,b]$. Then, as is easy to verify, the function

$$u(t,x) = \frac{a+b}{2} + \frac{b-a}{2}\sin\bigl(2\pi(\tau t+\xi\cdot x)\bigr)$$

is a periodic e.s. of (1.1), which does not satisfy the decay property. The obtained contradiction shows that condition (1.7) holds. We conclude that this condition is necessary and sufficient for the decay property. □

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