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Continuous dependence for NLS in fractional order spaces

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Abstract

For the nonlinear Schrödinger equation $iu_t + \Delta u + \lambda|u|^\alpha u = 0$ in $\mathbb{R}^N$, local existence of solutions in H^s is well known in the H^s-subcritical and critical cases $0 < \alpha \leqslant 4/(N-2s)$, where $0 < s < \min\{N/2, 1\}$. However, even though the solution is constructed by a fixed-point technique, continuous dependence in H^s does not follow from the contraction mapping argument. In this paper, we show that the solution depends continuously on the initial value in the sense that the local flow is continuous $H^s \to H^s$. If, in addition, $\alpha \geqslant 1$ then the flow is locally Lipschitz.

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1. Introduction

In this paper, we study the continuity of the solution map $\varphi \mapsto u$ for the nonlinear Schrödinger equation

$$\begin{cases} iu_t + \Delta u + g(u) = 0, \\ u(0) = \varphi, \end{cases} \tag{NLS}$$

in $H^s(\mathbb{R}^N)$, where $N \geqslant 1$ and

$$0 < s < \min\left\{1, \frac{N}{2}\right\}. \tag{1.1}$$

We assume that the nonlinearity g satisfies

$$g \in C^1(\mathbb{C}, \mathbb{C}), \quad g(0) = 0, \tag{1.2}$$

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and

$$\left|g'(u)\right| \leqslant A + B|u|^\alpha, \tag{1.3}$$

for all $u \in \mathbb{C}$, where

$$0 < \alpha \leqslant \frac{4}{N-2s}. \tag{1.4}$$

Under these assumptions, for every initial value $\varphi \in H^s(\mathbb{R}^N)$, there exists a local solution $u \in C([0,T], H^s(\mathbb{R}^N))$ of (NLS), which is unique under an appropriate auxiliary condition. More precisely, the following result is well known. (Here and in the rest of the paper, a pair (q,r) is called admissible if $2 \leqslant r < 2N/(N-2)$ ($2 \leqslant r < \infty$ if $N = 1, 2$) and $2/q = N(1/2 - 1/r)$.)

Theorem 1.1. *(See [8,13,7].) Assume* (1.1)–(1.4) *and let (γ, ρ) be the admissible pair defined by*

$$\rho = \frac{N(\alpha+2)}{N+s\alpha}, \qquad \gamma = \frac{4(\alpha+2)}{\alpha(N-2s)}. \tag{1.5}$$

Given $\varphi \in H^s(\mathbb{R}^N)$, there exist $0 < T_{\max} \leqslant \infty$ and a unique, maximal solution $u \in C([0,T_{\max}), H^s(\mathbb{R}^N)) \cap L^\gamma_{\text{loc}}([0,T_{\max}), B^s_{\rho,2}(\mathbb{R}^N))$ of (NLS)*, in the sense that*

$$u(t) = e^{it\Delta}\varphi + i\int_0^t e^{i(t-s)\Delta} g\big(u(s)\big)\, ds, \tag{1.6}$$

for all $0 \leqslant t < T_{\max}$, where $(e^{it\Delta})_{t\in\mathbb{R}}$ is the Schrödinger group. Moreover, $u \in L^q_{\text{loc}}([0,T_{\max}), B^s_{r,2}(\mathbb{R}^N))$ for every admissible pair (q,r) and u depends continuously on φ in the following sense. There exists a time $T \in (0, T_{\max})$ such that if $\varphi_n \to \varphi$ in $H^s(\mathbb{R}^N)$ and if u_n denotes the solution of (NLS) *with the initial value φ_n, then $T_{\max}(\varphi_n) > T$ for all sufficiently large n and u_n is bounded in $L^q((0,T), B^s_{r,2}(\mathbb{R}^N))$ for any admissible pair (q,r). Moreover, $u_n \to u$ in $L^q((0,T), B^{s-\varepsilon}_{r,2}(\mathbb{R}^N))$ as $n \to \infty$ for all $\varepsilon > 0$ and all admissible pairs (q,r). In particular, $u_n \to u$ in $C([0,T], H^{s-\varepsilon}(\mathbb{R}^N))$ for all $\varepsilon > 0$.*

Theorem 1.1 goes back to [8,13]. The precise statement which we give here is taken from [7, Theorems 4.9.1 and 4.9.7]. The admissible pair (γ,ρ) corresponds to one particular choice of an auxiliary space, which ensures that Eq. (NLS) makes sense and that the solution is unique. See [8,13,7] for details. Under certain conditions on α and s, an auxiliary space is not necessary: the equation makes sense by Sobolev's embedding and uniqueness in $C([0,T], H^s(\mathbb{R}^N))$ holds "unconditionally", see [13,11,19,26].

We note that the continuous dependence statement in Theorem 1.1 is weaker than the expected one (i.e. with $\varepsilon = 0$). Indeed, Theorem 1.1 is proved by applying a fixed point argument to Eq. (1.6), so one would expect that the dependence of the solution on the initial value is locally Lipschitz. However, the metric space in which one applies Banach's fixed point theorem involves Sobolev (or Besov) norms of order s, while the distance only involves Lebesgue norms. (The reason for that choice of the distance is that the nonlinearity need not be locally Lipschitz for Sobolev or Besov norms of positive order.) Thus the flow is locally Lipschitz for Lebesgue norms, and continuous dependence in the sense of Theorem 1.1 follows by interpolation inequalities. See [8,13] for details.

In this paper, we show that continuous dependence holds in H^s in the standard sense under the assumptions of Theorem 1.1 (with an extra condition in the critical case $\alpha = \frac{4}{N-2s}$). More precisely, our main result is the following.

Theorem 1.2. *Assume* (1.1)–(1.4) *and suppose further that*

$$A = 0 \quad \textit{if } \alpha = \frac{4}{N-2s}, \tag{1.7}$$

where A is the constant in (1.3)*. The solution of* (NLS) *given by Theorem* 1.1 *depends continuously on φ in the following sense.*

(i) *The mapping $\varphi \mapsto T_{\max}(\varphi)$ is lower semicontinuous $H^s(\mathbb{R}^N) \to (0,\infty]$.*

(ii) *If $\varphi_n \to \varphi$ in $H^s(\mathbb{R}^N)$ and if u_n (respectively, u) denotes the solution of* (NLS) *with the initial value φ_n (respectively, φ), then $u_n \to u$ in $L^q((0,T), B^s_{r,2}(\mathbb{R}^N))$ for all $0 < T < T_{\max}(\varphi)$ and all admissible pairs (q,r). In particular, $u_n \to u$ in $C([0,T], H^s(\mathbb{R}^N))$.*

Theorem 1.2 applies in particular to the model case $g(u) = \lambda|u|^\alpha u$ with $\lambda \in \mathbb{C}$ and $0 < \alpha \leqslant 4/(N-2s)$. If, in addition, $\alpha \geqslant 1$ then the dependence is in fact locally Lipschitz. We summarize the corresponding results in the following corollary.

Corollary 1.3. *Assume* (1.1) *and let $g(u) = \lambda|u|^\alpha u$ with $\lambda \in \mathbb{C}$ and $0 < \alpha \leqslant \frac{4}{N-2s}$. It follows that the solution of* (NLS) *given by Theorem* 1.1 *depends continuously on the initial value in the sense of Theorem* 1.2*. If, in addition, $\alpha \geqslant 1$ then the dependence is locally Lipschitz. More precisely, let $\varphi \in H^s(\mathbb{R}^N)$ and let u be the corresponding solution of* (NLS)*. Given $0 < T < T_{\max}(\varphi)$ there exists $\delta > 0$ such that if $\psi \in H^s(\mathbb{R}^N)$ satisfies $\|\varphi - \psi\|_{H^s} \leqslant \delta$ and v is the corresponding solution of* (NLS)*, then for every admissible pair (q,r)*

$$\|u - v\|_{L^q((0,T),B^s_{r,2})} \leqslant C\|\varphi - \psi\|_{H^s}, \tag{1.8}$$

where C depends on φ, T, q, r. In particular, $\|u - v\|_{L^\infty((0,T),H^s)} \leqslant C\|\varphi - \psi\|_{H^s}$.

We are not aware of any previous continuous dependence result for (NLS) in H^s with noninteger s. For integer s, the known results in the model case $g(u) = \lambda|u|^\alpha u$, $\lambda \in \mathbb{C}$ are the following. Continuous dependence in $L^2(\mathbb{R}^N)$ follows from [24] in the subcritical case $N\alpha < 4$ and from [8] in the critical case $N\alpha = 4$. Continuous dependence in $H^1(\mathbb{R}^N)$ is proved in [12] in the subcritical case $(N-2)\alpha < 4$ and in [8,16,22,17] in the critical case $(N-2)\alpha = 4$. Continuous dependence in $H^2(\mathbb{R}^N)$ follows from [12] in the subcritical case $(N-4)\alpha < 4$.

Our proof of Theorem 1.2 is based on the method used by Kato [12] to prove continuous dependence in $H^1(\mathbb{R}^N)$. We briefly recall Kato's argument. Convergence in Lebesgue spaces holds by the contraction mapping estimates, so the tricky part is the gradient estimate. By applying Strichartz estimates, this amounts in controlling $\nabla[g(u_n) - g(u)]$ in some L^p space. However, $\nabla[g(u_n) - g(u)] = g'(u_n)[\nabla u_n - \nabla u] + [g'(u_n) - g'(u)]\nabla u$. The term $g'(u_n)[\nabla u_n - \nabla u]$ is easily absorbed by the left-hand side of the inequality, and the key observation is that the remaining term $[g'(u_n) - g'(u)]\nabla u$ is of lower order, in the sense that the convergence of u_n to u in appropriate L^p spaces implies that this last term converges to 0. We prove Theorem 1.2 by applying the same idea. The key argument of the proof is an estimate which shows that $g(u) - g(v)$ is bounded in Besov spaces by a Lipschitz term (i.e. with a factor $u - v$) plus some lower order term. (See Lemmas 2.1 and 2.2 below.) In the critical case $\alpha = 4/(N-2s)$, a further argument is required in order to show that $u_n \to u$ in the appropriate L^p space (estimate (4.14)). For this, following Tao and Visan [22], we use a Strichartz-type estimate for a non-admissible pair, and this is where we use the assumption (1.7). Local Lipschitz continuity in Corollary 1.3 follows from a different, much simpler argument: in this case, the nonlinearity is locally Lipschitz in the appropriate Besov spaces.

In Theorem 1.2 we assume $s < \min\{1, N/2\}$. The assumption $s < N/2$ is natural, but the restriction $s < 1$ is technical. When $N \geqslant 3$, local existence in $H^s(\mathbb{R}^N)$ is known to hold for $0 < s < N/2$, in particular when $g(u) = \lambda|u|^\alpha u$, under some extra assumption on α and s that ensures that g is sufficiently smooth. See [8,13,18]. The limitation $s < 1$ first appears in our Besov space estimates. These could possibly be extended to $s > 1$. It also appears in a more subtle way. For example, the introduction of certain exponents (4.21) in the critical case explicitly requires $s < 1$. It is not impossible that the idea in [22] of using intermediate Besov spaces of lower order can be applied when $s > 1$.

We next mention a few open questions. As observed above, the fixed point argument used in [8,13,7] to prove Theorem 1.1 does not show that the flow is locally Lipschitz in $H^s(\mathbb{R}^N)$. On the other hand, it does not show either that it is not locally Lipschitz. This raises the following question: under the assumptions (1.1)–(1.4), is the flow of (NLS) locally Lipschitz in H^s? We suspect that the answer might be negative. Note that in the pure power case $g(u) = \lambda|u|^\alpha u$, Remark 2.3 implies that the nonlinearity is locally Hölder continuous in the appropriate Besov spaces of order s. Therefore it is natural to ask if the flow also is locally Hölder continuous in H^s. Finally, we note that in the critical case $\alpha = \frac{4}{N-2s}$ Theorem 1.2 imposes the restriction $A = 0$ in (1.3). Can this restriction be removed?

The rest of this paper is organized as follows. In Section 2 we establish estimates of $g(u) - g(v)$ in Besov spaces. We complete the proof Theorem 1.2 in Section 3 in the subcritical case $\alpha < \frac{4}{N-2s}$ and in Section 4 in the critical case $\alpha = \frac{4}{N-2s}$. Section 5 is devoted to the proof of Corollary 1.3.

Notation. Given $1 \leqslant p \leqslant \infty$, we denote by p' its conjugate given by $1/p' = 1 - 1/p$ and we consider the standard (complex-valued) Lebesgue spaces $L^p(\mathbb{R}^N)$. Given $1 \leqslant p, q \leqslant \infty$ and $s \in \mathbb{R}$ we consider the usual (complex-valued) Sobolev and Besov spaces $H^s(\mathbb{R}^N)$ and $B^s_{p,q}(\mathbb{R}^N)$ and their homogeneous versions $\dot{H}^s(\mathbb{R}^N)$ and $\dot{B}^s_{p,q}(\mathbb{R}^N)$. See for example [1,23] for the definitions of these spaces and the corresponding Sobolev's embeddings. We denote by $(e^{it\Delta})_{t\in\mathbb{R}}$ the Schrödinger group and we will use without further reference the standard Strichartz estimates, see [21,27,14,15].

2. Superposition operators in Besov spaces

The study of superposition operators in Besov spaces has a long history and necessary conditions (sometimes necessary and sufficient conditions) on g are known so that $u \mapsto g(u)$ is a bounded map of certain Besov spaces. (See e.g. [20] and the references therein.) On the other hand, few works study the continuity of such maps, among which [2–6], but none of these works applies to such cases as $g(u) = |u|^\alpha u$, which is a typical nonlinearity for (NLS). Besides, for our proof of Theorem 1.2 we do not need only continuity but instead a rather specific property, namely that $g(u) - g(v)$ can be estimated in a certain Besov space of order s by a Lipschitz term (i.e. involving $u - v$ in some other Besov space of order s) plus a lower order term. We establish such estimates in the two following lemmas. The first one concerns functions g such that $|g'(u)| \leqslant C|u|^\alpha$, and the second functions g that are globally Lipschitz. (A function g satisfying (1.2)–(1.3) can be decomposed as the sum of two such functions.) The proofs of these lemmas are based on the choice of an equivalent norm on $\dot{B}^s_{p,q}$ defined in terms of finite differences, and on rather elementary calculations.

Lemma 2.1. *Let $0 < s < 1$, $\alpha > 0$, $1 \leqslant q < \infty$ and $1 \leqslant p < r \leqslant \infty$, and let $0 < \sigma < \infty$ be defined by*

$$\frac{\alpha}{\sigma} = \frac{1}{p} - \frac{1}{r}. \tag{2.1}$$

We use the convention that $\|u\|_{L^\sigma} = (\int_{\mathbb{R}^N} |u|^\sigma)^{\frac{1}{\sigma}}$ even if $\sigma < 1$. Let $g \in C^1(\mathbb{C}, \mathbb{C})$ satisfy

$$\bigl|g'(u)\bigr| \leqslant C|u|^\alpha, \tag{2.2}$$

for all $u \in \mathbb{C}$. It follows that there exists a constant C such that if $u, v \in \dot{B}^s_{r,q}(\mathbb{R}^N)$ and $\|u\|_{L^\sigma}, \|v\|_{L^\sigma} < \infty$, then

$$\bigl\|g(v) - g(u)\bigr\|_{\dot{B}^s_{p,q}} \leqslant C\|v\|^\alpha_{L^\sigma}\|v - u\|_{\dot{B}^s_{r,q}} + K(u, v), \tag{2.3}$$

where $K(u, v)$ satisfies

$$K(u, v) \leqslant C\bigl(\|u\|^\alpha_{L^\sigma} + \|v\|^\alpha_{L^\sigma}\bigr)\|u\|_{\dot{B}^s_{p,q}}, \tag{2.4}$$

and

$$K(u, u_n) \xrightarrow[n\to\infty]{} 0, \tag{2.5}$$

if $(u_n)_{n\geqslant 1} \subset \dot{B}^s_{r,q}(\mathbb{R}^N)$ and $u \in \dot{B}^s_{r,q}(\mathbb{R}^N)$ are such that $\|u\|_{L^\sigma} < \infty$ and $\|u_n - u\|_{L^\sigma} \to 0$ as $n \to \infty$.

Proof. We denote by τ_y the translation operator defined for $y \in \mathbb{R}^N$ by $\tau_y u(\cdot) = u(\cdot - y)$ and we recall that (see e.g. Section 5.2.3, Theorem 2, p. 242 in [23])

$$\|u\|_{\dot{B}^s_{p,q}} \approx \biggl(\int_{\mathbb{R}^N} \|\tau_y u - u\|^q_{L^p(\mathbb{R}^N)} |y|^{-N-sq}\, dy\biggr)^{\frac{1}{q}}. \tag{2.6}$$

Given $z_1, z_2 \in \mathbb{C}$, we have

$$g(z_1) - g(z_2) = (z_1 - z_2)\int_0^1 \partial_z g\bigl(z_2 + \theta(z_1 - z_2)\bigr)\, d\theta + (\overline{z_1 - z_2})\int_0^1 \partial_{\bar{z}} g\bigl(z_2 + \theta(z_1 - z_2)\bigr)\, d\theta,$$

which we write, in short,

$$g(z_1)-g(z_2)=(z_1-z_2)\int_0^1 g'\big(z_2+\theta(z_1-z_2)\big)\,d\theta. \tag{2.7}$$

We set

$$A(u,v,y)(\cdot)=\tau_y\big[g(v)-g(u)\big]-\big[g(v)-g(u)\big], \tag{2.8}$$

so that by (2.6)

$$\big\|g(v)-g(u)\big\|_{\dot B^s_{p,q}}\leqslant C\bigg(\int_{\mathbb{R}^N}\big\|A(u,v,y)\big\|^q_{L^p}|y|^{-N-sq}\,dy\bigg)^{\frac{1}{q}}. \tag{2.9}$$

We deduce from (2.8) and (2.7) that

$$\begin{aligned}A(u,v,y)&=\big[g(\tau_y v)-g(v)\big]-\big[g(\tau_y u)-g(u)\big]\\&=(\tau_y v-v)\int_0^1 g'\big(v+\theta(\tau_y v-v)\big)\,d\theta-(\tau_y u-u)\int_0^1 g'\big(u+\theta(\tau_y u-u)\big)\,d\theta;\end{aligned}$$

and so

$$\begin{aligned}A(u,v,y)\;&=\big[\tau_y(v-u)-(v-u)\big]\int_0^1 g'\big(v+\theta(\tau_y v-v)\big)\,d\theta\\&\quad+(\tau_y u-u)\int_0^1\big[g'\big(v+\theta(\tau_y v-v)\big)-g'\big(u+\theta(\tau_y u-u)\big)\big]\,d\theta\\&=:A_1(u,v,y)+A_2(u,v,y).\end{aligned} \tag{2.10}$$

It follows from (2.2) that

$$\big|A_1(u,v,y)\big|\leqslant C\big(|v|^\alpha+|\tau_y v|^\alpha\big)\big|\tau_y(v-u)-(v-u)\big|.$$

We deduce by Hölder's inequality (note that $\sigma/\alpha\geqslant 1$) that

$$\big\|A_1(u,v,y)\big\|_{L^p}\leqslant C\|v\|^\alpha_{L^\sigma}\big\|\tau_y(v-u)-(v-u)\big\|_{L^r},$$

from which it follows by applying (2.6) that

$$\bigg(\int_{\mathbb{R}^N}\big\|A_1(u,v,y)\big\|^q_{L^p}|y|^{-N-sq}\,dy\bigg)^{\frac{1}{q}}\leqslant C\|v\|^\alpha_{L^\sigma}\|v-u\|_{\dot B^s_{r,q}}. \tag{2.11}$$

We set

$$K(u,v)=\bigg(\int_{\mathbb{R}^N}\big\|A_2(u,v,y)\big\|^q_{L^p}|y|^{-N-sq}\,dy\bigg)^{\frac{1}{q}}, \tag{2.12}$$

and formula (2.3) follows from (2.9), (2.10), (2.11) and (2.12). Next, it follows from assumption (2.2) that

$$\big|A_2(u,v,y)\big|\leqslant C\big(|u|^\alpha+|\tau_y u|^\alpha+|v|^\alpha+|\tau_y v|^\alpha\big)|\tau_y u-u|. \tag{2.13}$$

Applying Hölder's inequality, we deduce as above that

$$\bigg(\int_{\mathbb{R}^N}\big\|A_2(u,v,y)\big\|^q_{L^p}|y|^{-N-sq}\,dy\bigg)^{\frac{1}{q}}\leqslant C\big(\|u\|^\alpha_{L^\sigma}+\|v\|^\alpha_{L^\sigma}\big)\|u\|_{\dot B^s_{r,q}}, \tag{2.14}$$

which proves (2.4). It remains to show (2.5). Let $(u_n)_{n\geqslant 1}$ and u be as in the statement, and assume by contradiction that there is a subsequence, still denoted by $(u_n)_{n\geqslant 1}$, and $\varepsilon > 0$ such that

$$K(u,u_n) \geqslant \varepsilon. \tag{2.15}$$

We set

$$B_n(y,\theta)(\cdot) = |\tau_y u - u| \big| g'\big(u_n + \theta(\tau_y u_n - u_n)\big) - g'\big(u + \theta(\tau_y u - u)\big)\big|, \tag{2.16}$$

and we deduce from (2.2) that

$$\big|B_n(y,\theta)\big| \leqslant C\big(|u|^\alpha + |\tau_y u|^\alpha + |u_n|^\alpha + |\tau_y u_n|^\alpha\big)|\tau_y u - u|. \tag{2.17}$$

It follows from (2.10) that

$$\big\|A_2(u,u_n,y)\big\|_{L^p}^p \leqslant \int_0^1 \int_{\mathbb{R}^N} B_n(y,\theta)^p \, dx \, d\theta. \tag{2.18}$$

Note that $|u_n - u|^\sigma \to 0$ in $L^1(\mathbb{R}^N)$. In particular, by possibly extracting a subsequence, we see that $u_n \to u$ as $n\to\infty$ a.e. and that there exists a function $w \in L^1(\mathbb{R}^N)$ such that $|u_n - u|^\sigma \leqslant w$ a.e. It follows that $|u_n|^\sigma \leqslant C(|u|^\sigma + w) \in L^1(\mathbb{R}^N)$. Moreover, by Young's inequality

$$B_n(y,\theta)^p \leqslant C\big(|u|^\sigma + |\tau_y u|^\sigma + |u_n|^\sigma + |\tau_y u_n|^\sigma + |\tau_y u - u|^r\big). \tag{2.19}$$

Since by (2.6) $\tau_y u - u \in L^r(\mathbb{R}^N)$ for a.a. $y \in \mathbb{R}^N$, we see that for a.a. $y \in \mathbb{R}^N$, $B_n(y,\theta)^p$ is dominated by an L^1 function. Furthermore, g' is continuous and $u_n \to u$ a.e., so that $B_n(y,\theta) \to 0$ a.e. as $n\to\infty$. We deduce by dominated convergence and (2.18) that

$$\big\|A_2(u,u_n,y)\big\|_{L^p}^p \xrightarrow[n\to\infty]{} 0, \tag{2.20}$$

for almost all $y \in \mathbb{R}^N$. Next, it follows from (2.13) and Hölder's inequality that

$$\big\|A_2(u,u_n,y)\big\|_{L^p} \leqslant C\big(\|u\|_{L^\sigma}^\alpha + \|u_n\|_{L^\sigma}^\alpha\big)\|\tau_y u - u\|_{L^r} \leqslant C\|\tau_y u - u\|_{L^r}.$$

Since

$$\|\tau_y u - u\|_{L^r}^q |y|^{-N-sq} \in L^1\big(\mathbb{R}^N\big),$$

by (2.6), we see that $\|A_2(u,u_n,y)\|_{L^p}^q |y|^{-N-sq}$ is dominated by an L^1 function. Applying (2.20), we deduce by dominated convergence that $K(u,u_n) \to 0$ as $n\to\infty$. This contradicts (2.15) and completes the proof. □

Lemma 2.2. *Let $0 < s < 1$ and $1 \leqslant p \leqslant \infty$, $1 \leqslant q < \infty$. Let $g \in C^1(\mathbb{C},\mathbb{C})$ with g' bounded. It follows that there exists a constant C such that if $u, v \in \dot B^s_{p,q}(\mathbb{R}^N)$, then*

$$\big\|g(v) - g(u)\big\|_{\dot B^s_{p,q}} \leqslant C\|v - u\|_{\dot B^s_{p,q}} + K(u,v), \tag{2.21}$$

where $K(u,v)$ satisfies

$$K(u,v) \leqslant C\|u\|_{\dot B^s_{p,q}}, \tag{2.22}$$

and

$$K(u,u_n) \xrightarrow[n\to\infty]{} 0, \tag{2.23}$$

if $(u_n)_{n\geqslant 1} \subset \dot B^s_{p,q}(\mathbb{R}^N)$ and $u \in \dot B^s_{p,q}(\mathbb{R}^N)$ are such that $u_n \to u$ in $L^\mu(\mathbb{R}^N)$ for some $1 \leqslant \mu \leqslant \infty$.

Proof. The proof of Lemma 2.1 is easily adapted. □

Remark 2.3. In the particular case when $g(u)=|u|^\alpha u$ with $\alpha>0$, the estimate (2.3) can be refined. More precisely,

$$\|g(v)-g(u)\|_{\dot B^s_{p,q}} \leqslant C\|v\|^\alpha_{L^\sigma}\|v-u\|_{\dot B^s_{r,q}} + C\|u\|_{\dot B^s_{r,q}}\|u-v\|^\alpha_{L^\sigma}, \tag{2.24}$$

if $0<\alpha\leqslant 1$ and

$$\|g(v)-g(u)\|_{\dot B^s_{p,q}} \leqslant C\|v\|^\alpha_{L^\sigma}\|v-u\|_{\dot B^s_{r,q}} + C\|u\|_{\dot B^s_{r,q}}\big(\|u\|^{\alpha-1}_{L^\sigma}+\|v\|^{\alpha-1}_{L^\sigma}\big)\|u-v\|_{L^\sigma}, \tag{2.25}$$

if $\alpha\geqslant 1$. Indeed, note that $\partial_z g(z)=(1+\frac{\alpha}{2})|z|^\alpha$ and $\partial_{\bar z} g(z)=\frac{\alpha}{2}|z|^{\alpha-2}z^2$. We claim that,

$$\big||z_1|^\alpha-|z_2|^\alpha\big| \leqslant \begin{cases} \alpha(|z_1|^{\alpha-1}+|z_2|^{\alpha-1})|z_1-z_2| & \text{if } \alpha\geqslant 1,\\ |z_1-z_2|^\alpha & \text{if } 0<\alpha\leqslant 1,\end{cases} \tag{2.26}$$

and

$$\big||z_1|^{\alpha-2}z_1^2-|z_2|^{\alpha-2}z_2^2\big| \leqslant \begin{cases} C(|z_1|^{\alpha-1}+|z_2|^{\alpha-1})|z_1-z_2| & \text{if } \alpha\geqslant 1,\\ C|z_1-z_2|^\alpha & \text{if } 0<\alpha\leqslant 1.\end{cases} \tag{2.27}$$

Assuming (2.26)–(2.27), estimates (2.24) and (2.25) follow from the argument used at the beginning of the proof of Lemma 2.1. Indeed, the left-hand side of (2.14) (where A_2 is defined by (2.10)) is easily estimated by applying (2.26)–(2.27) and Hölder's inequality. It remains to prove the claim (2.26)–(2.27). The first three estimates are quite standard, and we only prove the last one. We assume, without loss of generality, that $0<|z_2|\leqslant|z_1|$. Note first that

$$\begin{aligned}\big||z_1|^{\alpha-2}z_1^2-|z_2|^{\alpha-2}z_2^2\big| &= \left||z_1|^\alpha\left(\frac{z_1^2}{|z_1|^2}-\frac{z_2^2}{|z_2|^2}\right)+\frac{z_2^2}{|z_2|^2}\big(|z_1|^\alpha-|z_2|^\alpha\big)\right|\\ &\leqslant |z_1|^\alpha\left|\frac{z_1^2}{|z_1|^2}-\frac{z_2^2}{|z_2|^2}\right|+|z_1-z_2|^\alpha.\end{aligned}$$

Next,

$$\begin{aligned}|z_1|^\alpha\left|\frac{z_1^2}{|z_1|^2}-\frac{z_2^2}{|z_2|^2}\right| &\leqslant 2|z_1|^\alpha\left|\frac{z_1}{|z_1|}-\frac{z_2}{|z_2|}\right|\\ &= 2|z_1|^{\alpha-1}\left|z_1-z_2+\frac{z_2}{|z_2|}\big(|z_2|-|z_1|\big)\right| \leqslant 4|z_1|^{\alpha-1}|z_1-z_2|.\end{aligned}$$

If $|z_1-z_2|\leqslant|z_1|$, then (since $\alpha\leqslant 1$) $|z_1|^{\alpha-1}|z_1-z_2|\leqslant|z_1-z_2|^\alpha$. If $|z_1-z_2|\geqslant|z_1|$, then (recall that $|z_1-z_2|\leqslant |z_1|+|z_2|\leqslant 2|z_1|$) we see that $|z_1|^{\alpha-1}|z_1-z_2|\leqslant 2|z_1|^\alpha\leqslant 2|z_1-z_2|^\alpha$. This shows the second estimate in (2.27).

3. Proof of Theorem 1.2 in the subcritical case $\alpha<\frac{4}{N-2s}$

Throughout this section, we assume $\alpha<\frac{4}{N-2s}$ and we use the admissible pair (γ,ρ) defined by (1.5). We need only to show that, given $\varphi\in H^s(\mathbb{R}^N)$, there exists $T=T(\|\varphi\|_{H^s})>0$ such that if $\|\varphi\|_{H^s}<M$ then $T_{\max}(\varphi)>T$ and such that if $\varphi_n\to\varphi$ in $H^s(\mathbb{R}^N)$, then $T_{\max}(\varphi_n)>T$ for all sufficiently large n and the corresponding solution u_n of (NLS) satisfies $u_n\to u$ in $L^q((0,T),B^s_{r,2}(\mathbb{R}^N))$ for all admissible pairs (q,r). Since $T=T(\|\varphi\|_{H^s})$, properties (i) and (ii) easily follow by a standard iteration argument.

We note that by Theorem 1.1 there exists $T=T(\|\varphi\|_{H^s})>0$ such that if $\|\varphi\|_{H^s}<M$ then $T_{\max}(\varphi)>T$. Moreover, by possibly choosing T smaller (but still depending on $\|\varphi\|_{H^s}$), if $\varphi_n\to\varphi$ in $H^s(\mathbb{R}^N)$, then $T_{\max}(\varphi_n)>T$ for all sufficiently large n and the corresponding solution u_n of (NLS) satisfies

$$\sup_{n\geqslant 1}\|u_n\|_{L^q((0,T),B^s_{r,2})}<\infty, \tag{3.1}$$

and

$$u_n \xrightarrow[n\to\infty]{} u \quad \text{in } L^q\big((0,T),B^{s-\varepsilon}_{r,2}(\mathbb{R}^N)\big)\cap C\big([0,T],H^{s-\varepsilon}(\mathbb{R}^N)\big), \tag{3.2}$$

for all $\varepsilon>0$ and all admissible pairs (q,r). It is not specified in Theorem 1.1 that, in the subcritical case, T can be bounded from below in terms of $\|\varphi\|_{H^s}$ but this is immediate from the proof.

We set

$$\sigma = \frac{N(\alpha+2)}{N-2s} > \rho, \tag{3.3}$$

so that by Sobolev's embedding

$$\dot{B}^s_{\rho,2}\big(\mathbb{R}^N\big) \hookrightarrow L^\sigma\big(\mathbb{R}^N\big), \tag{3.4}$$

and we claim that

$$u_n \xrightarrow[n\to\infty]{} u \quad \text{in } L^{\gamma-\varepsilon}\big((0,T), L^\sigma\big(\mathbb{R}^N\big)\big), \tag{3.5}$$

for all $0 < \varepsilon \leqslant \gamma - 1$. This is a consequence of (3.2). (In fact, if we could let $\varepsilon = 0$ in (3.2), then we would obtain by (3.4) convergence in $L^\gamma((0,T), L^\sigma(\mathbb{R}^N))$.) Indeed, given $\eta > 0$ and small, we have $B^{s-\frac{\eta N}{\rho(\rho+\eta)}}_{\rho+\eta,2}(\mathbb{R}^N) \hookrightarrow L^\sigma(\mathbb{R}^N)$. If η is sufficiently small, $\rho + \eta < 2N/(N-2)^+$ so that there exists γ_η such that $(\gamma_\eta, \rho+\eta)$ is an admissible pair, and we deduce from (3.2) that $u_n \to u$ in $L^{\gamma_\eta}((0,T), L^\sigma(\mathbb{R}^N))$. Since $\gamma_\eta \to \gamma$ as $\eta \to 0$, we conclude that (3.5) holds.

We decompose g in the form $g = g_1 + g_2$ where $g_1, g_1 \in C^1(\mathbb{C},\mathbb{C})$, $g_1(0) = g_2(0) = 0$ and

$$\big|g_1'(u)\big| \leqslant C, \tag{3.6}$$

$$\big|g_2'(u)\big| \leqslant C|u|^\alpha, \tag{3.7}$$

for all $u \in \mathbb{C}$. By Strichartz estimates in Besov spaces (see Theorem 2.2 in [8]), given any admissible pair (q,r) there exists a constant C such that

$$\begin{aligned}\|u_n - u\|_{L^q((0,T),\dot{B}^s_{r,2})} &\leqslant C\|\varphi_n - \varphi\|_{\dot{H}^s} \\ &\quad + C\big\|g_1(u_n) - g_1(u)\big\|_{L^1((0,T),\dot{B}^s_{2,2})} + C\big\|g_2(u_n) - g_2(u)\big\|_{L^{\gamma'}((0,T),\dot{B}^s_{\rho',2})}.\end{aligned} \tag{3.8}$$

We estimate the last two terms in (3.8) by applying Lemmas 2.2 and 2.1, respectively. We first apply Lemma 2.2 to g_1 with $q = p = 2$ and we obtain

$$\big\|g_1(u_n) - g_1(u)\big\|_{\dot{B}^s_{2,2}} \leqslant C\|u_n - u\|_{\dot{B}^s_{2,2}} + K_1(u,u_n). \tag{3.9}$$

Next, we apply Lemma 2.1 to g_2 with $q = 2$, $r = \rho$ and $p = \rho'$, and we obtain

$$\big\|g_2(u_n) - g_2(u)\big\|_{\dot{B}^s_{\rho',2}} \leqslant C\|u_n\|^\alpha_{L^\sigma}\|u_n - u\|_{\dot{B}^s_{\rho,2}} + K_2(u,u_n), \tag{3.10}$$

where σ is defined by (3.3). Applying Hölder's inequality in time, we deduce from (3.9) that

$$\big\|g_1(u_n) - g_1(u)\big\|_{L^1((0,T),\dot{B}^s_{2,2})} \leqslant CT\|u_n - u\|_{L^\infty((0,T),\dot{B}^s_{2,2})} + \big\|K_1(u,u_n)\big\|_{L^1(0,T)}, \tag{3.11}$$

and from (3.10) that

$$\begin{aligned}\big\|g_2(u_n) - g_2(u)\big\|_{L^{\gamma'}((0,T),\dot{B}^s_{\rho',2})} &\leqslant CT^{\frac{4-\alpha(N-2s)}{4}}\|u_n\|^\alpha_{L^\gamma((0,T),L^\sigma)}\|u_n - u\|_{L^\gamma((0,T),\dot{B}^s_{\rho,2})} \\ &\quad + \big\|K_2(u,u_n)\big\|_{L^{\gamma'}(0,T)}.\end{aligned} \tag{3.12}$$

Note that $\|u_n\|^\alpha_{L^\gamma((0,T),L^\sigma)}$ is bounded by (3.1) and (3.4). Thus we see that, by possibly choosing T smaller (but still depending on $\|\varphi\|_{H^s}$), we can absorb the first term in the right-hand side of (3.11) by the left-hand side of (3.8) (with the choice $(q,r) = (\infty,2)$), and similarly for (3.12) (with the choice $(q,r) = (\gamma,\rho)$). By doing so, we deduce from (3.8) that

$$\begin{aligned}&\|u_n - u\|_{L^\gamma((0,T),\dot{B}^s_{\rho,2})} + \|u_n - u\|_{L^\infty((0,T),\dot{H}^s)} \\ &\quad \leqslant C\|\varphi_n - \varphi\|_{\dot{H}^s} + C\big\|K_1(u,u_n)\big\|_{L^1(0,T)} + C\big\|K_2(u,u_n)\big\|_{L^{\gamma'}(0,T)}.\end{aligned} \tag{3.13}$$

Next, we note that by (2.22) $K_1(u,u_n)\leqslant \|u(t)\|_{\dot{B}^s_{2,2}}$ for all $t\in(0,T)$ and that $u\in L^\infty((0,T),B^s_{2,2}(\mathbb{R}^N))$. Moreover, $u_n\to u$ in $C([0,T],L^2(\mathbb{R}^N))$, so that $u_n(t)\to u(t)$ in $L^2(\mathbb{R}^N)$ for all $t\in(0,T)$. Applying (2.23) (with $\mu=2$), we deduce that $K_1(u_n,u)\to 0$ for all $t\in(0,T)$. By dominated convergence, we conclude that

$$\big\|K_1(u,u_n)\big\|_{L^1(0,T)} \underset{n\to\infty}{\longrightarrow} 0. \tag{3.14}$$

We now show that

$$\big\|K_2(u,u_n)\big\|_{L^{\gamma'}(0,T)} \underset{n\to\infty}{\longrightarrow} 0. \tag{3.15}$$

Indeed, suppose by contradiction that there exist a subsequence, still denoted by $(u_n)_{n\geqslant 1}$, and $\delta>0$ such that

$$\big\|K_2(u,u_n)\big\|_{L^{\gamma'}(0,T)} \geqslant \delta. \tag{3.16}$$

We note that by (2.4) and Young's inequality

$$\begin{aligned}K_2(u_n,u)^{\gamma'} &\leqslant C\big(\|u_n\|_{L^\sigma}^{\alpha\gamma'}+\|u\|_{L^\sigma}^{\alpha\gamma'}\big)\|u\|_{\dot{B}^s_{\rho,2}}^{\gamma'}\\ &\leqslant C\big(\|u_n\|_{L^\sigma}^{\frac{\alpha\gamma}{\gamma-2}}+\|u\|_{L^\sigma}^{\frac{\alpha\gamma}{\gamma-2}}+\|u\|_{\dot{B}^s_{\rho,2}}^{\gamma}\big).\end{aligned} \tag{3.17}$$

Since $\alpha\gamma/(\gamma-2)<\gamma$, we deduce from (3.5) that, after possibly extracting a subsequence, $K_2(u_n,u)^{\gamma'}$ is dominated by an L^1 function. It also follows from (3.5) that, after possibly extracting a subsequence, $u_n(t)\to u(t)$ in $L^\sigma(\mathbb{R}^N)$ for a.a. $t\in(0,T)$ so that, applying (2.5),

$$K_2(u,u_n) \underset{n\to\infty}{\longrightarrow} 0, \tag{3.18}$$

for a.a. $t\in(0,T)$. By dominated convergence, $\|K_2(u,u_n)\|_{L^{\gamma'}(0,T)}\to 0$. This contradicts (3.16), thus proving (3.15).

Finally, it follows from (3.13), (3.14) and (3.15) that

$$\|u_n-u\|_{L^\gamma((0,T),\dot{B}^s_{\rho,2})}+\|u_n-u\|_{L^\infty((0,T),\dot{H}^s)} \underset{n\to\infty}{\longrightarrow} 0. \tag{3.19}$$

Given any admissible pair (q,r), we deduce from (3.8), (3.11), (3.12), (3.14), (3.15) and (3.19) that $\|u_n-u\|_{L^q((0,T),\dot{B}^s_{r,2})}\to 0$ as $n\to\infty$. This completes the proof.

4. Proof of Theorem 1.2 in the critical case $\alpha=\frac{4}{N-2s}$

Throughout this section, we assume $\alpha=\frac{4}{N-2s}$ and we use the admissible pair (γ,ρ) defined by (1.5). We recall that by assumption (1.7),

$$\big|g'(u)\big|\leqslant C|u|^\alpha. \tag{4.1}$$

The argument of the preceding section fails essentially at one point. More precisely, here $\alpha\gamma/(\gamma-2)=\gamma$, so that the convergence (3.5) does not imply as in Section 3 (see (3.17)) that $K_2(u_n,u)^{\gamma'}$ is dominated by an L^1 function. To overcome this difficulty, we show in Lemma 4.1 below that (3.5) holds with $\varepsilon=0$. This is the key difference with the subcritical case, and the rest of the proof is very similar.

We now go into details and we first recall some facts concerning the local theory. (See e.g. [8].) We define

$$\mathcal{G}(u)(t)=\int_0^t e^{i(t-s)\Delta}g\big(u(s)\big)\,ds. \tag{4.2}$$

If (γ,ρ) is the admissible pair defined by (1.5), then

$$\big\|g(u)\big\|_{L^{\gamma'}((0,T),B^s_{\rho',2})}\leqslant C\|u\|^{\alpha+1}_{L^\gamma((0,T),B^s_{\rho,2})},$$

and

$$\big\|g(v)-g(u)\big\|_{L^{\gamma'}((0,T),L^{\rho'})}\leqslant C\big(\|u\|^\alpha_{L^\gamma((0,T),B^s_{\rho,2})}+\|u\|^\alpha_{L^\gamma((0,T),B^s_{\rho,2})}\big)\|u-v\|_{L^\gamma((0,T),L^\rho)},$$

where the constant C is independent of $T > 0$. By Strichartz estimates in Besov spaces, it follows that the problem (1.6) can be solved by a fixed point argument in the set $\mathcal{E} = \{u \in L^\gamma((0,T), B^s_{\rho,2}(\mathbb{R}^N)); \|u\|_{L^\gamma((0,T),B^s_{\rho,2})} \leqslant \eta\}$ equipped with the distance $\mathrm{d}(u,v) = \|u-v\|_{L^\gamma((0,T),L^\rho)}$ for $\eta > 0$ sufficiently small. (Note that $(\mathcal{E}, \mathrm{d})$ is a complete metric space. Indeed, $L^\gamma((0,T), B^s_{\rho,2})$ is reflexive, so its closed ball of radius η is weakly compact.) By doing so, we see that there exists $\delta_0 > 0$ (independent of $\varphi \in H^s(\mathbb{R}^N)$ and $T > 0$) such that if

$$\left\|e^{i\cdot\Delta}\varphi\right\|_{L^\gamma((0,T),B^s_{\rho,2})} < \delta \leqslant \delta_0, \tag{4.3}$$

then $T_{\max}(\varphi) > T$ and the corresponding solution of (1.6) satisfies

$$\|u\|_{L^\gamma((0,T),B^s_{\rho,2})} \leqslant 2\delta. \tag{4.4}$$

Moreover, given any admissible pair (q,r), there exists a constant $C(q,r)$ such that

$$\|u\|_{L^q((0,T),B^s_{r,2})} \leqslant \delta C(q,r). \tag{4.5}$$

In addition, if φ, ψ both satisfy (4.3) and u, v are the corresponding solutions of (NLS), then

$$\|u-v\|_{L^\gamma((0,T),L^\rho)} \leqslant C\|\varphi-\psi\|_{L^2}. \tag{4.6}$$

We also recall the Strichartz estimate in Besov spaces

$$\left\|e^{i\cdot\Delta}\varphi\right\|_{L^q((0,T),B^s_{r,2})} \leqslant C(q,r)\|\varphi\|_{H^s}, \tag{4.7}$$

for every admissible pair (q,r). In particular,

$$\left\|e^{i\cdot\Delta}\psi\right\|_{L^\gamma((0,T),B^s_{\rho,2})} \leqslant \left\|e^{i\cdot\Delta}\varphi\right\|_{L^\gamma((0,T),B^s_{\rho,2})} + C\|\psi-\varphi\|_{H^s}, \tag{4.8}$$

for every $\varphi, \psi \in H^s(\mathbb{R}^N)$.

We now fix $\varphi \in H^s(\mathbb{R}^N)$ satisfying (4.3) and we consider $(\varphi_n)_{n\geqslant 1} \subset H^s(\mathbb{R}^N)$ such that

$$\|\varphi_n - \varphi\|_{H^s} \underset{n\to\infty}{\longrightarrow} 0. \tag{4.9}$$

We denote by u and $(u_n)_{n\geqslant 1}$ the corresponding solutions of (NLS).

We show that, by letting δ_0 in (4.3) possibly smaller (but still independent of T and φ),

$$\|u_n - u\|_{L^q((0,T),B^s_{r,2})} \underset{n\to\infty}{\longrightarrow} 0, \tag{4.10}$$

for all admissible pairs (q,r). Indeed, arguing as in the subcritical case, we see that (see the proof of (3.13))

$$\begin{aligned} \|u_n - u\|_{L^\gamma((0,T),\dot{B}^s_{\rho,2})} &\leqslant C\|\varphi_n-\varphi\|_{\dot{H}^s} \\ &\quad + C\|u_n\|^\alpha_{L^\gamma((0,T),L^\sigma)}\|u_n-u\|_{L^\gamma((0,T),\dot{B}^s_{\rho,2})} + \left\|K(u,u_n)\right\|_{L^{\gamma'}(0,T)}, \end{aligned} \tag{4.11}$$

where $K(u,u_n)$ is given by Lemma 2.1. We next observe that, using (4.4), (3.4) and possibly choosing δ_0 smaller, $C\|u_n\|^\alpha_{L^\gamma((0,T),L^\sigma)} \leqslant 1/2$ for n large. We then deduce from (4.11) that

$$\|u_n - u\|_{L^\gamma((0,T),\dot{B}^s_{\rho,2})} \leqslant C\|\varphi_n-\varphi\|_{\dot{H}^s} + C\left\|K(u,u_n)\right\|_{L^{\gamma'}(0,T)}. \tag{4.12}$$

We claim that

$$\left\|K(u,u_n)\right\|_{L^{\gamma'}(0,T)} \underset{n\to\infty}{\longrightarrow} 0. \tag{4.13}$$

For proving (4.13), we use the following lemma, whose proof is postponed until the end of this section.

Lemma 4.1. *Under the above assumptions, and for* δ_0 *in* (4.3) *possibly smaller* (*but still independent of* T *and* φ) *it follows that*

$$\|u_n - u\|_{L^\gamma((0,T),L^\sigma)} \underset{n\to\infty}{\longrightarrow} 0, \tag{4.14}$$

where σ *is defined by* (3.3).

Assuming Lemma 4.1, suppose by contradiction that there exist a subsequence, still denoted by $(u_n)_{n\geqslant 1}$ and $\beta > 0$ such that

$$\big\|K(u,u_n)\big\|_{L^{\gamma'}(0,T)} \geqslant \beta. \tag{4.15}$$

Note that (see (3.17)) $K(u,u_n)^{\gamma'} \leqslant C(\|u_n\|_{L^\sigma}^{\frac{\alpha\gamma}{\gamma-2}} + \|u\|_{L^\sigma}^{\frac{\alpha\gamma}{\gamma-2}} + \|u\|_{\dot B^s_{\rho,2}}^{\gamma})$. Since $\alpha\gamma/(\gamma-2) = \gamma$, this implies

$$K(u,u_n)^{\gamma'} \leqslant C\big(\|u_n\|_{L^\sigma}^{\gamma} + \|u\|_{L^\sigma}^{\gamma} + \|u\|_{\dot B^s_{\rho,2}}^{\gamma}\big). \tag{4.16}$$

It follows from (4.14) and (4.16) that, after possibly extracting a subsequence, $K(u,u_n)^{\gamma'}$ is bounded by an L^1 function. Moreover, we may also assume that $u_n(t) \to u(t)$ in $L^\sigma(\mathbb{R}^N)$ for a.a. $t \in (0,T)$, so that by (2.5) $K(u,u_n) \to 0$ for a.a. $t \in (0,T)$. By dominated convergence, we deduce that $\|K(u,u_n)\|_{L^{\gamma'}(0,T)} \to 0$. This contradicts (4.15), thus proving (4.13).

Applying (4.12), (4.9) and (4.13), we conclude that

$$\|u_n - u\|_{L^\gamma((0,T),\dot B^s_{\rho,2})} \underset{n\to\infty}{\longrightarrow} 0.$$

By Lemma 2.1 (and using again (4.13)), this implies that

$$\big\|g(u_n) - g(u)\big\|_{L^{\gamma'}((0,T),\dot B^s_{\rho',2})} \underset{n\to\infty}{\longrightarrow} 0.$$

Applying Strichartz estimates, we conclude that (4.10) holds.

To conclude the proof, we argue as follows. We let $\widetilde T$ be the supremum of all $0 < T < T_{\max}(\varphi)$ such that if $\varphi_n \to \varphi$ in $H^s(\mathbb{R}^N)$, then $T_{\max}(\varphi_n) > T$ for all sufficiently large n and $u_n \to u$ in $L^q((0,T), B^s_{r,2}(\mathbb{R}^N))$ as $n \to \infty$ for all admissible pairs (q,r). We have just shown that $\widetilde T > 0$. We claim that $\widetilde T = T_{\max}(\varphi)$. Indeed, otherwise $\widetilde T < T_{\max}$. Since $u \in C([0,\widetilde T], H^s(\mathbb{R}^N))$, it follows that $\bigcup_{0\leqslant t\leqslant \widetilde T}\{u(t)\}$ is a compact subset of $H^s(\mathbb{R}^N)$. Therefore, it follows from Strichartz estimates that there exists $T > 0$ such that

$$\sup_{0\leqslant t\leqslant \widetilde T} \big\|e^{i\cdot\Delta}u(t)\big\|_{L^\gamma((0,T),B^s_{\rho,2})} \leqslant \delta_0. \tag{4.17}$$

We fix $0 < \tau < \widetilde T$ such that $\tau + T > \widetilde T$. If $\varphi_n \to \varphi$ in $H^s(\mathbb{R}^N)$, it follows from the definition of $\widetilde T$ that $T_{\max}(\varphi_n) > \tau$ for all sufficiently large n and that $u_n(\tau) \to u(\tau)$ in $H^s(\mathbb{R}^N)$ as $n \to \infty$. We then deduce from (4.17) and what precedes that $T_{\max}(u_n(\tau)) > T$ for all large n and that $u_n(\tau + \cdot) \to u(\tau + \cdot)$ in $L^q((0,T), B^s_{r,2}(\mathbb{R}^N))$ for all admissible pairs (q,r). Thus we see that $T_{\max}(\varphi_n) > \tau + T$ for all large n and that $u_n \to u$ in $L^q((0,\tau+T), B^s_{r,2}(\mathbb{R}^N))$ for all admissible pairs (q,r). This contradicts the definition of $\widetilde T$. Thus $\widetilde T = T$, which proves properties (i) and (ii).

To complete the proof, it thus remains to prove Lemma 4.1.

Proof of Lemma 4.1. Given $2 \leqslant r < N/s$, we define $\nu(r) \in (r,\infty)$ by

$$\frac{1}{\nu(r)} = \frac{1}{r} - \frac{s}{N}, \tag{4.18}$$

so that

$$B^s_{r,2}\big(\mathbb{R}^N\big) \hookrightarrow L^{\nu(r)}\big(\mathbb{R}^N\big), \tag{4.19}$$

by Sobolev's embedding. Note also that if (γ,ρ) is the admissible pair given by (1.5), then $\nu(\rho) = \sigma$ given by (3.3).

We first observe that we need only to prove that

$$\|u_n - u\|_{L^{q_0}((0,T),L^{\nu(r_0)})} \underset{n\to\infty}{\longrightarrow} 0, \tag{4.20}$$

for some admissible pair (q_0, r_0) for which $r_0 < \frac{N}{s}$ (and $\nu(r_0)$ is defined by (4.18)). Indeed, suppose (4.20) holds. If $\sigma \geqslant \nu(r_0)$, then $\rho \geqslant r_0$. We fix $\rho < \tilde r < \frac{N}{s}$ and we deduce from (4.18) and Hölder's inequality in space and time that

$$\|u_n - u\|_{L^\gamma((0,T),L^\sigma)} \leqslant \|u_n - u\|^{\theta}_{L^{q_0}((0,T),L^{\nu(r_0)})} \|u_n - u\|^{1-\theta}_{L^{\tilde q}((0,T),L^{\nu(\tilde r)})},$$

where $1/\rho = \theta/r_0 + (1-\theta)/\tilde{r}$. Since $\|u_n - u\|^{1-\theta}_{L^{\tilde{q}}((0,T),L^{\nu(\tilde{r})})}$ is bounded by (4.5) and Sobolev's embedding (4.19), we see that (4.14) follows. In the case $\sigma < \nu(r_0)$, then $\rho < r_0$ and we apply the same argument, this time with $(\tilde{q}, \tilde{r}) = (\infty, 2)$.

We now prove (4.20) for (q_0, r_0) defined by

$$q_0 = \frac{2\alpha(\alpha+2)}{4-(N-2)\alpha}, \qquad r_0 = \frac{N(\alpha+2)}{N+s(\alpha+2)}. \tag{4.21}$$

It is straightforward to check that (q_0, r_0) is an admissible pair, that $r_0 < \frac{N}{s}$ and that

$$\nu(r_0) = \alpha + 2. \tag{4.22}$$

We now use a Strichartz-type estimate for the non-admissible pair $(q_0, \alpha+2)$. More precisely, it follows from Lemma 2.1 in [9][4] that there exists a constant C independent of n and T such that

$$\left\|\mathcal{G}(u_n) - \mathcal{G}(u)\right\|_{L^{q_0}(\mathbb{R},L^{\alpha+2})} \leqslant C\left\|g(u_n) - g(u)\right\|_{L^{\frac{q_0}{\alpha+1}}(\mathbb{R},L^{\frac{\alpha+2}{\alpha+1}})}. \tag{4.23}$$

On the other hand, we deduce from (4.1) and Hölder's inequality in space and time that

$$\left\|g(u_n) - g(u)\right\|_{L^{\frac{q_0}{\alpha+1}}(\mathbb{R},L^{\frac{\alpha+2}{\alpha+1}})} \leqslant C\big(\|u_n\|^{\alpha}_{L^{q_0}((0,T),L^{\alpha+2})} + \|u\|^{\alpha}_{L^{q_0}((0,T),L^{\alpha+2})}\big)\|u_n - u\|_{L^{q_0}((0,T),L^{\alpha+2})}.$$

Applying (4.19) and (4.22), we obtain

$$\left\|g(u_n) - g(u)\right\|_{L^{\frac{q_0}{\alpha+1}}(\mathbb{R},L^{\frac{\alpha+2}{\alpha+1}})} \leqslant C\big(\|u_n\|^{\alpha}_{L^{q_0}((0,T),B^s_{r_0,2})} + \|u\|^{\alpha}_{L^{q_0}((0,T),B^s_{r_0,2})}\big)\|u_n - u\|_{L^{q_0}((0,T),L^{\alpha+2})}. \tag{4.24}$$

For the pair (q_0, r_0) being admissible, we deduce from Strichartz estimate (4.7), Sobolev's embedding (4.19) and the identity (4.22) that

$$\left\|e^{i\cdot\Delta}(\varphi_n - \varphi)\right\|_{L^{q_0}(\mathbb{R},L^{\alpha+2})} \leqslant C\|\varphi_n - \varphi\|_{H^s}. \tag{4.25}$$

Using Eq. (1.6) for u_n and u, together with (4.25), (4.23) and (4.24), we see that

$$\begin{aligned}\|u_n - u\|_{L^{q_0}((0,T),L^{\alpha+2})} &\leqslant C\|\varphi_n - \varphi\|_{H^s} \\ &\quad + C\big(\|u_n\|^{\alpha}_{L^{q_0}((0,T),B^s_{r_0,2})} + \|u\|^{\alpha}_{L^{q_0}((0,T),B^s_{r_0,2})}\big)\|u_n - u\|_{L^{q_0}((0,T),L^{\alpha+2})}.\end{aligned} \tag{4.26}$$

We now observe that if φ satisfies (4.3), then φ_n also satisfies (4.3) for n large (see (4.8)), so that by (4.5) we have

$$\max\big\{\|u\|_{L^{q_0}((0,T),B^s_{r_0,2})}, \|u_n\|_{L^{q_0}((0,T),B^s_{r_0,2})}\big\} \leqslant \delta C(q_0, r_0), \tag{4.27}$$

for all sufficiently large n. Therefore, by possibly choosing δ_0 smaller, we can absorb the last term in (4.26) by the left-hand side, and we deduce that (4.20) holds. This completes the proof. □

5. Proof of Corollary 1.3

The first statement of Corollary 1.3 follows from Theorem 1.2. The Lipschitz dependence in the case $\alpha \geqslant 1$ follows from the estimate (2.25). Indeed, if ρ is defined by (1.5) then, applying (2.25) with $p = \rho'$, $r = \rho$ and $q = 2$, we obtain

$$\left\|g(v) - g(u)\right\|_{\dot{B}^s_{\rho',2}} \leqslant C\|v\|^{\alpha}_{L^\sigma}\|v - u\|_{\dot{B}^s_{\rho,2}} + C\|u\|_{\dot{B}^s_{\rho,2}}\big(\|u\|^{\alpha-1}_{L^\sigma} + \|v\|^{\alpha-1}_{L^\sigma}\big)\|u - v\|_{L^\sigma},$$

where σ is given by (3.3). Using the embedding (3.4), we deduce that

$$\left\|g(v) - g(u)\right\|_{\dot{B}^s_{\rho',2}} \leqslant C\big(\|u\|^{\alpha}_{\dot{B}^s_{\rho,2}} + \|v\|^{\alpha}_{\dot{B}^s_{\rho,2}}\big)\|v - u\|_{\dot{B}^s_{\rho,2}}. \tag{5.1}$$

In view of (5.1), it is easy to see that one can go through the local existence argument by using Banach's fixed point theorem with the distance $\mathrm{d}(u,v) = \|u - v\|_{L^\gamma((0,T),B^s_{\rho,2})}$ instead of $\mathrm{d}(u,v) = \|u - v\|_{L^\gamma((0,T),L^\rho)}$. See for example the proofs of Theorems 4.9.1 (subcritical case) and 4.9.7 (critical case) in [7] for details. It now follows from standard arguments that the resulting flow is Lipschitz in the sense of Corollary 1.3.

[4] For more general estimates of this type, see [14,10,25].

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