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Best constants in a borderline case of second-order Moser type inequalities

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Abstract

We study optimal embeddings for the space of functions whose Laplacian Δu belongs to $L^1(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain. This function space turns out to be strictly larger than the Sobolev space $W^{2,1}(\Omega)$ in which the whole set of second-order derivatives is considered. In particular, in the limiting Sobolev case, when $N = 2$, we establish a sharp embedding inequality into the Zygmund space $L_{exp}(\Omega)$. On one hand, this result enables us to improve the Brezis–Merle (Brezis and Merle (1991) [13]) regularity estimate for the Dirichlet problem $\Delta u = f(x) \in L^1(\Omega)$, $u = 0$ on $\partial\Omega$; on the other hand, it represents a borderline case of D.R. Adams' (1988) [1] generalization of Trudinger–Moser type inequalities to the case of higher-order derivatives. Extensions to dimension $N \geqslant 3$ are also given. Besides, we show how the best constants in the embedding inequalities change under different boundary conditions.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, be a bounded domain and consider, for $p \geqslant 1$, the second-order Sobolev space

$$W^{2,p}(\Omega) := \left\{ u \in L^p(\Omega) \;\middle|\; D^\alpha u \in L^p(\Omega), \text{ for all } |\alpha| \leqslant 2 \right\}$$

endowed with the standard norm $\|u\|_{2,p} = \sum_{|\alpha| \leqslant 2} \|D^\alpha u\|_p$. Then, provided $\partial\Omega$ is sufficiently smooth, the following continuous embeddings hold (see e.g. [3])

$$W^{2,p}(\Omega) \hookrightarrow L^{\frac{Np}{N-2p}}(\Omega), \quad \text{if } 1 \leqslant p < \frac{N}{2} \tag{1}$$

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In the so-called Sobolev *limiting case* $p = \frac{N}{2}$ we have a striking difference between the cases $p > 1$ and $p = 1$: by Pohožaev [36] and Strichartz [40] we have

$$W^{2,p}(\Omega) \hookrightarrow L^{\phi}(\Omega), \qquad \phi(u) \sim e^{|u|^{\frac{p}{p-1}}}, \quad \text{if } p = \frac{N}{2} \text{ and } \boxed{p > 1} \tag{2}$$

where $L^{\phi}(\Omega)$ denotes the Orlicz space generated by the Young function ϕ whose elements enjoy the integrability condition $\int_\Omega \phi(u)\,dx < \infty$, while

$$W^{2,p}(\Omega) \hookrightarrow L^{\infty}(\Omega), \quad \text{if } p = \frac{N}{2} \text{ and } \boxed{p = 1} \tag{3}$$

Optimality issues in the above embeddings concern mainly the following two aspects:

- finding the *smallest* target space for which the embedding holds
- obtaining the *sharp* form of the underlying inequalities by exhibiting the best constants.

The target spaces in the embeddings (1) and (2) turn out to be optimal within the L^p resp. Orlicz space framework, in the sense that they cannot be replaced by a smaller space within these classes of spaces. However, they are not always the best possible; indeed, the question whether the embeddings (1) and (2) may be improved by finding optimal target spaces has been addressed by many authors: the process ends up for $p < \frac{N}{2}$ in the Lorentz space framework [32,35,5], see also [43] for a survey and [37] for the case $p = 1$, while in the case $1 < p = \frac{N}{2}$ the optimal setting is given by the Hansson–Brezis–Wainger spaces [23,15]. In both cases the target space turns out to be optimal among all rearrangement invariant spaces (loosely speaking, the largest class of function spaces in which membership of a function depends only on its degree of summability, see Section 2.2) as established in [17,19,26] (see also [9,31] for further generalizations which however drop the linear space structure). Note that for $p < \frac{N}{2}$, the best constant in the corresponding embedding inequality for (1) is explicitly known only in the case $p = 2$ (see [20,41,44]). In the Sobolev limiting case, namely $p = \frac{N}{2}$, $p > 1$, the following sharp version of Trudinger–Moser type of the embedding (2) has been established by D.R. Adams [1]:

$$\sup_{u \in \mathcal{C}_0^2(\Omega),\, \|\Delta u\|_p \leqslant 1} \int_\Omega e^{\beta |u|^{\frac{p}{p-1}}}\,dx \leqslant c_0(\beta)|\Omega| \begin{cases} < \infty, & \text{if } \beta \leqslant \beta_N \\ = \infty, & \text{if } \beta > \beta_N \end{cases} \tag{4}$$

where β_N is explicitly known.

The embeddings (1) and (2) are closely related to the regularity problem for solutions of second-order PDE's. Indeed, consider as a reference model the following equation

$$\begin{cases} -\Delta u = f(x), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{5}$$

where $f \in L^p(\Omega)$, $p \geqslant 1$.

If $p > 1$ then Eq. (5) has a unique (*weak*) solution $u \in W_0^{1,p}(\Omega)$ (see e.g. [22]), and elliptic regularity theory (see [4]) then yields $u \in W^{2,p}(\Omega)$ from which the sharp maximal degree of summability for the solutions follows from (1), (2) and the aforementioned optimal improvements.

In the case $p = 1$, i.e. $f \in L^1(\Omega)$, there is still a unique (*very weak* or *distributional*) solution $u \in L^1(\Omega)$ of Eq. (5). Regularity theory now yields that $u \in W_0^{1,1}(\Omega)$, and also that $\Delta u \in L^1(\Omega)$; however, in general it is *not true* that $u \in W^{2,1}(\Omega)$. This can be inferred from examples which show that in general $u \notin L^{\frac{N}{N-2}}$ $(N \geqslant 3)$, resp. $u \notin L^\infty$ $(N = 2)$, in contrast to (1) and (3). Thus, we cannot use (1) and (3) to determine the maximal degree of summability for solutions of (5). Indeed, from the work of Maz'ya [30] (see also [38,14]) in dimension $N \geqslant 3$ and Brezis–Merle [13] in the limiting case $N = 2$ we have the following summability estimates for solutions u of (5):

$$\sup_{\|f\|_1 = 1} \int_\Omega |u|^q\,dx \leqslant C(q) \begin{cases} < \infty, & \text{if } 1 \leqslant q < \frac{N}{N-2}, \\ = \infty, & \text{if } q = \frac{N}{N-2}, \end{cases} \qquad N \geqslant 3 \tag{6}$$

$$\sup_{\|f\|_1 = 1} \int_\Omega e^{\beta|u|}\,dx \leqslant C(\beta) \begin{cases} < \infty, & \text{if } \beta < 4\pi, \\ = \infty, & \text{if } \beta = 4\pi, \end{cases} \qquad N = 2 \tag{7}$$

We mention that the optimality of (6) (within the class of L^p spaces) was proved in [34]; see also [12] for a self contained survey on related problems with L^1 data.

The above mentioned solvability setting of Eq. (5) suggests to introduce a new function space which we define by completion as follows

$$W^{2,p}_{\Delta}(\Omega) := cl\{u \in \mathcal{C}^{\infty}(\Omega) \cap \mathcal{C}^0(\overline{\Omega}),\ u|_{\partial\Omega} = 0\colon\ \|\Delta u\|_p < \infty\}$$

where we have included the Dirichlet boundary condition. $W^{2,p}_{\Delta}$ with the norm $\|\Delta \cdot\|_p$ is a Banach space. Note that by the above

$$W^{2,1}(\Omega) \cap W^{1,1}_0(\Omega) \subsetneq W^{2,1}_{\Delta}(\Omega)$$

which is in contrast to the case $p > 1$ where we always have

$$W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) = W^{2,p}_{\Delta}(\Omega)$$

with equivalence of the two norms. The non-equivalence of the norms in $W^{2,1}$ and $W^{2,1}_{\Delta}$ follows also from important results by D. Ornstein [33] which, applied to our situation, say that the L^1-norm of the mixed second-order derivatives cannot be bounded by $\|\Delta \cdot\|_1$ (cf. also V.P. Ill'n [24]); for more details see Section 2.2.

The main purpose of our paper is to study *optimal* embeddings for the space $W^{2,1}_{\Delta}(\Omega)$. We begin with dimension $N = 2$: denoting by $L_{exp}(\Omega)$ the *Zygmund space* (whose elements u satisfy $\int_\Omega e^{\lambda u}\,dx < \infty$, for some $\lambda = \lambda(u) > 0$, see Section 2.1), we prove the following

Theorem 1. *Let $N = 2$ and Ω be a bounded domain in $\mathbb{R}^2$. Then, the following embedding holds*

$$W^{2,1}_{\Delta}(\Omega) \hookrightarrow L_{exp}(\Omega) \tag{8}$$

namely

$$\|u\|_{L_{exp}} \leqslant \frac{1}{4\pi}\|\Delta u\|_1 \tag{9}$$

for any $u \in W^{2,1}_{\Delta}(\Omega)$. Moreover, the constant appearing in (9) *is sharp for any bounded domain $\Omega \subset \mathbb{R}^2$.*

As a byproduct, we obtain *optimal* summability bounds for solutions of Eq. (5) which should be compared with the bounds in (7):

Corollary 2. *Let Ω be a bounded domain in $\mathbb{R}^2$. Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function such that $e^{4\pi t}\Phi(t)$ is increasing as $t \to +\infty$. Then there exists a constant $C = C(\Phi) > 0$ such that*

$$\sup_{u \in W^{2,1}_{\Delta}(\Omega),\ \|\Delta u\|_1 = 1} \int_\Omega e^{4\pi|u|}\Phi(|u|)\,dx \leqslant C|\Omega| \tag{10}$$

if and only if $\Phi(t)$ is integrable near infinity.

These ideas extend also to higher dimensions $N \geqslant 3$. Denoting by $L^{p,\infty}(\Omega)$ the classical *weak-L^p space* and by ω_{N-1} the measure of the unit sphere in $\mathbb{R}^N$, we prove the following optimal result.

Theorem 3. *Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, be a bounded domain. Then, the following embedding holds*

$$W^{2,1}_{\Delta}(\Omega) \hookrightarrow L^{\frac{N}{N-2},\infty}(\Omega) \tag{11}$$

namely

$$\|u\|_{\frac{N}{N-2},\infty} \leqslant \frac{1}{\omega_{N-1}^{2/N}(N-2)N^{\frac{N-2}{N}}}\|\Delta u\|_1 \tag{12}$$

for any $u \in W^{2,1}_{\Delta}(\Omega)$. Moreover, the constant appearing in (12) *is sharp for any bounded domain $\Omega \subset \mathbb{R}^2$.*

As a consequence, one has the following improvement of (6):

Corollary 4. *Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, be a bounded domain. Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function such that $t^{\frac{N}{N-2}}\Phi(t)$ is increasing as $t \to +\infty$. Then, there exists a constant $C = C(\Phi) > 0$ such that*

$$\sup_{u \in W^{2,1}_{\Delta}(\Omega),\ \|\Delta u\|_1 = 1} \int_{\Omega} |u|^{\frac{N}{N-2}} \Phi\big(|u|\big)\, dx \leqslant C|\Omega|$$

if and only if $\Phi(t)/t$ is integrable near infinity.

The embeddings (1)–(3) are proved by first considering smooth functions with compact support in Ω, and then by completion in the space $W^{2,p}_0(\Omega)$ whose elements vanish together with their gradients on the boundary. Then, provided the boundary $\partial\Omega$ is sufficiently smooth, the embeddings (1)–(3) for the space $W^{2,p}(\Omega)$ (regardless of boundary conditions) are obtained by extending the $\mathcal{C}^\infty_0(\Omega)$ functions outside Ω to functions belonging to $W^{2,1}(\mathbb{R}^N)$ and controlling the extension by means of extension theorems, see [16]. However, this procedure does not preserve the embedding constants. Also, it is a rather difficult issue to establish whether different boundary conditions affect the best embedding constants.

We give here an *optimal* embedding result also for the space

$$W^{2,1}_{\Delta,0}(\Omega) := cl\big\{u \in \mathcal{C}^\infty_c(\Omega):\ \|\Delta u\|_1 < \infty\big\}$$

which consists of functions which vanish together with their gradient on the boundary.

Theorem 5. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain containing the origin. Then, for all radially symmetric $u \in W^{2,1}_{\Delta,0}(\Omega)$, the following inequalities hold*

$$\|u\|_{L_{exp}} \leqslant \frac{1}{8\pi}\|\Delta u\|_1, \quad N = 2 \tag{13}$$

$$\|u\|_{\frac{N}{N-2},\infty} \leqslant \frac{1}{2\omega_{N-1}^{2/N}(N-2)N^{\frac{N-2}{N}}}\|\Delta u\|_1, \quad N \geqslant 3 \tag{14}$$

where the constants in (13) *and* (14) *are the best possible, independently of the domain.*

From Theorem 5 we obtain, analogous to Corollary 2, the following maximal degree of summability for radial functions belonging to the space $W^{2,1}_{\Delta,0}(\Omega)$:

Corollary 6. *Let $N = 2$ and $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function, such that*

$$e^{8\pi t}\Phi(t) \text{ is monotone increasing as } t \to +\infty$$

Then there exists a constant $C = C(\Phi) > 0$ such that

$$\sup_{u \in W^{2,1}_{\Delta,0}(\Omega),\ \|\Delta u\|_1 = 1} \int_{\Omega} e^{8\pi|u|} \Phi\big(|u|\big)\, dx \leqslant C|\Omega|$$

if and only if $\Phi(t)$ is integrable near infinity.

Remark 7. One actually proves a stronger result, as Theorem 5 and Corollary 6 hold in any bounded domain $\Omega \subset \mathbb{R}^N$ provided $u \in W^{2,1}_{\Delta,0}(\Omega)$ lies in the positive cone.

Remark 8. It is remarkable, and somewhat surprising, that the best embedding constant for the space $W^{2,1}_{\Delta}(\Omega)$ turns out to be exactly twice the best constant for the space $W^{2,1}_{\Delta,0}(\Omega)$.

Remark 9. In particular, we have the following uniform bound which has to be compared with the Brezis–Merle result (7), at least for radial or positive functions: there exists $C_\alpha > 0$ such that

$$\sup_{u \in W^{2,1}_{\Delta,0}(\Omega),\ \|\Delta u\|_1 = 1} \int_\Omega e^{\alpha|u|}\,dx < C_\alpha |\Omega|$$

if and only if $\alpha < 8\pi$.

Remark 10. Corollary 6 can also be viewed as a natural extension of (4) to the limiting case $p = 1$.

We point out that the knowledge of the best embedding constant $\frac{1}{4\pi}$ in inequality (9) is crucial to obtain the improvement (10) of the Brezis–Merle estimate (7). Further discussions on these topics are carried out in Section 6.

The target space in (11) and (14) is the best possible among all rearrangement invariant spaces; this is a consequence of what is proved in [21], in the more general setting of rearrangement invariant quasi-norms. In particular, the embedding (11) is not new, but the methods developed so far do not provide the best constant in the corresponding inequality. The embedding (11) can also be obtained by joining some results of [8] and [32], but again the methods involved are not suitable to obtain sharp constants. Here we give a new and more direct proof of the above embeddings which on one hand covers the natural situation of functions having just zero boundary conditions (i.e. belonging to $W^{2,1}_\Delta(\Omega)$), and which on the other hand enables us to trace the best constants in the embedding inequalities for $W^{2,1}_\Delta(\Omega)$ as well as for $W^{2,1}_{\Delta,0}(\Omega)$.

The proof of the above results proceeds along the following lines: first we derive uniform bounds for positive and super-harmonic radially symmetric functions. Then the Talenti comparison principle will enable us to remove these restrictions and thus to cover the general case. Finally, we show by constructing explicit counterexamples that the estimates are sharp.

The paper is organized as follows: in Section 2, we first recall some classical function space settings, and then we collect in a unified fashion some well-known results on equivalence and non-equivalence of Sobolev norms, which justify as well as motivate the introduction of the space $W^{2,1}_\Delta(\Omega)$. In Section 3 we focus on the limiting case $N = 2$ for which we develop the main ideas which lead to the proof of Theorem 1. Similar ideas are then exploited in Section 4 to handle the case of compactly supported functions. In Section 5 we show how the previously developed techniques extend to the higher dimensional case $N \geqslant 3$. In the final section, Section 6, we discuss further consequences of the achieved results.

2. Preliminaries

2.1. Some classical function spaces

We recall some basic definitions, for which our main reference is [11]. We start with the rearrangement of functions in the sense of Hardy and Littlewood (see also [25]).

Let $\phi : \Omega \to \mathbb{R}$ be a measurable function; denoting by $|S|$ the Lebesgue measure of a measurable set $S \subset \mathbb{R}^N$, let

$$\mu_\phi(t) = \left|\left\{x \in \Omega:\ \left|\phi(x)\right| > t\right\}\right|, \quad t \geqslant 0$$

be the distribution function of ϕ. The *decreasing rearrangement* $\phi^*(s)$ of ϕ is defined as the distribution function of μ_ϕ, that is

$$\phi^*(s) = \left|\left\{t \in [0, +\infty):\ \mu_\phi(t) > s\right\}\right| = \sup\left\{t > 0:\ \mu_\phi(t) > s\right\}, \quad s \in \left[0, |\Omega|\right]$$

the *spherically symmetric rearrangement* $\phi^\sharp(x)$ of ϕ is defined as

$$\phi^\sharp(x) = \phi^*\left(\frac{\omega_{N-1}}{N}|x|^N\right), \quad x \in \Omega^\sharp$$

where $\Omega^\sharp \subset \mathbb{R}^N$ is the open ball with center in the origin which satisfies $|\Omega^\sharp| = |\Omega|$. Clearly, u^* is a non-negative, non-increasing and right-continuous function on $[0, \infty)$; moreover, the (non-linear) rearrangement operator has the following properties:

(i) Positively homogeneous: $(\lambda u)^* = |\lambda| u^*$, $\lambda \in \mathbb{R}$.
(ii) Sub-additive: $(u+v)^*(t+s) \leqslant u^*(t) + v^*(s)$, $t, s \geqslant 0$.
(iii) Monotone: $0 \leqslant u(x) \leqslant v(x)$ a.e. $x \in \Omega \Rightarrow u^*(t) \leqslant v^*(t)$, $t \in (0, |\Omega|)$.
(iv) u and u^* are equidistributed, and in particular (a version of the Cavalieri Principle):

$$\int_\Omega A(|u(x)|)\, dx = \int_0^{|\Omega|} A(u^*(s))\, ds$$

for any continuous function $A : [0, \infty] \to [0, \infty]$, non-decreasing and such that $A(0) = 0$.
(v) The following inequality holds (Hardy–Littlewood):

$$\int_\Omega u(x) v(x)\, dx \leqslant \int_0^{|\Omega|} u^*(s) v^*(s)\, ds$$

provided the integrals are defined.
(vi) The map $u \mapsto u^*$ preserves Lipschitz regularity, namely $^* : Lip(\Omega) \to Lip(0, |\Omega|)$.

The *Lorentz space* $L^{p,\infty}(\Omega)$ is the collection of measurable functions $u(x)$ on Ω which satisfy $\|u\|_{p,\infty} < \infty$, where

$$\|u\|_{p,\infty} := \sup_{t>0} t^{\frac{1}{p}} u^*(t), \quad 1 \leqslant p \leqslant \infty$$

These spaces are sometimes called weak-L^p spaces, as it can be shown that $u \in L^{p,\infty}(\Omega)$ if and only if $u^\sharp(x) \leqslant C|x|^{N/p}$; hence, u belongs to $L^q(\Omega)$ for all $q < p$, but not to $L^p(\Omega)$. Notice that $\|\cdot\|_{p,\infty}$ is just a quasi-norm, however it turns out to be equivalent to a genuine norm.

The *Zygmund space* $L_{exp}(\Omega)$ consists of all measurable functions $u(x)$ on Ω for which there is a constant $\lambda = \lambda(u)$ such that

$$\int_\Omega e^{\lambda |u|}\, dx < \infty \tag{15}$$

The integral appearing in (15) does not satisfy the properties of a norm. However, the quantity

$$\|u\|_{L_{exp}} = \sup_{t \in (0,|\Omega|)} \frac{u^*(t)}{1 + \log(\frac{|\Omega|}{t})} \tag{16}$$

defines a quasi-norm on $L_{exp}(\Omega)$ which turns out to be equivalent to a real norm. We mention that the Zygmund space $L_{exp}(\Omega)$ appears as a limiting case of the Marcinkiewicz interpolation theorem (see also [10]) and coincides with the borderline case of the Hansson–Brezis–Wainger spaces [15,23]. Notice that the Zygmund space $L_{exp}(\Omega)$ can also be viewed as a weighted Lorentz $L^{1,\infty}$ space, more precisely, following Lorentz [29] we can set

$$\Lambda^{1,\infty}(w) = \left\{ u \text{ measurable: } \sup_{t \in (0,|\Omega|)} u^*(t) w(t) < \infty \right\}$$

with respect to the positive, increasing weight function

$$w(t) = \left[1 + \log\left(\frac{|\Omega|}{t} \right) \right]^{-1}$$

Thus, clearly $L_{exp} = \Lambda^{1,\infty}(w)$. The Lorentz space $L^{p,\infty}(\Omega)$ and the Zygmund space $L_{exp}(\Omega)$ are related to each another and to the L^p spaces by the following continuous embeddings

$$L^\infty(\Omega) \hookrightarrow L_{exp}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow L^{p,\infty}(\Omega) \hookrightarrow L^1(\Omega), \quad 1 < p < \infty$$

The *Orlicz class* $L^{\phi}(\Omega)$, generated by a Young function ϕ (a non-increasing convex function from $[0,\infty)$ into $[0,\infty]$ such that $\phi(0)=0$), is defined as the class of all measurable functions u on Ω which satisfy

$$\int_{\Omega} \phi\left(\frac{|u|}{\lambda}\right) dx < \infty$$

for some $\lambda > 0$; a norm on $L^{\phi}(\Omega)$ is given by (Luxemburg norm)

$$\|u\|_{L^{\phi}(\Omega)} = \inf\left\{\lambda > 0\colon \int_{\Omega} \phi\left(\frac{|u|}{\lambda}\right) dx \leqslant 1\right\}$$

All function spaces recalled so far are examples of rearrangement invariant spaces, which we define next: a Banach space $(X(\Omega), \|\cdot\|_X)$ of real-valued measurable functions in $\Omega \subseteq \mathbb{R}^N$ is called a *rearrangement invariant space* (r.i. space) provided:

(i) $0 \leqslant v \leqslant u$ a.e. in Ω then $u \in X(\Omega) \Rightarrow v \in X(\Omega)$ and $\|v\|_X \leqslant \|u\|_X$,
(ii) $0 \leqslant u_n \nearrow u \in X(\Omega)$ a.e. then $\|u_n\|_X \nearrow \|u\|_X$,
(iii) $\|\mathbf{1}_G\|_X < \infty$ for every $G \subset \Omega$ such that $|G| < \infty$,
(iv) for every $G \subset \Omega$ with $|G| < \infty$ there exists $C > 0$ such that $\int_G u\,dx < C\|u\|_X$,
(v) if $u \in X(\Omega)$ and $u^* = v^*$, then $v \in X(\Omega)$ and $\|v\|_X = \|u\|_X$.

Let $X(\Omega)$ be an r.i. space over a domain with finite measure; then the following embeddings hold

$$L^{\infty}(\Omega) \hookrightarrow X(\Omega) \hookrightarrow L^1(\Omega), \quad |\Omega| < \infty$$

2.2. *The space* $W^{2,1}_{\Delta}(\Omega)$

As mentioned above, the introduction of the space $W^{2,1}_{\Delta}(\Omega)$ becomes necessary due to the failure of the equivalence $W^{2,p}_{\Delta}(\Omega) \equiv W^{2,p} \cap W^{1,p}_0(\Omega)$ which holds only for $1 < p < \infty$. Indeed, the equivalence between the full Sobolev norm $\|u\|_{2,p}$ and $\|Lu\|_p$ $(1 < p < \infty)$ for any second-order strictly elliptic operator of the form $Lu = \sum_{i,j=1}^{N} a_{i,j}(x) D_{i,j} u + b_i(x) D_i u + c(x) u$, where $a_{i,j} \in \mathcal{C}^0(\overline{\Omega})$, $b_i, c \in L^{\infty}(\Omega)$ and $c \leqslant 0$, can be proved by standard elliptic theory, see [22, Lemma 9.17].

For $p = 1$ the above equivalence property between the Sobolev norm and the norm $\|\Delta \cdot\|_1$ breaks down; actually, it was proved by D. Ornstein [33] that the quantity $\|D^{\alpha} u\|_1$, where α is a multi index of length $|\alpha| = m \geqslant 2$, cannot be uniformly bounded by any linear combination of the L^1-norm of the remaining derivatives of order m. As a consequence one has

$$W^{2,1}(\Omega) \subsetneq W^{2,1}_{\mathcal{M}}(\Omega) \quad \text{and} \quad W^{2,1}(\Omega) \subsetneq W^{2,1}_{\Delta}(\Omega)$$

where $\mathcal{M} := \{\text{off-diagonal derivatives of length } m\}$. Note that the first inclusion is strict also for $p > 1$ as a consequence of [24] and the Marcinkiewicz interpolation theorem. On the other hand, the strictness of the second inclusion is closely connected to the value $p = 1$. This feature can be seen also in the very general setting of an *r.i. space* $(X, \|\cdot\|_X)$; define on measurable functions on $[0, |\Omega|]$ a dilation operator as follows

$$E_t u := \begin{cases} u(st), & 0 \leqslant s \leqslant \min\{|\Omega|, |\Omega|/t\}, \\ 0, & \min\{|\Omega|, |\Omega|/t\} < s \leqslant |\Omega|, \end{cases} \qquad t > 0$$

and denote by $h_X(t)$ the operator norm of E_t. Then, a device to measure how close $\|\cdot\|_X$ is to the borderline L^1 case, is provided by the Boyd indices defined as follows: the *lower Boyd index* is given by

$$i_B(X) := \lim_{t \to 0} \frac{\log(1/t)}{\log[h_X(t)]}$$

Analogously, the *upper Boyd index* $I_B(X)$ is obtained by taking the above limit as $t \to +\infty$. Then, the inequality

$$\|D^2 u\|_X \leqslant C\|\Delta u\|_X$$

for a positive constant C holds if and only if $I_B(X)$ is finite and the lower Boyd index satisfies $i_B(X) > 1$. In the case of L^p spaces, one has $h_X(t) = (1/t)^{1/p}$ hence $I_B(L^p) = i_B(L^p) = 1/p$, and thus in our case we encounter the borderline situation $i_B(L^1) = 1$; see [11] and also [18] for a detailed discussion on this subject.

The theorem by Ornstein concerns the question of Sobolev embeddings in the case of missing derivatives in which some of the highest order derivatives are neglected, that is one looks for so-called reduced Sobolev inequalities; in this respect, the following somewhat surprising result was proved in [2]:

$$\|u\|_{p^*} \leqslant C \sum_{\alpha\in\mathcal{M}} \left\|D^\alpha u\right\|_p, \quad u \in \mathcal{C}_c^\infty\left(\mathbb{R}^N\right),\ p^* := \begin{cases} \frac{Np}{N-mp}, & N > mp \\ \infty, & p = 1 \text{ and } N = m \end{cases}$$

In particular, the target space of the standard Sobolev embedding is preserved even for the space $W^{2,1}_{\mathcal{M}}(\Omega)$ in which only completely mixed derivatives are considered; the case $N = 2$ and $p = 1$ was actually remarked in [39] whereas the analogous result for the Sobolev limiting case $N = mp$, $p > 1$ is established in [28]. In striking contrast to the *off-diagonal* case, (6)–(7) show that the target space for the embedding of $W^{2,1}_\Delta(\Omega)$ is strictly larger than the target space of $W^{2,1}(\Omega)$.

We conclude this section by proving the following Meyers–Serrin type result for $W^{2,1}_\Delta(\Omega)$.

Proposition 11. *Let us define*

$$E^{2,1}(\Omega) := \left\{u \in W^{1,1}_0(\Omega):\ \Delta u \in L^1(\Omega)\right\}$$

Then, $E^{2,1}(\Omega)$ endowed with the norm $\|\Delta\cdot\|_1$ is a Banach space and $E^{2,1}(\Omega) = W^{2,1}_\Delta(\Omega)$.

Proof. If $\{u_n\} \subset E^{2,1}$ is a Cauchy sequence, then Δu_n converges to some $g \in L^1(\Omega)$ and thus $\|\Delta u_n - \Delta u\|_1 \to 0$ where u is the unique solution to the problem $-\Delta u = g(x)$, $x \in \Omega$, subject to the Dirichlet boundary condition. Since distributional and classical partial derivatives coincide whenever the latter exist, the set

$$\left\{u \in C^\infty(\Omega) \cap C^0(\overline{\Omega}), u|_{\partial\Omega} = 0:\ \|\Delta u\|_1 < \infty\right\}$$

is contained in $E^{2,1}(\Omega)$; since $E^{2,1}(\Omega)$ is complete, one has

$$W^{2,1}_\Delta(\Omega) \subseteq E^{2,1}(\Omega)$$

On the other hand, let $u \in E^{2,1}(\Omega)$ and set $g := \Delta u \in L^1(\Omega)$. Let $g_n \in \mathcal{C}_c^\infty(\Omega)$ such that $g_n \to g$ in $L^1(\Omega)$ and let u_n be the unique solution of

$$\begin{cases} -\Delta u_n = g_n(x), & x \in \Omega \\ u_n = 0, & x \in \partial\Omega \end{cases}$$

By standard elliptic regularity, $u_n \in \mathcal{C}^\infty(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ and satisfies $u_{n|_{\partial\Omega}} = 0$; by definition, $\|\Delta(u_n - u)\|_1 \to 0$ and hence $u \in W^{2,1}_\Delta(\Omega)$. $\square$

3. The limiting Sobolev case $N = 2$: Proof of Theorem 1

3.1. The radially symmetric case

Here we assume $\Omega = B_R$, the open ball of radius R centered at the origin of $\mathbb{R}^2$ and let us denote by $W^{2,1}_{\Delta,rad}(B_R)$ the radial part of $W^{2,1}_\Delta(B_R)$, namely the subspace consisting of radially symmetric functions. We have

Proposition 12. *The following inequality holds*

$$\|v\|_{L_{exp}(B_R)} \leqslant \frac{1}{4\pi}\|v\|_{W^{2,1}_{\Delta,rad}(B_R)} \tag{17}$$

for any $v \in W^{2,1}_{\Delta,rad}(B_R)$ such that $\Delta v \leqslant 0$ and $v \geqslant 0$.

Proof. By a standard density argument, we may assume $v \in \mathcal{C}^2(B_R) \cap \mathcal{C}^0(\overline{B_R})$ and vanishing at the boundary ∂B_R. Let us define

$$w(t) := 4\pi v\big(Re^{-t/2}\big), \qquad r = Re^{-t/2} \in (0, R]$$

Then $w \in C^2(0,\infty)$, $w(0) = 0$ and we have

$$\begin{aligned}
&w'(t) = -2\pi v_r\big(Re^{-t/2}\big)Re^{-t/2}\\
&w'(\infty) = 0\\
&w''(t) = \pi v_{rr}\big(Re^{-t/2}\big)R^2e^{-t} + \pi v_r\big(Re^{-t/2}\big)Re^{-t/2} = \pi R^2 e^{-t}\left[v_{rr}\big(Re^{-t/2}\big) + \frac{v_r(Re^{-t/2})}{Re^{-t/2}}\right]\\
&\qquad = \pi R^2 e^{-t}\Delta v|_{|x|=Re^{-t/2}} \leqslant 0
\end{aligned}$$

so that

$$\begin{aligned}
\|\Delta v\|_1 &= \int_{B_R} -\Delta v\,dx = 2\pi\int_0^R -\left(v_{rr} + \frac{v_r}{r}\right)r\,dr = \pi\int_0^\infty -\big[R^2e^{-t}v_{rr}\big(Re^{-t/2}\big) + Re^{-t/2}v_r\big(Re^{-t/2}\big)\big]\,dt\\
&= \int_0^\infty -w''(t)\,dt
\end{aligned}$$

Next notice that

$$\begin{aligned}
w(t) &= \int_0^t w'(s)\,ds = \int_0^t\left(-\int_s^\infty w''(z)\,dz\right)ds = -t\int_t^\infty w''(z)\,dz - \int_0^t zw''(z)\,dz\\
&= t\int_0^\infty -w''(z)\,dz + \int_0^t (t-z)w''(z)\,dz \leqslant t\|\Delta v\|_1 \qquad (18)
\end{aligned}$$

since $w''(z) \leqslant 0$. Therefore

$$v(r) = \frac{1}{4\pi}w\left(2\log\left(\frac{R}{r}\right)\right) \leqslant \frac{\|\Delta v\|_1}{2\pi}\log\left(\frac{R}{r}\right) \qquad (19)$$

Since the decreasing rearrangement is monotone and positively homogeneous (cf. Section 2.1), we obtain from (19)

$$v^*(t) \leqslant \frac{\|\Delta v\|_1}{2\pi}\left[\log\left(\frac{R}{r}\right)\right]^*(t) = \frac{\|\Delta v\|_1}{4\pi}\log\left(\frac{\pi R^2}{t}\right)$$

which implies by (16) directly (17). □

3.2. The general case

So far, the embedding (8) holds true if restricted to spherically symmetric, non-negative and super-harmonic functions. However, let us recall the following comparison principle by G. Talenti [42]: let u, v be weak solutions respectively of problems

$$(P)\quad \begin{cases} -\Delta u = f, & \text{in } \Omega\\ u = 0, & \text{on } \partial\Omega \end{cases} \qquad \text{and} \qquad (P^*)\quad \begin{cases} -\Delta v = f^\sharp, & \text{in } \Omega^\sharp\\ v = 0, & \text{on } \partial\Omega^\sharp \end{cases}$$

where $f \in L^p(\Omega)$, $p > N/2$. Then, we have the pointwise estimate

$$v(x) \geqslant u^\sharp(x)$$

and hence

$$v^*(t) \geqslant u^*(t)$$

Now let $u \in \mathcal{C}^\infty(\Omega) \cap C^0(\overline{\Omega})$ such that $u = 0$ on $\partial\Omega$ and set $f := -\Delta u$. By invariance under rearrangement of the L^1-norm, we have

$$\|\Delta u\|_1 = \|f\|_1 = \left\|f^\sharp\right\|_1 = \|\Delta v\|_1$$

Hence, from the monotonicity of rearrangement invariant norms (cf. Section 2.1) and Proposition 12, and since v is super-harmonic by construction, we get

$$\|u\|_{L_{exp}(\Omega)} \leqslant \|v\|_{L_{exp}(\Omega)} \leqslant \frac{1}{4\pi}\|\Delta v\|_1 = \frac{1}{4\pi}\|\Delta u\|_1$$

thus the embedding (8) is proved.

3.3. Optimal constant

The constant appearing in (9) is sharp, in the sense that it cannot be replaced by any smaller. Suppose by contradiction that

$$\|v\|_{L_{exp}} \leqslant \frac{1-\varepsilon}{4\pi}\|\Delta v\|_1$$

for any $v \in W^{2,1}_{\Delta,rad}(\Omega)$, with $0 < \varepsilon < 1$. Fix $\alpha \in (4\pi, \frac{4\pi}{1-\varepsilon})$, then we have for all $u \in W^{2,1}_{\Delta,rad}(\Omega)$

$$\int_\Omega e^{\alpha|u|}\,dx = \int_0^{|\Omega|} e^{\alpha u^*}\,dt$$

and by (16)

$$u^*(t) \leqslant \frac{1-\varepsilon}{4\pi}\left(1 + \log\left(\frac{|\Omega|}{t}\right)\right)$$

so that

$$\int_0^{|\Omega|} e^{\alpha u^*}\,dt \leqslant \int_0^{|\Omega|} e^{\alpha\frac{1-\varepsilon}{4\pi}(1+\log(|\Omega|/t))}\,dt = \big(|\Omega|e\big)^{\alpha\frac{1-\varepsilon}{4\pi}} \int_0^{|\Omega|} t^{-\alpha\frac{1-\varepsilon}{4\pi}}\,dt < C$$

since we have chosen $\alpha\frac{1-\varepsilon}{4\pi} < 1$ and $\|\Delta u\|_1 \leqslant 1$. Next we reach a contradiction by showing that this cannot occur, since one has

$$\sup_{\substack{u\in W^{2,1}_{\Delta,rad}(\Omega)\\ \|\Delta u\|_1\leqslant 1}} \int_\Omega \big(e^{4\pi\beta|u|} - 1\big)\,dx = +\infty \tag{20}$$

for any $\beta \geqslant 1$. Let us first consider the case $\Omega = B_R$ for which we construct the following counterexample:

$$w_n(t) := \begin{cases} t, & 0 \leqslant t \leqslant 2n \\ -2e^{2n-t} + 2e^{n-\frac{t}{2}} + 2n, & 2n \leqslant t \leqslant 2n + 2\log 2 \\ 2n + \frac{1}{2}, & t \geqslant 2n + 2\log 2 \end{cases} \tag{21}$$

Then $w_n(0) = 0$ and

$$w_n'(t) = \begin{cases} 1, & 0 \leqslant t \leqslant 2n \\ 2e^{2n-t} - e^{n-\frac{t}{2}}, & 2n \leqslant t \leqslant 2n + 2\log 2 \\ 0, & t \geqslant 2n + 2\log 2 \end{cases}$$

together with

$$w_n''(t) = \begin{cases} 0, & 0 < t < 2n \\ -2e^{2n-t} + \frac{1}{2}e^{n-\frac{t}{2}}, & 2n < t < 2n + 2\log 2 \\ 0, & t > 2n + 2\log 2 \end{cases}$$

so that w_n belongs to $\mathcal{C}^1(\mathbb{R}^+; \mathbb{R}^+)$ and posseses at least piecewise $\mathcal{C}^2$-regularity. Set $r = |x|$ and define

$$u_n(r) := \frac{1}{4\pi} w_n\left(2\log\frac{R}{r}\right) \tag{22}$$

thus

$$u_n'(r) = -\frac{1}{2\pi r} w_n'\left(2\log\left(\frac{R}{r}\right)\right)$$
$$\Delta u_n(r) = \frac{1}{\pi r^2} w_n''\left(2\log\left(\frac{R}{r}\right)\right)$$

in particular, $u_n \in W^{2,1}_{\Delta,0}(B_R)$. Let us evaluate

$$\int_{B_R} |\Delta u_n|\,dx = \int_0^\infty |w_n''|\,dt = \int_{2n}^{2n+2\log 2} \left|-2e^{2n-t} + \frac{1}{2}e^{n-\frac{t}{2}}\right| dt = 1$$

On the other hand, one has

$$\int_{B_R} e^{4\pi\beta|u_n|}\,dx = \pi R^2 \int_0^\infty e^{\beta w_n - t}\,dt \geqslant \pi R^2 \int_0^{2n} e^{(\beta-1)t}\,dt = \frac{\pi}{\beta-1}\left(e^{2(\beta-1)n} - 1\right) \to +\infty$$

for any $\beta > 1$, whence for $\beta = 1$ one has

$$\int_{B_R} e^{4\pi|u_n|}\,dx = \pi R^2 \int_0^\infty e^{w_n - t}\,dt \geqslant \pi R^2 \int_0^{2n} dt = 2n\pi \to +\infty, \quad \text{as } n \to \infty$$

In order to establish (20) for any bounded domain $\Omega \subset \mathbb{R}^2$, one may proceed by approximation, as in Brezis and Merle [13, Remark 3], by considering the following problems:

$$\begin{cases} -\Delta u_n = f_n, & \text{in } \Omega \\ u_n = 0, & \text{on } \partial\Omega \end{cases}$$

where $\|f_n\|_1 \leqslant 1$ and such that $f_n \to \delta_{x_0}$. Then $u_n \to u$, where $u(x) \simeq \frac{1}{2\pi}\log\frac{1}{|x-x_0|}$, as $x \to x_0$, and this yields (20).

Remark 13. Notice that the sequence defined in (22) allows to conclude the proof directly in the case of a ball. Since the u_n are radially decreasing, we have

$$u_n^*(s) = u_n(\sqrt{s/\pi}) = \frac{1}{4\pi} w_n\left(\log\left(\frac{\pi R^2}{s}\right)\right)$$

and

$$\|u_n\|_{L_{exp}(B_R)} = \sup_{s\in(0,\pi R^2)} \frac{u_n^*(s)}{1 + \log(\frac{\pi R^2}{s})} = \frac{1}{4\pi} \sup_{t\in(0,+\infty)} \frac{w_n(t)}{1+t} \to \frac{1}{4\pi}, \quad \text{as } n \to \infty$$

that is, (u_n) approaches the best constant.

3.4. Proof of Corollary 2

Let $u \in W^{2,1}_{\Delta}(\Omega)$, $\|\Delta u\|_1 = 1$, and let $\Phi(s):\mathbb{R}^+ \to \mathbb{R}^+$ be continuous function which is integrable at infinity, and such that $e^{4\pi s}\Phi(s)$ is monotone increasing for any $s \geqslant s_0$. The Cavalieri Principle yields

$$\int_{\Omega} e^{4\pi|u|}\Phi(|u|)\,dx = \int_{\Omega^\sharp} e^{4\pi u^\sharp}\Phi(u^\sharp)\,dx = \int_0^{|\Omega|} e^{4\pi u^*(t)}\Phi(u^*(t))\,dt$$
$$\leqslant |\Omega| e^{4\pi s_0}\max_{s\in[0,s_0]}\Phi(s) + \int_0^{t_0} e^{4\pi u^*(t)}\Phi(u^*(t))\,dt$$

where $t_0 = \inf\{t:\ u^*(t) \leqslant s_0\}$. In $[0,t_0]$ we have $u^*(t) > s_0$, so that the map $t \mapsto e^{4\pi u^*(t)}\Phi(u^*(t))$ is monotone increasing; on the other hand by (9), $u^*(t) \leqslant \frac{1}{4\pi}[1+\log(|\Omega|/t)]$, thus (denoting in the sequel with C possibly different positive constants)

$$\int_{\Omega} e^{4\pi|u|}\Phi(|u|)\,dx \leqslant C|\Omega| + \int_0^{t_0} e^{1+\log(|\Omega|)-\log t}\Phi\left(\frac{1}{4\pi}\log\left(\frac{|\Omega|}{t}\right)\right)dt \leqslant C|\Omega| + e|\Omega| \int_0^{t_0} \frac{\Phi(\frac{1}{4\pi}\log(\frac{|\Omega|}{t}))}{t}\,dt$$
$$= C|\Omega| + e|\Omega| \int_{\frac{1}{4\pi}\log(|\Omega|/t_0)}^{+\infty} \Phi(s)\,ds < C_0|\Omega| \tag{23}$$

that is the first part of the claim.

It remains to show that (23) does not hold if $\Phi(t)$ fails to be integrable at infinity. To this aim, we proceed as in Section 3.3 by considering first the case of the ball, for which the sequence of functions $\{u_n\}$ as defined in (22) gives

$$\int_{B_R} e^{4\pi|u_n|}\Phi(|u_n|)\,dx = \pi R^2 \int_0^{+\infty} e^{w_n(t)-t}\Phi(w_n(t))\,dt \geqslant \pi R^2 \int_0^{2n} \Phi(t)\,dt \to +\infty, \quad \text{as } n \to \infty$$

The general case follows by the approximation argument as at the end of Section 3.3. The proof of the Corollary 2 is thus complete.

4. The case of compactly supported functions

Let us recall from the introduction the following definition

$$W^{2,1}_{\Delta,0}(\Omega) := \overline{\mathcal{C}^\infty_0(\Omega)}^{\|\Delta\cdot\|_1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain. We assume the boundary $\partial\Omega$ to be sufficiently smooth, and so the function space $W^{2,1}_{\Delta,0}(\Omega)$ consists of functions u with $\Delta u \in L^1(\Omega)$ and which vanish together with their gradient (in the sense of trace) on the boundary $\partial\Omega$. As we already pointed out, the class of smooth and compactly supported functions are the usual setting for higher-order Sobolev embedding inequalities [3] (including the limiting cases [1]). However, we should bear in mind that if we consider applications to boundary value problems, then the boundary conditions play an important role. Indeed, in order to avoid that Dirichlet problems like (5) are over-determined (which induces the lack of an existence argument for solutions), the function space setting does not allow for an extra boundary conditions other than zero. In view of this it is legitimate to ask how different boundary conditions affect the embeddings of the corresponding function spaces. As we are going to see, the answer is quite surprising: we show that different boundary conditions do affect the best constants in the embedding inequalities, though the target space in the related embedding is preserved.

Proposition 14. *Let Ω be a bounded domain in $\mathbb{R}^2$. Then*

$$\|u\|_{L_{exp}} \leqslant \frac{1}{8\pi}\|u\|_{W^{2,1}_{\Delta,0}(\Omega)} \tag{24}$$

for all radially symmetric functions $u \in W^{2,1}_{\Delta,0}(\Omega)$, with Ω containing the origin, or for all $u \in W^{2,1}_{\Delta,0}(\Omega)$, with $u \geqslant 0$ in Ω. Furthermore, the constant appearing in (24) *is sharp for any domain.*

Proof. Let us divide the proof into three steps:

Step 1. Let us prove first the result for radial functions $u \in W^{2,1}_{\Delta,0}(\Omega)$ and let $B_R = \operatorname{supp}(u)$. By the change of variable

$$w(t) = 4\pi v\big(Re^{-t/2}\big)$$

we have $w \in C^2(0,\infty)$, $w(0) = w'(0) = 0$ and

$$\|\Delta v\|_1 = \int_0^\infty \big|w''(t)\big|\,dt$$

Thanks to the homogeneous boundary conditions we also have

$$w'(t) = -\int_t^{+\infty} w''(z)\,dz = \int_0^t w''(z)\,dz$$

since $w'(0) = w'(\infty) = 0$. Therefore,

$$2\big|w'(t)\big| = \left|-\int_t^{+\infty} w''(z)\,dz + \int_0^t w''(z)\,dz\right| \leqslant \int_t^{+\infty}\big|w''(z)\big|\,dz + \int_0^t \big|w''(z)\big|\,dz = \|\Delta v\|_1$$

from which

$$\big|w'(t)\big| \leqslant \frac{\|\Delta v\|_1}{2}$$

and in turn

$$\big|w(t)\big| \leqslant \int_0^t \big|w'(s)\big|\,ds \leqslant \frac{t}{2}\|\Delta v\|_1$$

Eventually we get

$$\big|v(r)\big| = \frac{1}{4\pi}\left|w\left(2\log\left(\frac{R}{r}\right)\right)\right| \leqslant \frac{\|\Delta v\|_1}{4\pi}\log\left(\frac{R}{r}\right)$$

Since the decreasing rearrangement is order-preserving, we obtain

$$v^*(t) \leqslant \frac{\|\Delta v\|_1}{4\pi}\left[\log\left(\frac{R}{r}\right)\right]^*(t) = \frac{\|\Delta v\|_1}{8\pi}\log\left(\frac{\pi R^2}{t}\right)$$

which implies

$$\|v\|_{L_{exp}(\Omega)} \leqslant \frac{1}{8\pi}\|\Delta v\|_1$$

Step 2. Here we show that the constant in (24) cannot be improved; in the case $\Omega = B_R$ we achieve this by constructing an explicit extremal sequence. Let $\varepsilon_n, \alpha_n > 0$ be such that

$$\alpha_n \to 0, \qquad \varepsilon_n = o(\alpha_n) \quad \text{and} \quad \alpha_n^2 = o(\varepsilon_n), \quad \text{as } n \to \infty$$

for example, an admissible pair $(\varepsilon_n, \alpha_n)$ is given by $\alpha_n = 1/n^2$, $\varepsilon_n = 1/n^3$. Let us define

$$z_n(t) := \begin{cases} A_n e^t - 2A_n e^{t/2} + A_n, & 0 \leqslant t \leqslant \varepsilon_n \\ -(e^{\alpha_n} + F_n e^{\alpha_n/2})\frac{e^{-t}}{2} + F_n e^{-t/2} + \frac{\alpha_n+1}{2} - \frac{F_n}{2} e^{-\alpha_n/2}, & \varepsilon_n \leqslant t \leqslant \alpha_n \\ \frac{t}{2}, & \alpha_n \leqslant t \leqslant 2n \\ -\frac{e^{2n+1}}{2(e-1)} e^{-t} + \frac{e^n}{e-1} e^{-t/2} + n - \frac{1}{2(e-1)} + \frac{1}{2}, & 2n \leqslant t \leqslant 2n+2 \\ n - \frac{1}{2e} + \frac{1}{2}, & 2n+2 \leqslant t \end{cases} \tag{25}$$

where

$$F_n = \frac{e^{\frac{3\varepsilon_n}{2}}(\alpha_n+1) - 2e^{\frac{\varepsilon_n}{2}+\alpha_n} + e^{\alpha_n}}{(e^{\frac{\varepsilon_n}{2}} - e^{\frac{\alpha_n}{2}})[e^{\varepsilon_n} - 2e^{\frac{\varepsilon_n+\alpha_n}{2}} + e^{\frac{\alpha_n}{2}}]} e^{\frac{\alpha_n}{2}}$$

$$A_n = -\frac{e^{\frac{\alpha_n-\varepsilon_n}{2}}[e^{\frac{\varepsilon_n}{2}}(\alpha_n+1) - e^{\frac{\alpha_n}{2}}]}{2(e^{\frac{\varepsilon_n}{2}} - 1)[e^{\varepsilon_n} - 2e^{\frac{\varepsilon_n+\alpha_n}{2}} + e^{\frac{\alpha_n}{2}}]}$$

and thus

$$z_n'(t) = \begin{cases} A_n e^t - A_n e^{t/2}, & 0 \leqslant t \leqslant \varepsilon_n \\ \frac{e^{\alpha_n} + F_n e^{\alpha_n/2}}{2} e^{-t} - \frac{F_n}{2} e^{-t/2}, & \varepsilon_n \leqslant t \leqslant \alpha_n \\ \frac{1}{2}, & \alpha_n \leqslant t \leqslant 2n \\ \frac{e^{2n+1}}{2(e-1)} e^{-t} - \frac{e^n}{2(e-1)} e^{-t/2}, & 2n \leqslant t \leqslant 2n+2 \\ 0, & 2n+2 \leqslant t \end{cases}$$

and

$$z_n''(t) = \begin{cases} A_n e^t - \frac{A_n}{2} e^{t/2}, & 0 < t < \varepsilon_n \\ -\frac{e^{\alpha_n} + F_n e^{\alpha_n/2}}{2} e^{-t} + \frac{F_n}{4} e^{-t/2}, & \varepsilon_n < t < \alpha_n \\ 0, & \alpha_n < t < 2n \\ -\frac{e^{2n+1}}{2(e-1)} e^{-t} + \frac{e^n}{4(e-1)} e^{-t/2}, & 2n < t < 2n+2 \\ 0, & 2n+2 < t \end{cases}$$

Note that

$$A_n \sim \varepsilon_n^{-1} \quad \text{and} \quad F_n \sim 2\frac{\varepsilon_n}{\alpha_n^2}, \quad \text{as } n \to \infty$$

so that eventually $A_n, F_n > 0$. It is easy to verify that $z_n(t)$ is piecewise $\mathcal{C}^2(0, +\infty)$ and $z_n(0) = z_n'(0) = 0$. Note that $z_n(t)$ is also a non-negative and non-decreasing function on $[0, +\infty)$. Set

$$v_n(r) := \frac{1}{4\pi} z_n\left(2\log\left(\frac{R}{r}\right)\right) \tag{26}$$

and thus

$$v_n'(r) = -\frac{1}{2\pi r} z_n'\left(2\log\left(\frac{R}{r}\right)\right)$$

$$\Delta v_n(r) = \frac{1}{\pi r^2} z_n''\left(2\log\left(\frac{R}{r}\right)\right)$$

in particular, $v_n \in W^{2,1}_{\Delta,0}(B_R)$. Let us evaluate

$$\begin{aligned} \|\Delta v_n\|_1 &= 2\pi \int_0^R |\Delta v_n| r\, dr = \int_0^{+\infty} |z_n''(t)|\, dt = \int_0^{\varepsilon_n} z_n''(t)\, dt + \int_{\varepsilon_n}^{\alpha_n} -z_n''(t)\, dt + \int_{2n}^{2n+2} -z_n''(t)\, dt \\ &= 2z_n'(\varepsilon_n) = 2A_n e^{\varepsilon_n/2}(e^{\varepsilon_n/2} - 1) \to 1 \end{aligned}$$

since $A_n \sim 1/\varepsilon_n$, as $n \to \infty$. Notice that v_n is also piecewise $\mathcal{C}^2(B_R)$ so that $v_n = v_n^\sharp$ since v_n is non-increasing, thus

$$v_n^*(s) = v_n\left(\sqrt{\frac{s}{\pi}}\right) = \frac{1}{4\pi} z_n\left(\log\left(\frac{\pi R^2}{s}\right)\right)$$

and finally

$$\frac{\|v_n\|_{L_{exp}(\Omega)}}{\|\Delta v_n\|_1} = \frac{1}{\|\Delta v_n\|_1} \sup_{s\in(0,\pi R^2)} \frac{v_n^*(s)}{1+\log(\frac{\pi R^2}{s})} = \frac{1}{\|\Delta v_n\|_1}\frac{1}{4\pi} \sup_{t\in(0,+\infty)} \frac{z_n(t)}{1+t} \to \frac{1}{8\pi}$$

as $n \to +\infty$; this proves the first part of Theorem 5, namely inequality (13).

Step 3. In order to deal with the case $u \in W^{2,1}_{\Delta,0}(\Omega)$, $u \geqslant 0$, for a bounded domain $\Omega \subset \mathbb{R}^2$, one may proceed as in Section 3.2 by replacing the positive Schwarz rearrangement with the rearrangement with sign, see [27, Theorem 3.1.1]. Indeed, the restriction to the positive cone enables us to use the Talenti comparison principle within the framework of compactly supported functions.

Finally, we point out that the constant is optimal for any domain Ω, which we may assume, up to translations, containing the origin. We argue by contradiction and assume that there exists a domain $\widetilde{\Omega}$ for which the constant may be improved and hence, arguing as in Section 3.3, there exists a constant $C > 0$, independent of u, such that for all $u \in W^{2,1}_{\Delta,0}(\Omega)$, $\|\Delta u\|_1 = 1$, the following holds

$$\int_{\widetilde{\Omega}} e^{\alpha|u|\,dx} \leqslant C, \quad \text{for some } \alpha \geqslant 8\pi$$

Thus a contradiction, since $\widetilde{\Omega} \supset B_R$ for some ball $B_R \subset \mathbb{R}^N$, and exploiting the extension by zero of the sequence (26) we get

$$\int_{\widetilde{\Omega}} \left(e^{8\pi|v_n|} - 1\right) dx \to +\infty, \quad \text{as } n \to +\infty \qquad \square$$

Remark 15. The proof of Corollary 6 may be achieved following line by line the proof of Corollary 2 given in Section 3.4, in which the constant 4π is replaced by 8π and the sequence $\{u_n\}$ by the sequence $\{v_n\}$ defined in (26); we only point out that optimality in the case of a general domain now follows by an extension argument as at the end of the proof of Proposition 14.

5. The Sobolev case $N \geqslant 3$

The approach developed so far in dimension two carries over with some technical changes into the case of higher dimensions in which optimal embeddings for the space $W^{2,1}_{\Delta}(\Omega)$ take place in the Lorentz function space setting. In particular, our method yields the best constant in the corresponding embedding, which actually is not new; it follows as a particular case of [21] where the authors study optimal pairs of rearrangement invariant quasi-norms, or more simply it can be also derived from some results of [8] (see also [6]) as we next explain. Indeed, the authors in [8] prove among other things that if u is a solution of

$$\begin{cases} -\Delta u = f, & \Omega \\ u = 0, & \partial\Omega \end{cases}$$

where $f \in L^1(\Omega)$, then

$$\|\nabla u\|_{\frac{N}{N-1},\infty} \leqslant C_1\|f\|_1 \tag{27}$$

and thus u belongs to the Lorentz–Sobolev space $W_0^1 L^{\frac{N}{N-1},\infty}(\Omega)$ which can be defined as the completion of compactly supported functions with respect to the Lorentz quasi-norm $\|\nabla\cdot\|_{\frac{N}{N-1},\infty}$ (cf. Section 2.1). As a consequence[1] of [32] the following embedding holds

$$W_0^1 L^{\frac{N}{N-1},\infty}(\Omega) \hookrightarrow L^{\frac{N}{N-2},\infty}(\Omega)$$

namely

$$\|u\|_{\frac{N}{N-2},\infty} \leqslant C_2 \|\nabla u\|_{\frac{N}{N-1},\infty} \tag{28}$$

Combining inequalities (27) and (28) one has

$$\|u\|_{\frac{N}{N-2},\infty} \leqslant C\|f\|_1 \tag{29}$$

and a fortiori the embedding (11); however, the sharp constants in (27) and (28) are not known. Here we give a new, short proof of (11) and we exhibit the best constant in (29).

5.1. Proof of Theorem 3

Let us first prove the embedding (11) in the radial case, namely

$$W^{2,1}_{\Delta,rad}(B_R) \hookrightarrow L^{\frac{N}{N-2},\infty}(B_R) \tag{30}$$

Let $v \in \mathcal{C}^2(B_R) \cap \mathcal{C}^0(\overline{B_R})$ be vanishing at the boundary and assume $\Delta v \leqslant 0$ and $v \geqslant 0$. Set

$$w(t) := \omega_{N-1}(N-2)R^{N-2} v\left(\frac{R}{t^{\frac{1}{N-2}}}\right), \quad t \geqslant 1 \tag{31}$$

Then we have

$$w''(t) = \omega_{N-1} \frac{R^N}{(N-2)t^{2\frac{N-1}{N-2}}} \Delta v_{\big| |x|=Rt^{\frac{-1}{N-2}}} \leqslant 0$$

so that

$$\|\Delta v\|_1 = \int_1^\infty -w''(t)\,dt$$

Moreover, the analogous of representation formula (18) now reads

$$w(t) = (t-1)\int_1^\infty -w''(z)\,dz + \int_1^t (t-z)w''(z)\,dz \leqslant (t-1)\|\Delta v\|_1 \tag{32}$$

since $w''(t) \leqslant 0$. From (31) and (32) we get

$$v(r) \leqslant \frac{\|\Delta v\|_1}{\omega_{N-1}(N-2)R^{N-2}}\left(\frac{R^{N-2}}{r^{N-2}} - 1\right)$$

and hence

$$\begin{aligned} v^*(s) &\leqslant \frac{\|\Delta v\|_1}{\omega_{N-1}(N-2)R^{N-2}}\left(\frac{R^{N-2}}{r^{N-2}} - 1\right)^*(s) \\ &= \frac{\|\Delta v\|_1}{\omega_{N-1}(N-2)R^{N-2}}\left[\left(\frac{\omega_{N-1}}{N}\right)^{\frac{N-2}{N}} \frac{R^{N-2}}{s^{\frac{N-2}{N}}} - 1\right], \quad s \in \left(0, \frac{\omega_{N-1}}{N}R^N\right) \end{aligned}$$

[1] See also Appendix in [7] for a survey on embeddings for Lorentz–Sobolev spaces.

Therefore,

$$\|v\|_{\frac{N}{N-2},\infty} = \sup_{s\in(0,|\Omega|)} v^*(s)s^{\frac{N-2}{N}} \leqslant \frac{1}{\omega_{N-1}^{2/N}(N-2)N^{\frac{N-2}{N}}}\|\Delta v\|_1$$

which yields (30). To pass from non-negative, super-harmonic and radial functions to the general case one may proceed exactly as in 3.2 by using the Talenti comparison principle. It remains to show that the constant appearing in (33) is actually sharp. We achieve this by means of the following sequence

$$u_n(r) := \frac{1}{\omega_{N-1}(N-2)R^{N-2}} w_n\left(\left(\frac{R}{r}\right)^{N-2} - 1\right) \tag{33}$$

where w_n is defined by (21). Then we have

$$u_n'(r) = -\frac{1}{\omega_{N-1}r^{N-1}} w_n'\left(\left(\frac{R}{r}\right)^{N-2} - 1\right)$$

$$\Delta u_n(r) = \frac{(N-2)R^{N-2}}{\omega_{N-1}r^{2N-2}} w_n''\left(\left(\frac{R}{r}\right)^{N-2} - 1\right)$$

so that $u_n \in W^{2,1}_\Delta(B_R)$ and

$$\|\Delta u_n\|_1 = \int_0^\infty \left|w_n''(t)\right| dt = 1$$

Notice that the sequence u_n is radially decreasing, thus

$$u_n^*(s) = u_n\left((sN/\omega_{N-1})^{1/N}\right) = \frac{1}{\omega_{N-1}(N-2)R^{N-2}} w_n\left(\left(\frac{\omega_{N-1}R^N}{Ns}\right)^{\frac{N-2}{N}} - 1\right)$$

and we eventually evaluate

$$\begin{aligned}\|u_n\|_{\frac{N}{N-2},\infty} &= \sup_{s\in(0,\frac{\omega_{N-1}}{N}R^N)} u_n^*(s)s^{\frac{N-2}{N}} = \frac{1}{\omega_{N-1}(N-2)R^{N-2}} \sup_{s\in(0,\frac{\omega_{N-1}}{N}R^N)} w_n\left(\left(\frac{\omega_{N-1}R^N}{Ns}\right)^{\frac{N-2}{N}} - 1\right)s^{\frac{N-2}{N}} \\ &= \frac{1}{(N-2)N^{\frac{N-2}{N}}\omega_{N-1}^{2/N}} \sup_{t\in(0,\infty)} \frac{w_n(t)}{1+t} \to \frac{1}{(N-2)N^{\frac{N-2}{N}}\omega_{N-1}^{2/N}}\end{aligned}$$

as $n \to \infty$ and the claim follows in the case $\Omega = B_R$. The case of a general bounded domain Ω can be obtained as in the proof of Theorem 1.

5.2. *Proof of Corollary 4*

The proof easily follows by mimicking the proof of Corollary 2; we just remark that optimality can be achieved by using the sequence of functions $\{u_n\}$ defined in (33). Indeed, let us set for simplicity

$$\kappa_N := \frac{\omega_{N-1}}{[\omega_{N-1}(N-2)R^{N-2}]^{\frac{N}{N-2}}}$$

We have

$$\begin{aligned}\int_{B_R} |u_n|^{\frac{N}{N-2}} \Phi\left(|u_n|\right) dx &= \frac{\kappa_N R^N}{N-2} \int_0^{+\infty} w_n^{\frac{N}{N-2}}(t)\Phi\left(w_n(t)\right) \frac{dt}{(t+1)^{\frac{2(N-1)}{N-2}}} \geqslant \frac{\kappa_N R^N}{N-2} \int_1^{2n} \left(\frac{t}{t+1}\right)^{\frac{2(N-1)}{N-2}} \frac{\Phi(t)}{t} dt \\ &\geqslant C \int_1^{2n} \frac{\Phi(t)}{t} dt \to \infty, \quad \text{as } n \to \infty\end{aligned}$$

since by hypothesis $\Phi(t)/t$ is not integrable at infinity.

5.3. The case of compactly supported functions

In this section we complete the proof of Theorem 5 by proving the sharp inequality (14) which we recall in the following

Proposition 16. *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geqslant 3$. Then*

$$\|u\|_{\frac{N}{N-2},\infty} \leqslant \frac{1}{2\omega_{N-1}^{2/N}(N-2)N^{\frac{N-2}{N}}}\|\Delta u\|_1, \quad N \geqslant 3 \tag{34}$$

for all $u \in W^{2,1}_{\Delta,0}(\Omega)$, $u \geqslant 0$, or radially symmetric functions on Ω containing the origin. Furthermore, the constant appearing in (34) *is the best possible for any domain.*

Proof. Let us consider just the radial version of (34) as to pass from the radially symmetric case to the more general case one may follow line by line, with obvious changes, the proof carried out in dimension two, namely Step 3 in the proof of Proposition 14. Let $v \in \mathcal{C}_0^2(B_R)$ and $w(t)$ as in (31), so that $w(1) = w'(1) = 0$ and

$$\|\Delta v\|_1 = \int_1^\infty |w''(t)|\,dt$$

Moreover, since also $w'(1) = w'(\infty) = 0$ one has

$$2|w'(t)| \leqslant \int_t^{+\infty} |w''(z)|\,dz + \int_1^t |w''(z)|\,dz = \|\Delta v\|_1$$

hence

$$|w(t)| \leqslant \int_1^t |w'(s)|\,ds \leqslant \frac{t-1}{2}\|\Delta v\|_1$$

Therefore

$$|v(r)| \leqslant \frac{\|\Delta v\|_1}{2\omega_{N-1}(N-2)R^{N-2}}\left(\frac{R^{N-2}}{r^{N-2}} - 1\right)$$

and thus

$$v^*(s) \leqslant \frac{\|\Delta v\|_1}{2\omega_{N-1}(N-2)R^{N-2}}\left[\left(\frac{\omega_{N-1}}{N}\right)^{\frac{N-2}{N}}\frac{R^{N-2}}{s^{\frac{N-2}{N}}} - 1\right], \quad s \in \left(0, \frac{\omega_{N-1}}{N}R^N\right)$$

Finally we get

$$\|v\|_{\frac{N}{N-2},\infty} = \sup_{s\in(0,|\Omega|)} v^*(s)s^{\frac{N-2}{N}} \leqslant \frac{1}{2\omega_{N-1}^{2/N}(N-2)N^{\frac{N-2}{N}}}\|\Delta v\|_1$$

The fact that the constant in (34) cannot be replaced by any other smaller one, it is readily seen by considering the extremal sequence

$$v_n(r) := \frac{1}{\omega_{N-1}(N-2)R^{N-2}} z_n\left(\left(\frac{R}{r}\right)^{N-2} - 1\right)$$

where z_n is defined in (25); this concludes the proof of Theorem 5. $\square$

6. Final remarks

Remark 17. As mentioned in the introduction, differently from the higher dimensional case, in dimension $N = 2$ the knowledge of the best constant in the embedding $W^{2,1}_\Delta(\Omega) \hookrightarrow L_{exp}(\Omega)$ enables us to improve the Brezis–Merle a priori estimate (7), as established in Corollary 2. In particular, the $L_{exp}(\Omega)$ membership of a function in $W^{2,1}_\Delta(\Omega)$ yields a stronger summability property than (7), as highlighted in the next straightforward calculation:

$$\int_\Omega e^{\frac{\alpha|u|}{\|\Delta u\|_1}}\,dx = \int_0^{|\Omega|} e^{\frac{\alpha u^*}{\|\Delta u\|_1}}\,dt \leqslant \int_0^{|\Omega|} e^{\frac{\alpha\|u\|_{L_{exp}}(1+\log\frac{|\Omega|}{t})}{\|\Delta u\|_1}}\,dt = \int_0^{|\Omega|} \big(|\Omega|e\big)^{\frac{\alpha\|u\|_{L_{exp}}}{\|\Delta u\|_1}} \frac{1}{t^{\frac{\alpha\|u\|_{L_{exp}}}{\|\Delta u\|_1}}} < \infty, \quad \text{if } \alpha < 4\pi \tag{35}$$

by inequality (9). Observe that from D.R. Adams' result (4), in the case $p = \frac{N}{2} > 1$ the exponential integrability condition (4) gives a better result than the corresponding L_{exp} maximal integrable growth, by arguing as in (35); in fact, in this case the extremal value β_0 (which plays the role of 4π in (35)) is achieved!

Remark 18. As remarked in [13], in our notation functions belonging to the space $W^{2,1}_\Delta(\Omega)$ enjoy the stronger integrability condition (cf. (15))

$$\int_\Omega e^{\lambda|u|}\,dx < \infty, \quad \forall\lambda > 0$$

This class of functions equipped with the norm $\|\cdot\|_{L_{exp}}$ turns out to be a strictly smaller space than $L_{exp}(\Omega)$, and is called Lorentz–Zygmund space $L_0^{\infty\infty}(\log L)^{-1}$ or simply the Zygmund space Z_0^1; see [10]. As a consequence, all the results derived so far still hold with the target space Z_0^1 in place of L_{exp}.

Remark 19. It is worth to point out that our extremal sequences have a pointwise limit which does not lie within the space $W^{2,1}_\Delta(\Omega)$; this reflects somehow the lack of reflexivity for which the space is not complete with respect to the weak topology. For example, consider in dimension two the sequence $\{u_n\}$ defined in (22) and which satisfies

$$u_n \to u_0(r) = -\log r, \quad \text{as } n \to \infty$$

in $W^{1,1}_0(B)$ but $u_0 \notin W^{2,1}_\Delta(B)$, since $-\Delta u_0 = \delta_0$. Indeed notice that u_0 is the fundamental solution for the Laplace operator in the unit ball. Nevertheless, for any $\phi \in \mathcal{C}_0^\infty(\Omega)$ one has

$$\int_\Omega \nabla u_0 \nabla\phi\,dx = \phi(0)$$

so that Δu_0 actually defines a measure with bounded variation on Ω, namely $\Delta u_0 \in M_b(\Omega)$. Hence,

$$u \in BL(\Omega) := \big\{u \in W^{1,1}_0(\Omega):\ \Delta u \in M_b(\Omega)\big\}$$

Denoting by $|\cdot|_T$ the total variation, the embedding $W^{2,1}_\Delta(\Omega) \hookrightarrow L_{exp}(\Omega)$ can be extended to the space $BL(\Omega)$ where the following inequality holds

$$\|u\|_{L_{exp}} \leqslant \frac{1}{4\pi}|\Delta u|_T \tag{36}$$

moreover, the best constant in (36) is attained by the function u_0, as one can easily check.

Remark 20. An explicit example of function satisfying Corollary 2 is given by

$$\Phi(s) = \frac{1}{(1+s)^\alpha}, \quad \alpha > 1$$

so that

$$\sup_{\substack{u\in W^{2,1}_\Delta(\Omega)\\ \|\Delta u\|_1=1}} \int_\Omega \frac{e^{4\pi|u|}}{(|u|+1)^\alpha}\,dx \leqslant C_\alpha|\Omega|, \quad \forall\alpha > 1 \tag{37}$$

If $\alpha = 1$ in (37), we can choose

$$\Phi(s) = \frac{1}{(1+s)[1+\log(1+s)]^{\alpha_1}}, \quad \alpha_1 > 1$$

and more in general one may keep on adding additional corrections by the following iterative procedure: define for any $t \geqslant 0$

$$\begin{cases} \log_0(t) = t+1 \\ \log_n(t) = \log\bigl(1+\log_{n-1}(t)\bigr), \quad n \geqslant 1 \end{cases}$$

and then define

$$\begin{cases} g_1(t) := \log_0 t = 1+t \\ g_n(t) := g_{n-1}(t) \cdot \bigl(1+\log_{n-1}(t)\bigr), \quad n \geqslant 2 \end{cases}$$

It is easy to verify that, for any $n \geqslant 2$ and $\alpha > 1$, the function

$$\Phi_{n,\alpha}(t) := g_{n-1}(t)(1+\log_n t)^\alpha$$

satisfies the hypotheses of Corollary 2, hence

$$\sup_{\substack{u \in W^{2,1}_\Delta(\Omega) \\ \|\Delta u\|_1 = 1}} \int_\Omega \frac{e^{4\pi|u|}}{g_{n-1}(|u|)(1+\log_n |u|)^\alpha}\, dx \leqslant C_\alpha |\Omega|, \quad \forall \alpha > 1$$

We merely mention that explicit examples in the spirit of the above, can be given also in the context of Corollaries 4 and 6.

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