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Probabilités/*Probability Theory*

# On self attracting/repelling diffusions

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**Abstract** We present an almost sure ergodic theorem for a class of self-interacting diffusions on a compact Riemannian manifold. ***To cite this article: M. Benaim, O. Raimond, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 541–544.***



### Diffusions auto attractives/repulsives

**Résumé** Nous présentons un résultat de type théorème ergodique presque sûr pour une classe de diffusions *inter-agissantes* sur une variété Riemanienne compacte. ***Pour citer cet article : M. Benaim, O. Raimond, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 541–544.***



A *self interacting diffusion* is a continuous time stochastic process living on a compact connected Riemannian manifold $M$ which can be typically described as a solution to a stochastic differential equation (SDE) of the form

$$\mathrm{d}X_t = \sum_i F_i(X_t) \circ \mathrm{d}B_t^i - \frac{\alpha}{2t}\left(\int_0^t \nabla V_{X_s}(X_t)\,\mathrm{d}s\right)\mathrm{d}t, \tag{1}$$

where $(B^i)_i$ is a family of independent Brownian motions, $(F_i)_i$ is a family of smooth vector fields on $M$ such that $\sum_i F_i(F_i f) = \Delta f$ (for $f \in C^\infty(M)$) where $\Delta$ denotes the Laplacian on $M$, and $(u,x) \in M \times M \mapsto V_u(x) \in \mathbb{R}$ is a smooth (at least $C^3$) "potential". The parameter $\alpha$ is real and measures the strength of the interaction.

Such a process is characterized by the fact that the drift term in Eq. (1) depends both on the position of the process and its empirical occupation measure:

$$\mu_t = \frac{1}{t}\int_0^t \delta_{X_s}\,\mathrm{d}s. \tag{2}$$

In [2] it is shown that the asymptotic behavior of $\{\mu_t\}$ can be precisely described in terms a certain deterministic semi-flow $\Psi = \{\Psi_t\}_{t \geqslant 0}$ defined on the space of Borel probability measures on $M$. For instance, there are situations (depending on the shape of $V$) in which $\{\mu_t\}$ converges almost surely to

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an equilibrium point $\mu^*$ of $\Psi$ and other situations where the limit set of $\{\mu_t\}$ coincides almost surely with a periodic orbit for $\Psi$ (see the examples in Section 4 of [2]).

The purpose of this note is to announce new results showing that for a certain class of potentials, $\{\mu_t\}$ converges almost surely (up to a change of variable) to the critical set of an "energy" function. This encompasses most of the examples considered in [2] and enlightens the results of [2]. It also allows to give a sensible definition of *self-attracting* or *repelling* diffusions. In particular, we can show that under a natural assumption (Hypothesis 1.2 below) there is a critival value $\alpha_c < 0$ such that $\mathbb{P}(\mu_t \to \lambda) > 0$ for $\alpha > \alpha_c$ and $\mathbb{P}(\mu_t \to \lambda) = 0$ for $\alpha < \alpha_c$; where $\lambda$ stands for the Riemannian probability on $M$.

While some of the proofs are sketched here, the details will be given in [3].

## 1. Hypotheses

The main assumption is the following:

HYPOTHESIS 1.1 (Standing assumption). – There exists a compact space $C$, a Borel probability measure $\nu$ over $C$, a continuous function $G : C \times M \to \mathbb{R}$, and a real number $\beta$ such that

$$V(x, y) = \int_C G(u, x) G(u, y) \nu(\mathrm{d}u) + \beta.$$

A process (1) satisfying 1.1 will be called *self-attracting* for $\alpha \leqslant 0$ and *self-repelling* otherwise.

We sometime use the following additional hypothesis:

HYPOTHESIS 1.2 (Occasional assumption). – The mapping

$$V\lambda : x \mapsto V\lambda(x) = \int_M V(x, y) \lambda(\mathrm{d}y)$$

is constant.

This later condition has the interpretation that if the empirical occupation measure of $X_t$ is (close to) $\lambda$ then the drift term in (1) is (close to) zero. In other words, if the process has visited $M$ "uniformly" between times 0 and $t$, then it has no preferred directions and behaves like a Brownian motion.

Several examples of potentials satisfying Hypotheses 1.1 and 1.2 are given in [3].

*Remark.* – The class of potential verifying Hypothesis 1.1 belong to a more general class introduced by Ben Arous and Brunaud in [4].

## 2. Statement of main results

Let $\mathcal{M}(M)$ denote the space of bounded Borel measures on $M$. For $\mu \in \mathcal{M}(M)$ we let $G\mu \in \mathrm{C}^0(C)$ denote the function defined by

$$G\mu(u) = \int_M G(u, x) \mu(\mathrm{d}x). \tag{3}$$

If $g \in \mathrm{L}^2(\lambda)$ we write $Gg$ for $G(g\lambda)$, where $g\lambda$ stands for the measure whose Radon–Nikodym derivative with respect to $\lambda$ is $g$. Associated to $G$ is the operator $G^* : \mathrm{L}^2(\nu) \to \mathrm{L}^2(\lambda)$, defined by

$$G^* f(x) = \int_C G(u, x) f(u) \nu(\mathrm{d}u). \tag{4}$$

Let $\mathcal{M}_0(M) \subset \mathcal{M}(M)$ be the set consisting of measures $\mu$ such that $\mu(M) = 0$ and let $\mathcal{H} \subset \mathrm{L}^2(\nu)$ denote the closure of $G(\mathcal{M}_0(M))$ in $\mathrm{L}^2(\nu)$. Then $\mathcal{H}$ (equipped with the $\mathrm{L}^2(\nu)$ topology) is an Hilbert space. Define $\mathcal{B}$ to be the Hilbert affine space parallel to $\mathcal{H}$ containing $G\lambda$:

$$\mathcal{B} = \left\{ f \in \mathrm{L}^2(\nu) : f - G\lambda \in \mathcal{H} \right\}.$$

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DEFINITION 2.1. – The *"energy function"* associated to the data $((C,\nu),G,\alpha)$ is the functional $J:\mathcal{B}\to\mathbb{R}$ defined by

$$J(f)=\frac{1}{2}\|f\|^2_{L^2(\nu)}+\frac{1}{\alpha}\log\left[\int_M e^{-\alpha(G^*f)(x)}\lambda(dx)\right]. \quad (5)$$

We let

$$\mathrm{crit}(J)=\left\{f\in\mathcal{B}:\nabla J(f)=0\right\}$$

denote the *critical set* of $J$.

Let $\mathcal{P}(M)\subset\mathcal{M}(M)$ be the set of Borel probabilities over $M$, equipped with the topology of weak* convergence. The *limit set* of $\{\mu_t\}$ denoted $L(\{\mu_t\})$ is the set of limits (in $\mathcal{P}(M)$) of convergent sequences $\{\mu_{t_k}\},\ t_k\to\infty$.

The following theorem describes $L(\{\mu_t\})$ in terms of $\mathrm{crit}(J)$.

THEOREM 2.1. – *Assume Hypothesis* 1.1. *Then the following properties hold with probability one*:

(i) $L(\{\mu_t\})$ *is a compact connected subset of* $\mathcal{P}(M)$.

(ii) *Let* $\mu\in L(\{\mu_t\})$. *Then* $\mu$ *has a smooth* ($C^k$ *if* $V$ *is* $C^k$) *density with respect to* $\lambda$ *characterized by*

$$f=G\mu\in\mathrm{crit}(J),$$

*and*

$$\frac{d\mu}{d\lambda}=\xi(\alpha G^*f),$$

*where* $\xi:C^0(M)\to C^0(M)$ *is the function defined by*

$$\xi(f)(x)=\frac{e^{-f(x)}}{\int_M e^{-f(y)}\lambda(dy)}. \quad (6)$$

Given $\mu\in\mathcal{P}(M)$ let $\Pi(\mu)$ denote the Borel probability measure absolutely continuous with respect to $\lambda$ whose Radon–Nikodym density is

$$\frac{d\Pi(\mu)}{d\lambda}=\xi(\alpha V\mu), \quad (7)$$

where $V\mu$ is defined like $G\mu$ with $V$ instead of $G$. Since $\xi(\alpha V\mu)=\xi(\alpha G^*G\mu)$, Theorem 2.1 can be rephrased as follows:

COROLLARY 2.2. – *With probability one* $L(\{\mu_t\})$ *is a compact connected subset of*

$$\mathrm{Fix}(\Pi)=\left\{\mu\in\mathcal{P}(M):\mu=\Pi(\mu)\right\}.$$

*Sketch of the proof of Theorem* 2.1. – The vector field $F$ defined on $\mathcal{M}(M)$ by $F(\mu)=-\mu+\Pi(\mu)$ induces a continuous semi-flow $\{\Psi_t\}$ on $\mathcal{P}(M)$ (see Section 3 in [2]). By Theorem 3.8 in [2] $L=L(\{\mu_t\})$ is almost surely an *attractor free set* for $\Psi$. In other words, it is a compact invariant set for $\Psi$ and $\Psi|L$ ($\Psi$ restricted to $L$) is a *chain-transitive flow* in the sense of Conley [5]. Now let $\Phi=\{\Phi_t\}$ be the local flow induced by the vector field $X=-\nabla J$. The change of variable $f=G\mu$ shows that $G\circ\Psi_t=\Phi_t\circ G$. Hence $G(L)$ is a compact invariant set for $\Phi$ and $\Phi|G(L)$ is chain-transitive. The last step is the observation that $X=-\nabla J$ is a Fredholm vector field (*see* [6]). Thus, by a theorem of Tromba [6] (extending Sard's lemma to functionals whose gradient is Fredholm) the set of critical values of $J$ has empty interior. This implies that any chain-transitive set for $\Phi$ consists of critical points (see Proposition 6.4 of [1]). □

With Theorem 2.1 in hands, it is now clear that our description of self-interacting diffusions (satisfying Hypothesis 1.1) on $M$ relies on our understanding of the critical point structure of $J$. A first step in this direction is the observation that $J$ is convex for $\alpha$ large enough.

THEOREM 2.3. – *Let*

$$W^* = \sup_{x,y\in M}\left(\frac{V(x,x)+V(y,y)}{2} - V(x,y)\right).$$

*Assume*

$$\alpha > -\frac{1}{W^*}.$$

*Then $J$ is strictly convex,* $\mathsf{Fix}(\Pi)$ *reduces to a singleton* $\{\mu^*\}$ *and* $\lim_{t\to\infty}\mu_t = \mu^*$ *almost surely. If we furthermore assume that Hypothesis* 1.2 *holds, then* $\mu^* = \lambda$.

*Sketch of proof.* – The Hessian of $J$ is definite positive for $\alpha > -1/W^*$. □

If $\alpha \leqslant -1/W^*$ the functional $J$ may have several critical points.

THEOREM 2.4. – *Let* $\mu^* \in \mathsf{Fix}(\Pi)$. *Assume that* $f^* = G\mu^*$ *is a non-degenerate critical point of* $J$. *Then*

$$\mathbb{P}\Big(\lim_{t\to\infty}\mu_t = \mu^*\Big) > 0$$

*if and only if* $f^*$ *is a local minimum of* $J$.

A consequence of this result is the following "localization" theorem.

THEOREM 2.5. – *Suppose that both Hypotheses* 1.1 *and* 1.2 *hold. Let*

$$\rho(V) = \sup\big\{\langle Vg, g\rangle_{\mathrm{L}^2(\lambda)} : g \in \mathrm{L}^2(\lambda),\ \langle g, 1\rangle_{\mathrm{L}^2(\lambda)} = 0,\ \|g\|_{\mathrm{L}^2(\lambda)} = 1\big\}.$$

*Then*

$$\mathbb{P}\Big(\lim_{t\to\infty}\mu_t = \lambda\Big) > 0$$

*if* $1 + \alpha\rho(V) > 0$; *and*

$$\mathbb{P}\Big(\lim_{t\to\infty}\mu_t = \lambda\Big) = 0$$

*if* $1 + \alpha\rho(V) < 0$.

*Sketch of the proof.* – The condition $1 + 2\alpha\rho(V) \neq 0$ makes $G\lambda$ a non-degenerate critical point of $J$. Such a critical point is a local minimum provided $1 + 2\alpha\rho(V) > 0$. □

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## References

[1] M. Benaïm, Dynamics of stochastic approximation algorithms, in: Séminaire de Probabilités XXXIII, in: Lecture Notes in Math., Vol. 1709, Springer, 1999, pp. 1–68.
[2] M. Benaïm, M. Ledoux, O. Raimond, Self-interacting diffusions, Probab. Theory Related Fields 122 (2002) 1–41.
[3] M. Benaïm, O. Raimond, Self-interacting diffusions III: The gradient case, in preparation.
[4] G. Ben Arous, M. Brunaud, Méthode de Laplace : étude variationnelle des fluctuations de type champ moyen, Stochastic 31 (1990) 79–144.
[5] C.C. Conley, Isolated Invariant Sets and the Morse Index, in: CBMS Regional Conf. Ser. in Math., Vol. 38, American Mathematical Society, Providence, 1978.
[6] A.J. Tromba, The Morse–Sard–Brown theorem for functionals and the problem of Plateau, Amer. J. Math. 99 (6) (1977) 1251–1256.