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Interpolation orbits in couples of L_p spaces

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Abstract We consider linear operators T mapping a couple of weighted L_p spaces $\{L_{p_0}(U_0), L_{p_1}(U_1)\}$ into $\{L_{q_0}(V_0), L_{q_1}(V_1)\}$ for any $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$, and describe the interpolation orbit of any $a \in L_{p_0}(U_0) + L_{p_1}(U_1)$ that is we describe a space of all $\{Ta\}$, where T runs over all linear bounded mappings from $\{L_{p_0}(U_0), L_{p_1}(U_1)\}$ into $\{L_{q_0}(V_0), L_{q_1}(V_1)\}$. We show that interpolation orbit is obtained by the Lions–Peetre method of means with functional parameter as well as by the K-method with a weighted Orlicz space as a parameter. ***To cite this article: V.I. Ovchinnikov, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 881–884.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Orbites d'interpolation pour les couples d'espaces L_p

Résumé Nous considérons les opérateurs T partant d'un couple d'espaces L_p à poids $\{L_{p_0}(U_0), L_{p_1}(U_1)\}$ à valeurs dans $\{L_{q_0}(V_0), L_{q_1}(V_1)\}$, où $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$, et donnons une description de l'orbite d'interpolation de tout élément $a \in L_{p_0}(U_0) + L_{p_1}(U_1)$; autrement dit nous décrivons l'espace de toutes les images $\{Ta\}$, où T parcourt l'espace des opérateurs linéaires bornés de $\{L_{p_0}(U_0), L_{p_1}(U_1)\}$ dans $\{L_{q_0}(V_0), L_{q_1}(V_1)\}$. Nous montrons que l'orbite d'interpolation est obtenue par la méthode des moyennes de Lions–Peetre avec un paramètre fonctionnel, et aussi par la K-méthode avec un espace d'Orlicz à poids comme paramètre fonctionnel. ***Pour citer cet article : V.I. Ovchinnikov, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 881–884.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

This paper is devoted to description of interpolation orbits with respect to linear operators mapping an arbitrary couple of L_p spaces with weights $\{L_{p_0}(U_0), L_{p_1}(U_1)\}$ into an arbitrary couple $\{L_{q_0}(V_0), L_{q_1}(V_1)\}$, where $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$. By $L_p(U)$ we denote the space of measurable functions f on a measure space $\mathfrak{M}$ such that $fU \in L_p$ with the norm $\|f\|_{L_p(U)} = \|fU\|_{L_p}$.

Let $\{X_0, X_1\}$ and $\{Y_0, Y_1\}$ be two Banach couples, $a \in X_0 + X_1$. The space $\mathrm{Orb}(a, \{X_0, X_1\} \to \{Y_0, Y_1\})$ is a Banach space of $y \in Y_0 + Y_1$ such that $y = Ta$, where T is a linear operator mapping the couple $\{X_0, X_1\}$ into the couple $\{Y_0, Y_1\}$. This space is called an interpolation orbit of the element a.

We are going to describe the spaces $\mathrm{Orb}(a, \{L_{p_0}(U_0), L_{p_1}(U_1)\} \to \{L_{q_0}(V_0), L_{q_1}(V_1)\})$ for any a, any $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$ and any positive weights U_0, U_1, V_0, V_1.

Fundamental results on description of these spaces in separate cases are well known since 1965. The key role was played by the J. Peetre K-functional.

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Let $\{X_0, X_1\}$ be a Banach couple, $x \in X_0 + X_1$, $s > 0$, $t > 0$. Denote by

$$K\bigl(s,t,x;\{X_0,\ X_1\}\bigr) = \inf_{x=x_0+x_1} s\|x_0\|_{X_0} + t\|x_1\|_{X_1},$$

where infimum is taken over all representations of x as a sum of $x_0 \in X_0$ and $x_1 \in X_1$. The function $K(s,t)$ is concave and is uniquely defined by the function $K(1,t,x;\{X_0,X_1\})$ which is also denoted by $K(t,x;\{X_0,X_1\})$.

If $1 \leqslant p_0 \leqslant q_0 \leqslant \infty$, $1 \leqslant p_1 \leqslant q_1 \leqslant \infty$, the orbits $\mathrm{Orb}(a,\{L_{p_0}(U_0),L_{p_1}(U_1)\} \to \{L_{q_0}(V_0),L_{q_1}(V_1)\})$ were described as the generalized Marcinkiewicz spaces, i.e.,

$$\mathrm{Orb}\bigl(a,\bigl\{L_{p_0}(U_0),L_{p_1}(U_1)\bigr\} \to \bigl\{L_{q_0}(V_0),L_{q_1}(V_1)\bigr\}\bigr) = \left\{y : \sup_{s,t} \frac{K(s,t,y;\{L_{q_0}(V_0),L_{q_1}(V_1)\})}{K(s,t,a;\{L_{p_0}(U_0),L_{p_1}(U_1)\})} < \infty\right\}$$

for any $a \in L_{p_0}(U_0) + L_{p_1}(U_1)$. The decisive steps were done by Sparr in [10,11] and Dmitriev in [3]. In particular Sparr showed that if

$$K\bigl(s,t,y;\bigl\{L_{p_0}(V_0),L_{p_1}(V_1)\bigr\}\bigr) \leqslant CK\bigl(s,t,a;\bigl\{L_{p_0}(U_0),L_{p_1}(U_1)\bigr\}\bigr),$$

then there exists a linear operator $T:\{L_{p_0}(U_0),L_{p_1}(U_1)\} \to \{L_{p_0}(V_0),L_{p_1}(V_1)\}$ such that $y = Ta$.

Dmitriev in [3] had also found a description of orbits in the case of arbitrary $1 \leqslant p_0, p_1 \leqslant \infty$ and $q_0 = q_1 = 1$ as well as in the case of arbitrary $1 \leqslant p_1, q_0 \leqslant \infty$ and $p_0 = q_1 = 1$.

The result we are going to present here goes up to the paper [6] where some optimal interpolation theorems were found. Developing this approach the following hypothesis was formulated in [7]. Roughly speaking it states that the space $\mathrm{Orb}(a,\{L_{p_0}(U_0),L_{p_1}(U_1)\} \to \{L_{q_0}(V_0),L_{q_1}(V_1)\})$ is situated between $L_{q_0}(V_0)$ and $L_{q_1}(V_1)$ exactly in the same place as the Calderon–Lozanovskii space $\varphi(L_{r_0}(W_0),L_{r_1}(W_1))$ between $L_{r_0}(W_0)$ and $L_{r_1}(W_1)$, where $\varphi(s,t) = K(s,t,a,\{L_{p_0}(U_0),L_{p_1}(U_1)\})$ and $r_0^{-1} = (q_0^{-1} - p_0^{-1})_+$, $r_1^{-1} = (q_1^{-1} - p_1^{-1})_+$. This hypothesis was partially confirmed in [8]. Now we show that hypothesis from [7] is true for any $a \in L_{p_0}(U_0) + L_{p_1}(U_1)$. We also present a slightly modified description of interpolation orbits which resembles Dmitriev's description from [3].

1. The method of means for any quasi-concave functional parameter

Let $\varphi(s,t)$ be interpolation function, that is let $\rho(t) = \varphi(1,t)$ be quasi-concave and $\varphi(s,t)$ be homogeneous of the degree one. Assume that $\varphi \in \Phi_0$ which means that $\varphi(1,t) \to 0$ and $\varphi(t,1) \to 0$ as $t \to 0$. Denote by $\{t_n\}$ the sequence invented by K. Oskolkov and introduced to interpolation by S. Janson. The sequence is constructed by induction $\min(\rho(t_{n+1})/\rho(t_n), t_{n+1}\rho(t_n)/t_n\rho(t_{n+1})) = q > 1$. (For simplicity in the sequel we suppose that $\{t_n\}$ is two-sided.)

The main property of this sequence is the following

$$K\bigl(s,t,\bigl\{\rho(t_n)\bigr\},\bigl\{l_{p_0},l_{p_1}\bigl(t_n^{-1}\bigr)\bigr\}\bigr) \asymp \varphi(s,t) \tag{1}$$

for any $1 \leqslant p_0, p_1 \leqslant \infty$.

Definition 1. – Let $\{X_0,\ X_1\}$ be any Banach couple, $\rho(t)$ be a quasi-concave function such that $\varphi \in \Phi_0$ and $1 \leqslant p_0, p_1 \leqslant \infty$. Denote by $\varphi(X_0,\ X_1)_{p_0,p_1}$ the space of $x \in X_0 + X_1$ such that

$$x = \sum_{n\in\mathbb{Z}} \rho(t_n) w_n \quad \text{(convergence in } X_0 + X_1\text{)}, \tag{2}$$

where $w_n \in X_0 \cap X_1$ and $\{\|w_n\|_{X_0}\} \in l_{p_0}$, $\{t_n\|w_n\|_{X_1}\} \in l_{p_1}$.

The norm in $\varphi(X_0, X_1)_{p_0,p_1}$ is naturally defined. In the case of $\varphi(s,t) = s^{1-\theta}t^{\theta}$, where $0 < \theta < 1$, these spaces were introduced by Lions and Peetre in [5] and were called the spaces of means.

Note that $\varphi(X_0, X_1)_{\infty,\infty}$ coincides with the generalized Marcinkiewicz space $M_\varphi(X_0, X_1)$ as well as with the space $(X_0, X_1)_{\rho,\infty}$ (see, for instance, [9]).

Let $\{X_0, X_1\}$ be a couple of Banach lattices. Recall that $\varphi(X_0, X_1)$ is the space of all elements from $X_0 + X_1$ such that $|x| = \varphi(|x_0|, |x_1|)$, where $x_0 \in X_0,\ x_1 \in X_1$.

LEMMA 1. – *Let $1 \leqslant p_0, p_1 \leqslant \infty$, then $\varphi(L_{p_0}(U_0), L_{p_1}(U_1)) = \varphi(L_{p_0}(U_0), L_{p_1}(U_1))_{p_0,p_1}$.*
(Note that if $U_0 = 1$ and $U_1 = 1$, then $\varphi(L_{p_0}, L_{p_1})$ is an Orlicz space.)

Recall that interpolation function φ is called non-degenerate if the ranges of the functions $\varphi(t,1)$ and $\varphi(1,t)$ where $t > 0$ coincide with $(0,\infty)$.

LEMMA 2. – *If φ is non-degenerate, then for any Banach couple the space $\varphi(X_0, X_1)_{p_0,p_1}$ consists of $x \in X_0 + X_1$ for which $\{K(u_m, x, \{X_0, X_1\}\} \in \varphi(l_{p_0}, l_{p_1}(u_m^{-1}))$, where $\{u_m\}$ is the Oskolkov–Janson sequence for the function $K(t, x, \{X_0, X_1\})$.*

We omit the proof. Note however that the proof is based on the K-divisibility (*see* [2]) and Lemma 1. With the help of K-divisibility for the couple $\{l_{p_0}, l_{p_1}(u_m^{-1})\}$ the expansion (2) of $x \in \varphi(X_0, X_1)_{p_0,p_1}$ in the couple $\{X_0, X_1\}$ generates the analogous expansion of the sequence $\{K(u_m, x, \{X_0, X_1\}\} \in \varphi(l_{p_0}, l_{p_1}(u_m^{-1}))_{p_0,p_1} = \varphi(l_{p_0}, l_{p_1}(u_m^{-1}))$ in the couple $\{l_{p_0}, l_{p_1}(u_m^{-1})\}$, and vice versa.

Remark. – Note that the spaces $\varphi(X_0, X_1)_{p,p}$ coincide with the space $(X_0, X_1)_{\rho,p}$ introduced by Janson (*see* [4]). Lemma 2 gives us a new description of these spaces as well.

2. The main theorem

THEOREM. – *Let $\{L_{p_0}(U_0), L_{p_1}(U_1)\}$ and $\{L_{q_0}(V_0), L_{q_1}(V_1)\}$ be two Banach couples, where $1 \leqslant p_0$, p_1, q_0, $q_1 \leqslant \infty$, and $a \in L_{p_0}(U_0) + L_{p_1}(U_1)$ such that $\varphi(s,t) = K(s,t,a,\{L_{p_0}(U_0), L_{p_1}(U_1)\}) \in \Phi_0$, then*

$$\operatorname{Orb}\bigl(a, \bigl\{L_{p_0}(U_0), L_{p_1}(U_1)\bigr\} \to \bigl\{L_{q_0}(V_0), L_{q_1}(V_1)\bigr\}\bigr) = \varphi\bigl(L_{q_0}(V_0), L_{q_1}(V_1)\bigr)_{r_0,r_1},$$

where $r_0^{-1} = (q_0^{-1} - p_0^{-1})_+$ and $r_1^{-1} = (q_1^{-1} - p_1^{-1})_+$. (As usual x_+ denotes the positive part of x.)

The rest cases $\varphi(s,t) \notin \Phi_0$ can be easily reduced to $\varphi(s,t) \in \Phi_0$ as it was done in [9] where the analogous situation takes place for $p_0 \leqslant q_0$ and $p_1 \leqslant q_1$.

The proof is a combination of the following propositions.

PROPOSITION 1. – *For any $1 \leqslant p_0$, p_1, q_0, $q_1 \leqslant \infty$ and any weights U_0, U_1, V_0, V_1, and for any $a \in L_{p_0}(U_0) + L_{p_1}(U_1)$*

$$\operatorname{Orb}\bigl(a, \bigl\{L_{p_0}(U_0), L_{p_1}(U_1)\bigr\} \to \bigl\{L_{q_0}(V_0), L_{q_1}(V_1)\bigr\}\bigr) \subset \varphi\bigl(L_{q_0}(V_0), L_{q_1}(V_1)\bigr)_{r_0,r_1}.$$

Proof. – Let $b = Ta$, where $T : \{L_{p_0}(U_0), L_{p_1}(U_1)\} \to \{L_{q_0}(V_0), L_{q_1}(V_1)\}$. Recall that $\rho(t) = \varphi(1,t)$. Denote $a_\rho = \{\rho(t_n)\}$, $\psi(u) = K(u, b, \{L_{q_0}(V_0), L_{q_1}(V_1)\})$ and $b_\psi = \{\psi(u_m)\}$, where u_m is the Oskolkov–Janson sequence for $\psi(u)$. The Sparr theorem implies that there exists a linear operator $S : \{l_{p_0}, l_{p_1}(t_n^{-1})\} \to \{l_{q_0}, l_{q_1}(u_m^{-1})\}$ such that $Sa_\rho = b_\psi$.

We consider the embedding $\{l_{q_0}, l_{q_1}(u_m^{-1})\} \subset \{l_\infty, l_\infty(u_m^{-1})\}$. It is known that the embedding $l_{q_i} \subset l_\infty$ are $(1, q_i)$-summing operators (by the Karl–Bennett theorem, *see* [1]). Hence if $q_0 < p_0$, then the image of the standard basis sequence in l_{p_0} with respect to $S : l_{p_0} \to l_\infty$ is l_{r_0}-sequence, that is $\{\|S(e_n)\|_{l_\infty}\} \in l_{r_0}$, where $r_0^{-1} = q_0^{-1} - p_0^{-1}$. Analogously $\{t_n\|S(e_n)\|_{l_\infty(u_m^{-1})}\} \in l_{r_1}$, where $r_1^{-1} = q_1^{-1} - p_1^{-1}$. Hence in any case we have $\{\|S(e_n)\|_{l_\infty}\} \in l_{r_0}$, and $\{t_n\|S(e_n)\|_{l_\infty(u_m^{-1})}\} \in l_{r_1}$, where $r_0^{-1} = (q_0^{-1} -$

$p_0^{-1})_+$ and $r_1^{-1} = (q_1^{-1} - p_1^{-1})_+$. Therefore by definition $b_\psi \in \varphi(l_\infty, l_\infty(u_m^{-1}))_{r_0,r_1}$. By Lemma 2 this means $\{K(v_m, b_\psi, \{l_\infty, l_\infty(u_m^{-1})\})\} \in \varphi(l_{r_0}, l_{r_1}(v_m^{-1}))$, where $\{v_m\}$ is the Oskolkov–Janson sequence for $K(v, b_\psi, \{l_\infty, l_\infty(u_m^{-1})\}) \asymp K(v, b_\psi, \{l_{q_0}, l_{q_1}(u_m^{-1})\}) = \psi(v)$. Hence $v_m = u_m$, and by (1)

$$K\big(u_m, b, \big\{L_{q_0}(V_0), L_{q_1}(V_1)\big\}\big) \asymp K\big(u_m, b_\psi, \big\{l_{q_0}, l_{q_1}\big(u_m^{-1}\big)\big\}\big) \asymp K\big(u_m, b_\psi, \big\{l_\infty, l_\infty\big(u_m^{-1}\big)\big\}\big).$$

So $\{K(u_m, b, \{L_{q_0}(V_0), L_{q_1}(V_1)\})\} \in \varphi(l_{r_0}, l_{r_1}(u_m^{-1}))$. By Lemma 2 proposition is proved.

The following propositions are devoted to the inverse inclusion

$$\varphi\big(L_{q_0}(V_0), L_{q_1}(V_1)\big)_{r_0,r_1} \subset \mathrm{Orb}\big(a, \big\{L_{p_0}(U_0), L_{p_1}(U_1)\big\} \to \big\{L_{q_0}(V_0), L_{q_1}(V_1)\big\}\big).$$

For any $b \in \psi(L_{q_0}(V_0), L_{q_1}(V_1))_{r_0,r_1}$ we must find an operator $T \in \{L_{p_0}(U_0), L_{p_1}(U_1)\} \to \{L_{q_0}(V_0), L_{q_1}(V_1)\}$ such that $b = Ta$. Again with the help of the Sparr theorem we substitute a by a_ρ and b by b_ψ as well as initial couples by $\{l_{p_0}, l_{p_1}(t_n^{-1})\}$ and $\{l_{q_0}, l_{q_1}(u_m^{-1})\}$, respectively.

PROPOSITION 2. – *Let* $\{\psi(u_m)\} \in \varphi(l_{r_0}, l_{r_1}(u_m^{-1}))$, *then there exist sequences* $\{\beta_m^0\} \in l_{r_0}$ *and* $\{\beta_m^1\} \in l_{r_1}$ *such that* $K(s, t, \{\psi(u_m)\}, \{l_1(1/\beta_m^0), l_1(1/\beta_m^1 u_m)\}) \leqslant C\varphi(s, t)$.

PROPOSITION 3. – *Let* $b_\psi = \{\psi(u_m)\} \in \varphi(l_{r_0}, l_{r_1}(u_m^{-1}))$, *then* $b_\psi = S(a_\rho)$ *for some linear operator* $S: \{l_{p_0}, l_{p_1}(t_n^{-1})\} \to \{l_{q_0}, l_{q_1}(u_m^{-1})\}$.

Proof. – Without loss of generality we assume that $p_0 \geqslant q_0$, $p_1 \geqslant q_1$. By Proposition 2 we can find $\beta^0 \in l_{r_0}$ and $\beta^1 \in l_{r_1}$. Consider the embedding

$$\big\{l_1\big(1/\beta_m^0\big), l_1\big(1/\beta_m^1 u_m\big)\big\} \subset \big\{l_{p_0}\big(1/\beta_m^0\big), l_{p_1}\big(1/\beta_m^1 u_m\big)\big\} \subset \big\{l_{q_0}, l_{q_1}\big(u_m^{-1}\big)\big\} \tag{3}$$

and the element b_ψ. By Proposition 2 we have $K(s, t, b_\psi, \{l_{p_0}(1/\beta_m^0), l_{p_1}(1/\beta_m^1 u_m)\}) \leqslant C\varphi(s, t)$. Since $\varphi(s, t) \asymp K(s, t, a_\rho, \{l_{p_0}, l_{p_1}(t_n^{-1})\})$, by the Sparr theorem there exists an operator $S: \{l_{p_0}, l_{p_1}(t_n^{-1})\} \to \{l_{p_0}(1/\beta_m^0), l_{p_1}(1/\beta_m^1 u_m)\}$ mapping a_ρ into b_ψ.

The composition of S and the right-hand side embedding in (3) is the desired mapping. Thus proposition and theorem are proved.

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