Available online at www.sciencedirect.com
SCIENCE @ DIRECT®
C. R. Acad. Sci. Paris, Ser. I 336 (2003) 111–116
COMPTES RENDUS MATHEMATIQUE

Combinatorics/Computer Science

Invertible substitutions on a three-letter alphabet ☆

Substitutions inversibles sur un alphabet de trois lettres

Bo Tan, Zhi-Xiong Wen, Yiping Zhang

Department of Mathematics, Wuhan University, 430072 Wuhan, Hubei, PR China

Received 29 October 2002; accepted 5 November 2002

Presented by Jean-Pierre Kahane

Abstract

We study the structure of invertible substitutions on a three-letter alphabet. We show that there exists a finite set $\mathbb{S}$ of invertible substitutions such that any invertible substitution can be written as $I_w \circ \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_k$, where I_w is the inner automorphism associated with w, and $\sigma_j \in \mathbb{S}$ for $1 \leqslant j \leqslant k$. As a consequence, M is the matrix of an invertible substitution if and only if it is a finite product of non-negative elementary matrices. ***To cite this article: B. Tan et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Nous étudions la structure des substitutions inversibles sur un alphabet à trois lettres. Nous prouvons qu'il existe un ensemble fini $\mathbb{S}$ de substitutions inversibles tel que toute substitution inversible puisse être écrite comme $I_w \circ \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_k$, où I_w est l'automorphisme intérieur associé à w et où $\sigma_j \in \mathbb{S}$ pour $1 \leqslant j \leqslant k$. Comme conséquence, M est la matrice d'une substitution inversible si et seulement si elle est un produit fini de matrices élémentaires non-négatives. ***Pour citer cet article : B. Tan et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Soit $A = \{a, b, c\}$ un alphabet de trois lettres. On désigne par A^* et Γ_A les monoïde et groupe libres engendrés par A. Une substitution σ sur A (c'est-à-dire un endomorphisme de A^*) est dite inversible si elle se prolonge en un automorphisme de Γ_A. On peut identifier σ au triplet de mots $(\sigma(a), \sigma(b), \sigma(c))$. On pose $\pi_1 = (b, a, c)$, $\pi_2 = (c, b, a)$, $\phi_l = (ba, b, c)$, $\phi_r = (ab, b, c)$. Le monoïde des substitutions inversibles sur A sera noté $\mathcal{IS}(A^*)$.

Si $w \in \Gamma_A$, on désigne par I_w l'automorphisme intérieur de Γ_A associé à w : $I_w(u) := wuw^{-1}$.

La structure du monoïde des substitutions inversibles sur un alphabet de deux lettres est connue : il est engendré par une permutation et deux substitutions de Fibonacci [12]. Quand l'alphabet a plus de deux lettres, la situation

☆ Research supported by NSFC and by the Special Funds for Major State Basic Research Projects of China.
E-mail addresses: tanbo@colmath.whu.edu.cn (B. Tan), zhxwen@whu.edu.cn (Z.-X. Wen), ypzhang@whu.edu.cn (Y. Zhang).

1631-073X/03/$ – see front matter
doi:10.1016/S1631-073X(02)00006-7

est beaucoup plus compliquée. Dans [15], il est prouvé que le monoïde $\mathcal{IS}(A^*)$ n'est pas de type fini, mais sa structure restait jusqu'à présent inconnue.

Dans cette Note, nous prouvons les théorèmes suivants.

Théorème 1. *Soit σ est une substitution inversible. Il existe $w \in A^*$ ou $w^{-1} \in A^*$ et $\sigma_1, \ldots, \sigma_k \in \{\pi_1, \pi_2, \phi_l, \phi_r\}$ tels que*

(1) $\sigma = I_w \circ \sigma_1 \circ \cdots \circ \sigma_k$.
(2) *De plus, on peut choisir pour w (ou w^{-1}) un suffixe (ou un préfixe) commun aux mots $\sigma(a)$, $\sigma(b)$ et $\sigma(c)$.*

Théorème 2. *Une matrice 3×3 à coefficients entiers positifs ou nuls est la matrice d'une substitution inversible si et seulement si c'est un produit fini de matrices élémentaires à coefficients positifs.*

1. Introduction

The study of substitutions (the endomorphisms of free monoids of finite type) plays an important role in finite automata, symbolic dynamics, and fractal geometry [1,2,4]. It has various applications to quasicrystals, computational complexity, information theory. In addition, substitution is also a fundamental object studied in combinatorial group theory [6–8].

Plenty of results have been obtained for substitutions over a two-letter alphabet [3,11,13]. The notion of invertible substitution appears in [10]: these are the substitutions which extend as automorphisms of the corresponding free group. Since then, they have been studied by many authors (for instance, see [3,5]). The invertible substitutions over a two-letter alphabet form a monoid whose structure in known [12]: this monoid can be generated by a permutation and two so-called Fibonacci substitutions. This result has had many applications, namely to the study of local isomorphisms of fixed points of substitutions [12,14] and to the study of trace maps [10].

When the alphabet has more than two letters, the situation is much more complicated. In [15], it was shown, by enumerating infinitely many so-called indecomposable substitutions, that the monoid of invertible substitutions over three letters (which will be denoted by $\mathcal{IS}(A^*)$) is not finitely generated. But, up to now the structure of $\mathcal{IS}(A^*)$ remained unknown. Here, we elucidate this structure. We show that if σ is an invertible substitution over a three-letter alphabet, there exists a word w such that $I_w \circ \sigma$ or $I_{w^{-1}} \circ \sigma$ is the composition of finitely many Fibonacci substitutions and permutations (Theorem 3.1). As a consequence, the matrix of an invertible substitution is positively decomposable (Theorem 3.2).

2. Preliminaries and notations

Let us first recall some basic definitions and notations in the theory of substitutions (see [1,8] for a general theory).

Set $A = \{a, b, c\}$ and $\bar{A} = \{a, b, c, a^{-1}, b^{-1}, c^{-1}\}$. Let A^* (resp. Γ_A) be the free monoid (resp. the free group) generated by A (the unit element is the empty word ε). The elements of A^* will be called "positive words" or simply "words" and those of Γ_A "signed" or "mixed" words. The inverse of a positive word will be said to be "negative".

Let $w \in \Gamma_A$: $w = x_1 \cdots x_k$ with $x_i \in \bar{A}$ $(i = 1, 2, \ldots, k)$. If $x_i x_{i+1} \neq \varepsilon$ (for $i = 1, \ldots, k-1$), we say that $x_1 \cdots x_k$ is in the reduced form and that the length of w is k. This length will be denoted by $|w|$. Let w, w_j $(j = 1, \ldots, k) \in \Gamma_A$. If $w = w_1 w_2 \cdots w_k$ satisfies $|w| = |w_1| + \cdots + |w_k|$, we say that $w_1 w_2 \cdots w_k$ is a reduced expression of w. Then, we say that w_1 is a prefix of w and that w_k is a suffix of w, and we write $w_1 \lhd w$ and $w_k \rhd w$.

A substitution σ over A is a morphism σ of A^*. Such a morphism extends in a natural way to an endomorphism of Γ_A; If this extension is an automorphism of Γ_A, the substitution σ is said to be invertible. The set of substitutions (resp. invertible substitutions) is denoted by $\mathcal{S}(A^*)$ (resp. by $\mathcal{IS}(A^*)$).

We often identify an endomorphism σ of Γ_A with the triple $(\sigma(a), \sigma(b), \sigma(c))$ of (maybe mixed) words. We define the length of σ to be $|\sigma| = |\sigma(a)| + |\sigma(b)| + |\sigma(c)|$.

If U is a subset of a monoid, $\langle U \rangle$ stands for the sub-monoid (not the sub-group, even when dealing inside a group) generated by U. The use of the same notation for different groups and monoids will not generate any confusion.

We shall use the following basic invertible substitutions and automorphisms.

Permutations: Let $\mathcal{P} = \langle \pi_1, \pi_2 \rangle$, where $\pi_1 = (b, a, c)$ and $\pi_2 = (c, b, a)$. Note that $\mathcal{P}$ is a subgroup isomorphic to the symmetric group on A.

Fibonacci type: Set $\phi_l = (ba, b, c), \phi_r = (ab, b, c)$. Set $\mathcal{L} = \{\pi \circ \phi_l \circ \pi';\ \pi, \pi' \in \mathcal{P}\}, \mathcal{R} = \{\pi \circ \phi_r \circ \pi';\ \pi, \pi' \in \mathcal{P}\}$, and $\mathcal{F} = \mathcal{R} \cup \mathcal{L}$. The elements in $\mathcal{F}$ are called substitutions of Fibonacci type or simply Fibonacci substitutions.

Simple substitutions: Set $S = \langle \pi_1, \pi_2, \phi_l, \phi_r \rangle = \langle \mathcal{P}, \mathcal{L}, \mathcal{R} \rangle \subset \mathcal{IS}(A^*)$. The elements in S will be called simple substitutions.

Involutions: $\iota_1 = (a^{-1}, b, c)$, $\iota_2 = (a, b^{-1}, c)$, $\iota_3 = (a, b, c^{-1})$, $\mathcal{I} = \{\iota_i;\ 1 \leqslant i \leqslant 3\}$.

According to Nielsen's theory [8], $\mathrm{Aut}(\Gamma_A) = \langle \mathcal{P}, \mathcal{F}, \mathcal{I} \rangle = \langle \pi_1, \pi_2, \phi_l, \phi_r, \iota_1 \rangle$.

Definition 2.1. An invertible substitution σ is said to be trivial if $\sigma \in \mathcal{P}$. If there exist non-trivial invertible substitutions σ_1 and σ_2 such that $\sigma = \sigma_1 \circ \sigma_2$, we say that σ is decomposable.

The invertible substitutions which are not in $\mathcal{P} \cup \mathcal{F}$ and not decomposable will be called non-simple indecomposable. As a matter of fact, there exist infinitely many such substitutions, hence $\mathcal{IS}(A^*)$ is not finitely generated [15].

Inner automorphisms: Let $z \in \Gamma_A$. $I_z \in \mathrm{Aut}(\Gamma_A)$ is defined as follows: $I_z(w) = zwz^{-1}$ $(w \in \Gamma_A)$. That is $I_z = (zaz^{-1}, zbz^{-1}, zcz^{-1})$. We have $I_z \circ (w_1, w_2, w_3) = (zw_1z^{-1}, zw_2z^{-1}, zw_3z^{-1})$ and $\sigma \circ I_z = I_{\sigma(z)} \circ \sigma$. Notice also that $I_\varepsilon = I$.

If $w \in \Gamma_A$, one denotes by $|w|_a$ (resp. $|w|_b$ or $|w|_c$) the number (the algebraic sum of exponents) of appearances of a (resp. b or c) in w (e.g., $|a^{-1}ba|_a = 0$). Let $\sigma \in \mathcal{S}(A^*)$. One associates a matrix M_σ with σ: $M_\sigma = (|\sigma(\beta)|_\alpha)_{\alpha,\beta \in A}$. Let $\sigma, \tau \in \mathcal{S}(A^*)$. One has $M_{\sigma\circ\tau} = M_\sigma M_\tau$. If $\sigma \in \mathcal{IS}(A^*)$, then $\det(M_\sigma) = \pm 1$.

The above definitions and equalities can be extended to the case when σ and τ are endomorphisms of Γ_A.

3. Main theorems

The following decomposition theorem characterizes the structure of $\mathcal{IS}(A^*)$: any invertible substitution is a simple one up to an inner automorphism.

Theorem 3.1. *Let $\sigma \in \mathcal{IS}(A^*)$. There exists $w \in A^*$ or $w^{-1} \in A^*$ such that*

(1) *$I_w \circ \sigma$ is a simple substitution. That is, there exist $\sigma_1, \ldots, \sigma_k \in \{\pi_1, \pi_2, \phi_l, \phi_r\}$ such that $\sigma = I_{w^{-1}} \circ \sigma_1 \cdots \sigma_k$.*

(2) *Furthermore, we can take* w (*or* w^{-1}) *to be a common suffix* (*or prefix*) *of* $\sigma(a)$, $\sigma(b)$, *and* $\sigma(c)$.

Remark 3.1. We point out that the decomposition theorem for the case of two-letter alphabet [12] is an easy consequence of Theorem 3.1.

As consequence we have:

Theorem 3.2. *A* 3×3*-matrix with non-negative integer coefficients is the matrix of some invertible substitution if and only if it is a finite product of non-negative elementary matrices.*

4. Proofs

The symbol "$+$" (resp. "$-$") will represent various non-empty positive words (resp. non-empty negative words). We also use the symbols like "$+-+$" to represent various types of mixed words. As an example, $u = +-+$ means that $u = u_1 u_2^{-1} u_3$, where $u_j \in A^*$ and $u_1 u_2^{-1} u_3$ is a reduced expression. The meanings of "$u = + \cdots$", "$u = \cdots -$", "$u = + \cdots -$" etc. are now clear.

The following lemma is a simple version of Nielsen's cancellation procedure that we use in the proofs. For the details, we refer the reader to [8,9].

Lemma 4.1. *Let* $\sigma = (w_1, w_2, w_3) \in \mathrm{Aut}(\Gamma_A)$. *There exist* $k \geqslant 0$ *and* $\tau_1, \ldots, \tau_k \in \{\pi_1, \pi_2, \phi_l, \phi_r, \iota_1\}$ *such that, if one sets* $\sigma_0 = \sigma$ *and* $\sigma_i = \sigma_{i-1} \circ \tau_i$ (for $i = 1, \ldots, k$), *one has* $|\sigma_i| \leqslant |\sigma_{i-1}|$ ($i = 1, \ldots, k$) *and* σ_k *is the identity.*

Definition 4.1. We say that a non-trivial substitution $\sigma = (w_1, w_2, w_3)$ is mixed if it satisfies $w_i w_k^{-1} w_j = +-+$ for all (i, j, k) such that $i \neq k$ and $j \neq k$.

Lemma 4.2 [15]. *Any mixed substitution is non-invertible.*

Corollary 4.1. *Let* $\sigma = (w_1, w_2, w_3)$ *be an invertible substitution, suppose that for all* $i \neq j$, w_i *is neither a prefix nor a suffix of* w_j. *There exist* $\pi \in \mathcal{P}$ *and non-empty words* u, v, x, y *such that either* $\sigma \circ \pi = (ux, uv, yv)$ *or* $\sigma \circ \pi = (uxv, uv, y)$.

Lemma 4.3. *Let* $u \in \Gamma_A$ *and* $x, y \in A^*$ *non-empty. Suppose that* $ux = yu$ *and* $|ux| = |u| + |x|$. *Then we have either* $u \in A^*$ *or* $u^{-1} \in A^*$.

When $w = \alpha_1 \alpha_2 \cdots \alpha_{|w|} \in \Gamma_A$ ($\alpha_i \in \bar{A}$), we shall use the following notations:

$$h_i(w) = \alpha_i \quad \big(i = 1, 2, \ldots, |w|\big), \quad h_\infty(w) = h_{|w|}(w).$$

The following lemmas study the "cancellation properties" between special words.

Lemma 4.4. *Let* u, v, x *and* y *be non-empty words satisfying*

$$h_1(u) \neq h_1(y) \quad \text{and} \quad h_\infty(x) \neq h_\infty(v); \tag{4.1}$$

$$x \text{ (resp. } v\text{) is not a prefix of } v \text{ (resp. } x\text{)}; \tag{4.2}$$

$$y \text{ (resp. } u\text{) is not a suffix of } u \text{ (resp. } y\text{)}. \tag{4.3}$$

Set $w_1 = ux$, $w_2 = uv$, $w_3 = yv$ *and consider the following mixed word*:

$$w = w_{i_0}^{\varepsilon} w_{i_1}^{-\varepsilon} \cdots w_{i_k}^{(-1)^k \varepsilon}, \tag{4.4}$$

where $k \geqslant 0$, $\varepsilon \in \{+1, -1\}$, $i_m \in \{1, 2, 3\}$ $(m = 0, 1, \ldots, k)$ *and* $i_m \neq i_{m+1}$ $(m = 0, 1, \ldots, k-1)$. *Then we have*

$$h_1(w) = h_1(w_{i_0}^{\varepsilon}),\ h_\infty(w) = h_\infty\big(w_{i_k}^{(-1)^k\varepsilon}\big), \tag{4.5}$$

$$|w| \geqslant 2. \tag{4.6}$$

The above lemma will be used to study substitutions of the form (ux, uv, yv). The following lemma is the corresponding version for (uxv, uv, y).

Lemma 4.5. *Let* u, v, x *and* y *be non-empty words such that* $h_1(u) \neq h_1(y)$, $h_\infty(v) \neq h_\infty(y)$, v *is not a prefix of* xv *and* u *is not a suffix of* ux.

Set $w_1 = uxv$, $w_2 = uv$, $w_3 = y$ *and consider the mixed words of the form* (4.4). *Then*

(i) (4.5) *holds*;
(ii) *If* $|y| > 1$, (4.6) *holds*;
(iii) *If* $|y| = 1$, (4.6) *holds except when* $k = 0$, *and* $w_{i_0} = w_3$.

Lemma 4.6. *Under the notations and conditions of Lemma* 4.4 (*resp. Lemma* 4.5), *the substitution* $\sigma = (w_1, w_2, w_3)$ *is not invertible.*

Proposition 4.1. *Suppose that* $\sigma = (w_1, w_2, w_3)$ *is a non-simple indecomposable substitution. Then we have either* $h_1(w_1) = h_1(w_2) = h_1(w_3)$ *or* $h_\infty(w_1) = h_\infty(w_2) = h_\infty(w_3)$. *In other words,* w_1, w_2, *and* w_3 *must have a common non-empty prefix or suffix.*

Lemma 4.7. *Let* $z, u, v, x, y \in A^*$. *Assume that* u, v, x, y *satisfy* (4.1), (4.2) *and* (4.3). *Set* $w_1 = ux$, $w_2 = uv$, $w_3 = yv$, *then both substitutions* (zw_1, zw_2, zw_3) *and* (w_1z, w_2z, w_3z) *are not invertible.*

Remark 4.1. When x, y, u, v, w_1, w_2, w_3 are given as in Lemma 4.5, conclusions similar to those of the above lemma hold.

Lemma 4.8. *Suppose that* $\sigma = (zu_1, zu_2, zu_3)$ (*resp.* $\sigma = (u_1z, u_2z, u_3z)$) *is a non-simple indecomposable substitution, where* z *is a non-empty word. Suppose that* u_1, u_2, u_3 *have no common prefix and no common suffix. Then there exist a non-trivial invertible substitution* σ', *a permutation* $\pi \in \mathcal{P}$ *and a Fibonacci* $f \in \mathcal{F}$, *such that* $I_{z^{-1}} \circ \sigma = \sigma' \circ f \circ \pi$ (*resp.* $I_z \circ \sigma = \sigma' \circ f \circ \pi$).

Lemma 4.9. *Let* σ *be a non-simple indecomposable substitution. Then there exists* $z \in \Gamma_A$ *such that* $I_z \circ \sigma$ *is decomposable. That is, there exist non-trivial invertible substitutions* σ_1 *and* σ_2 *such that* $I_z \circ \sigma = \sigma_1 \circ \sigma_2$. *Furthermore,* $|\sigma_i| < |\sigma|$ $(i = 1, 2)$.

Now we are ready to prove our theorems.

Proof of Theorem 3.1. First we prove Theorem 3.1(1). Let σ be an invertible substitution. Then there exist $k \geqslant 1$ and indecomposable substitutions $\sigma_1, \ldots, \sigma_k$ such that $\sigma = \sigma_1 \circ \cdots \circ \sigma_k$.

If some σ_i is non-simple, then by Lemma 4.9, there exist $z \in \Gamma_A$, invertible substitutions σ_i' and σ_i'' such that $\sigma_i = I_{z^{-1}} \circ \sigma_i' \circ \sigma_i''$ and that

$$|\sigma_i'| < |\sigma_i|, \qquad |\sigma_i''| < |\sigma_i|. \tag{4.7}$$

Then we repeat such decomposition for σ_i' (resp. σ_i'') and so on. By (4.7), such decomposition will terminate after a finite of steps. Finally every factor will be a simple substitution. Hence we can write $\sigma = I_{w_1} \circ \tau_1 \circ I_{w_2} \circ \tau_2 \circ$

$\cdots \circ I_{w_n} \circ \tau_n$, where τ_i $(i = 1, \ldots, n)$ is a simple substitution and $w_i \in \Gamma_A$ (put $w_i = \varepsilon$ if necessary). Thus we may write $\sigma = I_w \circ \tau_1 \circ \tau_2 \circ \cdots \circ \tau_n$, where $w \in \Gamma_A$.

Let $\tau = \tau_1 \circ \tau_2 \circ \cdots \circ \tau_n$. It is clear that τ is a simple substitution.

Finally, since $I_{w^{-1}} \circ \sigma = \tau$, Lemma 4.3 implies that $w \in A^*$ or $w^{-1} \in A^*$. Theorem 3.1(1) is thus proved.

For Theorem 3.1(2), denoting $|\sigma|_{\min} = \min\{|\sigma(a)|, |\sigma(b)|, |\sigma(c)|\}$, we only need to prove the following equivalent statement.

Claim. *We can choose* w *in* (1) *such that* $|w| \leqslant |\sigma|_{\min}$.

Let us prove the claim by induction on $k = |\sigma|$.

If $k \leqslant 4$ the claim can be verified simply by enumerating all cases.

Suppose that the claim is true for $k \leqslant n$ and that $|\sigma| = n + 1$. By the conclusion (1) of the theorem, there exists $w \in A^*$ or $w^{-1} \in A^*$ such that $I_w \circ \sigma$ is simple. To be specific, we can suppose that $w^{-1} \in A^*$, $\sigma = (w_1, w_2, w_3)$ and that $|w_1| = |\sigma|_{\min}$.

If $|w| \leqslant |\sigma|_{\min}$, nothing needs to be proven. Suppose $|w| > |\sigma|_{\min} = |w_1|$. It is easy to see that w_1 is a common prefix of w_1, w_2, w_3. Let then $w_2 = w_1 w_2'$. We have $(w_1, w_2, w_3) = (w_1, w_2', w_2) \circ (a, ab, c)$, that is, $\sigma = \sigma' \circ f$, where $\sigma' = (w_1, w_2', w_2)$ is obviously an invertible substitution and $f = (a, ab, c)$ is a Fibonacci substitution.

It is trivial that $|\sigma'| < |\sigma| = n + 1$ and that $|\sigma'|_{\min} \leqslant |\sigma|_{\min}$, hence by the hypothesis of induction we have the following fact:

There exists $z \in A^*$ (or $z^{-1} \in A^*$) such that $|z| \leqslant |\sigma'|_{\min}$ and that $g := I_z \circ \sigma'$ is simple.

Then it follows that $I_z \circ \sigma = I_z \circ \sigma' \circ f = g \circ f$.

Since $|z| \leqslant |\sigma'|_{\min} \leqslant |\sigma|_{\min}$ and $g \circ f$ is (by definition) simple, the conclusion is proved.

Acknowledgement

The authors would like to thank Professors S. Ito, J. Peyrière, and Z.-Y. Wen for helpful discussions.

References

[1] P. Arnoux et al., Introduction to finite automata and substitution dynamical systems, in: Lecture Notes in Math., Springer-Verlag, to appear.
[2] P. Arnoux, S. Ito, Pisot substitutions and Rauzy fractals, in: Journées Montoises d'Informatique Théorique (Marne-la-Vallée, 2000), Bull. Belgian Math. Soc. Simon Stevin 8 (2) (2001) 181–207.
[3] J. Berstel, Recent results in Sturmian words, in: Developments in Language Theory, II (Magdeburg, 1995), World Sci. Publishing, River Edge, NJ, 1996, pp. 13–24.
[4] A. Cobham, Uniform tag sequences, Math. System Theory 6 (1972) 164–192.
[5] H. Ei, S. Ito, Decomposition theorem for invertible substitution, Osaka J. Math. 34 (1998) 821–834.
[6] M. Lothaire, Algebraic combinatorics on words, Cambridge University Press, 2002, to appear. Available at http://www-igm.univ-mlv.fr/~berstel/Lothaire.
[7] R.C. Lyndon, P.E. Schupp, Combinatorial Group Theory, Spring-Verlag, Berlin, 1977.
[8] W. Magnus, A. Karrass, D. Solitar, Combinatorial Group Theory: Presentations of Groups in Terms of Generators and Relations, Second revised edition, Dover, 1976.
[9] J. Nielsen, Die Isomorphismen der freien Gruppen, Math. Ann. 91 (1924) 169–209.
[10] J. Peyrière, Z.-X. Wen, Z.-Y. Wen, Polynomês associées aux endomorphismes de groupes libres, Enseign. Math. 39 (1993) 153–175.
[11] P. Séébold, Fibonacci morphisms and Sturmian words, Theoret. Comput. Sci. 88 (1991) 365–384.
[12] Z.-X. Wen, Z.-Y. Wen, Local isomorphism of the invertible substitutions, C. R. Acad. Sci. Paris Sér. I Math. 318 (1994) 299–304.
[13] Z.-X. Wen, Z.-Y. Wen, Some properties of the singular words of the Fibonacci word, European J. Combin. 15 (1994) 587–598.
[14] Z.-X. Wen, Z.-Y. Wen, J. Wu, Invertible substitutions and local isomorphisms, C. R. Acad. Sci. Paris Sér. I Math. 334 (2002) 629–634.
[15] Z.X. Wen, Y. Zhang, Some remarks on invertible substitutions on three letter alphabet, Chinese Sci. Bull. 44 (19) (1999) 1755–1760.