

FOURIER INTERPOLATION ON THE REAL LINE

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ABSTRACT

In this paper we construct an explicit interpolation formula for Schwartz functions on the real line. The formula expresses the value of a function at any given point in terms of the values of the function and its Fourier transform on the set $\{0, \pm\sqrt{1}, \pm\sqrt{2}, \pm\sqrt{3}, \dots\}$. The functions in the interpolating basis are constructed in a closed form as an integral transform of weakly holomorphic modular forms for the theta subgroup of the modular group.

1. Introduction

Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be an integrable function and let \widehat{f} be the Fourier transform of f :

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx.$$

The classical Whittaker-Shannon interpolation formula (see [19], [15]) states that if the Fourier transform \widehat{f} is supported in $[-w/2, w/2]$, then

$$(1) \quad f(x) = \sum_{n \in \mathbf{Z}} f(n/w) \operatorname{sinc}(wx - n),$$

where $\operatorname{sinc}(x) = \sin(\pi x)/(\pi x)$ is the cardinal sine function. In other words, the functions $s_n(x) = \operatorname{sinc}(wx - n)$ form an interpolation basis on the set $\frac{1}{w}\mathbf{Z}$ for the space of functions whose Fourier transform is supported in $[-w/2, w/2]$ (the so-called Paley-Wiener space PW_w). For a nice overview of history of the Whittaker-Shannon formula, its generalizations and other related results, see [7].

Note that it is not possible to apply the Whittaker-Shannon formula directly to functions whose Fourier transform \widehat{f} has unbounded support, say, to $f(x) = \exp(-\pi x^2)$. The main goal of this paper is to prove an interpolation formula that can be applied to arbitrary Schwartz functions on the real line.

Theorem 1. — *There exists a collection of even Schwartz functions $a_n: \mathbf{R} \rightarrow \mathbf{R}$ with the property that for any even Schwartz function $f: \mathbf{R} \rightarrow \mathbf{R}$ and any $x \in \mathbf{R}$ we have*

$$(2) \quad f(x) = \sum_{n=0}^{\infty} a_n(x) f(\sqrt{n}) + \sum_{n=0}^{\infty} \widehat{a}_n(x) \widehat{f}(\sqrt{n}),$$

where the right-hand side converges absolutely.

As immediate corollary of Theorem 1, we get the following.

Corollary 1. — *Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be an even Schwartz function that satisfies*

$$f(\sqrt{n}) = \widehat{f}(\sqrt{n}) = 0, \quad n \in \mathbf{Z}_{\geq 0}.$$

Then f vanishes identically.

Denote by \mathfrak{s} the vector space of all rapidly decaying sequences of real numbers, i.e., sequences $(x_n)_{n \geq 0}$ such that for all $k > 0$ we have $n^k x_n \rightarrow 0$, $n \rightarrow \infty$. If we denote by $\mathcal{S}_{\text{even}}$ the space of even Schwartz functions on \mathbf{R} (see Section 6 for a formal definition), then there is a well-defined map $\Psi: \mathcal{S}_{\text{even}} \rightarrow \mathfrak{s} \oplus \mathfrak{s}$ given by

$$\Psi(f) = (f(\sqrt{n}))_{n \geq 0} \oplus (\widehat{f}(\sqrt{n}))_{n \geq 0}.$$

Together with Theorem 1 the following result gives a complete description of what values an even Schwartz function and its Fourier transform can take at $\pm\sqrt{n}$ for $n \geq 0$.

Theorem 2. — *The map Ψ is an isomorphism of the space of even Schwartz functions onto the vector space $\ker L \subset \mathfrak{s} \oplus \mathfrak{s}$, where $L: \mathfrak{s} \oplus \mathfrak{s} \rightarrow \mathbf{R}$ is the linear functional*

$$L((x_n)_{n \geq 0}, (y_n)_{n \geq 0}) = \sum_{n \in \mathbf{Z}} x_{n^2} - \sum_{n \in \mathbf{Z}} y_{n^2}.$$

In the proof of Theorem 1 we will give an explicit construction of the interpolating basis $\{a_n(x)\}_{n \geq 0}$. For instance, the Fourier invariant part of a_n will be given by

$$a_n(x) + \widehat{a}_n(x) = \int_{-1}^1 g_n(z) e^{i\pi x^2 z} dz,$$

where g_n is a certain weakly holomorphic modular form of weight $3/2$, and the integral is over a semicircle in the upper half-plane. The anti-invariant part $a_n(x) - \widehat{a}_n(x)$ will be defined by a similar expression. For an explicit example, we define $a_0(x)$ by

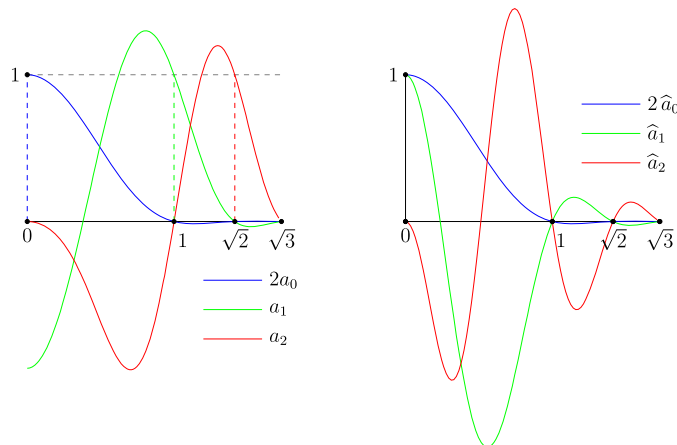
$$a_0(x) = \frac{1}{4} \int_{-1}^1 \theta^3(z) e^{i\pi x^2 z} dz,$$

where $\theta(z)$ is the classical theta series

$$(3) \quad \theta(z) = \sum_{n \in \mathbf{Z}} e^{i\pi n^2 z}.$$

The modular transformation property of g_n is chosen in such a way that it complements the action of the Fourier transform on Gaussian functions:

$$\widehat{e}_z(\xi) = \frac{1}{\sqrt{-iz}} e_{-1/z}(\xi),$$

FIG. 1. — Plots of a_n and \hat{a}_n for $n = 0, 1, 2$

where $e_z(x) = e^{i\pi zx^2}$, and the square root is chosen to be positive when z lies on the imaginary axis (this comment also applies whenever expression $(-iz)^\alpha$ occurs throughout the paper; note that z belongs to the upper half-plane). For instance, using the identity

$$\theta\left(-\frac{1}{z}\right) = \sqrt{-iz}\theta(z)$$

and applying the change of variable $z \mapsto -1/z$ in the integral that defines $a_0(x)$ we see that $\hat{a}_0 = a_0$. The general definition of a_n needs some preparation and will be given in Section 4. The plots of the first three functions are shown in Figure 1.

An analogue of Theorem 1 holds also for odd Schwartz functions, but we postpone its formulation until Section 7. It is possible to combine the two results into a general interpolation theorem, but it is more convenient to work with the two cases separately.

Remark. Another way to interpret equation (2) is to think of it as a “deformation” of the classical Poisson summation formula

$$(4) \quad \sum_{n \in \mathbf{Z}} f(n) = \sum_{n \in \mathbf{Z}} \hat{f}(n),$$

which will be a special case of (2) for $x = 0$ (more precisely, $-a_{n^2}(0) = \hat{a}_{n^2}(0) = 1$ for $n \geq 1$, $a_0(0) = \hat{a}_0(0) = 1/2$, and all other values are zero). Note also that equation (2) gives a continuous family of measures μ_x such that μ_x is a tempered distribution, and both μ_x and $\hat{\mu}_x$ have locally finite support. Such measures are called crystalline measures, for general discussion and some interesting examples see [11], [8].

Our general approach fits into the framework of Eichler cohomology (see [6]; some relevant results can also be found in [9] and [10]) but for the most part we avoid using its general results and terminology. In our case we prefer to obtain explicit estimates by direct methods, and this also allows us to keep the proofs relatively self-contained.

Let us also remark that functions with properties similar to that of a_n have recently been used in [17] and [3] to solve the sphere packing problem in dimensions 8 and 24. The functions constructed there, motivated by the Cohn-Elkies optimization problem [2], were also solutions to a very special case of an interpolation problem closely related to (2) that also involved the values of the first derivative. Similarly, in the Paley-Wiener space, an analogue of (1) for second-order interpolation (i.e., interpolation of values of the function and the values of its first derivative) plays important role in optimization problems of Beurling and Selberg, see [16].

The paper is organized as follows. In Section 2 we recall some known facts about modular forms for the theta group Γ_θ . In Section 3 we compute an explicit basis of a certain space of weakly holomorphic modular forms of weight $3/2$ for the group Γ_θ . Then, in Section 4 we use these modular forms to construct an interpolation basis for the even Schwartz functions and prove some of its properties. In the next section we prove an estimate on the growth of this sequence functions; this is by far the most technical part of the paper. In Section 6 we prove the main result for even functions, and in Section 7 we define the interpolation basis and formulate corresponding statements for the odd functions.

2. The theta group

In this section we set up notation and collect facts about the theta group and related modular forms. Most of the material from this section can be found, in much greater detail, in [13]. For a motivated general introduction to the theory of modular forms, see [21].

2.1. Upper half-plane and the action of $\mathrm{SL}_2(\mathbf{R})$. — Denote by \mathfrak{H} the complex upper half-plane $\{z \in \mathbf{C} : \mathrm{Im}(z) > 0\}$. The group $\mathrm{SL}_2(\mathbf{R})$ of 2×2 matrices with real coefficients and determinant 1 acts on the upper half-plane on the left by Moebius transformations

$$\gamma z = \frac{az + b}{cz + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{R}).$$

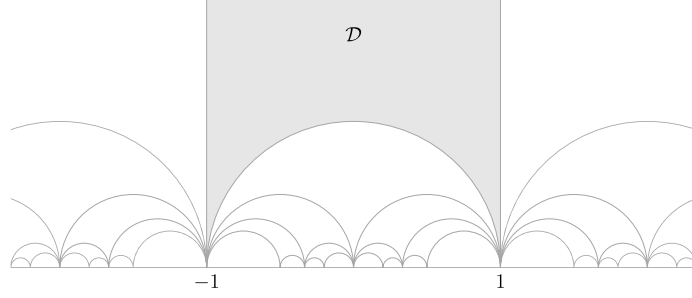
The kernel of this action coincides with the center $\{\pm I\}$ of $\mathrm{SL}_2(\mathbf{R})$ and thus we can work with the action of $\mathrm{PSL}_2(\mathbf{R}) = \mathrm{SL}_2(\mathbf{R})/\{\pm I\}$ instead.

We will use special notation for the following elements of $\mathrm{SL}_2(\mathbf{Z})$ (or, by abuse of notation, of $\mathrm{PSL}_2(\mathbf{Z})$):

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Recall that $\Gamma(2) \subset \mathrm{SL}_2(\mathbf{Z})$ is defined as

$$\Gamma(2) = \left\{ A \in \mathrm{SL}_2(\mathbf{Z}) \mid A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{2} \right\},$$

FIG. 2. — Fundamental domain for Γ_θ

and the theta group Γ_θ is the subgroup of $\mathrm{SL}_2(\mathbf{Z})$ generated by S and T^2 , or, equivalently,

$$\Gamma_\theta = \left\{ A \in \mathrm{SL}_2(\mathbf{Z}) \mid A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pmod{2} \right\}.$$

Note the obvious inclusions $\mathrm{SL}_2(\mathbf{Z}) \supset \Gamma_\theta \supset \Gamma(2)$. The group $\Gamma(2)$ has three cusps 0 , 1 , and ∞ , while the group Γ_θ has only two cusps: 1 and ∞ . The standard fundamental domain for the theta group is (see Figure 2)

$$(5) \quad \mathcal{D} = \{ \tau \in \mathfrak{H} : |\tau| > 1, \operatorname{Re}(\tau) \in (-1, 1) \}.$$

Finally, we are going to use the “ θ -automorphy factor” on the group Γ_θ , which we define for all $z \in \mathfrak{H}$ and $\gamma \in \Gamma_\theta$ by

$$(6) \quad j_\theta(z, \gamma) = \frac{\theta(z)}{\theta(\gamma z)}.$$

From the definition it immediately follows that $j_\theta(z, \gamma_1 \gamma_2) = j_\theta(z, \gamma_2) j_\theta(\gamma_2 z, \gamma_1)$, so j_θ is indeed an automorphy factor on Γ_θ . We have $j_\theta(z, T^2) = 1$ and $j_\theta(z, S) = (-iz)^{-1/2}$, and in general we have $j_\theta(z, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \zeta \cdot (cz + d)^{-1/2}$ for some suitable 8-th root of unity ζ (an explicit expression for ζ can be found in [13, Th. 7.1]). Using this automorphy factor we define the following slash operator in weight $k/2$ (that acts on holomorphic functions defined on the upper half-plane \mathfrak{H})

$$(7) \quad (f|_{k/2} A)(z) = j_\theta(z, A)^{kf} \left(\frac{az + b}{cz + d} \right),$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\theta$. More generally, for $\varepsilon \in \{-, +\}$ define a slash operator $|_{k/2}^\varepsilon$ by

$$(8) \quad f|_{k/2}^\varepsilon A = \chi_\varepsilon(A) f|_{k/2} A,$$

where $\chi_\varepsilon : \Gamma_\theta \rightarrow \{\pm 1\}$ is the homomorphism defined by $\chi_\varepsilon(S) = \varepsilon$ and $\chi_\varepsilon(T^2) = 1$. The slash operator defines a group action, that is, $f|AB = (f|A)|B$. Another fact that we will

use is that for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{R})$ we have

$$(9) \quad \mathrm{Im}\left(\frac{a\tau + b}{c\tau + d}\right) = \frac{\mathrm{Im}(\tau)}{|c\tau + d|^2}.$$

For any real number a we will denote by q^a the analytic function

$$q^a = q^a(z) = \exp(2\pi i az).$$

Any N -periodic holomorphic function on \mathfrak{H} admits an expansion in powers of $q^{1/N}$ (in general as a Laurent series, but in our case such expansions will have only finitely many negative powers). We will be using subscripts to indicate the main variable of q , i.e., q_τ^a is the same as $q^a(\tau)$; by default the variable of q^a is z .

2.2. Modular forms for the group Γ_θ . — We begin by defining the classical Jacobi theta series (the so-called Thetanullwerte):

$$\begin{aligned} \Theta_2(z) &= \sum_{n \in \mathbf{Z} + \frac{1}{2}} q^{\frac{1}{2}n^2} = 2 \frac{\eta(2z)^2}{\eta(z)}, \\ \Theta_3(z) &= \sum_{n \in \mathbf{Z}} q^{\frac{1}{2}n^2} = \frac{\eta(z)^5}{\eta(z/2)^2 \eta(2z)^2} \quad (= \theta(z)), \\ \Theta_4(z) &= \sum_{n \in \mathbf{Z}} (-1)^n q^{\frac{1}{2}n^2} = \frac{\eta(z/2)^2}{\eta(z)}, \end{aligned}$$

where $\eta(z) = q^{1/24} \prod_{n \geq 1} (1 - q^n)$ is the Dedekind eta function. The functions Θ_2^4 , Θ_3^4 , and Θ_4^4 generate the ring of holomorphic modular forms on $\Gamma(2)$ and satisfy the Jacobi identity

$$(10) \quad \Theta_3^4 = \Theta_2^4 + \Theta_4^4.$$

The q -expansions of these forms at the cusp $i\infty$ are as follows:

$$\begin{aligned} \Theta_2^4(z) &= 16q^{1/2} + 64q^{3/2} + 96q^{5/2} + O(q^3), \\ \Theta_3^4(z) &= 1 + 8q^{1/2} + 24q + 32q^{3/2} + 24q^2 + 48q^{5/2} + O(q^3), \\ \Theta_4^4(z) &= 1 - 8q^{1/2} + 24q - 32q^{3/2} + 24q^2 - 48q^{5/2} + O(q^3). \end{aligned}$$

Under the action of $\mathrm{SL}_2(\mathbf{Z})$ the theta functions transform as follows. Under the action of S we have

$$(11) \quad \begin{aligned} (-iz)^{-1/2} \Theta_2(-1/z) &= \Theta_4(z), \\ (-iz)^{-1/2} \Theta_3(-1/z) &= \Theta_3(z), \\ (-iz)^{-1/2} \Theta_4(-1/z) &= \Theta_2(z), \end{aligned}$$

and under the action of T we have

$$\begin{aligned} \Theta_2(z+1) &= e^{i\pi/4} \Theta_2(z), \\ (12) \quad \Theta_3(z+1) &= \Theta_4(z), \\ \Theta_4(z+1) &= \Theta_3(z) \end{aligned}$$

Together with the q -series for Θ_2 , Θ_3 , and Θ_4 , these transformations allow us to compute the q -series expansion of any expression in theta functions at any of the three cusps of $\Gamma(2)$.

Using these theta functions we can define the classical modular lambda invariant

$$\lambda(z) = \frac{\Theta_2^4(z)}{\Theta_3^4(z)} = 16q^{1/2} - 128q + 704q^{3/2} + \dots,$$

which is a Hauptmodul for $\Gamma(2)$. In particular, we have

$$\lambda\left(\frac{az+b}{cz+d}\right) = \lambda(z), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{2},$$

and any meromorphic function with these transformation properties and with appropriate behavior at the cusps can be expressed as a rational function of λ . From (10)–(12) we see that under the action of $\text{PSL}_2(\mathbf{Z})$ the function $\lambda(z)$ transforms as follows:

$$\begin{aligned} (13) \quad \lambda\left(-\frac{1}{z}\right) &= 1 - \lambda(z), \\ \lambda(z+1) &= \frac{\lambda(z)}{\lambda(z) - 1}. \end{aligned}$$

Since Θ_3 , Θ_2 , and Θ_4 do not vanish in \mathfrak{H} (by the product expression in terms of $\eta(z)$), we get the well-known fact that $\lambda(z)$ omits the values 0 and 1.

Using $\lambda(z)$, define a Hauptmodul J for the group Γ_θ

$$(14) \quad J(z) = \frac{1}{16} \lambda(z) (1 - \lambda(z)) = \frac{\Theta_2^4(z) \Theta_4^4(z)}{16 \Theta_3^8(z)} = q^{1/2} - 24q + 300q^{3/2} + \dots$$

Note that $J(z) = \eta(z/2)^{24} \eta(2z)^{24} \eta(z)^{-48}$, hence it does not have zeros in \mathfrak{H} . This function satisfies the transformation laws

$$\begin{aligned} J\left(-\frac{1}{z}\right) &= J(z), \\ J(z+2) &= J(z), \end{aligned}$$

and it maps the fundamental domain \mathcal{D} conformally onto the cut plane $\mathbf{C} \setminus [1/64, +\infty)$. Finally, note that $1/J$ vanishes at the cusp 1 since a simple calculation shows that

$$(15) \quad \frac{1}{J(1 - 1/z)} = -4096q - 98304q^2 + O(q^3).$$

2.3. Asymptotic notation. — We freely use the standard big O notation. In addition, we also use Vinogradov's “ \ll ” sign

$$f \ll_{\varepsilon, \delta, \dots} g \quad \Leftrightarrow \quad f = O_{\varepsilon, \delta, \dots}(g).$$

Notationally, we prefer to use “ O ” for sequences and additive remainders, while for most inequalities with implied constants we use “ \ll ”.

3. Weakly holomorphic modular forms on Γ_θ of weight $3/2$

We begin by constructing a basis for a certain space of weakly holomorphic modular forms of weight $3/2$. Namely, let $\{g_n^+(z)\}_{n \geq 0}$ and $\{g_n^-(z)\}_{n \geq 1}$ be two collections of holomorphic functions on the upper half-plane \mathfrak{H} that satisfy the transformation properties

$$(16) \quad \begin{aligned} g_n^\varepsilon(z+2) &= g_n^\varepsilon(z), \\ (-iz)^{-3/2} g_n^\varepsilon(-1/z) &= \varepsilon g_n^\varepsilon(z), \end{aligned}$$

as well as the following behavior at the cusps

$$(17) \quad \begin{aligned} g_n^+(z) &= q^{-n/2} + O(q^{1/2}), \quad z \rightarrow i\infty, \\ g_n^-(z) &= q^{-n/2} + O(1), \quad z \rightarrow i\infty, \\ g_n^\varepsilon(1+i/t) &\rightarrow 0, \quad t \rightarrow \infty. \end{aligned}$$

The reason behind these conditions will be made clear in the next section. We make the following ansatz:

$$(18) \quad \begin{aligned} g_n^+(z) &= \theta^3(z) P_n^+(J^{-1}(z)), \\ g_n^-(z) &= \theta^3(z) (1 - 2\lambda(z)) P_n^-(J^{-1}(z)), \end{aligned}$$

where $P_n^\pm \in \mathbf{Q}[x]$ are monic polynomials of degree n and $P_n^-(0) = 0$. The polynomials P_n^\pm are uniquely determined by the first two conditions in (17), since J^{-1} has q -expansion starting with $q^{-1/2} + 24 + O(q^{1/2})$, and thus the coefficients of P_n^\pm can be found by inverting an upper-triangular matrix. The transformation properties (16) follow from the properties of $J(z)$ and $\lambda(z)$. The first few of these functions are

$$\begin{aligned} g_0^+ &= \theta^3, & g_1^- &= \theta^3 \cdot (1 - 2\lambda) \cdot (J^{-1}), \\ g_1^+ &= \theta^3 \cdot (J^{-1} - 30), & g_2^- &= \theta^3 \cdot (1 - 2\lambda) \cdot (J^{-2} - 22J^{-1}), \\ g_2^+ &= \theta^3 \cdot (J^{-2} - 54J^{-1} + 192), & g_3^- &= \theta^3 \cdot (1 - 2\lambda) \\ & & & \cdot (J^{-3} - 46J^{-2} + 252J^{-1}). \end{aligned}$$

Note that the polynomials P_n^+ are the Faber polynomials associated to the function $1/J$, viewed as a function on the unit disk (see [4]). In the next theorem we give closed form expressions for generating functions of $\{g_n^\pm\}$.

Theorem 3. — *The generating functions for $\{g_n^+(z)\}_{n \geq 0}$ and $\{g_n^-(z)\}_{n \geq 1}$ are given by*

$$(19) \quad \begin{aligned} \sum_{n=0}^{\infty} g_n^+(z) e^{i\pi n \tau} &= \frac{\theta(\tau)(1 - 2\lambda(\tau))\theta^3(z)J(z)}{J(z) - J(\tau)} =: K_+(\tau, z), \\ \sum_{n=1}^{\infty} g_n^-(z) e^{i\pi n \tau} &= \frac{\theta(\tau)J(\tau)\theta^3(z)(1 - 2\lambda(z))}{J(z) - J(\tau)} =: K_-(\tau, z). \end{aligned}$$

Here $K_\pm(\tau, z)$ is a meromorphic function with poles at $\tau \in \Gamma_\theta z$, and the series on the left-hand side converges for all τ with large enough $\text{Im}(\tau)$.

Proof. — The proof follows the same lines as the proof of Theorem 2 from [5]. We only prove the statement for g_n^+ , since the case of g_n^- is almost identical. From the q -expansion of J^{-1} and the fact that

$$\frac{J(z)}{J(z) - J(\tau)} = \sum_{n \geq 0} J^n(\tau) J^{-n}(z),$$

it is clear that the g_n^+ defined by (19) are also of the form $\theta^3(z)P_n(J^{-1}(z))$ for some monic polynomial P_n of degree n . The only thing that we need to check is that they satisfy

$$g_n^+(z) = q^{-n/2} + O(q^{1/2}), \quad z \rightarrow i\infty,$$

or, equivalently, that $P_n = P_n^+$. By Cauchy's theorem we know that

$$g_n^+(z) = \frac{1}{2} \int_{\tau_0}^{\tau_0+2} K_+(\tau, z) q_\tau^{-n/2} d\tau = \frac{1}{2\pi i} \oint_C K_+(\tau, z) q_\tau^{-(n+1)/2} d(q_\tau^{1/2}),$$

where $\tau_0 \in \mathfrak{H}$ has sufficiently large imaginary part and C is a small enough loop around 0 in the $q_\tau^{1/2}$ -plane. Using the identity

$$(20) \quad q_\tau^{1/2} \frac{dJ}{d(q_\tau^{1/2})}(\tau) = \frac{J'(\tau)}{\pi i} = \theta^4(\tau)(1 - 2\lambda(\tau))J(\tau)$$

we get that

$$K_+(\tau, z) = \frac{q_\tau^{1/2} \frac{dJ}{d(q_\tau^{1/2})}(\tau)}{J(z) - J(\tau)} \cdot \frac{\theta^3(z)J(z)}{\theta^3(\tau)J(\tau)},$$

and thus changing the variable of integration we get

$$g_n^+(z) = \frac{1}{2\pi i} \oint_{\tilde{C}} \frac{(q_\tau^{1/2}(j))^{-n}}{J(z) - j} \cdot \frac{\theta^3(z)J(z)}{\theta^3(\tau)j} dj.$$

(We write $q_\tau^{1/2}(j)$ to emphasized dependence on j .) Now recall that $\theta^3(z)P_n^+(J^{-1}(z)) = q^{-n/2} + O(q^{1/2})$, so that $(\theta^3(\tau)P_n^+(j^{-1}) - q_\tau^{-n/2}(j))/j$ is holomorphic in some small neighborhood of 0 in the j -plane. Therefore, for some small loop \tilde{C} around zero, we have

$$\begin{aligned} g_n^+(z) &= \frac{1}{2\pi i} \oint_{\tilde{C}} \frac{(q_\tau^{1/2}(j))^{-n}}{J(z) - j} \cdot \frac{\theta^3(z)J(z)}{\theta^3(\tau)j} dj = \frac{\theta^3(z)}{2\pi i} \oint_{\tilde{C}} \frac{P_n^+(j^{-1})}{j(J(z) - j)} J(z) dj \\ &= -\frac{\theta^3(z)}{2\pi i} \oint_{\tilde{C}} \frac{P_n^+(j^{-1})}{(j^{-1} - J^{-1}(z))} dj^{-1} = \theta^3(z)P_n^+(J^{-1}(z)). \end{aligned}$$

The last sign is changed since the contour for j^{-1} in the last application of Cauchy's formula has the opposite orientation. \square

Remark. From (20) it also follows that $K_\varepsilon(\tau, z)$ has a simple pole at $z = \tau$ with residue $\frac{1}{i\pi}$ for all $\tau \in \mathfrak{H}$. We also record here the following identities for K_ε :

$$\begin{aligned} (21) \quad K_\varepsilon(\tau, -1/z) &= \varepsilon(-iz)^{3/2} K_\varepsilon(\tau, z), \\ K_\varepsilon(-1/\tau, z) &= -\varepsilon(-i\tau)^{1/2} K_\varepsilon(\tau, z). \end{aligned}$$

Note that generating functions very similar to (19) have also been used in [20] in the computation of traces of singular moduli.

4. Interpolation basis for even functions

Let us define a function $b_m^\varepsilon: \mathbf{R} \rightarrow \mathbf{R}$ by the integral

$$(22) \quad b_m^\varepsilon(x) = \frac{1}{2} \int_{-1}^1 g_m^\varepsilon(z) e^{i\pi x^2 z} dz,$$

where the path of integration is chosen to lie in the upper half-plane and orthogonal to the real line at the endpoints 1 and -1 . Since g_m^ε has exponential decay at ± 1 , the above integral converges. Note that b_m^ε is defined for $m \geq 0$ if $\varepsilon = +1$ and for $m \geq 1$ if $\varepsilon = -1$; for convenience let us also define $b_0^-(x) = 0$.

Recall that Schwartz functions are C^∞ -smooth functions that, together with all of their derivatives, decay faster than any inverse power of x .

Proposition 1. — *The function $b_m^\varepsilon: \mathbf{R} \rightarrow \mathbf{R}$ is an even Schwartz function that satisfies*

$$\widehat{b}_m^\varepsilon(x) = \varepsilon b_m^\varepsilon(x)$$

and

$$b_m^\varepsilon(\sqrt{n}) = \delta_{n,m}, \quad n \geq 1, m \geq 0,$$

where $\delta_{n,m}$ is the Kronecker delta. In addition, we have $b_0^+(0) = 1$.

Proof. — Clearly, b_m^ε is an even function, since $e_z(x) = e^{i\pi x^2 z}$ is even. That it indeed takes real values for $x \in \mathbf{R}$ can be seen by taking the integral over the semicircle $z = e^{it}$, $t \in (0, \pi)$, making a change of variables $z \mapsto -\bar{z}$, and noting that $\overline{g_m^\varepsilon(z)} = g_m^\varepsilon(-\bar{z})$. Let us prove that b_m^ε belongs to the Schwartz class. We will only consider the case “ $\varepsilon = +$ ”, but the same argument will work also in the case “ $\varepsilon = -$ ”. Since $g_n^+(z) = \theta^3(z)P_n^+(J^{-1}(z))$, it is enough to prove that for each $n \in \mathbf{N}$ the integral

$$\beta_n(x) = \frac{1}{2} \int_{-1}^1 \theta^3(z) J^{-n}(z) e^{i\pi x^2 z} dz$$

is a Schwartz function. On the circle arc from -1 to 1 the function $1/J(z)$ takes real values between 0 and 64 , and moreover

$$J^{-1}(\pm 1 + i/t) \leq C \exp(-2\pi t), \quad t \rightarrow \infty, \operatorname{Re}(t) > 0.$$

By taking the k -th derivative of $\beta_n(x)$ with respect to x under the integral we obtain

$$\beta_n^{(k)}(x) = \frac{1}{2} \int_{-1}^1 \theta^3(z) J^{-n}(z) Q_k(x, z) e^{i\pi x^2 z} dz,$$

where $Q_k(x, z)$ are polynomials defined by

$$(23) \quad \frac{\partial^k}{\partial x^k} e^{i\pi x^2 z} = Q_k(x, z) e^{i\pi x^2 z}.$$

Clearly, there exists a constant C_k such that

$$|Q_k(x, z)| \leq C_k (1 + |x|^2)^k (1 + |z|^2)^k,$$

thus we get

$$|\beta_n^{(k)}(x)| \leq \pi 2^{k+3} C_k (1 + |x|^2)^k \int_0^{1/2} J^{-n}(e^{i\pi t}) e^{-\pi x^2 \sin(\pi t)} dt.$$

Here we used a rather crude estimate $|\theta(e^{i\pi t})| < 2$ for $t \in (0, 1/2)$. When $|x|$ is small, we estimate the above integral by 64^n , for all other values of x we estimate the integral by splitting it into two parts (where we take $\delta = (\sqrt{\pi}x)^{-1}$):

$$\begin{aligned}
& \int_0^{1/2} J^{-n}(e^{i\pi t}) e^{-\pi x^2 \sin(\pi t)} dt \\
&= \int_0^\delta J^{-n}(e^{i\pi t}) e^{-\pi x^2 \sin(\pi t)} dt + \int_\delta^{1/2} J^{-n}(e^{i\pi t}) e^{-\pi x^2 \sin(\pi t)} dt \\
&\leq C\delta e^{-2/\delta} + 64^n e^{-2\pi\delta x^2} = e^{-2\sqrt{\pi}x} (64^n + C/(x\sqrt{\pi})),
\end{aligned}$$

from which it follows that β_n is a Schwartz function.

To check that $\widehat{b}_m^\varepsilon = \varepsilon b_m^\varepsilon$ we will use the fact that $\widehat{e}_z = (-iz)^{-1/2} e_{-1/z}$ and the transformation property (16):

$$\begin{aligned}
\widehat{b}_m^\varepsilon(x) &= \frac{1}{2} \int_{-1}^1 g_m^\varepsilon(z) (-iz)^{-1/2} e^{i\pi x^2 (-1/z)} dz \\
&= \frac{1}{2} \int_{-1}^1 -g_m^\varepsilon(z) (-iz)^{3/2} e^{i\pi x^2 (-1/z)} d(-1/z) \\
&= \frac{1}{2} \int_1^{-1} \varepsilon g_m^\varepsilon(-1/z) e^{i\pi x^2 (-1/z)} d(-1/z) = \varepsilon b_m^\varepsilon(x).
\end{aligned}$$

In the above computations we always choose the branch of $(-iz)^{k/2}$ that takes positive values for z on the imaginary semiaxis. Finally, note that

$$b_m^\varepsilon(\sqrt{n}) = \frac{1}{2} \int_{-1}^1 g_m^\varepsilon(z) e^{i\pi n z} dz$$

is simply the coefficient of $q^{-n/2}$ in the q -expansion of g_m^ε , so that (17) immediately implies $b_m^\varepsilon(\sqrt{n}) = \delta_{n,m}$ and $b_0^+(0) = 1$. \square

Remark. Note that (17) also implies that $b_m^+(0) = \delta_{m,0}$, and using the explicit formula (19) for the kernel K_- , we also get

$$b_m^-(0) = \begin{cases} -2, & m \geq 1 \text{ is a square,} \\ 0, & \text{otherwise.} \end{cases}$$

Alternatively, this last equation follows from the Poisson summation formula

$$\sum_{n \in \mathbf{Z}} b_m^-(n) = \sum_{n \in \mathbf{Z}} \widehat{b}_m^-(n) = - \sum_{n \in \mathbf{Z}} b_m^-(n).$$

To establish other properties of the sequences $\{b_m^\varepsilon(x)\}_m$ we will need to work with generating functions. Let \mathcal{D} be the standard fundamental domain for the group Γ_θ (as defined in (5)). For a fixed x define a function $F_\varepsilon(\tau, x)$ on the set

$$\{\tau \in \mathfrak{H} : \forall k \in \mathbf{Z}, |\tau - 2k| > 1\} \supset \mathcal{D} + 2\mathbf{Z}$$

by

$$(24) \quad F_\varepsilon(\tau, x) = \frac{1}{2} \int_{-1}^1 K_\varepsilon(\tau, z) e^{i\pi x^2 z} dz,$$

where the contour is the semicircle in the upper half-plane that passes through -1 and 1 . Note that for $\text{Im}(\tau) > 1$ we have

$$(25) \quad F_\varepsilon(\tau, x) = \sum_{n=0}^{\infty} b_n^\varepsilon(x) e^{i\pi n \tau},$$

and the series converges absolutely. Our next task is to show that F_ε can be analytically continued to \mathfrak{H} (and hence (25) also holds for all $\tau \in \mathfrak{H}$).

Proposition 2. — For any $\varepsilon \in \{+, -\}$ and $x \in \mathbf{R}$ the function $F_\varepsilon(\tau, x)$ admits an analytic continuation to \mathfrak{H} . Moreover, the analytic continuation satisfies the functional equations

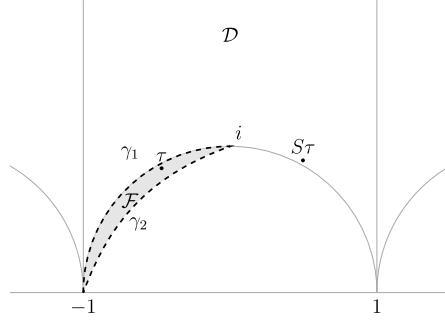
$$(26) \quad \begin{aligned} F_\varepsilon(\tau, x) - F_\varepsilon(\tau + 2, x) &= 0, \\ F_\varepsilon(\tau, x) + \varepsilon(-i\tau)^{-1/2} F_\varepsilon\left(-\frac{1}{\tau}, x\right) &= e^{i\pi \tau x^2} + \varepsilon(-i\tau)^{-1/2} e^{i\pi(-1/\tau)x^2}. \end{aligned}$$

Proof. — To prove the theorem, it is enough to show that there exists an analytic continuation to some open set Ω containing the relative closure of \mathcal{D} , on which (26) holds. Indeed, we can then choose a smaller Ω in such a way that $\Omega \cap \gamma^{-1}(\Omega) \neq \emptyset$ if and only if $\gamma \in \{T^2, T^{-2}, S, I\}$. Since $\bigcup_{g \in \Gamma_\theta} g\Omega = \mathfrak{H}$, we can construct a continuation inductively by repeatedly using (26) to pass to the neighboring sets $g\Omega$ that have not been covered yet. Since Γ_θ is generated by S and T^2 , and the only relation is $S^2 = 1$, this process indeed gives a single-valued analytic continuation (the main reason is that there are no cycles of neighboring domains; this is also clear from Figure 2).

The first functional equation in (26) is clearly satisfied, since the integral that defines F_ε automatically defines a 2-periodic function on the open set $\{\tau \in \mathfrak{H} : \forall k \in \mathbf{Z}, |\tau - 2k| > 1\}$ that contains the vertical lines $\text{Im}(\tau) = \pm 1$.

Hence, we only need to deal with the second functional equation. We can get an analytic continuation of F_ε to some neighborhood of $\{z \in \mathfrak{H} : |z| = 1, z \neq i\}$ by changing the contour of integration in (24). First, we rewrite the integral as

$$(27) \quad \begin{aligned} 2F_\varepsilon(\tau, x) &= \int_{-1}^i K_\varepsilon(\tau, z) e^{i\pi x^2 z} dz + \int_i^1 K_\varepsilon(\tau, z) e^{i\pi x^2 z} dz \\ &= \int_{-1}^i K_\varepsilon(\tau, z) e^{i\pi x^2 z} dz - \int_{-1}^i K_\varepsilon(\tau, -1/z) e^{i\pi x^2(-1/z)} z^{-2} dz \\ &= \int_{-1}^i K_\varepsilon(\tau, z) (e^{i\pi x^2 z} + \varepsilon(-iz)^{-1/2} e^{i\pi x^2(-1/z)}) dz, \end{aligned}$$

FIG. 3. — Fundamental domain for Γ_θ and the contour of integration

where we have used the transformation property (21). Note, that if τ belongs to $\overline{D} \cup \overline{SD}$, then the only poles of $K_\varepsilon(\tau, z)$ (as a function of z) inside $\overline{D} \cup \overline{SD}$ are at $z = \tau$ and $z = -1/\tau$. Let γ_1 be the circle arc from -1 to i , and let γ_2 be a simple smooth path from -1 to i that lies inside \overline{SD} and strictly below γ_1 . Denote by \mathcal{F} the region enclosed between γ_1 and γ_2 (see Figure 3). We will now build a continuation of F_ε to \mathcal{F} and show that it satisfies the functional equation. We define a continuation by the contour integral

$$\tilde{F}_\varepsilon(\tau, x) = \frac{1}{2} \int_{\gamma_2} K_\varepsilon(\tau, z) (e^{i\pi x^2 z} + \varepsilon(-iz)^{-1/2} e^{i\pi x^2(-1/z)}) dz.$$

Clearly, $F_\varepsilon = \tilde{F}_\varepsilon$ for τ with big enough imaginary part, so \tilde{F} indeed defines an analytic continuation to \mathcal{F} . For $\tau \in \mathcal{F}$ we compute

$$\begin{aligned} & \tilde{F}_\varepsilon(\tau, x) + \frac{\varepsilon}{\sqrt{-i\tau}} F_\varepsilon\left(-\frac{1}{\tau}, x\right) \\ &= \tilde{F}_\varepsilon(\tau, x) - \frac{1}{2} \int_{\gamma_1} \frac{\varepsilon K_\varepsilon(-1/\tau, z)}{-\sqrt{-i\tau}} \left(e^{i\pi x^2 z} + \frac{\varepsilon e^{i\pi x^2(-1/z)}}{\sqrt{-iz}} \right) dz \\ &= \tilde{F}_\varepsilon(\tau, x) - \frac{1}{2} \int_{\gamma_1} K_\varepsilon(\tau, z) (e^{i\pi x^2 z} + \varepsilon(-iz)^{-1/2} e^{i\pi x^2(-1/z)}) dz \\ &= \frac{1}{2} \int_{\partial\mathcal{F}} K_\varepsilon(\tau, z) (e^{i\pi x^2 z} + \varepsilon(-iz)^{-1/2} e^{i\pi x^2(-1/z)}) dz \\ &= i\pi \sum_{w \in \mathcal{F}} \text{Res}_{z=w} (K_\varepsilon(\tau, z) (e^{i\pi x^2 z} + \varepsilon(-iz)^{-1/2} e^{i\pi x^2(-1/z)})) \\ &= e^{i\pi x^2 \tau} + \varepsilon(-i\tau)^{-1/2} e^{i\pi x^2(-1/\tau)}, \end{aligned}$$

which is precisely the functional equation that we needed. Similar computation works for the arc from i to 1 . The only thing that is left is to check that F_ε has no singularity at $\tau = i$. For $\varepsilon = 1$ this follows from the second functional equation, while for $\varepsilon = -1$ both

$2\lambda(z) - 1$ and $e^{i\pi z^2} + \varepsilon(-iz)^{-1/2} e^{i\pi(-1/z)^2}$ vanish at $z = i$, so that they cancel the double pole at i coming from $J(i) - J(z)$, and hence the integral (27) converges at $\tau = i$. \square

As an immediate corollary, we obtain that formula (25) is valid for all $\tau \in \mathfrak{H}$. This already implies that for all $\delta > 0$ we have $b_n^\varepsilon(x) = O((1 + \delta)^n)$. In the next section we prove a much stronger estimate.

Note that the only properties of K_ε that were used in the proof are the modularity in τ and in z , as well as the fact that the only poles are at $z \in \Gamma_\theta \tau$, and that the residue at $z = \tau$ is equal to $1/(i\pi)$.

5. Growth estimate

The main result of this section is the following.

Theorem 4. — *For any $\varepsilon \in \{+, -\}$ the numbers $b_n^\varepsilon(x)$ satisfy*

$$|b_n^\varepsilon(x)| = O(n^2)$$

uniformly in x .

To prove this we will use the following general result that goes back to Hecke (see, for example, [1, Lemma 2.2, (ii)]).

Lemma 1. — *If a 2-periodic analytic function $f: \mathfrak{H} \rightarrow \mathbf{C}$ has a Fourier expansion $f(\tau) = \sum_{n \geq 0} a_n e^{i\pi n \tau}$ and for some $\alpha > 0$ it satisfies*

$$|f(\tau)| \leq C \operatorname{Im}(\tau)^{-\alpha} \text{ for } \operatorname{Im}(\tau) < c,$$

then for all sufficiently large n we have

$$|a_n| \leq C \left(\frac{e\pi}{\alpha} \right)^\alpha n^\alpha.$$

To prove Theorem 4 we will apply this lemma to the generating function $F_\varepsilon(\tau, x)$. To simplify notation, we will write $F_\varepsilon(\tau)$ instead of $F_\varepsilon(\tau, x)$. The estimate of $|F_\varepsilon(\tau)|$ naturally splits into two parts: combinatorial (estimating $F_\varepsilon(\tau) - (F_\varepsilon|_A)(\tau)$ using functional equations) and analytic (estimating $F_\varepsilon(\tau)$ using the defining contour integral).

To deal with the first part, we define functions $\phi_A(\tau)$ for $A \in \Gamma_\theta$:

$$(28) \quad \phi_A(\tau) := F_\varepsilon(\tau) - (F_\varepsilon|_{1/2}^\varepsilon A)(\tau).$$

From the functional equations (26) for F_ε we have

$$(29) \quad \begin{aligned} \phi_{T^2}(\tau) &= 0, \\ \phi_S(\tau) &= e^{i\pi x^2 \tau} + \varepsilon(-i\tau)^{-1/2} e^{i\pi x^2(-1/\tau)}. \end{aligned}$$

Moreover, the functions ϕ_A satisfy the cocycle relation $\phi_{AB} = \phi_B + \phi_A|B$ (where we write $|$ for $|\cdot|_{1/2}^{-\varepsilon}$). In other words, the collection $\{\phi_A\}_{A \in \Gamma_\theta}$ forms what is usually called a Γ_θ -cocycle (see, for example, [10]).

First, we need the following elementary lemma.

Lemma 2. — *For any $\tau \in \mathfrak{H}$ with $|\tau| \geq 1$ and any sequence of non-zero integers $\{n_j\}_{j \geq 1}$ define a sequence of numbers $\tau_j \in \mathfrak{H}$ as follows:*

$$\begin{aligned} \tau_0 &= \tau, \\ \tau_j &= 2n_j - \frac{1}{\tau_{j-1}}, \quad j \geq 1. \end{aligned}$$

Then the sequence $\{\text{Im}(\tau_j)\}_{j \geq 0}$ is strictly decreasing and $\text{Im}(\tau_j) \leq \frac{1}{2j-1}$ for all $j \geq 1$.

Proof. — First, observe that $|\tau_j| > 1$ for all $j \geq 1$ (the proof is by induction). The inequality $\text{Im}(\tau_j) \geq \text{Im}(\tau_{j+1})$ then follows from $\text{Im}(\tau_{j+1}) = \text{Im}(\tau_j)/|\tau_j|^2 < \text{Im}(\tau_j)$.

For $a, b \in \mathbf{R}$ denote by $D(a, b)$ the half-disk with center $(a+b)/2$ whose boundary semicircle passes through a and b . Let D be any such half-disk that does not intersect $D(-1, 1)$ and set $D' = SD$. Then a simple calculation shows that

$$\text{diam}(D') \leq \frac{\text{diam}(D)}{1 + \text{diam}(D)}.$$

Note that $\tau_1 \in \bigcup_{n \neq 0} D(2n-1, 2n+1)$, so τ_1 lies in some half-disk of diameter 2. Denote this half-disk by D_1 , and define $D_{j+1} = 2n_j + SD_j$. Then $\tau_j \in D_j$ and no D_j intersects $D(-1, 1)$. By repeatedly applying the above inequality we get that D_j has diameter at most $2/(2j-1)$, thus $\text{Im}(\tau_j) \leq 1/(2j-1)$. \square

The following lemma allows us to estimate values of certain cocycles.

Lemma 3. — *Let $\{\psi_A\}_{A \in \Gamma_\theta}$ be a cocycle (with respect to $| := |\cdot|_{k/2}^{-\varepsilon}$) such that*

$$\begin{aligned} \psi_{T^2} &= 0, \\ |\psi_S(\tau)| &\leq |\tau|^\alpha + \text{Im}(\tau)^{-\beta} \end{aligned}$$

for some $\alpha, \beta \geq 0$. Let $\tau' \in \mathcal{D}$, $A \in \Gamma_\theta$, and $\tau = A\tau' \in \mathfrak{H}$ and suppose that $\text{Im}(\tau) \leq 1$. Then

$$|\psi_A(\tau')| \leq |\tau|^\alpha + \text{Im}(\tau)^{-\alpha-1} + 2 \text{Im}(\tau)^{-\beta-1}.$$

Proof. — Let us consider the case when

$$A = ST^{2n_m}ST^{2n_{m-1}}S \dots T^{2n_1}S.$$

By applying the cocycle relation repeatedly, we get that

$$\psi_A = \psi_S + \psi_S|A_1 + \psi_S|A_2 + \cdots + \psi_S|A_m,$$

where we write $A_j = T^{2n_j}S \dots T^{2n_1}S$. Hence

$$|\psi_A(\tau')| \leq \sum_{j=0}^m \frac{|\psi_S(\tau_j)|}{|c_j\tau' + d_j|^k},$$

where $A_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$ and τ_j are defined by

$$\begin{cases} \tau_0 = \tau', \\ \tau_j = 2n_j - 1/\tau_{j-1}. \end{cases}$$

Under these definitions $\tau_j = \frac{a_j\tau' + b_j}{c_j\tau' + d_j}$ and $\tau = -1/\tau_m$. Multiplying both sides of the above inequality by $\text{Im}(\tau')^{k/2}$ we get

$$\text{Im}(\tau')^{k/2} |\psi_A(\tau')| \leq \sum_{j=0}^m \text{Im}(\tau_j)^{k/2} |\psi_S(\tau_j)|$$

Lemma 2 implies that $\text{Im}(\tau)^{-1} \geq 2m - 1$ and $\text{Im}(\tau_j) \geq \text{Im}(\tau)$ for $j = 0, \dots, m$. We also have $|\tau_j| \leq \text{Im}(\tau)^{-1}$ for $j = 0, \dots, m - 1$, since $\text{Im}(\tau) \leq \text{Im}(\tau_{j+1}) = \text{Im}(\tau_j)/|\tau_j|^2 \leq |\tau_j|^{-1}$. Therefore

$$\begin{aligned} \text{Im}(\tau')^{k/2} |\psi_A(\tau')| &\leq \sum_{j=0}^m \text{Im}(\tau_j)^{k/2} (|\tau_j|^\alpha + \text{Im}(\tau_j)^{-\beta}) \\ &\leq \text{Im}(\tau')^{k/2} \sum_{j=0}^m (|\tau_j|^\alpha + \text{Im}(\tau_j)^{-\beta}) \\ &\leq \text{Im}(\tau')^{k/2} (|\tau|^\alpha + m \text{Im}(\tau)^{-\alpha} + (m+1) \text{Im}(\tau)^{-\beta}) \\ &\leq \text{Im}(\tau')^{k/2} (|\tau|^\alpha + \text{Im}(\tau)^{-\alpha-1} + 2 \text{Im}(\tau)^{-\beta-1}), \end{aligned}$$

where in the last line we used $m+1 \leq 4m-2 \leq \text{Im}(\tau)^{-1}$.

The proof in the other cases (i.e., when A is of the form $T^{2n_k}ST^{2n_{k-1}}S \dots T^{2n_1}S$, $ST^{2n_k}ST^{2n_{k-1}}S \dots T^{2n_1}$, or $T^{2n_k}ST^{2n_{k-1}}S \dots T^{2n_1}$) can be completed using similar estimates. \square

Next, we deal with the analytic part of the estimate. For Theorem 1 the case $n = 0$ of the lemma below will suffice, but we need the general form for the proof of Theorem 2.

Lemma 4. — *For each $n, k \geq 0$ there exists an absolute constant $C_{n,k} > 0$ such that the inequality*

$$\left| x^k \frac{d^n}{dx^n} F_\varepsilon(\tau, x) \right| \leq C_{n,k} (1 + \operatorname{Im}(\tau))^{-(n+k+1)/2}$$

holds for all $\tau \in \mathcal{D}$.

Proof. — Let τ be any point in \mathcal{D} . Since $F_\varepsilon(it)$ is real for all $t > 0$, from the Schwarz reflection principle we get that

$$(30) \quad F_\varepsilon(-\bar{\tau}) = \overline{F_\varepsilon(\tau)}.$$

Using this symmetry we reduce the inequality to the case $\tau \in \mathcal{D}_1$, where $\mathcal{D}_1 = \{\tau \in \mathcal{D} : \operatorname{Re}(\tau) \in (-1, 0)\}$. Observe that $\operatorname{Im}(J(\tau)) < 0$ for all $\tau \in \mathcal{D}_1$ and $\operatorname{Im}(J(\tau)) \geq 0$ for all $\tau \in \mathcal{D} \setminus \mathcal{D}_1$. Indeed, since J is a Hauptmodul, the map $J : \mathcal{D} \rightarrow \mathbf{C}$ is injective. The identity (30) for J implies that for $\tau \in \mathcal{D}$ the value $J(\tau)$ is real if and only if τ lies on the imaginary axis. It is easy to see from (14) that $\operatorname{Im}(J(\tau)) < 0$ for $\tau \in \mathcal{D}_1$ and $\operatorname{Im}(\tau) \gg 1$. Hence, this inequality also holds for all $\tau \in \mathcal{D}_1$.

Define

$$L = \{w \in \mathbf{C} \mid \operatorname{Re}(w) = J(i) = 1/64, \operatorname{Im}(w) > 0\},$$

and let ℓ be the preimage of L under the map $J : \mathcal{D} \rightarrow \mathbf{C}$ (see Figure 4). Then ℓ is a smooth path contained in $\mathcal{D} \setminus \mathcal{D}_1$ and goes from i to 1 . We set γ to be the path $S\ell \cup \ell$ that goes from -1 to 1 . Note that $|z|$ and $|z|^{-1}$ are bounded on γ and that γ has finite length (this fact will follow from the computations below).

As in the proof of Proposition 1 let $Q_\varepsilon(x, z)$ be a polynomial defined by (23). We have

$$x^k \frac{d^n}{dx^n} F_\varepsilon(\tau, x) = \frac{1}{2} \int_{-1}^1 K_\varepsilon(\tau, z) x^k Q_\varepsilon(x, z) e^{i\pi x^2 z} dz.$$

From (21) we find

$$\begin{aligned} x^k \frac{d^n}{dx^n} F_\varepsilon(\tau, x) &= \frac{1}{2} \int_i^1 K_\varepsilon(\tau, z) x^k (Q_\varepsilon(x, z) e^{i\pi x^2 z} \\ &\quad + \varepsilon(-iz)^{-1/2} Q_\varepsilon(x, -1/z) e^{i\pi x^2 (-1/z)}) dz. \end{aligned}$$

Without loss of generality, we may assume $x \geq 0$. Since $|z|$ is bounded for $z \in \gamma$, any monomial $z^\alpha x^\beta$ with $0 \leq \beta \leq n$ is majorized by $1 + x^n$, and thus for all such z we have $|x^k Q_\varepsilon(x, z)| \ll_{n,k,\gamma} 1 + x^{n+k}$. Then

$$\begin{aligned}
& \left| x^k \frac{d^n}{dx^n} F_\varepsilon(\tau, x) \right| \\
& \ll \int_\ell |K_\varepsilon(\tau, z)| x^k |Q_\nu(x, z) e^{i\pi x^2 z} \\
& \quad + \varepsilon(-iz)^{-1/2} Q_\nu(x, -1/z) e^{i\pi x^2(-1/z)}| |dz| \\
(31) \quad & \ll \int_\ell |K_\varepsilon(\tau, z)| (1 + x^{k+n}) (e^{-\pi x^2 \text{Im}(z)} + |z|^{-1/2} e^{-\pi x^2 \text{Im}(-1/z)}) |dz|.
\end{aligned}$$

Next, we observe that

$$(1 + x^{k+n}) e^{-\pi x^2 \text{Im}(z)} \ll_{k+n} 1 + \text{Im}(z)^{\frac{-k-n}{2}}.$$

Note, that $1 \leq |z| \ll 1$ for $z \in \ell$. Hence, we get

$$\begin{aligned}
& \left| x^k \frac{d^n}{dx^n} F_\varepsilon(\tau, x) \right| \\
& \ll \int_\ell |K_\varepsilon(\tau, z)| (1 + \text{Im}(z)^{\frac{-k-n}{2}} + |z|^{-1/2} \text{Im}\left(\frac{-1}{z}\right)^{\frac{-k-n}{2}}) |dz| \\
& = \int_\ell |K_\varepsilon(\tau, z)| (1 + \text{Im}(z)^{\frac{-k-n}{2}} + |z|^{k+n-1/2} \text{Im}(z)^{\frac{-k-n}{2}}) |dz| \\
(32) \quad & \ll \int_\ell |K_\varepsilon(\tau, z)| (1 + \text{Im}(z)^{\frac{-k-n}{2}}) |dz|.
\end{aligned}$$

Without loss of generality, we may also assume that $|\tau - i| \geq 1/10$, since we can recover the inequality of the Lemma in the region $|\tau - i| < 1/10$ by applying the maximum modulus principle together with the functional equation for F_ε .

For τ with $\text{Im}(\tau) \geq 1/2$ and $|\tau - i| > 1/10$ we can estimate $|K_\varepsilon(\tau, z)| \ll |\theta(z)|^3$ with a constant independent of τ . Since $|\theta^3(z)|$ behaves like $\text{Im}(z)^{-2} e^{-\pi/\text{Im}(z)}$ as z approaches 1, by splitting the integral into $\{z: \text{Im}(z) \geq 1/x\}$ and $\{z: \text{Im}(z) < 1/x\}$ we obtain

$$|F_\varepsilon(\tau, x)| \ll (1 + x^2) e^{-c\pi x},$$

which clearly implies the needed inequality.

Now let $\text{Im}(\tau) < 1/2$. To bound $|K_\varepsilon(\tau, z)|$ we use the following estimates

$$\begin{aligned}
|\theta(z)| & \ll |J(z)|^{-1/8} \text{Im}(z)^{-1/2}, \\
|1 - 2\lambda(z)| & \ll |J(z)|^{1/2},
\end{aligned}$$

which hold for all $z \in \mathcal{D}$ near the cusp 1 (such z correspond to large values of $|J(z)|$). The first inequality follows from the fact that $\theta^8(z)J(z)$ is a holomorphic modular form

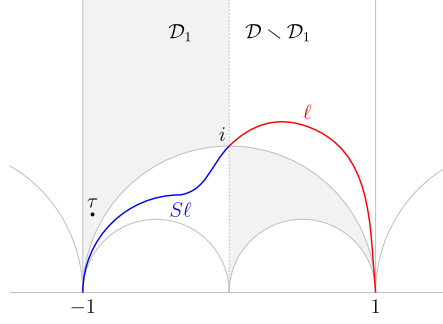


FIG. 4. — Deforming the contour of integration

of weight 4 for Γ_θ (the term $\text{Im}(z)^{-1/2}$ comes from the modular transformation). To prove the second inequality, simply note that $(1 - 2\lambda(z))^2 = 1 - 64J(z)$. Thus, we get

$$(33) \quad \begin{aligned} |K_+(\tau, z)| &\ll \text{Im}(\tau)^{-1/2} \frac{|J(\tau)|^{3/8} |J(z)|^{5/8} \text{Im}(z)^{-3/2}}{|J(z) - J(\tau)|}, \\ |K_-(\tau, z)| &\ll \text{Im}(\tau)^{-1/2} \frac{|J(\tau)|^{7/8} |J(z)|^{1/8} \text{Im}(z)^{-3/2}}{|J(z) - J(\tau)|}. \end{aligned}$$

From now on, we make all estimates for $z \in \ell$ with $\text{Im}(z) < 1/2$, and we define $t > 0$ in such a way that $J(z) = 1/64 + it$. For such z we can use the following simple geometric estimate (recall that $\text{Im}(J(\tau)) < 0$)

$$(34) \quad |J(\tau) - J(z)| \gg \sqrt{|J(\tau)|^2 + |J(z)|^2}.$$

Let $w: \mathbf{C} \setminus [0, \frac{1}{64}) \rightarrow \mathcal{D}$ be the inverse of J on \mathcal{D} , so that $z = w(1/64 + it)$. We have $J'(\tau) = i\pi f(\tau)J(\tau)$, where $f(\tau) = \theta^4(\tau)(1 - 2\lambda(\tau))$ is a holomorphic modular form of weight 2. Since f does not vanish at the cusp 1, we have that $|f(z)| \gg \text{Im}(z)^{-2}$, and thus

$$(35) \quad |dz| = |w'(1/64 + it)| |dt| = \frac{|dt|}{|J'(w(\frac{1}{64} + it))|} \ll \frac{|dt|}{|J(z)| \cdot \text{Im}(z)^{-2}}.$$

Note that this last estimate readily implies that ℓ has finite length.

From inequality (32) it follows that it is enough to find a bound for

$$\int_{\ell} |K_{\varepsilon}(\tau, z)| \text{Im}(z)^{-m} |dz| \quad \text{for } m \geq 0.$$

From inequalities (33), (34), (35) we deduce

$$\int_{\ell} |K_+(\tau, z)| \text{Im}(z)^{-m} |dz| \ll \int_0^\infty \frac{|J(\tau)|^{3/8} t^{-3/8} \text{Im}(z)^{1/2-m}}{\text{Im}(\tau)^{1/2} \sqrt{t^2 + |J(\tau)|^2}} dt.$$

We will also need the estimate $|J(z)| \gg e^{\pi/\text{Im}(z)}$ for $\text{Im}(z)$ small enough. Indeed, this inequality follows from the q -expansion (15) of $J(z)$ at the cusp 1. This implies that $\text{Im}(z)^{-m} \ll_m \log^m(1 + |J(z)|)$. Thus, we have

$$\begin{aligned} \int_{\ell} |K_+(\tau, z)| |\text{Im}(z)^{-m}| |dz| &\ll \text{Im}(\tau)^{-1/2} \int_0^{\infty} \frac{|J(\tau)|^{3/8} t^{-3/8} \log^m(1+t) dt}{\sqrt{|J(\tau)|^2 + t^2}} \\ &= \text{Im}(\tau)^{-1/2} \int_0^{\infty} \frac{t^{-3/8} \log^m(1+t|J(\tau)|) dt}{\sqrt{1+t^2}}. \end{aligned}$$

By using an obvious inequality $\log(1+ab) \leq \log(1+a) + \log(1+b)$, we estimate the last integral by

$$\begin{aligned} &\text{Im}(\tau)^{-1/2} \sum_{j=0}^m \binom{m}{j} \log^j(1 + |J(\tau)|) \int_0^{\infty} \frac{t^{-3/8} \log^{m-j}(1+t) dt}{\sqrt{1+t^2}} \\ &\ll \sum_{j=0}^m c_{j,m} \text{Im}(\tau)^{-j-1/2}, \end{aligned}$$

where $c_{j,m} = \binom{m}{j} \int_0^{\infty} (1+t^2)^{-1/2} t^{-3/8} \log^{m-j}(1+t) dt$ are finite constants, and we have used the inequality $\log(1 + |J(\tau)|) \ll \text{Im}(\tau)^{-1}$ that follows from (15).

The estimates in the case “ $\varepsilon = -$ ” are completely analogous, except that we need to change the exponent $3/8$ to $7/8$. \square

We are now ready to prove Theorem 4.

Proof of Theorem 4. — Let $\tau \in \mathfrak{H}$ be an arbitrary point in the upper half-plane with $\text{Im}(\tau) \leq 1$ that does not lie on the boundary of the fundamental domain \mathcal{D} or any of its translates by elements of Γ_{θ} . Let $\tau = \frac{a\tau' + b}{c\tau' + d}$, where $\tau' \in \mathcal{D}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{\theta}$. By (28) we have

$$\chi_{\varepsilon}(A) j_{\theta}(\tau', A) F_{\varepsilon}\left(\frac{a\tau' + b}{c\tau' + d}\right) = F_{\varepsilon}(\tau') - \phi_A(\tau').$$

Combining the results of Lemma 4 and Lemma 3 (which we apply to $\psi_A = \phi_A$ with $\alpha = 0$ and $\beta = 1/2$) we obtain

$$\begin{aligned} |F_{\varepsilon}(\tau)| &\leq \frac{\text{Im}(\tau')^{1/4}}{\text{Im}(\tau)^{1/4}} |F_{\varepsilon}(\tau')| + \frac{\text{Im}(\tau')^{1/4}}{\text{Im}(\tau)^{1/4}} |\phi_A(\tau')| \\ &\leq C_0 \frac{\text{Im}(\tau')^{1/4} + \text{Im}(\tau')^{-1/4}}{\text{Im}(\tau)^{1/4}} \\ &\quad + \text{Im}(\tau')^{1/4} (1 + \text{Im}(\tau)^{-5/4} + 2 \text{Im}(\tau)^{-7/4}). \end{aligned}$$

(Here C_0 is the constant from Lemma 4.) If $c = 0$, then $\text{Im}(\tau') = \text{Im}(\tau)$ and thus

$$|F_\varepsilon(\tau)| \leq C_0(1 + \text{Im}(\tau)^{-1/2}) + \text{Im}(\tau)^{1/4} + \text{Im}(\tau)^{-1} + 2\text{Im}(\tau)^{-3/2}.$$

If, on the other hand, $c \neq 0$, then we have $\text{Im}(\tau) < \text{Im}(\tau')$ and

$$\text{Im}(\tau)\text{Im}(\tau') = \frac{\text{Im}(\tau')^2}{|c\tau' + d|^2} \leq 1,$$

and we get the estimate

$$|F_\varepsilon(\tau)| \leq 2C_0\text{Im}(\tau)^{-1/2} + \text{Im}(\tau)^{-1/4} + \text{Im}(\tau)^{-3/2} + 2\text{Im}(\tau)^{-2}.$$

Therefore, an application of Lemma 1 gives

$$|b_n^\varepsilon(x)| \ll n^2.$$

□

The exponent “2” in Theorem 4 is not optimal, but for the proof of Theorem 1 any polynomial bound would suffice.

6. Proof of the main results

Now that we know that $b_n^\varepsilon(x)$ have polynomial growth in n , the proof of Theorem 1 and Theorem 2 is not hard.

Recall the definition of Schwartz functions:

$$\mathcal{S} = \{f \in C^\infty(\mathbf{R}) : \|f\|_{\alpha,\beta} < \infty \forall \alpha, \beta \geq 0\},$$

where the seminorms $\|\cdot\|_{\alpha,\beta}$ are defined by

$$\|f\|_{\alpha,\beta} = \sup_{x \in \mathbf{R}} |x^\alpha f^{(\beta)}(x)|.$$

Convergence in \mathcal{S} is defined in terms of this family of seminorms, i.e., $f_n \rightarrow f$ if and only if $\|f_n - f\|_{\alpha,\beta} \rightarrow 0$ for all $\alpha, \beta \geq 0$.

Proof of Theorem 1. — Let $\mathcal{S}_{\text{even}}$ be the space of even Schwartz functions. Let us define

$$a_n(x) := \frac{b_n^+(x) + b_n^-(x)}{2}.$$

Lemma 1 implies that

$$\widehat{a}_n(x) = \frac{b_n^+(x) - b_n^-(x)}{2}.$$

Our aim is to show that (2) holds for all $f \in \mathcal{S}_{\text{even}}$. Theorem 4 implies that the series

$$\sum_{n=0}^{\infty} a_n(x) f(\sqrt{n}) + \sum_{n=0}^{\infty} \widehat{a}_n(x) \widehat{f}(\sqrt{n})$$

converges absolutely. Moreover, it follows from the definition of b_n^ε and the functional equations (26) that for any $\tau \in \mathfrak{H}$ we have

$$(36) \quad e_\tau(x) = \sum_{n=0}^{\infty} a_n(x) e_\tau(\sqrt{n}) + \sum_{n=0}^{\infty} \widehat{a}_n(x) \widehat{e}_\tau(\sqrt{n}),$$

where $e_\tau(x) = e^{i\pi\tau x^2}$.

For $x \geq 0$ consider the linear functional ϕ_x on $\mathcal{S}_{\text{even}}$ given by

$$\phi_x(f) := f(x) - \sum_{n=0}^{\infty} a_n(x) f(\sqrt{n}) - \sum_{n=0}^{\infty} \widehat{a}_n(x) \widehat{f}(\sqrt{n}).$$

It follows from Theorem 4 that ϕ_x is a tempered distribution, i.e., it is continuous with respect to convergence in $\mathcal{S}_{\text{even}}$. From equation (36) we see that ϕ_x vanishes on the subspace spanned by $\{e_\tau\}_{\tau \in \mathfrak{H}}$. Our goal is to show that ϕ_x vanishes on the whole $\mathcal{S}_{\text{even}}$.

Let \mathcal{C} be the space of compactly supported even C^∞ functions on \mathbf{R} . Recall, that \mathcal{C} dense in $\mathcal{S}_{\text{even}}$ (see [18, pp. 74–75]). Therefore, it suffices to show (2) for $f \in \mathcal{C}$. Let f be a function in \mathcal{C} . We may assume that

$$f(x) = F(x^2) e^{-\pi x^2}$$

where F is a C^∞ function with compact support on \mathbf{R} . Consider the one-dimensional Fourier transform of F

$$\widehat{F}(s) := \int_{-\infty}^{\infty} F(t) e^{-2\pi i s t} dt.$$

Note, that \widehat{F} is a Schwartz function. By the Fourier inversion formula we have

$$f(x) = F(x^2) e^{-\pi x^2} = \int_{-\infty}^{\infty} \widehat{F}(s) e^{2\pi i s x^2 - \pi x^2} ds = \int_{-\infty}^{\infty} \widehat{F}(s) e_{i+2s}(x) ds.$$

Define

$$h_T := \int_{-T}^T \widehat{F}(s) e_{i+2s}(x) ds.$$

It is easy to see that for all seminorms $\|\cdot\|_{\alpha,\beta}$

$$\|f - h_T\|_{\alpha,\beta} \rightarrow 0 \quad \text{as } T \rightarrow \infty.$$

Therefore, for all $x \geq 0$

$$\phi_x(f - h_T) \rightarrow 0 \quad \text{as } T \rightarrow \infty.$$

On the other hand, we have

$$\phi_x(h_T) = \int_{-T}^T \widehat{F}(s) \phi_x(e_{i+2s}) ds = 0.$$

This finishes the proof of Theorem 1. □

We are also ready to prove Theorem 2.

Proof of Theorem 2. — First, we observe that the image of Ψ is contained in the kernel of L . Indeed, the Poisson summation formula implies

$$\sum_{n \in \mathbf{Z}} f(n) = \sum_{n \in \mathbf{Z}} \widehat{f}(n)$$

for all $f \in \mathcal{S}$ as well as $f \in \mathcal{S}_{\text{even}}$. This identity is equivalent to $L \circ \Psi(f) = 0$.

Next, we construct the function $\Phi : \ker L \rightarrow \mathcal{S}_{\text{even}}$ such that $\Psi \circ \Phi = \mathbf{I}_{\ker L}$. To this end we consider the map

$$\Phi : \ker L \rightarrow \mathcal{S}_{\text{even}}, \quad ((x_n), (y_n)) \mapsto \sum_n x_n a_n(x) + y_n \widehat{a}_n(x).$$

We need to show that Φ is well-defined. Since \mathcal{S} is complete with respect to the family of norms $\|\cdot\|_{\alpha, \beta}$ it is enough to prove that for any fixed $\alpha, \beta \geq 0$ the sequences $(\|a_n\|_{\alpha, \beta})_n$ and $(\|\widehat{a}_n\|_{\alpha, \beta})_n$ have at most polynomial growth in n . Equivalently, it is enough to prove that the sequences $(\|b_n^\varepsilon\|_{\alpha, \beta})_n$ have polynomial growth.

As before, let $Q_k(x, z)$ be the polynomial defined by (23). Let $U(\tau, x)$ be the generating function

$$U(\tau, x) = x^\alpha \frac{d^\beta}{dx^\beta} F_\varepsilon(\tau, x) = x^\alpha \sum_{n=0}^{\infty} \frac{d^\beta}{dx^\beta} b_n^\varepsilon(x) e^{i\pi n \tau}.$$

Then, following the proof of Proposition 2, we see that the generating function U satisfies the functional equation

$$U(\tau) - (U|_{1/2}^{-\varepsilon} A)(\tau) = \phi_A(\tau),$$

where ϕ_A is the cocycle defined by

$$\phi_{T^2}(\tau) = 0,$$

$$\phi_S(\tau) = x^\alpha Q_\beta(x, \tau) e^{i\pi x^2 \tau} + \varepsilon(-i\tau)^{-1/2} x^\alpha Q_\beta(x, -1/\tau) e^{i\pi x^2(-1/\tau)}.$$

Using the estimates

$$|x^k \tau^l e^{i\pi x^2 \tau}| \ll |\tau|^l \operatorname{Im}(\tau)^{-k/2} < \operatorname{Im}(\tau)^{-k} + |\tau|^{2l}$$

and

$$|x^k \tau^{-l} e^{i\pi x^2 (-1/\tau)}| \ll |\tau|^{k-l} \operatorname{Im}(\tau)^{-k/2} < \operatorname{Im}(\tau)^{-k} + |\tau|^{2k-2l},$$

and in case $k < l$ the replacing $|\tau|^{2k-2l}$ by $\operatorname{Im}(\tau)^{2k-2l}$, we see that Lemma 3 can be applied to $\{\phi_A\}_{A \in \Gamma_\theta}$ (for some choice of α and β in Lemma 3). Lemma 4 implies that for $\tau \in \mathcal{D}$ we have

$$U(\tau, x) \ll 1 + \operatorname{Im}(\tau)^{-(\alpha+\beta+1)/2}.$$

Arguing the same way as in the proof of Theorem 4 we obtain that for some $C > 0$ and all $\tau \in \mathfrak{H}$ with $\operatorname{Im}(\tau) < 1$ we have $|U(\tau, x)| \ll \operatorname{Im}(\tau)^{-C}$, which implies that $\|b_n^\varepsilon\|_{\alpha, \beta} \ll n^C$. Therefore, the map Φ is well-defined.

Now Theorem 1 implies that $\Phi \circ \Psi = \mathbf{I}_{\mathcal{S}_{\text{even}}}$ and Proposition 1 implies that $\Psi \circ \Phi = \mathbf{I}_{\ker L}$. This finishes the proof. \square

7. Interpolation basis for odd functions

The case of odd Schwartz functions is very similar to the even case. The proofs are easy enough to adapt to this case, so we will just give the general outline. The role of the Gaussian $e_\tau(x) = e^{i\pi \tau x^2}$ is played by the Schwartz function

$$o_\tau(x) = x e^{i\pi \tau x^2},$$

that satisfies

$$\widehat{o_\tau}(\xi) = -i(-i\tau)^{-3/2} o_{-1/\tau}(\xi).$$

To construct the interpolation basis for odd Schwartz functions we use the same idea as before: to get an eigenfunction we integrate o_τ over τ with some “modular weight”. More precisely, let $h_n^\varepsilon: \mathfrak{H} \rightarrow \mathbf{C}$ be holomorphic functions with the following properties:

$$\begin{aligned} h_n^\varepsilon(z+2) &= h_n^\varepsilon(z), \\ (-iz)^{-1/2} h_n^\varepsilon(-1/z) &= \varepsilon h_n^\varepsilon(z), \\ h_n^+(z) &= q^{-n/2} + O(q^{1/2}), \quad z \rightarrow i\infty, \\ h_n^-(z) &= q^{-n/2} + O(1), \quad z \rightarrow i\infty, \\ h_n^\varepsilon(1+i/t) &\rightarrow 0, \quad t \rightarrow \infty. \end{aligned}$$

Once again, we may assume that they are of the form

$$(37) \quad \begin{aligned} h_n^+(z) &= \theta(z) Q_n^+(J^{-1}(z)), \\ h_n^-(z) &= \theta(z) (1 - 2\lambda(z)) Q_n^-(J^{-1}(z)), \end{aligned}$$

where $Q_n^\pm \in \mathbf{Q}[x]$ are monic of degree n and Q_n^- has no constant term. The first few of these functions are

$$\begin{aligned} h_0^+ &= \theta, & h_1^- &= \theta \cdot (1 - 2\lambda) \cdot (J^{-1}), \\ h_1^+ &= \theta \cdot (J^{-1} - 26), & h_2^- &= \theta \cdot (1 - 2\lambda) \cdot (J^{-2} - 18J^{-1}), \\ h_2^+ &= \theta \cdot (J^{-2} - 50J^{-1} + 76), & h_3^- &= \theta \cdot (1 - 2\lambda) \cdot (J^{-3} - 42J^{-2} + 168J^{-1}). \end{aligned}$$

By the same arguments as in the even case, we establish generating functions for h_n^ε , which turn out to be the same, except for switching the roles of τ and z .

Theorem 5. — *The generating functions for $\{h_n^+(z)\}_{n \geq 0}$ and $\{h_n^-(z)\}_{n \geq 1}$ are given by*

$$(38) \quad \begin{aligned} \sum_{n=0}^{\infty} h_n^+(z) e^{i\pi n \tau} &= \frac{\theta^3(\tau)(1 - 2\lambda(\tau))\theta(z)J(z)}{J(z) - J(\tau)} = -K_-(z, \tau), \\ \sum_{n=1}^{\infty} h_n^-(z) e^{i\pi n \tau} &= \frac{\theta^3(\tau)J(\tau)\theta(z)(1 - 2\lambda(z))}{J(z) - J(\tau)} = -K_+(z, \tau). \end{aligned}$$

Similarly to the even case, define $d_m^\varepsilon: \mathbf{R} \rightarrow \mathbf{R}$ by

$$d_m^\varepsilon(x) = \frac{1}{2} \int_{-1}^1 h_m^\varepsilon(z) x e^{i\pi x^2 z} dz.$$

Proposition 3. — *The function $d_m^\varepsilon: \mathbf{R} \rightarrow \mathbf{R}$ is odd, belongs to the Schwartz class, and satisfies*

$$\widehat{d}_m^\varepsilon(x) = (-i\varepsilon) d_m^\varepsilon(x)$$

and

$$d_m^\varepsilon(\sqrt{n}) = \delta_{n,m} \sqrt{n}, \quad n \geq 1,$$

where $\delta_{n,m}$ is the Kronecker delta. Moreover,

$$\lim_{x \rightarrow 0} \frac{d_m^+(x)}{x} = \delta_{m,0}.$$

Furthermore, we have the following estimate on the growth of $d_n^\pm(x)$ as a function of n .

Theorem 6. — For any $\varepsilon \in \{+, -\}$ the numbers $d_n^\varepsilon(x)$ satisfy

$$d_n^\varepsilon(x) = O(n^{5/2})$$

uniformly in x .

The proof of this estimate is also based on estimating the growth for $\text{Im}(\tau) \rightarrow 0$ of the generating function

$$G_\varepsilon(\tau, x) = \sum_{n \geq 0} d_n^\varepsilon(x) e^{i\pi n \tau}.$$

The functional equations for G_ε are

$$(39) \quad \begin{aligned} G_\varepsilon(\tau, x) - G_\varepsilon(\tau + 2, x) &= 0, \\ G_\varepsilon(\tau, x) + \varepsilon(-i\tau)^{-3/2} G_\varepsilon\left(-\frac{1}{\tau}, x\right) &= x e^{i\pi \tau x^2} + \varepsilon(-i\tau)^{-3/2} x e^{i\pi(-1/\tau)x^2}. \end{aligned}$$

The difference in exponents of $(-i\tau)$ come from the fact that the weight of $K_\varepsilon(z, \tau)$ in variable τ is now $3/2$, but with appropriate changes the proof still goes through. Finally, we get the following interpolation theorem for odd Schwartz functions.

Theorem 7. — For any odd Schwartz function $f: \mathbf{R} \rightarrow \mathbf{R}$ and any $x \in \mathbf{R}$ we have

$$(40) \quad f(x) = d_0^+(x) \frac{f'(0) + \widehat{if}'(0)}{2} + \sum_{n=1}^{\infty} c_n(x) \frac{f(\sqrt{n})}{\sqrt{n}} - \sum_{n=1}^{\infty} \widehat{c}_n(x) \frac{\widehat{f}(\sqrt{n})}{\sqrt{n}},$$

where $c_n(x) = (d_n^+(x) + d_n^-(x))/2$.

As in the even case, the functional equations for G_ε show that (40) holds for $o_\tau(x)$, so one only needs to show that o_τ are dense in the space of odd Schwartz functions, which can be done by an approximation argument, similarly to the proof of Theorem 1.

Let us also note that the even interpolation basis $\{a_n(x)\}_n$ is defined using the kernel $K(\tau, z) := K_+(\tau, z) + K_-(\tau, z)$, and the odd interpolation basis $\{c_n(x)\}_n$ is defined using the kernel $\widetilde{K}(\tau, z) := -K(z, \tau)$. Thus, even though we have dealt with even and odd interpolation problems separately, there is a nice duality between the two.

Remark. As in the even case, using the explicit formula for the kernels, we get

$$\begin{aligned} d_m^{+'}(0) &= \delta_{m,0}, \\ d_m^{-'}(0) &= -r_3(m), \quad m \geq 1, \end{aligned}$$

where $r_3(m)$ is the number of representations of m as the sum of squares of 3 integers. Taking $x = 0$ in (40) we get the following identity

$$f'(0) + \sum_{n=1}^{\infty} \frac{r_3(n)f(\sqrt{n})}{\sqrt{n}} = \widehat{f}'(0) + \sum_{n=1}^{\infty} \frac{r_3(n)\widehat{f}(\sqrt{n})}{\sqrt{n}},$$

valid for arbitrary odd Schwartz functions. As was pointed out to us by Yves Meyer, this formula was previously found by Guinand [8, p. 265].

8. Open questions and concluding remarks

Let us indicate some further directions and observations related to Theorem 1.

Function space. — In this paper we have only worked with the space of Schwartz functions, but it is interesting to ask in what generality the interpolation formula (2) holds. The best possible scenario would be a positive answer to the following question.

Question 1. — *Do the results of Theorems 1 and 7 hold whenever the sum on the right-hand side is well-defined and converges absolutely?*

Even to find explicit conditions for when the convergence is absolute, one would need to obtain exact bounds on the growth of $b_n^\varepsilon(x)$, which appears to be difficult. Let us outline a simple approximation argument that shows that the interpolation formula is true whenever both f and \widehat{f} decay sufficiently fast:

Proposition 4. — *Let f be an even integrable function. If $f(x)$ and $\widehat{f}(x)$ are both bounded by $(1 + |x|)^{-13}$, then the summation formula (2) holds.*

Proof sketch. — Indeed, for every $T > 0$ consider the following linear operator \mathcal{R}_T that takes values in \mathcal{S} :

$$\mathcal{R}_T(f)(x) = T^{1/2} e_{i/T} \cdot (e_{i/T} * f)(x) = T^{1/2} e^{-\pi x^2/T} \int_{-\infty}^{\infty} f(x-y) e^{-\pi T y^2} dy.$$

The Fourier transform is then given by

$$\widehat{\mathcal{R}_T(f)}(x) = T^{1/2} e_{i/T} * (e_{i/T} \cdot \widehat{f})(x) = T^{1/2} \int_{-\infty}^{\infty} f(x-y) e^{-\pi T y^2 - \pi(x-y)^2/T} dy.$$

Then a routine calculation shows that

$$|\mathcal{R}_T(f)(x) - f(x)| \ll (1 - e^{-\pi x^2/T}) |f(x)| + T^{-1/2} \max_{y \in [x-1, x+1]} |f'(y)|,$$

and similarly

$$|\widehat{\mathcal{R}_T(f)}(x) - \widehat{f}(x)| \ll \left(1 - \frac{e^{-\pi x^2 T/(1+T^2)}}{\sqrt{1+T^{-2}}}\right) |\widehat{f}(x)| + T^{-1/2} \max_{y \in [x-1, x+1]} |\widehat{f}'(y)|.$$

By summing up these estimates for $x = \sqrt{n}$ over $n \geq 1$ and taking the limit as $T \rightarrow \infty$ we see that the proof will be complete if we can show that $f'(x)$ and $\widehat{f}'(x)$ decay as $(1 + |x|)^{-l}$ for some $l > 6$ (since $a_n(x) = O(n^2)$). We consider only bounding $f'(x)$, since one can obtain the other estimate by interchanging f and \widehat{f} . It was pointed out to the authors by Emanuel Carneiro that this can be done using the following simple observation: if g is a C^2 -smooth function on $[1, \infty)$ that satisfies $|g(x)| \ll x^{-k}$ and $|g''(x)| \ll 1$ then $|g'(x)| \ll x^{-k/2}$. Indeed, then by the Fourier inversion formula we have $|f''(x)| \ll 1$, so we can apply the observation to get $|f'(x)| \ll (1 + |x|)^{-13/2}$, and thus we are done.

To prove the above observation: let $|g''(x)| \leq 1$ and $|g(x)| \leq Cx^{-k}$. Then Taylor's theorem with remainder in the Lagrange form implies that for any $\Delta \geq 0$ we have

$$|g(x + \Delta) - g(x) - g'(x)\Delta| \leq \frac{\Delta^2}{2},$$

from which we get, taking $\Delta = 2\sqrt{Cx^{-k}}$, that

$$|g'(x)| \leq \frac{\Delta}{2} + \frac{2Cx^{-k}}{\Delta} = 2\sqrt{C} x^{-k/2},$$

as required. \square

Note that the number “13” in the above proposition can be improved by using more careful estimates.

Relation to the Laplace transform. — The basis functions that we have constructed are all of the shape

$$f(x) = \frac{1}{2} \int_{-1}^1 g(z) e^{i\pi x^2 z} dz$$

for some weakly holomorphic modular form g (in the odd case, f is multiplied by x). To get an alternative expression for f we can shift the contour of integration to the rectangular line passing through -1 , $-1 + iT$, $1 + iT$, and 1 . A simple computation then shows that

$$f(x) = \sin(\pi x^2) \int_0^T g(1 + it) e^{-\pi x^2 t} dt + e^{-\pi x^2 T} \int_{-1}^1 g(s + iT) e^{i\pi s x^2} ds.$$

If we take T to infinity, then we see that for all x^2 greater than the order of the pole of g at $i\infty$ we have

$$f(x) = \sin(\pi x^2) \int_0^\infty g(1 + it) e^{-\pi x^2 t} dt.$$

The integral on the right is simply the Laplace transform of $g(1 + it)$ evaluated at πx^2 . This can be used to show that all but finitely many real zeros of $b_m^\pm(x)$ are of the form $\pm\sqrt{n}$. Combined with the q -expansion of $g(1 + z)$ at infinity, this also implies that b_m^\pm extends analytically to an entire function. Alternatively, this also follows directly from the definition (22).

Sine-sinh ratio. — The function $d_0^+(x)$ is quite special. Recall that it is defined by

$$d_0^+(x) = \frac{1}{2} \int_{-1}^1 \theta(z) x e^{i\pi x^2 z} dz.$$

Changing the contour of integration as before, we get

$$d_0^+(x) = x \sin(\pi x^2) \int_0^\infty \theta(1 + it) e^{-\pi x^2 t} dt.$$

Next, integrating the q -expansion of θ termwise and using the identity

$$\sum_{n \in \mathbf{Z}} \frac{(-1)^n}{\pi(x^2 + n^2)} = \frac{1}{x \sinh(\pi x)}$$

we find that $d_0^+(x)$ is, in fact, an elementary function:

$$d_0^+(x) = \frac{\sin(\pi x^2)}{\sinh(\pi x)}.$$

Note that $d_0^+(x)$ and its Fourier transform $\widehat{d_0^+}(x) = (-i)d_0^+(x)$ both vanish at $x = \pm\sqrt{n}$ for all $n \geq 0$. It follows from Theorems 1 and 7 that any Schwartz function with this property is of the form αd_0^+ .

It appears that this function was first considered by Ramanujan in [14], where he studies a number of integrals involving similar expressions, and, in particular, shows the Fourier invariance of d_0^+ (see [14, Eq. 34]). It is also directly related to the so-called Mordell integral [12], which played an important role in Zwegers's seminal work on mock theta functions [22].

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