EFFECTIVITY OF IITAKA FIBRATIONS AND PLURICANONICAL SYSTEMS OF POLARIZED PAIRS

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ABSTRACT

For every smooth complex projective variety W of dimension d and nonnegative Kodaira dimension, we show the existence of a universal constant m depending only on d and two natural invariants of the very general fibres of an Iitaka fibration of W such that the pluricanonical system $|mK_W|$ defines an Iitaka fibration. This is a consequence of a more general result on polarized adjoint divisors. In order to prove these results we develop a generalized theory of pairs, singularities, log canonical thresholds, adjunction, etc.

CONTENTS

1. Introduction

We work over the complex number field **C**. However, our results hold over any algebraically closed field of characteristic zero.

Effectivity of Iitaka fibrations. — Let W be a smooth projective variety of Kodaira dimension $\kappa(W) \geq 0$. Then by a well-known construction of Iitaka, there is a birational morphism $V \to W$ from a smooth projective variety V, and a contraction $V \to X$ onto a projective variety X such that a (very) general fibre F of $V \to X$ is smooth with Kodaira dimension zero, and $\dim X$ is equal to the Kodaira dimension $\kappa(W)$. The map $W \dashrightarrow X$ is referred to as an *Iitaka fibration* of W, which is unique up to birational equivalence. For any sufficiently divisible natural number m, the pluricanonical system $|mK_W|$ defines an Iitaka fibration.

When $\dim W = 2$, in 1970, Iitaka [12] proved that if m is any natural number divisible by 12 and $m \geq 86$, then $|mK_W|$ defines an Iitaka fibration (Fabrizio Catanese informed us that Iitaka proved this result for compact complex surfaces but the algebraic case goes back to Enriques). It has since been a question whether something similar holds in higher dimension. More precisely (cf. [8]):

Conjecture **1.1** *(Effective Iitaka fibration).* — *Let* W *be a smooth projective variety of dimension d and Kodaira dimension* $\kappa(W) \geq 0$. *Then there is a natural number* m_d *depending only on d such that the pluricanonical system* $|m K_W|$ *defines an Iitaka fibration for any natural number m divisible by* m_d.

In this paper, we show a version of the conjecture as formulated in [24, Question 0.1] holds, that is, by assuming that some invariants of the very general fibres of the Iitaka fibration are bounded. Without these extra assumptions the above conjecture seems out of reach at the moment because most likely one needs the abundance conjecture to deal with the very general fibres. For example, when $\kappa(W) = 0$, the conjecture is equivalent to the effective nonvanishing $h^0(W, m_d K_W) \neq 0$ which is obviously related to the abundance conjecture. Note that there is also a log version of the conjecture for pairs: see [11, Conjecture 1.2, Theorem 1.4] and the references therein, where the authors confirmed this log version when the boundary divisor is big over the generic point of the base of the log Iitaka fibration.

We recall some definitions before stating our result. Using the notation above, let W be a smooth projective variety of Kodaira dimension $\kappa(W) \geq 0$ and $V \to X$ an Iitaka fibration from a resolution V of W. For a very general fibre F of $V \to X$, let

$$b_F := \min\{u \in \mathbf{N} \mid |u K_F| \neq \emptyset\}.$$

Let $\widetilde{F}$ be a smooth model of the $\mathbf{Z}/(b_F)$-cover of F ramified over the unique divisor in $|b_F K_F|$. Then $\widetilde{F}$ still has Kodaira dimension zero, but with $|K_{\widetilde{F}}| \neq \emptyset$. Note that

$$\dim \widetilde{F} = \dim F = \dim W - \dim X = \dim W - \kappa(W)$$

and we denote this number by d_F. We call the Betti number

$$\beta_{\widetilde{F}} := \dim H^{d_F}(\widetilde{F}, \mathbf{C})$$

the middle Betti number of $\widetilde{F}$.

Theorem **1.2.** — *Let* W *be a smooth projective variety of dimension d and Kodaira dimension* $\kappa(W) \geq 0$. *Then there is a natural number* $m(d, b_F, \beta_{\widetilde{F}})$ *depending only on d, b_F and $\beta_{\widetilde{F}}$ such that the pluricanonical system* $|m K_W|$ *defines an Iitaka fibration whenever the natural number m is divisible by* $m(d, b_F, \beta_{\widetilde{F}})$.

The theorem is an almost immediate consequence of 1.3 below. The proof is given at the end of Section 8. When X is of general type, the numbers b_F, $\beta_{\widetilde{F}}$ do not play any role so $m(d, b_F, \beta_{\widetilde{F}})$ depends only on d.

Here is a brief history of partial cases of Theorem 1.2:
- when $\dim W = 2$ [12],
- when $\kappa(W) = 1$ [7],
- when W is of general type [8, 21] (see also [23]),

- when $\kappa(W) = 2$ [24] (see also [22]),
- when $\dim W = 3$ [7, 8, 14, 21, 24] (see also [6]),

• when X is non-uniruled, $V \to X$ has maximal variation and its general fibres have good minimal models [19] (see also [5]),

• when $V \to X$ has zero variation and its general fibres have good minimal models [13].

Note that the above references show that Conjecture 1.1 holds when $\dim W \le 3$.

Effective birationality for polarized pairs of general type. — Let W be a smooth projective variety of nonnegative Kodaira dimension. After replacing W birationally we can assume the Iitaka fibration $W \to X$ is a morphism. Applying the canonical bundle formula of [7] (which is based on [16]), perhaps after replacing W and X birationally, there is a **Q**-boundary B and a nef **Q**-divisor M on X such that for any natural number m divisible by b_F we have a natural isomorphism between $H^0(W, mK_W)$ and $H^0(X, m(K_X + B + M))$. In particular, if $|m(K_X + B + M)|$ defines a birational map, then $|mK_W|$ defines an Iitaka fibration. Moreover, the coefficients of B belong to a DCC set and the Cartier index of M is bounded in terms of b_F and $\beta_{\widetilde{F}}$. Therefore we can derive Theorem 1.2 from the next result.

Theorem **1.3.** — *Let Λ be a DCC set of nonnegative real numbers, and d, r natural numbers. Then there is a natural number $m(\Lambda, d, r)$ depending only on Λ, d, r such that if:*

(i) (X, B) *is a projective lc pair of dimension d,*
(ii) *the coefficients of B are in Λ,*
(iii) rM *is a nef Cartier divisor, and*
(iv) $K_X + B + M$ *is big,*

then the linear system $|m(K_X + B + M)|$ defines a birational map if $m \in \mathbf{N}$ is divisible by $m(\Lambda, d, r)$.

We call $(X, B+M)$ a *polarized pair*. When $M = 0$, the theorem is [10, Theorem 1.3]. Note that for an **R**-divisor D, by $|D|$ and $H^0(X, D)$ we mean $|\lfloor D \rfloor|$ and $H^0(X, \lfloor D \rfloor)$.

Generalized polarized pairs. — In order to prove Theorem 1.3 we need to generalize the definitions of pairs, singularities, lc thresholds, adjunction, etc. We develop this theory, which is of independent interest, in some detail in Section 4 but for now we only give the definition of generalized polarized pairs.

Definition **1.4.** — *A* generalized polarized pair *consists of a normal variety X' equipped with projective morphisms $X \xrightarrow{f} X' \to Z$ where f is birational and X is normal, an **R**-boundary B', and an **R**-Cartier divisor M on X which is nef $/Z$ such that $K_{X'} + B' + M'$ is **R**-Cartier, where $M' := f_*M$. We call B' the* boundary part *and M the* nef part.

Note that the definition is flexible with respect to X, M. To be more precise, if $g \colon Y \to X$ is a projective birational morphism from a normal variety, then there is no harm in replacing X with Y and replacing M with g^*M.

For us the most interesting case is when $M = \sum \mu_j M_j$ where $\mu_j \geq 0$ and M_j are *nef* /Z Cartier divisors. In many ways $B' + M'$ behaves like a boundary, that is, it is as if the M_j' were components of the boundary with coefficient μ_j. Although the coefficients of B_i' belong to the real interval $[0, 1]$ the coefficients μ_j are only assumed to be nonnegative. Moreover, the M_j are not necessarily distinct. See Section 4 for more details.

When $X \to X'$ is the identity morphism, we recover the definition of polarized pairs which was formally introduced in [4] but appeared earlier in the subadjunction formula of [16]. If moreover $M = 0$, then (X', B') is just a pair in the traditional sense.

ACC for generalized lc thresholds. — The next result shows that the generalized lc thresholds satisfy ACC under suitable assumptions. We suggest the reader looks at Definitions 4.1 and 4.3 before continuing.

Theorem 1.5. — *Let Λ be a DCC set of nonnegative real numbers and d a natural number. Then there is an ACC set Θ depending only on Λ, d such that if $(X', B' + M')$, M, N, and D' are as in Definition 4.3 satisfying*

(i) $(X', B' + M')$ *is generalized lc of dimension d,*
(ii) $M = \sum \mu_j M_j$ *where M_j are nef /Z Cartier divisors and $\mu_j \in \Lambda$,*
(iii) $N = \sum \nu_k N_k$ *where N_k are nef /Z Cartier divisors and $\nu_k \in \Lambda$, and*
(iv) *the coefficients of B' and D' belong to Λ,*

then the generalized lc threshold of $D' + N'$ with respect to $(X', B' + M')$ belongs to Θ.

Note that the theorem is a local statement over X', so Z does not play any role and we could simply assume $X' \to Z$ is the identity map.

When $X \to X'$ is the identity map, $M = 0$, and $N = 0$, the theorem is the usual ACC for lc thresholds [10, Theorem 1.1].

Global ACC. — The proof of the previous result requires the following global ACC. We will also use this to bound pseudo-effective thresholds (Theorem 8.1) which is in turn used in the proof of Theorem 1.3.

Theorem 1.6. — *Let Λ be a DCC set of nonnegative real numbers and d a natural number. Then there is a finite subset $\Lambda^0 \subseteq \Lambda$ depending only on Λ, d such that if $(X', B' + M')$, $X \to X' \to Z$ and M are as in Definition 1.4 satisfying*

(i) $(X', B' + M')$ *is generalized lc of dimension d,*
(ii) Z *is a point,*

(iii) $M = \sum \mu_j M_j$ *where* M_j *are nef Cartier divisors and* $\mu_j \in \Lambda$,

(iv) $\mu_j = 0$ *if* $M_j \equiv 0$,

 (v) *the coefficients of* B' *belong to* Λ, *and*

(vi) $K_{X'} + B' + M' \equiv 0$,

then the coefficients of B' *and the* μ_j *belong to* Λ^0.

When $X \to X'$ is the identity map and $M = 0$, the theorem is [10, Theorem 1.5].

About this paper. — It is not hard to reduce Theorems 1.3 and 1.5 to Theorem 1.6. So most of the difficulties we face have to do with 1.6. Since the statement of Theorems 1.3, 1.5, and 1.6 involve nef divisors which may not be semi-ample (or effectively semi-ample), there does not seem to be any easy way to reduce them to the traditional versions (i.e. without nef divisors) proved in [10] or to mimic the arguments in [10]. Instead we need to develop new ideas and arguments and this occupies much of this paper.

We briefly explain the organization of the paper. In Section 3, we prove a special case of Theorem 1.3 (Proposition 3.4) by closely following [10]. In Section 4, we introduce generalized singularities and generalized lc thresholds, discuss the log minimal model program for generalized polarized pairs, and treat generalized adjunction. In Section 5, we give bounds, both in the local and global situations, on the numbers of components in the boundary and nef parts of generalized polarized pairs, under appropriate assumptions. These bounds will be used in the proof of Proposition 7.2 which serves as the key inductive step toward the proof of Theorem 1.6. In Section 6, we reduce Theorem 1.5 to Theorem 1.6 in lower dimension by adapting a standard argument. In Section 7, we treat Theorem 1.6 inductively where we apply Proposition 3.4; a sketch of the main ideas is included below in this introduction. In Section 8, we give the proofs of our main results. Theorems 1.5 and 1.6 follow immediately from Sections 6 and 7. To prove Theorem 1.3, we use Theorem 1.6 to bound certain pseudo-effective thresholds (Theorem 8.1) and use the concept of potential birationality [10] to reduce to the special case of Proposition 3.4. Finally, we extend Theorem 1.3 to allow more general coefficients in the nef part of the pair (see Theorem 8.2), and deduce Theorem 1.2 from Theorem 1.3 as in [7, 24].

A few words about the proof of Theorem 1.6. — We try to explain, briefly, some of the ideas used in the proof of 1.6. By [10, 1.5] we can assume $M \not\equiv 0$. The basic strategy is to modify $(X', B' + M')$ so that the nef part has one less coefficient μ_j and then repeat this to reach the case $M = 0$. Running appropriate LMMP's we can reduce the problem to the case when X' is a **Q**-factorial klt Fano variety with Picard number one. Moreover, some lengthy arguments show that the number of the μ_j is bounded (Section 5). If $(X', B' + M')$ is not generalized klt, one can do induction: for example if $\lfloor B' \rfloor \neq 0$, then we let S' be the normalization of a component of $\lfloor B' \rfloor$ and use generalized adjunction (see Definition 4.7) to write

$$K_{S'} + B_{S'} + M_{S'} = \left(K_{X'} + B' + M' \right)\big|_{S'}$$

and apply induction to the generalized lc polarized pair $(S', B_{S'} + M_{S'})$. So we can assume $(X', B' + M')$ is generalized klt.

Although we cannot use the arguments of [10] to prove Theorem 1.6 but there is an exception: if we take $n \in \mathbf{N}$ to be sufficiently large, then following [10] closely one can show that there is $m \in \mathbf{N}$ depending only on Λ, d such that $|m(K_X + B + \sum nM_j|)|$ defines a birational map (Proposition 3.4) where B is the sum of the birational transform of B' and the reduced exceptional divisor of $X \to X'$. One can then show that there is an $\mathbf{R}$-divisor D such that

$$0 \le D \sim_{\mathbf{R}} K_X + B + \sum nM_j$$

where the coefficients of D belong to some DCC set depending only on Λ, d. Then the pushdown D' of D satisfies

$$D' \sim_{\mathbf{R}} K_{X'} + B' + \sum nM'_j \equiv \sum (n - \mu_j)M'_j \equiv \rho M'_1$$

for some number ρ. Changing the indexes one can assume that ρ belongs to some ACC set depending only on Λ, d. Let $N = M - \mu_1 M_1$. Now the idea is to take s, t, with s maximal, so that

$$K_{X'} + B' + sD' + N' + tM'_1 \equiv K_{X'} + B' + M'$$

and that $(X', B' + sD' + N' + tM'_1)$ is generalized lc. If it happens to have $t = 0$, then s would belong to some DCC set and we can replace B' with $B' + sD'$ and replace M with N which has one less summand, and repeat the process. But if $t > 0$, then $(X', B' + sD' + N' + tM'_1)$ is generalized lc but not generalized klt. We cannot simply apply induction because the s, t may not belong to a DCC set. For simplicity assume $\lfloor B' + sD' \rfloor \ne 0$ and let S' be one of its components and assume S' is normal. The idea is to keep S' but to remove the other components of D' and increase t instead so that we get

$$K_{X'} + B' + \tilde{s}S' + N' + \tilde{t}M'_1 \equiv K_{X'} + B' + M'$$

for some $\tilde{s}$ and $\tilde{t} \ge t$ where S' is a component of $\lfloor B' + \tilde{s}S' \rfloor$. Now it turns out $\tilde{t}$ belongs to some DCC set and we can apply induction by restricting to S'.

2. Preliminaries

Notation and terminology. — All the varieties in this paper are quasi-projective over $\mathbf{C}$ unless stated otherwise. For definitions and basic properties of singularities of pairs such as log canonical (lc), Kawamata log terminal (klt), divisorially log terminal (dlt), purely log terminal (plt), and the log minimal model program (LMMP) we refer to [18]. We recall some notation:

- The sets of natural, integer, rational, and real numbers are respectively denoted as $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$.
- Divisors on normal varieties are always Weil $\mathbf{R}$-divisors unless otherwise stated.
- Let $X \to Z$ be a projective morphism from a normal variety. *Linear equivalence*, $\mathbf{Q}$-*linear equivalence*, $\mathbf{R}$-*linear equivalence*, and *numerical equivalence* over Z, between two $\mathbf{R}$-divisors D_1, D_2 on X are respectively denoted as $D_1 \sim D_2/Z$, $D_1 \sim_{\mathbf{Q}} D_2/Z$, $D_1 \sim_{\mathbf{R}} D_2/Z$, and $D_1 \equiv D_2/Z$. If Z is a point, we usually drop the Z.
- If $\phi \colon X \dashrightarrow X'$ is a birational morphism whose inverse does not contract divisors, and D is an $\mathbf{R}$-divisor on X, we usually write D' for $\phi_* D$. If X' is replaced by X'' (resp. Y) we usually write D'' (resp. D_Y) for $\phi_* D$.
- Let X, Y be normal varieties projective over some base Z, and $\phi \colon X \dashrightarrow Y$ a birational map/Z whose inverse does not contract any divisor. Let D be an $\mathbf{R}$-Cartier divisor on X such that D_Y is also $\mathbf{R}$-Cartier. We say ϕ is D-*negative* if there is a common resolution $g \colon W \to X$ and $h \colon W \to Y$ such that $E := g^* D - h^* D_Y$ is effective and exceptional/Y, and $\mathrm{Supp}\, g_* E$ contains all the exceptional divisors of ϕ.

ACC and DCC sets. — A sequence $\{a_i\}$ of numbers is *increasing* (resp. *strictly increasing*) if $a_i \le a_{i+1}$ (resp. $a_i < a_{i+1}$) for all i. The definition of a decreasing or strictly decreasing sequence is similar. A set Λ of real numbers satisfies DCC (descending chain condition) if it does not contain a strictly decreasing infinite sequence. A set Ω of real numbers satisfies ACC (ascending chain condition) if it does not contain a strictly increasing infinite sequence.

Lemma **2.1.** *— Let Λ and Ω be sets of nonnegative real numbers. Define*

$$\Lambda + \Omega = \{a + b \mid a \in \Lambda, b \in \Omega\}$$

and

$$\Lambda \cdot \Omega = \{ab \mid a \in \Lambda, b \in \Omega\}.$$

Then the following hold:

(1) If Λ and Ω are both ACC sets (resp. DCC sets), then $\Lambda + \Omega$ and $\Lambda \cdot \Omega$ are also ACC sets (resp. DCC sets).

(2) Let $\{a_i\} \subseteq \Lambda$ and $\{b_i\} \subseteq \Omega$ be sequences of numbers. Assume that both sequences are increasing and that one of them is strictly increasing. Then the sequences $\{a_i + b_i\}$ and $\{a_i b_i\}$ are strictly increasing.

(3) A statement similar to (2) holds if we replace 'increasing' by 'decreasing'.

(4) Let $m, l \in \mathbf{N}$. Assume that Λ is a DCC set and that $a \le l$ for every $a \in \Lambda$. Then the set $\{\langle ma \rangle \mid a \in \Lambda\}$ also satisfies DCC, where $\langle ma \rangle := ma - \lfloor ma \rfloor$, that is, the fractional part of ma.

Proof. — The proof is left to the reader. $\square$

Lemma **2.2.** — *Let d, r be natural numbers. Let X_i be a sequence of normal projective varieties of dimension d and Picard number one. Assume that $D_{1,i}, \ldots, D_{r,i}$ are nonzero **R**-Cartier divisors on X_i. Let $\lambda_{j,i}$ be the numbers such that $D_{j,i} \equiv \lambda_{j,i} D_{1,i}$. Then possibly after replacing the sequence with an infinite subsequence and rearranging the indexes, the sequence $\lambda_{j,i}$ is a decreasing sequence for each j.*

Proof. — Let $\rho_{j,k,i}$ be the numbers such that $D_{j,i} \equiv \rho_{j,k,i} D_{k,i}$. Replacing the sequence we may assume that for each j, k the sequence $\rho_{j,k,i}$ is increasing or decreasing. If $\rho_{j,k,i}$ is decreasing we write $j \trianglelefteq k$. This relation is associative, that is, if $j \trianglelefteq k$ and $k \trianglelefteq l$, then $j \trianglelefteq l$ because $\rho_{j,l,i} = \rho_{j,k,i} \rho_{k,l,i}$. So we can order the sequences of divisors according to this relation. Changing the indexes we may assume that $r \trianglelefteq \cdots \trianglelefteq 1$ which in particular means that the $\lambda_{j,i} = \rho_{j,1,i}$ form a decreasing sequence for each j. $\qquad\square$

Minimal models and Mori fibre spaces. — Let $X \to Z$ be a projective morphism of normal varieties and D an **R**-Cartier divisor on X. A normal variety Y projective over Z together with a birational map $\phi\colon X \dashrightarrow Y/Z$ whose inverse does not contract any divisor is called a *minimal model of* D *over* Z if:

(1) Y is **Q**-factorial,

(2) $D_Y = \phi_* D$ is *nef* /Z, and

(3) ϕ is D-negative.

If one can run an LMMP on D over Z which terminates with a **Q**-factorial model Y on which D_Y is *nef* /Z, then Y is a minimal model of D over Z.

On the other hand, we call Y a *Mori fibre space* of D over Z if Y satisfies the above conditions with condition (2) replaced by:

(2)$'$ there is an extremal contraction $Y \to T/Z$ such that $-D_Y$ is ample/T.

In practice, we consider minimal models and Mori fibre space for $K_{X'} + B' + M'$ where $(X', B' + M')$ is a generalized polarized pair.

Some notions and results of [10]. — For convenience we recall some technical notions and results of [10] which will be used in Section 3.

Let X be a normal projective variety, and let D be a big **Q**-Cartier **Q**-divisor on X. We say that D is *potentially birational* [10, Definition 3.5.3] if for any pair x and y of general points of X, possibly switching x and y, we can find $0 \le \Delta \sim_{\mathbf{Q}} (1 - \epsilon)D$ for some $0 < \epsilon < 1$ such that (X, Δ) is not klt at y but (X, Δ) is lc at x and $\{x\}$ is a non-klt centre.

Theorem **2.3** *[10, Theorem 3.5.4].* — *Let (X, B) be a klt pair, where X is projective of dimension d, and let H be an ample **Q**-divisor. Suppose there exist a constant $\gamma \ge 1$ and a family $V \to C$ of subvarieties of X with the following property: if x and y are two general points of X then, possibly switching x and y, we can find $c \in C$ and $0 \le \Delta_c \sim_{\mathbf{Q}} (1 - \delta)H$, for some $\delta > 0$, such that $(X, B + \Delta_c)$ is not klt at y and there is a unique non-klt place of $(X, B + \Delta_c)$ whose centre V_c contains x. Further assume there is a divisor D on W, the normalization of V_c, such that the linear system $|D|$ defines a birational map and $\gamma H|_W - D$ is pseudo-effective. Then mH is potentially birational, where $m = 2p^2\gamma + 1$ and $p = \dim V_c$.*

Theorem **2.4** *[10, Theorem 4.2]. — Let* Λ *be a subset of* $[0, 1]$ *which contains* 1. *Let* X *be a projective variety of dimension* d, *and let* V *be a subvariety, with normalization* W. *Suppose we are given an* **R**-*boundary* B *and an* **R**-*Cartier divisor* $G \geq 0$, *with the following properties:*

(1) *the coefficients of* B *belong to* Λ;

(2) (X, B) *is klt; and*

(3) *there is a unique non-klt place* v *for* $(X, B + G)$, *with centre* V.

Then there is an **R**-*boundary* B_W *on* W *whose coefficients belong to*

$$\big\{ a \,|\, 1 - a \in \mathrm{LCT}_{d-1}\big(D(\Lambda)\big) \big\} \cup \{1\}$$

such that the difference

$$(K_X + B + G)|_W - (K_W + B_W)$$

is pseudo-effective.

Now suppose that V *is the general member of a covering family of subvarieties of* X. *Let* $\psi :$ $U \to W$ *be a log resolution of* (W, B_W), *and let* B_U *be the sum of the birational transform of* B_W *and the reduced exceptional divisor of* ψ. *Then*

$$K_U + B_U \geq (K_X + B)|_U.$$

The notation $|_W$ *and* $|_U$ *mean pullback to* W *and* U *respectively.*

Remark **2.5.** — Assume that the Λ in 2.4 satisfies DCC. Then the hyperstandard set $D(\Lambda)$ also satisfies DCC, hence the set of lc thresholds $\mathrm{LCT}_{d-1}(D(\Lambda))$ satisfies ACC by the ACC for usual lc thresholds [10, Theorem 1.1]. Therefore, the set

$$\big\{ a \,|\, 1 - a \in \mathrm{LCT}_{d-1}\big(D(\Lambda)\big) \big\} \cup \{1\}$$

to which the coefficients of B_U belong, also satisfies DCC.

3. Effective birationality of $K + B + n M$

In this section, following [10] closely, we prove a special case of Theorem 1.3 (see 3.4) which will be used in Sections 7 and 8 in proving Theorems 1.6 and 1.3. This special case concerns effective birationality for big divisors of the form $K_X + B + n M$ where (X, B) is projective lc, rM is nef and Cartier, and n/r is large enough. Running an LMMP on $K_X + B + n M$ preserves the nef and Cartier properties of rM by boundedness of length of extremal rays [15] which allows one to apply the methods of [10]. In contrast if one runs an LMMP on $K_X + B + M$, the nef and Cartier properties of rM may be lost, hence one needs to consider generalized polarized pairs which will be discussed in later sections.

First we prove a few lemmas.

Lemma **3.1.** — *Let* X *be a normal projective variety,* D *a big* **Q***-Cartier* **Q***-divisor, and* G *a nef* **Q***-Cartier* **Q***-divisor on* X. *If* D *is potentially birational, then* D + G *is also potentially birational. In particular,* $|K_X + \lceil D + G \rceil|$ *defines a birational map.*

Proof. — Write $D \sim_{\mathbf{Q}} A + B$ with B effective and A ample. By definition, for any pair $x, y \in X$ of general points, possibly after switching x, y, there exist $\epsilon \in (0, 1)$ and a **Q**-divisor $0 \le \Delta \sim_{\mathbf{Q}} (1 - \epsilon)D$ such that (X, Δ) is not klt at y but it is lc at x and $\{x\}$ is a non-klt centre. Now if $\epsilon' \in (0, \epsilon)$ is rational, then we can find

$$0 \le \Delta' \sim_{\mathbf{Q}} \Delta + (\epsilon - \epsilon')B + (\epsilon - \epsilon')A + (1 - \epsilon')G \sim_{\mathbf{Q}} (1 - \epsilon')(D + G)$$

so that (X, Δ') satisfies the same above properties as (X, Δ) at x, y. So $D + G$ is potentially birational. To get the last claim, just apply [9, Lemma 2.3.4 (1)]. □

Lemma **3.2.** — *Let* Λ *be a DCC set of nonnegative real numbers, and* d, r *natural numbers. Then there is a real number* $t \in (0, 1)$ *depending only on* Λ, d, r *such that if:*

- (X, B) *is projective lc of dimension* d,
- *the coefficients of* B *are in* Λ,
- rM *is a nef Cartier divisor, and*
- $K_X + B + M$ *is a big divisor,*

then $K_X + tB + nM$ *is a big divisor for any natural number* $n > 2rd$.

Proof. — Since M is nef, it is enough to treat the case $n = 2rd + 1$. We can assume $1 \in \Lambda$. Let (X, B) and M be as in the statement of the lemma. Let $f : W \to X$ be a log resolution and let B_W be the sum of the birational transform of B and the reduced exceptional divisor of f, and let M_W be the pullback of M. Then we can replace (X, B) with (W, B_W) and replace M_W with M hence it is enough to only consider log smooth pairs.

We want to argue that, after extending Λ if necessary, it is enough to only consider the case when (X, B) is klt. If the lemma does not hold, then there is a sequence (X_i, B_i), M_i of log smooth lc pairs and nef **Q**-divisors satisfying the assumptions of the lemma but such that the pseudo-effective thresholds

$$b_i = \min\{a \ge 0 \mid K_{X_i} + aB_i + nM_i \text{ is pseudo-effective}\}$$

is a strictly increasing sequence of numbers approaching 1. Now by extending Λ and decreasing the coefficients in B_i which are equal to 1, we can assume that (X_i, B_i) are klt. To get a contradiction it is obviously enough to only consider this sequence hence we only need to consider the klt case.

Now let (X, B) and M be as in the statement of the lemma where we assume (X, B) is log smooth klt. Let b be the pseudo-effective threshold as defined above. We may assume $b > 0$. By Lemma 4.4(2) below, we can run an LMMP on $K_X + bB + nM$

which ends with a minimal model X' on which $K_{X'} + bB' + nM'$ is semi-ample defining a contraction $X' \to T'$. Since $b > 0$, a general fibre of $X' \to T'$ is positive-dimensional and the restriction of B' to it is big, by the bigness of $K_{X'} + B' + nM'$ and the definition of b. So relying on Lemma 4.4(2) once more, we can also run an LMMP/T' on $K_{X'} + nM'$ with scaling of bB' which terminates with a Mori fibre space. Denote the end result again by X' and the Mori fibre space structure by $X' \to S'$. By Lemma 4.4(3), both LMMP's are M-trivial and hence the Cartier and nefness of rM is preserved in the process. Now since $K_{X'} + bB' + nM' \equiv 0/S'$ and since $n > 2rd$, $M' \equiv 0/S'$ by boundedness of length of extremal rays [15]. In particular, if F' is a general fibre of $X' \to S'$, then

$$K_{F'} + \Delta' := \left(K_{X'} + bB'\right)\big|_{F'} \equiv \left(K_{X'} + bB' + nM'\right)\big|_{F'} \sim_{\mathbf{R}} 0.$$

By construction, $(K_{X'} + B' + M')|_{F'}$ is big and $M'|_{F'} \equiv 0$, so $B'|_{F'}$ is not zero and its coefficients belong to Λ. Therefore, b is bounded away from 1 otherwise we get a contradiction with the ACC property of [10, Theorem 1.5]. Thus there is $t_0 \in (0, 1)$ depending only on Λ, d, r such that $K_X + t_0 B + nM$ is pseudo-effective. Now take $t = \frac{t_0 + 1}{2}$.   □

We should point out that although we have used (and continue to use) Lemma 4.4 but its proof does not rely on any of the results of this section.

Lemma **3.3.** *— Let* X *be a normal projective variety,* L *a big* **R**-*divisor, and* M *a nef* **Q**-*divisor which is not numerically trivial. Then* $\mathrm{vol}(L + nM)$ *goes to* ∞ *as n goes to* ∞.

Proof. — We may write $L \sim_{\mathbf{R}} A + D$ where A is ample **Q**-Cartier and $D \geq 0$. Thus

$$\mathrm{vol}(L + nM) \geq \mathrm{vol}(A + nM) \geq n^\nu A^{d-\nu} \cdot M^\nu$$

where $d = \dim X$ and ν is the numerical dimension of M. Since A is ample and $\nu > 0$, $A^{d-\nu} \cdot M^\nu > 0$. Hence the above volume goes to infinity as n goes to infinity.   □

Proposition **3.4.** *— Let* $\Lambda \subset [0, 1]$ *be a DCC set of nonnegative real numbers and let* d, r *be natural numbers. Then there exists a natural number m depending only on* Λ, d, r *such that if:*

- *n is a natural number satisfying $n > 2rd$ and $r|n$,*
- *(X, B) is projective lc of dimension d,*
- *the coefficients of B are in Λ,*
- *rM is a nef Cartier divisor, and*
- *$K_X + B + M$ is a big divisor,*

then $|m(K_X + B + nM)|$ *defines a birational map.*

Proof. — *Step 1.* We prove the proposition by induction on d. In particular, we may assume that the proposition holds in dimension $< d$.

Fix $\beta > 0$. Pick (X, B), M, and n as in the proposition. Assume that $\mathrm{vol}(K_X + B + n\mathrm{M}) > \beta$. We first prove the result for such (X, B), M, and n. At the end, in Steps 6 and 7, we treat the general case.

As in the proof of Lemma 3.2, by extending Λ, by taking a log resolution of (X, B), and by decreasing the coefficients of B, we can assume that (X, B) is klt.

Step 2. By Lemma 3.2, $K_X + b\mathrm{B} + n\mathrm{M}$ is big for some $b \in (0, 1)$ depending only on Λ, d, r. Thus

$$\mathrm{vol}\left(K_X + \frac{1}{2}(b+1)\mathrm{B} + n\mathrm{M}\right)$$

$$= \mathrm{vol}\left(\frac{1}{2}(K_X + b\mathrm{B} + n\mathrm{M} + K_X + \mathrm{B} + n\mathrm{M})\right)$$

$$> \mathrm{vol}\left(\frac{1}{2}(K_X + \mathrm{B} + n\mathrm{M})\right) > \frac{1}{2^d}\beta.$$

Replacing b by $\frac{1}{2}(b+1)$ we may assume that

$$\mathrm{vol}(K_X + b\mathrm{B} + n\mathrm{M}) > \beta' := \frac{1}{2^d}\beta.$$

Moreover, there exists a natural number p depending only on Λ and b (and hence only on Λ, d, r) and there exists a boundary B$'$ such that pB$'$ is an integral divisor and bB $\leq$ B$' \leq$ B: this follows from the fact that we can find p so that $\lambda - b\lambda > \frac{1}{p}$ for every nonzero $\lambda \in \Lambda$ which in turn implies that for each λ we can find an integer $0 \leq i \leq p$ such that $b\lambda \leq \frac{i}{p} \leq \lambda$. By the calculation above, $\mathrm{vol}(K_X + \mathrm{B}' + n\mathrm{M}) > \beta'$. Replacing B with B$'$, and β with β', we can assume $\Lambda = \{i/p \mid 0 \leq i \leq p\}$ and that pB is integral.

Step 3. Applying Lemma 4.4(2) below, we can replace X with the lc model (= ample model) of $K_X + B + n\mathrm{M}$ so that we can assume that $K_X + B + n\mathrm{M}$ is ample keeping $r\mathrm{M}$ nef and Cartier. Since $\mathrm{vol}(K_X + B + n\mathrm{M}) > \beta$, there is a natural number $k > 0$ depending only on d, β, such that

$$\mathrm{vol}\big(k(K_X + B + n\mathrm{M})\big) > (2d)^d.$$

Applying [10, Lemma 7.1] to the log pair (X, B) and the big divisor $k(K_X + B + n\mathrm{M})$, we get a covering family V $\to$ C of subvarieties of X such that if x and y are two general points of X, then we may find $c \in$ C and

$$0 \leq \Delta_c \sim_{\mathbf{R}} k(K_X + B + n\mathrm{M})$$

such that (X, B $+ \Delta_c$) is not klt at y but it is lc at x and there is a unique non-klt place of (X, B $+ \Delta_c$) whose centre is equal to V$_c$ which contains x.

Step 4. Let $H := 2k(K_X + B + nM)$. In this step we make the necessary preparations in order to apply [10, Theorem 3.5.4] (= Theorem 2.3 above). To do this we need to find a natural number γ, depending only on Λ, d, r, and find a divisor D on the normalization W of V_c such that $\gamma H|_W - D$ is pseudo-effective and $|D|$ defines a birational map.

If $\dim W = \dim V_c = 0$, then γ, D exist trivially (and H is potentially birational). So assume that $\dim W \geq 1$. Now applying the adjunction formula of [10, Theorem 4.2] (= Theorem 2.4 above) to the klt pair (X, B) and the divisor Δ_c, and taking into account Remark 2.5, we can find a boundary B_W on W whose coefficients belong to a DCC set Λ' uniquely determined by Λ, d, such that the difference

$$(*) \quad (K_X + B + \Delta_c)|_W - (K_W + B_W)$$

is a pseudo-effective divisor. Further, let $\psi : U \to W$ be a log resolution of (W, B_W) and let B_U be the sum of the strict transform of B_W and the reduced exceptional divisor of ψ. Then

$$K_U + B_U \geq (K_X + B)|_U.$$

Denote by $M_U := M|_U$, the pullback of M to U, by the composition

$$U \to W \to V_c \hookrightarrow X$$

which is birational onto its image. Then

$$K_U + B_U + M_U \geq (K_X + B + M)|_U.$$

Hence $K_U + B_U + M_U$ is big because $(K_X + B + M)|_U$ is big being the pullback of the big divisor $K_X + B + M$ to a smooth model of the general subvariety V_c.

Since the coefficients of B_U belong to the DCC set Λ', since rM_U is a nef Cartier divisor, and since $n > 2rd$, the induction hypothesis implies that $|m(K_U + B_U + nM_U)|$ defines a birational map for some $m > 0$ depending only on Λ' (and hence on Λ) and d, r. Thus $|m(K_W + B_W + nM_W)|$ also defines a birational map since it contains the direct image of $|m(K_U + B_U + nM_U)|$ where M_W denotes the pullback of M to W.

Note that the difference

$$(K_X + B + nM + \Delta_c)|_W - (K_W + B_W + nM_W)$$

$$\sim_{\mathbf{R}} (k+1)(K_X + B + nM)|_W - (K_W + B_W + nM_W)$$

is a pseudo-effective divisor by $(*)$ above. Now let $D := m(K_W + B_W + nM_W)$ and let γ be the smallest natural number satisfying $\gamma \geq m(k+1)/2k$. Then $\gamma H|_W - D$ is a pseudo-effective divisor and $|D|$ defines a birational map as required.

Step 5. By Step 4 and Theorem 2.3,

$$m'H = 2m'k(K_X + B + nM)$$

is potentially birational for some

$$m' \leq 2(d-1)^2 \gamma + 1.$$

Thus by Lemma 3.1,

$$2m'kp(\mathrm{K_X} + \mathrm{B} + n\mathrm{M}) + n\mathrm{M}$$

is also potentially birational and

$$\left| \mathrm{K_X} + \left\lceil 2m'kp(\mathrm{K_X} + \mathrm{B} + n\mathrm{M}) + n\mathrm{M} \right\rceil \right|$$

defines a birational map where p is as in Step 2 (recall that $p\mathrm{B}$ is an integral divisor). Since

$$\mathrm{K_X} + \left\lceil 2m'kp(\mathrm{K_X} + \mathrm{B} + n\mathrm{M}) + n\mathrm{M} \right\rceil \leq \left\lfloor \left(2m'kp + 1\right)(\mathrm{K_X} + \mathrm{B} + n\mathrm{M}) \right\rfloor$$

the linear system

$$\left| \left\lfloor \left(2m'kp + 1\right)(\mathrm{K_X} + \mathrm{B} + n\mathrm{M}) \right\rfloor \right|$$

also defines a birational map. Now the number $m'' := 2m'kp + 1$ only depends on the data Λ, d, r, β.

Step 6. Now we go back to Step 1. We will show that there exist a natural number q and a real number $\alpha > 0$ depending only on Λ, d, r, such that if (X, B), M, n are as in the statement of the proposition and if $n \geq q$, then $\mathrm{vol}(\mathrm{K_X} + \mathrm{B} + n\mathrm{M}) > \alpha$. If this is not true, then we can find a sequence $(\mathrm{X}_i, \mathrm{B}_i)$, M_i, n_i satisfying the assumptions of the proposition such that the n_i form a strictly increasing sequence approaching ∞ and the $\mathrm{vol}(\mathrm{K}_{\mathrm{X}_i} + \mathrm{B}_i + n_i\mathrm{M}_i)$ approach 0. By replacing X_i with a minimal model of $\mathrm{K}_{\mathrm{X}_i} + \mathrm{B}_i + n_i\mathrm{M}_i$, we may assume that $\mathrm{K}_{\mathrm{X}_i} + \mathrm{B}_i + n_i\mathrm{M}_i$ is nef. We can also assume that ν, the numerical dimension of M_i, is independent of i. We may assume $\nu > 0$ otherwise we can get a contradiction using [10, Theorem 1.3].

By Lemma 3.3, for each i, there is n'_i the largest natural number divisible by r such that $\mathrm{vol}(\mathrm{K}_{\mathrm{X}_i} + \mathrm{B}_i + n'_i\mathrm{M}_i) < 1$. We show that the volume $\mathrm{vol}(\mathrm{K}_{\mathrm{X}_i} + \mathrm{B}_i + (2n'_i - 1)\mathrm{M}_i)$ is bounded from above. This follows from

$$\begin{aligned}
2^d &> \mathrm{vol}\big(2\big(\mathrm{K}_{\mathrm{X}_i} + \mathrm{B}_i + n'_i\mathrm{M}_i\big)\big) \\
&= \mathrm{vol}\big(\mathrm{K}_{\mathrm{X}_i} + \mathrm{B}_i + \big(2n'_i - 1\big)\mathrm{M}_i + \mathrm{K}_{\mathrm{X}_i} + \mathrm{B}_i + \mathrm{M}_i\big) \\
&> \mathrm{vol}\big(\mathrm{K}_{\mathrm{X}_i} + \mathrm{B}_i + \big(2n'_i - 1\big)\mathrm{M}_i\big)
\end{aligned}$$

where we use the assumption that $\mathrm{K}_{\mathrm{X}_i} + \mathrm{B}_i + \mathrm{M}_i$ is big.

On the other hand, since

$$\mathrm{vol}\big(\big(\mathrm{K}_{\mathrm{X}_i} + \mathrm{B}_i + \big(n'_i + r\big)\mathrm{M}_i\big)\big) \geq 1,$$

by Steps 2–5 above, we may assume that there is an m'' depending only on Λ, d, r such that

$$\left| m''\big(K_{X_i} + B_i + \big(n_i' + r\big)M_i\big) \right|$$

defines a birational map for every i. In particular, there exist resolutions $f_i \colon Y_i \to X_i$ such that

$$P_i := f_i^* m''\big(K_{X_i} + B_i + \big(n_i' + r\big)M_i\big) \sim H_i + G_i$$

where H_i is big and base point free and G_i is effective. So we can calculate

$$2^d\big(m''\big)^d > \mathrm{vol}\big(m''\big(K_{X_i} + B_i + \big(2n_i' - 1\big)M_i\big)\big)$$
$$= \big(P_i + m''\big(n_i' - r - 1\big)f_i^*M_i\big)^d \geq \big(m''\big(n_i' - r - 1\big)\big)^\nu H_i^{d-\nu} \cdot f_i^*M_i^\nu$$

which gives a contradiction as $\lim(n_i' - r - 1) = \infty$ and $H_i^{d-\nu} \cdot f_i^*M_i^\nu \geq \frac{1}{r^\nu}$.

Step 7. Let q, α be as in Step 6. In this step we show that there is $\beta > 0$ depending only on Λ, d, r such that $\mathrm{vol}(K_X + B + nM) > \beta$ for any (X, B), M, n as in the statement of the proposition. We may assume $q > n$ otherwise we can use Step 6. Let $s = \frac{n-1}{q-1}$. Then

$$\mathrm{vol}(K_X + B + nM) = \mathrm{vol}\big((1 - s)(K_X + B + M) + s(K_X + B + qM)\big)$$
$$\geq s^d \, \mathrm{vol}(K_X + B + qM) > s^d\alpha \geq \frac{\alpha}{(q-1)^d} =: \beta.$$

This completes the proof of the proposition. $\qquad\square$

4. Generalized polarized pairs

In this section, we define generalized lc and klt singularities, discuss some of their basic properties, and then define generalized lc thresholds for generalized polarized pairs. Next we consider running the log minimal model program for these pairs, and use it to extract divisors with generalized log discrepancy <1. Then we define generalized adjunction and discuss DCC and ACC properties of coefficients in the boundary and nef parts of generalized polarized pairs under this adjunction.

Generalized singularities. — We already defined generalized polarized pairs in the introduction. Now we define their singularities.

Definition **4.1.** — *Let* $(X', B' + M')$ *be a generalized polarized pair as in* 1.4 *which comes with the data* $X \xrightarrow{f} X' \to Z$ *and* M. *Let* E *be a prime divisor on some birational model of* X'. *We*

define the generalized log discrepancy *of* E *with respect to the above generalized polarized pair as follows. After replacing* X, *we may assume* E *is a prime divisor on* X. *We can write*

$$K_X + B + M = f^*(K_{X'} + B' + M')$$

for some **R**-*divisor* B. *The generalized log discrepancy of* E *is defined to be* $1 - b$ *where b is the coefficient of* E *in* B.

We say that $(X', B' + M')$ is generalized lc *(resp.* generalized klt*) if the generalized log discrepancy of any prime divisor is* ≥ 0 *(resp.* > 0*). If* f *is a log resolution of* (X', B'), *then generalized lc (resp. generalized klt) is equivalent to the coefficients of* B *being* ≤ 1 *(resp.* < 1*). If the generalized log discrepancy of* E *is* ≤ 0, *we call the image of* E *in* X′ *a* generalized non-klt centre. *If* $(X', B'+M')$ *is generalized lc, a non-klt centre is also referred to as a* generalized lc centre.

Remark **4.2.** — We use the notation of 4.1.

(1) Note that Z does not play any role in the definition of singularities. That is because singularities are local in nature over X′, so one can simply assume $X' \to Z$ is the identity map. The same applies to generalized lc thresholds defined below (4.3) and in general to notions and statements that are local.

(2) Assume that $(X', B' + M')$ is generalized klt. Let D′ be an effective **R**-Cartier divisor. Then from the definitions we can easily see that $(X', B' + \epsilon D' + M')$ is generalized klt with boundary part $B' + \epsilon D'$ and nef part M, for any small $\epsilon > 0$.

Now assume that D′ is ample/Z. Then for any $a > 0$ we can find a boundary

$$\Delta' \sim_{\mathbf{R}} B' + aD' + M'/Z$$

such that (X', Δ') is klt.

(3) Assume that $K_{X'} + B'$ is **R**-Cartier and write $K_X + \tilde{B} = f^*(K_{X'} + B')$ and $f^*M' = M + E$. By the negativity lemma [20, Lemma 1.1], $E \geq 0$. Thus $B = \tilde{B} + E \geq \tilde{B}$. Therefore, if $(X', B' + M')$ is generalized lc (resp. generalized klt), then (X', B') is lc (resp. klt).

(4) Assume that $M \sim_{\mathbf{R}} 0/X'$. Then $(X', B' + M')$ is generalized lc (resp. generalized klt) iff (X', B') is generalized lc (resp. generalized klt). Indeed in this case $M = f^*M'$ hence $K_X + B = f^*(K_{X'} + B')$ which implies the claim. In this situation M′ does not contribute to the singularities even if its coefficients are large. In contrast, the larger the coefficients of B, the worse the singularities.

(5) In general, M does contribute to singularities. For example, assume $X' = \mathbf{P}^2$ and that f is the blowup of a point x'. Let E be the exceptional divisor, L′ a line passing through x' and L the birational transform of L′.

If $B' = 0$ and $M = 2L$, then we can calculate $B = E$ hence $(X', B' + M')$ is generalized lc but not generalized klt. However, if $B' = L'$ and $M = 2L$, then $(X', B' + M')$ is not generalized lc because in this case $B = L + 2E$.

(6) Assume we are given a contraction $X' \to Y/Z$. We may assume f is a log resolution of (X', B'). Let F be a general fibre of $X \to Y$, F' the corresponding fibre of $X' \to Y$, and $g\colon F \to F'$ the induced morphism. Let

$$B_F = B|_F, \quad M_F = M|_F, \quad B_{F'} = g_*B_F, \quad M_{F'} = g_*M_F.$$

Then $(F', B_{F'} + M_{F'})$ is a generalized polarized pair with the data $F \to F' \to Z$ and M_F. Moreover,

$$K_{F'} + B_{F'} + M_{F'} = \left(K_{X'} + B' + M'\right)\big|_{F'}.$$

In addition, $B_{F'} = B'|_{F'}$ and $M_{F'} = M'|_{F'}$: note that since F' is a general fibre, B' and M' are **R**-Cartier along any codimension one point of F' hence we can define these restrictions.

(7) Let $\phi\colon X'' \to X'$ be a birational contraction from a normal variety. We can assume $X \dashrightarrow X''$ is a morphism. Let B'', M'' be the pushdowns of B, M. Then

$$K_{X''} + B'' + M'' = \phi^*\left(K_{X'} + B' + M'\right).$$

Now assume that B'' is a boundary. Then we can naturally consider $(X'', B'' + M'')$ as a generalized polarized pair with boundary part B'' and nef part M. One may think of $(X'', B'' + M'')$ as a crepant model of $(X', B' + M')$.

Definition **4.3.** — *Let $(X', B' + M')$ be a generalized polarized pair as in 1.4 which comes with the data $X \xrightarrow{f} X' \to Z$ and M. Assume that D' on X' is an effective* **R**-*divisor and that N on X is an* **R**-*divisor which is nef /Z and that $D' + N'$ is* **R**-*Cartier. The* generalized lc threshold *of $D' + N'$ with respect to $(X', B' + M')$ (more precisely, with respect to the above data) is defined as*

$$\sup\left\{s \mid \left(X', B' + sD' + M' + sN'\right) \text{ is generalized lc}\right\}$$

where the pair in the definition has boundary part $B' + sD'$ and nef part $M + sN$.

By the negativity lemma, $G := f^*(D' + N') - N \geq 0$. Thus we can write

$$K_X + B + M = f^*\left(K_{X'} + B' + M'\right)$$

and

$$K_X + B + sG + M + sN = f^*\left(K_{X'} + B' + sD' + M' + sN'\right).$$

In particular, if $(X', B' + M')$ is generalized lc, then the just defined generalized lc threshold is nonnegative. However, the threshold might be $+\infty$: this happens when $D' = 0$ and $N \sim_{\mathbf{R}} 0/X'$.

As pointed earlier, the generalized lc threshold is local over X', so we can usually assume $X' \to Z$ is the identity map. When $M = N = 0$, we recover the usual lc threshold of D' with respect to (X', B').

LMMP for generalized polarized pairs. — Let $(X', B' + M')$ be a **Q**-factorial generalized lc polarized pair with data $X \xrightarrow{f} X' \to Z$ and M. One can ask whether one can run an LMMP/Z on $K_{X'} + B' + M'$ and whether it terminates. We cannot answer this question in such generality but we will put some extra assumptions under which the answer would be yes.

Assume that $K_{X'} + B' + M' + A'$ is *nef* /Z for some **R**-Cartier divisor $A' \geq 0$ which is big/Z. Moreover, assume

(∗)   for any $s \in (0, 1)$ there is a boundary $\Delta' \sim_{\mathbf{R}} B' + sA' + M'/Z$ such that $(X', \Delta' + (1 - s)A')$ is klt.

Condition (∗) is automatically satisfied if A' is general ample/Z and either

 (i) $(X', B' + M')$ is generalized klt, or
(ii) $(X', B' + M')$ is generalized lc and $(X', 0)$ is klt.

We will show that we can run the LMMP/Z on $K_{X'} + B' + M'$ with scaling of A' (However, we do not know whether it terminates). Let

$$\lambda = \min\{t \geq 0 \mid K_{X'} + B' + M' + tA' \text{ is } nef /Z\}.$$

We may assume $\lambda > 0$. Replacing A' with $\lambda A'$ we may assume $\lambda = 1$. By assumption we can find a number $0 < s < 1$ and a boundary $\Delta' \sim_{\mathbf{R}} B' + sA' + M'/Z$ such that $(X', \Delta' + (1 - s)A')$ is klt. Now by [1, Lemma 3.1], there is an extremal ray R'/Z such that $(K_{X'} + \Delta') \cdot R' < 0$ and

$$\left(K_{X'} + \Delta' + (1 - s)A'\right) \cdot R' = 0.$$

In particular, $(K_{X'} + B' + M') \cdot R' < 0$ and

$$\left(K_{X'} + B' + M' + A'\right) \cdot R' = 0.$$

Moreover, R' can be contracted and its flip exists if it is of flipping type. If R' defines a Mori fibre space we stop. Otherwise let $X' \dashrightarrow X''$ be the divisorial contraction or the flip of R'.

Replacing X we may assume $X \dashrightarrow X''$ is a morphism. Then $(X'', B'' + M'')$ is naturally a generalized lc polarized pair with boundary part B'' and nef part M. Moreover, $K_{X''} + B'' + M'' + A''$ is *nef* /Z and (∗) is preserved. Repeating the process gives the LMMP.

Now we show the LMMP terminates under suitable assumptions.

Lemma **4.4.** *— Let* $(X', B' + M')$ *be a* **Q***-factorial generalized lc polarized pair of dimension* d *with data* $X \xrightarrow{f} X' \to Z$ *and* M. *Assume that* $(X', B' + M')$ *satisfies* (i) *or* (ii) *above. Run an LMMP/Z on* $K_{X'} + B' + M'$ *with scaling of some general ample/Z* **R***-Cartier divisor* $A' \geq 0$. *Then the following hold:*

(1) *Assume that $K_{X'} + B' + M'$ is not pseudo-effective$/Z$. Then the LMMP terminates with a Mori fibre space.*

(2) *Assume that*
- $K_{X'} + B' + M'$ *is pseudo-effective$/Z$,*
- $(X', B' + M')$ *is generalized klt, and that*
- $K_{X'} + (1 + \alpha)B' + (1 + \beta)M'$ *is $\mathbf{R}$-Cartier and big$/Z$ for some $\alpha, \beta \geq 0$.*

   Then the LMMP terminates with a minimal model X'' and $K_{X''} + B'' + M''$ is semi-ample$/Z$, hence it defines a contraction $\phi : X'' \to T''/Z$. If moreover a general fibre of ϕ is positive-dimensional and if the restriction of B'' to it is nonzero, then we can run the LMMP$/T''$ on $K_{X''} + M''$ with scaling of B'' which terminates with a Mori fibre space of $K_{X''} + M''$ over both T'' and Z.

(3) *Assume $X \to X'$ is the identity morphism and that $M = \sum \mu_j M_j$ where $\mu_j \geq 0$ and M_j are Cartier nef $/Z$ divisors. Pick j and assume $\mu_j > 2d$. Then the above LMMP's are M_j'-trivial. In particular, the LMMP's preserve the Cartier and the nefness$/Z$ of M_j'. Moreover, under the assumptions of (2) and assuming ϕ is birational, $M_j'' \equiv 0/T''$ and M_j'' is the pullback of some Cartier divisor on T''.*

Proof. — (1) Since $K_{X'} + B' + M'$ is not pseudo-effective$/Z$, the LMMP is also an LMMP on $K_{X'} + B' + \epsilon A' + M'$ with scaling of $(1 - \epsilon)A'$ for some $\epsilon > 0$. Now we can find a boundary

$$\Delta' \sim_{\mathbf{R}} B' + \epsilon A' + M'/Z$$

such that $(X', \Delta' + (1 - \epsilon)A')$ is klt. The claim then follows from [3] as the LMMP is an LMMP$/Z$ on $K_{X'} + \Delta'$ with scaling of $(1 - \epsilon)A'$.

(2) As

$$K_{X'} + (1 + \alpha)B' + (1 + \beta)M'$$

is big$/Z$, it is $\mathbf{R}$-linearly equivalent to some $P' + G'$ over Z where P' is ample and $G' \geq 0$. Now if $\epsilon > 0$ is small, then

$$(1 + \epsilon)\bigl(K_{X'} + B' + M'\bigr) \sim_{\mathbf{R}} K_{X'} + (1 - \epsilon\alpha)B' + (1 - \epsilon\beta)M' + \epsilon P' + \epsilon G'$$
$$\sim_{\mathbf{R}} K_{X'} + \Delta'/Z$$

for some Δ' such that (X', Δ') is klt and Δ' is big$/Z$. The LMMP is also an LMMP$/Z$ on $K_{X'} + \Delta'$ with scaling of $(1 + \epsilon)A'$ which terminates on some model X'' by [3]. By the base point free theorem for klt pairs with big boundary divisor [3, Corollary 3.9.2], $K_{X''} + \Delta''$ is semi-ample$/Z$ hence $K_{X''} + B'' + M''$ is semi-ample$/Z$ and so it defines a contraction $\phi : X'' \to T''$.

Now assume a general fibre of $\phi : X'' \to T''$ is positive-dimensional and the restriction of B'' to it is nonzero. In particular, this implies that $K_{X''} + M''$ is not pseudo-effective

over T''. Since

$$\frac{1}{1+\epsilon}\left(K_{X''}+\Delta''\right) \equiv K_{X''}+B''+M'' \equiv 0/T'',$$

running the LMMP/T'' on $K_{X''}+M''$ with scaling of B'' is the same as running the LMMP/T'' on $K_{X''}+\Delta''-\tau B''$ with scaling of $\tau B''$ for some small $\tau > 0$ and this terminates with a Mori fibre space over T'' and also over Z, by [3]. Note that, $\Delta''-\tau B'' \geq 0$ by construction.

(3) Each step of those LMMP's is M'_j-trivial and preserves the Cartier and the nefness/Z of M'_j by boundedness of the length of extremal rays and the cone theorem [15], [18, Theorem 3.7 (1) and (4)]. Under the assumptions of (2) and assuming ϕ is birational, to show that M'_j is the pullback of some Cartier divisor on T'', it is enough to show that $X'' \to T''$ decomposes into a sequence of extremal contractions which are negative with respect to certain klt pairs. We write this more precisely.

Since Δ' in the proof of (2) is big/Z, we can assume $\Delta'' \geq C''$ for some ample $\mathbf{Q}$-divisor C''. Since $K_{X''}+\Delta'' \equiv 0/T''$, if $X'' \to T''$ is not an isomorphism, then there is a $(K_{X''}+\Delta''-C'')$-negative extremal ray which gives a contraction $X'' \to X''_2/T''$. In particular M''_j is the pullback of a Cartier divisor on X''_2 [18, Theorem 3.7 (4)]. Now repeat the process with X''_2 and so on. Since ϕ is birational by assumption, the process ends with T'' hence we can indeed decompose $X'' \to T''$ into a sequence of extremal contractions as required.   □

We will apply the LMMP to birationally extract certain divisors for a generalized polarized pair.

Lemma **4.5.** — *Let* $(X', B'+M')$ *be a generalized lc polarized pair with data* $X \xrightarrow{f} X' \to Z$ *and* M. *Let* $S_1, \ldots, S_r$ *be prime divisors on birational models of* X' *which are exceptional/X' and whose generalized log discrepancies with respect to* $(X', B'+M')$ *are at most* 1. *Then perhaps after replacing* f *with a high resolution, there exist a* $\mathbf{Q}$-*factorial generalized lc polarized pair* $(X'', B''+M'')$ *with data* $X \xrightarrow{g} X'' \to Z$ *and* M, *and a projective birational morphism* $\phi \colon X'' \to X'$ *such that*

- $S_1, \ldots, S_r$ *appear as divisors on* X'',
- *each exceptional divisor of* ϕ *is one of the* S_i *or is a component of* $\lfloor B'' \rfloor$, *and*
- $K_{X''}+B''+M'' = \phi^*(K_{X'}+B'+M')$.

In particular, the exceptional divisors of ϕ *are exactly the* S_i *if* $(X', B'+M')$ *is generalized klt.*

Proof. — Replacing X we may assume the S_i are divisors on X and that f is a log resolution of (X', B'). Let $E_1, E_2, \ldots$ be the exceptional divisors of f where we can assume $E_i = S_i$ for $i \leq r$. Write

$$K_X+B+M = f^*\left(K_{X'}+B'+M'\right)$$

and let $\Delta = B + E$ where $E := \sum_{i>r} a_i E_i$ and a_i is the generalized log discrepancy of E_i (by definition a_i is equal to $1 - b_i$ where b_i is the coefficient of E_i in B). Then Δ is a boundary and

$$K_X + \Delta + M = f^*\left(K_{X'} + B' + M'\right) + E \equiv E/X'$$

with $E \geq 0$ exceptional$/X'$. By construction, none of the S_i are components of E.

Now run an LMMP$/X'$ on $K_X + \Delta + M$ with scaling of some ample divisor. This is also an LMMP$/X'$ on E. In the course of the LMMP we arrive at a model X'' on which $K_{X''} + \Delta'' + M''$ is a limit of movable$/X'$ divisors hence it is nef on the general curves$/X'$ of any exceptional divisor of $X'' \to X'$ where Δ'', M'' are the pushdowns of Δ, M. But since E'' is effective and exceptional$/X'$, $E'' = 0$ by the general negativity lemma (cf. [2, Lemma 3.3 and the proof of Theorem 3.4]).

Note that since the LMMP contracts E, we have $\Delta'' = B''$. So we can write

$$K_{X''} + B'' + M'' = \phi^*\left(K_{X'} + B' + M'\right)$$

where ϕ is the morphism $X'' \to X'$. By construction, none of the S_i is contracted by the LMMP. Moreover, any exceptional divisor of ϕ is one of the S_i or is a component of $\lfloor B'' \rfloor$. In particular, the exceptional divisors of ϕ are exactly the S_i if $(X', B' + M')$ is generalized klt. Note that X'' is **Q**-factorial by construction. $\qquad\square$

Lemma **4.6.** — *Under the notation and assumptions of Lemma 4.5, further assume that (X', C') is klt for some C', and that the generalized log discrepancies of the S_i with respect to $(X', B' + M')$ are < 1. Then we can construct ϕ so that in addition it satisfies:*

- *its exceptional divisors are exactly $S_1, \ldots, S_r$, and*
- *if $r = 1$ and X' is **Q**-factorial, then ϕ is an extremal contraction.*

Proof. — Since (X', C') is klt and $(X', B' + M')$ is generalized lc,

$$\left(X', (1 - \epsilon)B' + \epsilon C' + (1 - \epsilon)M'\right)$$

is generalized klt for any small $\epsilon > 0$ with boundary part $\Gamma' := (1 - \epsilon)B' + \epsilon C'$ and nef part $(1 - \epsilon)M$. Moreover, the generalized log discrepancies of the S_i with respect to $(X', \Gamma' + (1 - \epsilon)M')$ are still less than 1. So by Lemma 4.5, there is $\phi \colon X'' \to X'$ which extracts exactly the S_i.

Now further assume that $r = 1$ and that X' is **Q**-factorial. By construction, we can write

$$K_{X''} + \Gamma'' + (1 - \epsilon)M'' = \phi^*\left(K_{X'} + \Gamma' + (1 - \epsilon)M'\right)$$

where Γ'' is the sum of the birational transform of Γ' and sS_1'' for some $s \in (0, 1)$. Now run an LMMP$/X'$ on $K_{X''} + \Gamma'' + \delta S_1'' + (1 - \epsilon)M''$ for some small $\delta > 0$ which is also

an LMMP on S_1''. Since X' is **Q**-factorial, the last step of the LMMP is an extremal contraction $X''' \to X'$ which contracts S_1''', the pushdown of S_1'', and $X'' \dashrightarrow X'''$ is an isomorphism in codimension one. Thus replacing X'' with X''' we can assume ϕ is extremal. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Generalized adjunction. — We define an adjunction formula for generalized polarized pairs similar to the traditional one.

Definition **4.7.** — *Let* $(X', B' + M')$ *be a generalized polarized pair with data* $X \xrightarrow{f} X' \to Z$ *and* M. *Assume that* S' *is the normalization of a component of* $\lfloor B' \rfloor$ *and* S *is its birational transform on* X. *Replacing* X *we may assume* f *is a log resolution of* (X', B'). *Write*

$$K_X + B + M = f^*(K_{X'} + B' + M')$$

and let

$$K_S + B_S + M_S := (K_X + B + M)|_S$$

where $B_S = (B - S)|_S$ *and* $M_S = M|_S$. *Let* g *be the induced morphism* $S \to S'$ *and let* $B_{S'} = g_* B_S$ *and* $M_{S'} = g_* M_S$. *Then we get the equality*

$$K_{S'} + B_{S'} + M_{S'} = (K_{X'} + B' + M')\big|_{S'}$$

which we refer to as generalized adjunction. *It is obvious that* $B_{S'}$ *depends on both* B' *and* M.

Now assume that $(X', B' + M')$ *is generalized lc. By Remark [4.8] below* $B_{S'}$ *is a boundary divisor on* S', *i.e. its coefficients belong to* $[0, 1]$. *We consider* $(S', B_{S'} + M_{S'})$ *as a generalized polarized pair which is determined by the boundary part* $B_{S'}$, *the morphisms* $S \to S' \to Z$, *and the nef part* M_S. *It is also clear that* $(S', B_{S'} + M_{S'})$ *is generalized lc if* $(X', B' + M')$ *is so because then*

$$K_S + B_S + M_S = g^*(K_{S'} + B_{S'} + M_{S'})$$

and the coefficients of B_S *are at most* 1.

Remark **4.8.** — We will argue that the $B_{S'}$ defined in [4.7] is indeed a boundary divisor on S', if $(X', B' + M')$ is generalized lc. The lc property immediately implies that the coefficients of $B_{S'}$ do not exceed 1, hence we only have to show that $B_{S'} \geq 0$. Moreover, if $K_{X'} + B'$ is **R**-Cartier, then $B_{S'} \geq 0$ follows from the usual divisorial adjunction: indeed in this case if $\tilde{B}_{S'}$ is the divisor given by the adjunction

$$K_{S'} + \tilde{B}_{S'} = (K_{X'} + B')\big|_{S'}$$

then it is well-known that $\tilde{B}_{S'}$ is a boundary divisor, and it is also clear from our definitions that $\tilde{B}_{S'} \leq B_{S'}$.

In practice when we apply generalized adjunction, X' will be $\mathbf{Q}$-factorial, hence $K_{X'} + B'$ will be $\mathbf{R}$-Cartier. But for the sake of completeness we treat the general case, i.e. the non-$\mathbf{R}$-Cartier $K_{X'} + B'$ case. We will reduce the statement to the situation $\dim X' = 2$ in which case $K_{X'} + B'$ turns out to be $\mathbf{R}$-Cartier automatically. Assume $\dim X' > 2$. Let H' be a general hypersurface section and G' its pullback to S'. Adding H' to B' we may assume H' is a component of $\lfloor B' \rfloor$. Both H' and G' are normal varieties. Let $B_{H'}$ be given by the generalized adjunction

$$K_{H'} + B_{H'} + M_{H'} = \left(K_{X'} + B' + M'\right)\big|_{H'}.$$

Since H' is a general hypersurface section, $B_{H'}$ is simply the intersection of $B' - H'$ with H', that is, each component of $B_{H'}$ is a component of the intersection of some component of $B' - H'$ with H' inheriting the same coefficient. In particular, $B_{H'}$ is a boundary divisor and G' is a component of $\lfloor B_{H'} \rfloor$.

A further generalized adjunction and induction on dimension gives

$$K_{G'} + B_{G'} + M_{G'} = (K_{H'} + B_{H'} + M_{H'})|_{G'}$$

where $B_{G'}$ is a boundary. But $B_{G'}$ is equal to the intersection of $B_{S'} - G'$ with the ample divisor G' on S' which implies that $B_{S'}$ is a boundary divisor too.

Now we can assume $\dim X' = 2$. Since

$$K_X + B + M = f^*\left(K_{X'} + B' + M'\right) \equiv 0/X'$$

and since M is *nef* $/X'$, there is a divisor $\tilde{B} \le B$ such that $K_X + \tilde{B} \equiv 0/X'$ and $f_*\tilde{B} = B'$. Since each coefficient of B is at most 1, each coefficient of $\tilde{B}$ is also at most 1. Therefore (X', B') is *numerically lc* (see [18, Section 4.1]; note however that [18] only considers B' with rational coefficients but all the definitions and results that we need make sense and hold true for real coefficients as well). Now by [18, Section 4.1], (X', B') is lc. In particular, $K_{X'} + B'$ is $\mathbf{R}$-Cartier. So we are done by the above arguments.

Proposition **4.9.** — *Let d be a natural number and Λ a DCC set of nonnegative real numbers. Then there is a DCC set Ω of nonnegative real numbers depending only on d and Λ such that if $(X', B' + M')$ is a generalized lc polarized pair of dimension d with data $X \xrightarrow{f} X' \to Z$ and M, and S' is the normalization of a component of $\lfloor B' \rfloor$ satisfying*

- $M = \sum \mu_j M_j$ *where M_j are nef $/Z$ Cartier divisors and $\mu_j \in \Lambda$,*
- *the coefficients of B' belong to Λ, and*
- $B_{S'}$ *is given by the following generalized adjunction (as in 4.7)*

$$K_{S'} + B_{S'} + M_{S'} = \left(K_{X'} + B' + M'\right)\big|_{S'},$$

then the coefficients of $B_{S'}$ belong to Ω.

Proof. — If the statement does not hold, then there exist a sequence of generalized lc polarized pairs $(X_i', B_i' + M_i')$ and S_i', with data $X_i \xrightarrow{f_i} X_i' \to Z_i$ and $M_i = \sum \mu_{j,i} M_{j,i}$, satisfying the assumptions of the proposition but such that the set of the coefficients of all the $B_{S_i'}$ put together does not satisfy DCC. Note that since the problem is local, we may assume $X_i' \to Z_i$ is the identity map for each i. We may also assume f_i is a log resolution of (X', B').

Let $S_i \subset X_i$ be the birational transform of S_i'. We can assume that each $B_{S_i'}$ has a component V_i with coefficient a_i such that $\{a_i\}$ is a strictly decreasing sequence. Let $a = \lim a_i$.

We may assume that the $K_{X_i'} + B_i'$ are **R**-Cartier otherwise as in Remark 4.8, by taking hypersurface sections, we reduce the problem to dimension 2 in which case this **R**-Cartier property holds automatically. Let $\tilde{B}_{S_i'}$ be the divisor given by the adjunction

$$K_{S_i'} + \tilde{B}_{S_i'} = (K_{X_i'} + B_i')\big|_{S_i'}.$$

It is clear from our definitions that $\tilde{B}_{S_i'} \le B_{S_i'}$. If c_i is the coefficient of V_i in $\tilde{B}_{S_i'}$, then we may assume $c_i \le a_i \le a + \epsilon$ for some fixed $\epsilon > 0$ so that $a + \epsilon < 1$. Therefore, (X_i', B_i') is plt near the generic point of (the image of) V_i (this follows from inversion of adjunction on surfaces [20, Corollary 3.12]) and there is a natural number l depending only on $a + \epsilon$ such that for each i there is $l_i \le l$ so that for any Weil divisor D_i' on X_i' the divisor $l_i D_i'$ is Cartier near the (image of the) generic point of V_i [20, Proposition 3.9]. Moreover, by [20, Corollary 3.10] we can write

$$c_i = \frac{l_i - 1}{l_i} + \sum b_{k,i} \frac{d_{k,i}}{l_i}$$

for some nonnegative integers $d_{k,i}$ where $b_{k,i}$ are the coefficients of the components of B_i' other than (the image of) S_i' passing through V_i.

On the other hand, shrinking X_i' if necessary we can assume $M_{j,i}'$ is **Q**-Cartier for each j, i so we can write

$$f_i^* M_{j,i}' = M_{j,i} + E_{j,i}$$

where the exceptional divisor $E_{j,i}$ is effective by the negativity lemma. Since $l_i M_{j,i}'$ is Cartier near the (image of the) generic point of V_i, the multiplicity of the birational transform of V_i in $E_{j,i}|_{S_i}$ is equal to $\frac{e_{j,i}}{l_i}$ for some nonnegative integer $e_{j,i}$. Therefore,

$$a_i = \frac{l_i - 1}{l_i} + \sum b_{k,i} \frac{d_{k,i}}{l_i} + \sum \mu_{j,i} \frac{e_{j,i}}{l_i}.$$

This is a contradiction because the above expression and Lemma 2.1 show that the set $\{a_i\}$ satisfies DCC, while the a_i form a strictly decreasing sequence. $\square$

We will need the next technical lemma in the proof of Proposition 7.1 to treat Theorem 1.6 inductively.

Lemma **4.10.** — *Let d be a natural number and Λ be a DCC set of nonnegative real numbers. Let $(X_i', B_i' + M_i')$ be a sequence of generalized lc polarized pairs of dimension d with data $X_i \to X_i' \to Z_i$ and M_i. Let S_i' be the normalization of a component of $\lfloor B_i' \rfloor$ and consider the generalized adjunction formula*

$$K_{S_i'} + B_{S_i'} + M_{S_i'} = (K_{X_i'} + B_i' + M_i')\big|_{S_i'}.$$

Assume further that

(1) X_i' *is $\mathbf{Q}$-factorial and Z_i is a point,*
(2) $B_i' = \sum b_{k,i} B_{k,i}'$ *where $B_{k,i}'$ are distinct prime divisors and $b_{k,i} \in \Lambda$,*
(3) $M_i = \sum \mu_{j,i} M_{j,i}$ *where $M_{j,i}$ are nef Cartier divisors and $\mu_{j,i} \in \Lambda$,*
(4) *and one of the following holds:*
  (i) $\{b_{1,i}\}$ *is not finite, and $B_{1,i}'|_{S_i'} \not\equiv 0$ for each i, or*
  (ii) $\{\mu_{1,i}\}$ *is not finite, and $M_{1,i}'|_{S_i'} \not\equiv 0$ for each i.*

Then the set of the coefficients of all the $B_{S_i'}$ union the set $\{\mu_{j,i} \mid M_{j,i}|_{S_i} \not\equiv 0\}$ is not finite.

Proof. — Let V_i be a prime divisor on S_i'. As in the proof of Proposition 4.9, the coefficient of V_i in $B_{S_i'}$ is of the form

$$a_i = \frac{l_i - 1}{l_i} + \sum b_{k,i} \frac{d_{k,i}}{l_i} + \sum \mu_{j,i} \frac{e_{j,i}}{l_i}$$

where l_i is a natural number and $d_{k,i}, e_{j,i}$ are nonnegative integers which are contributed by the $B_{k,i}'$ and $M_{j,i}'$ respectively.

Now assume (i) of (4) holds. Since $\{b_{1,i}\}$ is not finite, we can assume $b_{1,i} < 1$ for each i which in particular means $B_{1,i}'$ is not equal to the image of S_i'. Thus $B_{1,i}'|_{S_i'}$ is a nonzero effective divisor for each i. Choose V_i to be a component of $B_{1,i}'|_{S_i'}$. Then the set $\{a_i\}$ cannot be finite by Lemma 2.1 because $\{b_{1,i}\}$ is not finite and $d_{1,i}$ is positive.

Next assume (ii) of (4) holds. Although $M_{1,i}'|_{S_i'}$ is not numerically trivial by assumption but $M_{1,i}|_{S_i}$ may be numerically trivial for some i. If $M_{1,i}|_{S_i}$ is not numerically trivial for infinitely many i, then obviously the set $\{\mu_{j,i} \mid M_{j,i}|_{S_i} \not\equiv 0\}$ is not finite and we are done. So we may assume $M_{1,i}|_{S_i}$ is numerically trivial for every i. Recall from the proof of Proposition 4.9 that we can assume $f_i^* M_{j,i}' = M_{j,i} + E_{j,i}$ with $E_{j,i} \geq 0$. Now we can choose V_i so that $e_{1,i} \neq 0$ for each i: indeed since $M_{1,i}'|_{S_i'} \not\equiv 0$ but $M_{1,i}|_{S_i} \equiv 0$, we deduce that $E_{1,i}|_{S_i} \neq 0$ and that its pushdown to S_i' is also not zero; thus the components of the pushdown of $E_{1,i}|_{S_i}$ are components of $B_{S_i'}$, hence we can choose V_i to be one of these components. Again this shows that $\{a_i\}$ cannot be finite because $\{\mu_{1,i}\}$ is not finite and $e_{1,i} > 0$. □

5. Bounds on the number of coefficients of B'_i and M'_i

A well-known fact says that if (X, B) is a lc pair, then near each point $x \in X$ the number of components of B with coefficient $\geq b > 0$ is bounded in terms of b and dimension of X. There is also a global version of this fact. In this section, we prove similar local and global statements bounding the number of the coefficients of B'_i and the μ_j in $M = \sum \mu_j M_j$ of a generalized lc polarized pair $(X', B' + M')$ under certain assumptions. These bounds will be used in the proof of Proposition 7.2.

We start with a global statement for pairs which can also be applied to generalized polarized pairs.

Proposition **5.1.** *— Let d be a natural number and b a positive real number. Let* (X, B) *be a projective lc pair of dimension d such that*

(i) $B \geq \sum_1^r B_k$ *where* $B_k \geq 0$ *are big* **R**-*Cartier divisors,*
(ii) $B_k = \sum b_{j,k} B_{j,k}$ *is the irreducible decomposition and* $b_{j,k} \geq b$ *for every* j, k, *and*
(iii) $K_X + B + P \equiv 0$ *for some pseudo-effective* **R**-*Cartier divisor* P.

Then the number of the B_k *is at most* $(d + 1)/b$, *that is,* $r \leq (d + 1)/b$.

Proof. — Let (Y, Δ) be a **Q**-factorial dlt model of $(X, B - \sum_1^r B_k)$ and $f \colon Y \to X$ the corresponding morphism. By definition, Δ is the sum of the reduced exceptional divisor of f and the birational transform of $B - \sum_1^r B_k$. Moreover, since (X, B) is lc, $\mathrm{Supp}(\sum_1^r B_k)$ does not contain the image of any exceptional divisor of f, hence $f^* B_k$ is equal to the birational transform of B_k. In particular, $f^* B_k$ is big and it inherits the same coefficients as B_k. Moreover, by letting $B_Y := \Delta + \sum_1^r f^* B_k$ we get

$$K_Y + B_Y + f^* P = K_Y + \Delta + \sum_1^r f^* B_k + f^* P = f^*(K_X + B + P) \equiv 0.$$

Now by replacing (X, B) with (Y, B_Y) and replacing P with $f^* P$ we can assume that $(X, 0)$ is **Q**-factorial klt. Moreover, by adding $B - \sum_1^r B_k$ to P we can assume $B = \sum_1^r B_k$.

If $P \not\equiv 0$, then $K_X + B$ is not pseudo-effective so we can run an LMMP on $K_X + B$ which terminates with a Mori fibre space, by Lemma 4.4(1). But if $P \equiv 0$, then K_X is not pseudo-effective as B is big, and we can run an LMMP on K_X which terminates with a Mori fibre space [3]. Note that in both cases the LMMP preserves the lc property of (X, B) and the **Q**-factorial klt property of $(X, 0)$: in the first case the klt property of $(X, 0)$ is preserved since the LMMP is also an LMMP on $K_X + \tilde{B}$ for some klt $(X, \tilde{B})$; in the second case the lc property of (X, B) is preserved as $K_X + B \equiv 0$. Also note that in either case the LMMP does not contract any B_k because B_k is big (although some of its components may be contracted). So in either case replacing X with the Mori fibre space obtained we may assume that we already have a K_X-negative Mori fibre structure $X \to T$.

Let F be a general fibre of $X \to T$. Since B_k is big, $B_k|_F$ is big too. Restricting to F and applying induction on dimension we can reduce the problem to the case $\dim T = 0$, that is, when X is a $\mathbf{Q}$-factorial klt Fano variety of Picard number one. Pick a small number $\epsilon > 0$. For each j, k take a rational number $b'_{j,k} \le b_{j,k}$ such that $b'_{j,k} \ge b - \epsilon$. Let $B' = \sum_k \sum_j b'_{j,k} B_{j,k}$. Then there is $P' \ge 0$ such that $K_X + B' + P' \equiv 0$ and $(X, B' + P')$ is lc. Now by [17, Corollary 18.24],

$$r(b - \epsilon) \le \sum_k \sum_j b'_{j,k} \le d + 1.$$

Therefore taking the limit when ϵ approaches 0 we get $rb \le d + 1$ hence $r \le (d+1)/b$. $\square$

Next we prove a result similar to Proposition 5.1, though not as sharp, for the nef part of generalized polarized pairs.

Proposition **5.2.** — *Let d be a natural number and b a positive real number. Assume that the ACC for generalized lc thresholds (Theorem 1.5) holds in dimension d. Then there is a natural number p depending only on d, b such that if $(X', B' + M')$ is a generalized lc polarized pair of dimension d with data $X \xrightarrow{f} X' \to Z$ and M satisfying*

(i) *Z is a point,*
(ii) $M = \sum_1^r \mu_j M_j$ *where M_j are nef Cartier divisors and $\mu_j \ge b$,*
(iii) M'_j *is a big $\mathbf{Q}$-Cartier divisor for every j, and*
(iv) $K_{X'} + B' + M' + P' \equiv 0$ *for some pseudo-effective $\mathbf{R}$-Cartier divisor P',*

then the number of the μ_j is at most p, that is, $r \le p$.

Before giving the proof we prove a related local statement.

Proposition **5.3.** — *Let d be a natural number and b a positive real number. Assume that Theorem 1.5 and Proposition 5.2 hold in dimension $< d$. Then there is a natural number q depending only on d, b such that if $(X', B' + M')$ is a $\mathbf{Q}$-factorial generalized lc polarized pair of dimension d with data $X \xrightarrow{f} X' \to Z$ and M, and if*

(i) $x' \in X'$ *is a (not necessarily closed) point,*
(ii) $M = \sum_1^r \mu_j M_j$ *where M_j are nef /Z Cartier divisors and $\mu_j \ge b$,*
(iii) M_j *is not relatively numerically zero over any neighborhood of x', for every j, and*
(iv) $(X', 0)$ *is klt,*

then the number of the μ_j is at most q, that is, $r \le q$.

Proof. — *Step 1.* Let C' be the closure of x'. By (iii), the codimension of C' in X' is at least two. By adding appropriate divisors to B' and shrinking X' we can assume C' is a generalized lc centre of $(X', B' + M')$: to be more precise, let W be the blowup of

X' along C'; we can assume $X \to X'$ factors through W; now take a general sufficiently ample divisor on W and let A be its pullback to X; if α is the generalized lc threshold of A' near x' with respect to $(X', B' + M')$, then $(X', B' + \alpha A' + M')$ is generalized lc near x' with boundary part $B' + \alpha A'$ and nef part M, and C' is a generalized lc centre of $(X', B' + \alpha A' + M')$; the point is that after shrinking X' we can assume $f^*A' = A + E$ where $E \neq 0$ is effective with large coefficients, and that every component of E maps onto C' so adding $\alpha A'$ creates deeper singularities only along C'. Now we may replace B' with $B' + \alpha A'$.

Step 2. By Step 1, we can assume that there is a prime divisor S on X mapping onto C' whose generalized log discrepancy with respect to $(X', B' + M')$ is 0. Since $(X', 0)$ is **Q**-factorial klt, by Lemma 4.6, there is an extremal birational contraction $\phi : X'' \to X'$ which extracts S'', the birational transform of S, and X'' is **Q**-factorial. We can write

$$K_{X''} + B'' + M'' = \phi^*\big(K_{X'} + B' + M'\big)$$

where B'' is the sum of S'' and the birational transform of B', and M'' is the pushdown of M. Writing

$$K_X + B + M = f^*\big(K_{X'} + B' + M'\big)$$

we can see that B'' is just the pushdown of B.

We claim that M_j'' is not numerically trivial over any neighborhood of x' for any j which in turn implies that M_j'' is ample/X'. If this is not true for some j, then we can write $f^*M_j' = M_j + \tilde{E}_j$ where $\tilde{E}_j \geq 0$ and S is not a component of $\tilde{E}_j$. But then for any general closed point $y' \in C'$, the fibre $f^{-1}\{y'\}$ is not inside $\operatorname{Supp} \tilde{E}_j$, so the fibre does not intersect $\operatorname{Supp} \tilde{E}_j$, by [18, Lemma 3.39(2)]. Therefore, $\tilde{E}_j = 0$ over the generic point of C', that is over x', hence M_j is numerically trivial over some neighborhood of x', a contradiction.

Step 3. We can assume the induced map $g: X \dashrightarrow X''$ is a morphism. To ease notation we replace S'' with its normalization and denote the induced morphism $S \to S''$ by h. By generalized adjunction and usual adjunction, we can write

$$K_{S''} + B_{S''} + M_{S''} = \big(K_{X''} + B'' + M''\big)\big|_{S''} \equiv 0/C'$$

and

$$K_{S''} + \Delta_{S''} = \big(K_{X''} + B''\big)\big|_{S''}.$$

Write $g^*M_j'' = M_j + E_j$ where $E_j \geq 0$ is exceptional/X''. Then

$$M_j''\big|_{S''} = h_*\big(M_j|_S\big) + h_*\big(E_j|_S\big)$$

and

$$M''\big|_{S''} = \sum \mu_j h_*\big(M_j|_S\big) + \sum \mu_j h_*\big(E_j|_S\big) = M_{S''} + \sum \mu_j h_*\big(E_j|_S\big)$$

and

$$B_{S''} = \Delta_{S''} + \sum \mu_j h_*(E_j|_S).$$

Let V be a prime divisor on S'' and b_V be its coefficient in $B_{S''}$. Then, by the proof of Proposition 4.9,

$$b_V \geq 1 - \frac{1}{l} + \sum \frac{\mu_j n_j}{l}$$

for some natural number l and integers $n_j \geq 0$. Moreover, $n_j > 0$ if V is a component of $h_*(E_j|_S)$. This in particular shows that there is a natural number s depending only on b such that V is a component of $h_*(E_j|_S)$ for at most s of the j because $\sum \mu_j n_j \leq 1$.

Step 4. Let F'' be a general fibre of the induced map $S'' \to C'$ and F the corresponding fibre of $S \to C'$. Restricting to F'' as in Remark 4.2(6), we get

$$K_{F''} + B_{F''} + M_{F''} = (K_{S''} + B_{S''} + M_{S''})|_{F''} \equiv 0.$$

Also we get

$$K_{F''} + \Delta_{F''} := (K_{S''} + \Delta_{S''})|_{F''}.$$

Denote the morphism $F \to F''$ by e. Since F'' is a general fibre, restricting Weil divisors on S'' to F'' makes sense, and if P is a Weil divisor on S, then we have $(h_*P)|_{F''} = e_*(P|_F)$. Therefore,

$$M_j''\big|_{F''} = e_*(E_j|_F) + e_*(M_j|_F), \qquad M_{F''} = \sum \mu_j e_*(M_j|_F),$$

and

$$B_{F''} = B_{S''}|_{F''} = \left(\Delta_{S''} + \sum \mu_j h_*(E_j|_S)\right)\bigg|_{F''} = \Delta_{F''} + \sum \mu_j e_*(E_j|_F).$$

Since F'' may not be **Q**-factorial, we need to make some further constructions. Let $(H'', \Delta_{H''})$ be a **Q**-factorial dlt model of $(F'', \Delta_{F''})$ and $\psi : H'' \to F''$ the corresponding morphism. By definition

$$K_{H''} + \Delta_{H''} = \psi^*(K_{F''} + \Delta_{F''})$$

and the exceptional divisors of ψ all appear with coefficient 1 in $\Delta_{H''}$. Moreover, we can write

$$K_{H''} + B_{H''} + M_{H''} = \psi^*(K_{F''} + B_{F''} + M_{F''}) \equiv 0$$

where $B_{H''}$ is the sum of the birational transform of $B_{F''}$ and the reduced exceptional divisor of ψ.

We can assume $c\colon \mathrm{F} \dashrightarrow \mathrm{H}''$ is a morphism. By construction,

$$\psi^*\big(\mathrm{M}_j''|_{\mathrm{F}''}\big) = c_*(\mathrm{E}_j|_{\mathrm{F}}) + c_*(\mathrm{M}_j|_{\mathrm{F}})$$

which is big, and

$$\mathrm{M}_{\mathrm{H}''} = \sum \mu_j c_*(\mathrm{M}_j|_{\mathrm{F}}) \quad \text{and} \quad \mathrm{B}_{\mathrm{H}''} = \Delta_{\mathrm{H}''} + \sum \mu_j c_*(\mathrm{E}_j|_{\mathrm{F}}).$$

Moreover, since the exceptional divisors of ψ are components of $\lfloor \Delta_{\mathrm{H}''} \rfloor$, the divisor $\sum \mu_j c_*(\mathrm{E}_j|_{\mathrm{F}})$ has no exceptional component, so it is just the birational transform of $\sum \mu_j e_*(\mathrm{E}_j|_{\mathrm{F}})$.

Step 5. Run an LMMP on $\mathrm{K}_{\mathrm{H}''}$. It terminates with a Mori fibre space $\overline{\mathrm{H}}'' \to \overline{\mathrm{T}}''$ and the generalized lc property of $(\mathrm{H}'', \mathrm{B}_{\mathrm{H}''} + \mathrm{M}_{\mathrm{H}''})$ is preserved by the LMMP. Since $c_*(\mathrm{E}_j|_{\mathrm{F}}) + c_*(\mathrm{M}_j|_{\mathrm{F}})$ is big, its pushdown to $\overline{\mathrm{H}}''$ is also big, hence ample over $\overline{\mathrm{T}}''$. Let $\overline{\mathrm{G}}''$ be a general fibre of the above Mori fibre space. Then restriction to $\overline{\mathrm{G}}''$ gives

$$\mathrm{K}_{\overline{\mathrm{G}}''} + \mathrm{B}_{\overline{\mathrm{G}}''} + \mathrm{M}_{\overline{\mathrm{G}}''} = (\mathrm{K}_{\overline{\mathrm{H}}''} + \mathrm{B}_{\overline{\mathrm{H}}''} + \mathrm{M}_{\overline{\mathrm{H}}''})|_{\overline{\mathrm{G}}''} \equiv 0.$$

By construction, $\mathrm{M}_{\overline{\mathrm{G}}''} = \sum \mu_j a_*(\mathrm{M}_j|_{\mathrm{F}})|_{\overline{\mathrm{G}}''}$ where we can assume $a\colon \mathrm{F} \dashrightarrow \overline{\mathrm{H}}''$ is a morphism. Applying Proposition 5.2 and rearranging the indexes, we can assume that there is a natural number t depending only on d, b such that $a_*(\mathrm{M}_j|_{\mathrm{F}})|_{\overline{\mathrm{G}}''} \equiv 0$ for every $j > t$. But then $a_*(\mathrm{E}_j|_{\mathrm{F}})|_{\overline{\mathrm{G}}''}$ is big for each $j > t$.

For each $j > t$ choose a component W_j of $a_*(\mathrm{E}_j|_{\mathrm{F}})$ which is ample over $\overline{\mathrm{T}}''$. By construction, W_j is the birational transform of a component U_j of $e_*(\mathrm{E}_j|_{\mathrm{F}}) = (h_*(\mathrm{E}_j|_{\mathrm{S}}))|_{\mathrm{F}''}$ and U_j in turn is a component of $\mathrm{V}_j \cap \mathrm{F}''$ for some component V_j of $h_*(\mathrm{E}_j|_{\mathrm{S}})$. Moreover, $\mathrm{W}_k = \mathrm{W}_j$ if and only if $\mathrm{U}_k = \mathrm{U}_j$ if and only if $\mathrm{V}_k = \mathrm{V}_j$. By Step 3, for each k, $\mathrm{V}_k = \mathrm{V}_j$ for at most s of the j. Thus for each k, $\mathrm{W}_k = \mathrm{W}_j$ for at most s of the j. On the other hand, by Steps 3 and 4, the V_j appear as components of $\mathrm{B}_{\mathrm{S}''}$ with coefficient $\geq \min\{b, \tfrac{1}{2}\}$, and there exist at least $\frac{r-t}{s}$ such components. Similarly the W_j appear as components of $\mathrm{B}_{\overline{\mathrm{H}}''}$ with coefficient $\geq \min\{b, \tfrac{1}{2}\}$, and there exist at least $\frac{r-t}{s}$ such components. Now apply Proposition 5.1 to $(\overline{\mathrm{G}}'', \mathrm{B}_{\overline{\mathrm{G}}''})$ to deduce that $\frac{r-t}{s}$ is bounded hence r is bounded by some q. $\qquad\qquad\square$

Proof of Proposition 5.2. — We argue by induction on the dimension d. The case $d = 1$ is clear. Suppose that the proposition holds in dimension $< d$.

Step 1. Since $(\mathrm{X}', \mathrm{B}' + \mathrm{M}')$ is generalized lc and $\mathrm{K}_{\mathrm{X}'} + \mathrm{B}'$ is $\mathbf{R}$-Cartier, $(\mathrm{X}', \mathrm{B}')$ is lc. Let $(\mathrm{X}'', \mathrm{B}'')$ be a $\mathbf{Q}$-factorial dlt model of $(\mathrm{X}', \mathrm{B}')$ and $\phi\colon \mathrm{X}'' \to \mathrm{X}'$ the corresponding morphism. We may assume $\mathrm{X} \dashrightarrow \mathrm{X}''$ is a morphism. For each j, we have $\phi^* \mathrm{M}_j' = \mathrm{M}_j'' + \mathrm{E}_j''$ where $\mathrm{E}_j'' \geq 0$ is exceptional$/\mathrm{X}'$ and M_j'' is the pushdown of M_j. So

$$\mathrm{K}_{\mathrm{X}''} + \mathrm{B}'' + \sum \mu_j \mathrm{E}_j'' + \mathrm{M}'' = \phi^*\big(\mathrm{K}_{\mathrm{X}'} + \mathrm{B}' + \mathrm{M}'\big)$$

where M″ is the pushdown of M. Since the exceptional divisors of ϕ are components of $\lfloor B'' \rfloor$ and since $(X', B' + M')$ is generalized lc, we deduce $E_j'' = 0$ for every j, hence $M_j'' = \phi^* M_j'$ for every j and $M'' = \phi^* M'$. Thus we may replace X' with X'', hence assume that (X', B') is **Q**-factorial dlt.

Step 2. If $P' \not\equiv 0$, then $K_{X'} + B' + M'$ is not pseudo-effective and so we can run an LMMP on $K_{X'} + B' + M'$ which terminates with a Mori fibre space, by Lemma 4.4(1). But if $P' \equiv 0$, then $K_{X'} + B'$ is not pseudo-effective as M' is big and so we can run an LMMP on $K_{X'} + B'$ which terminates with a Mori fibre space. Note that in both cases the generalized lc property of $(X', B' + M')$ is preserved: in the second case we use the fact $K_{X'} + B' + M' \equiv 0$. Also note that in both cases none of the M_j' is contracted by the LMMP since M_j' is big. In either case we can replace X' with the Mori fibre space hence we may assume we already have a Mori fibre structure $X' \to T'$. Let F' be a general fibre of this fibre space. Since M_j' is big, $M_j'|_{F'}$ is big too. Restricting to F' and applying induction on dimension we can reduce the problem to the case $\dim T' = 0$, that is, when X' is a Fano variety of Picard number one.

Step 3. Perhaps after changing the indexes we may write $M_j' \equiv \lambda_j M_1'$ such that $\lambda_j \geq 1$ for every j. Now we define $\tilde{\mu}_j$ as follows: initially let $\tilde{\mu}_j = \mu_j$; next decrease $\tilde{\mu}_2$ and instead increase $\tilde{\mu}_1$ as much as possible so that

$$\left(X', B' + \sum_{j \neq 2} \tilde{\mu}_j M_j' \right)$$

is generalized lc and

$$K_{X'} + B' + \sum \tilde{\mu}_j M_j' + P' \equiv 0.$$

Either we hit a generalized lc threshold, i.e. $(X', B' + \sum_{j \neq 2} \tilde{\mu}_j M_j')$ is generalized lc but not generalized klt, or that we reach $\tilde{\mu}_2 = 0$. If the first case happens, we stop. But if the second case happens we repeat the process by decreasing $\tilde{\mu}_3$ and increasing $\tilde{\mu}_1$, and so on.

We show that the above process involves only a bounded number of the μ_j. Let l be the smallest number such that $\tilde{\mu}_j = \mu_j$ for every $j > l$. We want to show that l is bounded depending only on d, b. We can assume $l > 1$. By construction,

$$\tilde{\mu}_1 \geq \sum_{j \leq l-1} \mu_j \lambda_j \geq \sum_{j \leq l-1} \mu_j \geq (l-1)b$$

so it is enough to show that $\tilde{\mu}_1$ is bounded depending only on d, b. If M_1 is not numerically trivial over X', then the generalized lc threshold of M_1' with respect to (X', B') is finite and bounded from above by Theorem 1.5, and this in turn implies boundedness of $\tilde{\mu}_1$. But if M_1 is numerically trivial over X', then again $\tilde{\mu}_1$ is bounded from above but for a different

reason: by the cone theorem X' can be covered by curves Γ' such that $-(K_{X'} + B') \cdot \Gamma' \leq 2d$ which in turn implies that $\tilde{\mu}_1 M'_1 \cdot \Gamma' \leq 2d$ hence $\tilde{\mu}_1 M_1 \cdot \Gamma \leq 2d$ where $\Gamma \subset X$ is the birational transform of Γ'. This is possible only if $\tilde{\mu}_1$ is bounded from above since M_1 is big and Cartier and hence $M_1 \cdot \Gamma \geq 1$.

If at the end of the process $\tilde{\mu}_j = 0$ for every $j \geq 2$, then the above arguments show that r is indeed bounded by some number p. But if $\tilde{\mu}_j > 0$ for some $j \geq 2$, then we replace M with $\sum_{j \neq l} \tilde{\mu}_j M_j$ and replace P' with $P' + \tilde{\mu}_l M'_l$ where l is as above, and rearrange the indexes. We can then assume that $(X', B' + M')$ is generalized lc but not generalized klt.

Step 4. The arguments of Step 3 show that, after replacing X, we can assume that there is a prime divisor S on X exceptional over X' whose generalized log discrepancy with respect to $(X', B' + M')$ is 0. Since $(X', 0)$ is **Q**-factorial klt, by Lemma 4.6, there is an extremal contraction $\phi: X'' \to X'$ which extracts S'', the birational transform of S. We can write

$$K_{X''} + B'' + M'' = \phi^*\big(K_{X'} + B' + M'\big)$$

where B'' is the sum of S'' and the birational transform of B' and M'' is the pushdown of M.

Since $\rho(X') = 1$ and ϕ is extremal, $\rho(X'') = 2$. Moreover,

$$K_{X''} + B'' + M'' + P'' \equiv 0$$

where P'' is the pullback of P' on X'. Since $\rho(X') = 1$, P' and so P'' is semi-ample, hence we may assume that $(X'', B'' + P'' + M'')$ is generalized lc with boundary part $B'' + P''$ and nef part M.

Since S'' is a component of $\lfloor B'' \rfloor$, $(X'', B'' - \delta S'' + P'' + M'')$ is generalized lc where $\delta > 0$ is small, and

$$-\delta S'' \equiv K_{X''} + B'' - \delta S'' + P'' + M''.$$

So by Lemma 4.4(1), we can run an LMMP on $-S''$ which ends up with a Mori fibre space $X''' \to T'''$. Note that by construction X''' has Picard number one or two: in any case one of the extremal rays of X''' corresponds to the Fano contraction $X''' \to T'''$ and S''' is positive on this ray.

We may assume that both $g: X \dashrightarrow X''$ and $h: X \dashrightarrow X'''$ are morphisms.

Step 5. Consider the case $\dim T''' > 0$. Then the Picard number $\rho(X''') = 2$, hence $X'' \dashrightarrow X'''$ is an isomorphism in codimension one. Moreover, by restricting to the general fibres of $X''' \to T'''$ and applying induction we may assume $M'''_j \equiv 0/T'''$ for all but a bounded number of j. For any such j, M'''_j is not big, hence M''_j is not big too. Thus M''_j is ample$/X'$ otherwise M''_j would be the pullback of M'_j which is big, a contradiction. Let $C' := \phi(S'')$ and let x' be the generic point of C'. Then $M'''_j \equiv 0/T'''$ implies that M_j is not numerically trivial over any neighborhood of x'. Now apply Proposition 5.3 to

$(X', B' + P' + M')$ at x' to bound the number of such j. Therefore r is indeed bounded by some number p depending only on d, b.

Step 6. We can now assume $\dim T''' = 0$. Let $\tilde{X}'' \to X'''$ be the last step of the LMMP which contracts some divisor $\tilde{R}''$. Let x''' be the generic point of the image of $\tilde{R}''$. For each j, either M_j'' is ample over X' or $\tilde{M}_j''$ is ample over X''' where $\tilde{M}_j''$ is the push-down of M_j via $X \dashrightarrow \tilde{X}''$ which we can assume to be a morphism. So either M_j is not numerically trivial over any neighborhood of x' or that it is not numerically trivial over any neighborhood of x'''. Now apply Proposition 5.3 to $(X', B' + P' + M')$ and $(X''', B''' + P''' + M''')$ at x' and x''' to bound r by some number p depending only on d, b. $\qquad\square$

6. ACC for generalized lc thresholds

In this section, we reduce the ACC for generalized lc thresholds (Theorem 1.5) to the Global ACC (Theorem 1.6) in lower dimension by adapting a standard argument due to Shokurov. We create an appropriate generalized lc centre of codimension one and restrict to it to do induction.

Proposition **6.1.** — *Assume that Theorem 1.6 holds in dimension $\leq d - 1$. Then Theorem 1.5 holds in dimension d.*

Proof. — Applying induction we may assume that Theorem 1.5 holds in dimension $\leq d - 1$. If Theorem 1.5 does not hold in dimension d, then there exist a sequence of generalized lc polarized pairs $(X_i', B_i' + M_i')$ of dimension d with data $X_i \xrightarrow{f_i} X_i' \to Z_i$ and $M_i = \sum \mu_{j,i} M_{j,i}$, and divisors D_i' and $N_i = \sum \nu_{k,i} N_{k,i}$ satisfying the assumptions of the theorem but such that the generalized lc thresholds t_i of $D_i' + N_i'$ with respect to $(X_i', B_i' + M_i')$ form a strictly increasing sequence of numbers. We may assume that $0 < t_i < \infty$ for every i. Since the problem is local over X_i', we can assume $X_i' \to Z_i$ is the identity morphism. Moreover, we can discard any $\mu_{j,i}$ and $\nu_{k,i}$ if they are zero.

By definition,

$$\left(X_i', B_i' + t_i D_i' + M_i' + t_i N_i' \right)$$

is generalized lc with boundary part $B_i' + t_i D_i'$ and nef part $M_i + t_i N_i$ but

$$\left(X_i', B_i' + a_i D_i' + M_i' + a_i N_i' \right)$$

is not generalized lc for any $a_i > t_i$.

If $\lfloor B_i' \rfloor \neq \lfloor B_i' + t_i D_i' \rfloor$ for infinitely many i, then we can easily get a contradiction as the t_i can be calculated in terms of the coefficients of B_i' and D_i'. Thus we may assume

that $\lfloor B_i' \rfloor = \lfloor B_i' + t_i D_i' \rfloor$ for every i. In particular, this means that there is a generalized lc centre of

$$\left(X_i', B_i' + t_i D_i' + M_i' + t_i N_i' \right)$$

of codimension ≥ 2 which is not a generalized lc centre of $(X_i', B_i' + M_i')$.

We may assume that the given morphism $f_i \colon X_i \to X_i'$ is a log resolution of $(X_i', B_i' + t_i D_i')$. Let $\Delta_i' := B_i' + t_i D_i'$ and let $R_i := M_i + t_i N_i$. We can write

$$K_{X_i} + \Delta_i + R_i = f_i^* \left(K_{X_i'} + \Delta_i' + R_i' \right) + E_i$$

where Δ_i is the sum of the birational transform of Δ_i' and the reduced exceptional divisor of f_i, and $E_i \geq 0$ is exceptional/X_i'. Then the pair (X_i, Δ_i) is lc but not klt; more precisely there is a component of $\lfloor \Delta_i \rfloor$ which is not a component of E_i; moreover, there is such a component which is exceptional/X_i' by the last paragraph. In addition, the set of the coefficients of all the Δ_i union with $\{ \mu_{j,i}, t_i \nu_{k,i} \}$ satisfies the DCC by Lemma 2.1.

Run an LMMP/X_i' on $K_{X_i} + \Delta_i + R_i$ with scaling of some ample divisor which is also an LMMP/X_i' on E_i. Since E_i is effective and exceptional/X_i', the LMMP ends on a model X_i'' on which $E_i'' = 0$ (as in the proof of Lemma 4.5). In particular,

$$K_{X_i''} + \Delta_i'' + R_i'' \equiv 0/X_i'.$$

Let S_i be a component of $\lfloor \Delta_i \rfloor$ exceptional/X_i' but not a component of E_i. Since the LMMP only contracts components of E_i, this S_i is not contracted/X_i''. Define $\Delta_{S_i''}$ by the generalized adjunction

$$K_{S_i''} + \Delta_{S_i''} + R_{S_i''} = \left(K_{X_i''} + \Delta_i'' + R_i'' \right) \big|_{S_i''}.$$

Then the set of the coefficients of all the $\Delta_{S_i''}$ satisfies DCC by Proposition 4.9. By construction

$$K_{S_i''} + \Delta_{S_i''} + R_{S_i''} \equiv 0/X_i'.$$

Let $S_i'' \to V_i'$ be the contraction given by the Stein factorization of $S_i'' \to X_i'$ and let F_i'' be a general fibre of $S_i'' \to V_i'$. We can write

$$K_{F_i''} + \Delta_{F_i''} + R_{F_i''} = (K_{S_i''} + \Delta_{S_i''} + R_{S_i''})|_{F_i''} \equiv 0$$

as in Remark 4.2 (6): here $\Delta_{F_i''} = \Delta_{S_i''}|_{F_i''}$ and $R_{F_i''} = R_{S_i''}|_{F_i''}$ is the pushdown of $R_i|_{F_i} = (M_i + t_i N_i)|_{F_i}$ where F_i is the fibre of $S_i \to V_i'$ corresponding to F_i''.

Suppose that we can choose the S_i such that

($*$) the set of the coefficients of all the $\Delta_{F_i''}$ together with $\{ \mu_{j,i} \mid M_{j,i}|_{F_i} \not\equiv 0 \} \cup \{ t_i \nu_{k,i} \mid N_{k,i}|_{F_i} \not\equiv 0 \}$ does not satisfy ACC.

But then $(*)$ contradicts Theorem 1.6. So it is enough to find the S_i so that $(*)$ holds. We will show that $(*)$ holds if for each i we can find S_i satisfying:

$(**)$ $\quad (D_i'' + N_i'')|_{F_i''}$ is not numerically trivial

where D_i on X_i is the birational transform of D_i' and D_i'' is the pushdown of D_i; here we can assume $g_i \colon X_i \dashrightarrow X_i''$ is a morphism. Indeed, let B_i be the sum of the birational transform of B_i' plus the reduced exceptional divisor of f_i, and B_i'' its pushdown on X_i''. By generalized adjunction we can write

$$K_{S_i''} + B_{S_i''} + M_{S_i''} = \left(K_{X_i''} + B_i'' + M_i''\right)\big|_{S_i''}.$$

Write $g_i^*(N_i'') = N_i + Q_i$. Then $N_i''|_{F_i''} = N_{F_i''} + Q_{F_i''}$ where $N_{F_i''}$ is the pushdown of $N_i|_{F_i}$ and $Q_{F_i''}$ is the pushdown of $Q_i|_{F_i}$. If $N_i|_{F_i} \not\equiv 0$ for every i, then $(*)$ is satisfied. So we can assume $N_i|_{F_i} \equiv 0$ for every i, hence by $(**)$ we have

$$\left(D_i'' + N_i''\right)\big|_{F_i''} \equiv D_{F_i''} + Q_{F_i''} \neq 0$$

for every i where $D_{F_i''} := D_i''|_{F_i''}$. But now $\Delta_i'' = B_i'' + t_i D_i''$ and $\Delta_{S_i''} = B_{S_i''} + t_i(D_{S_i''} + Q_{S_i''})$ where $D_{S_i''} := D_i''|_{S_i''}$ and $Q_{S_i''}$ is the pushdown of $Q_i|_{S_i}$. Moreover, since $D_{S_i''} + Q_{S_i''} \neq 0$ near F_i'', Proposition 4.9 and its proof show that the set of the coefficients of all the $\Delta_{S_i''}$ near F_i'' does not satisfy ACC. Thus the set of the coefficients of all the $\Delta_{F_i''}$ does not satisfies ACC, hence $(*)$ holds.

Finally we show that $(**)$ holds. By the negativity lemma, we can write

$$f_i^*\left(D_i' + N_i'\right) = D_i + N_i + P_i$$

where $P_i \geq 0$ is exceptional/X_i'. By the definition of t_i and the assumption $\lfloor B_i' \rfloor = \lfloor B_i' + t_i D_i' \rfloor$, there is a component of P_i which is a component of $\lfloor \Delta_i \rfloor$ but not a component of E_i. In fact any component of P_i not contracted/X_i'', is of this kind. Since $P_i'' \neq 0$ is exceptional/X_i', by the negativity lemma [3, Lemma 3.6.2], there is a component S_i'' of P_i'' with a covering family of curves C (contracted over X_i') such that $P_i'' \cdot C < 0$. So $(D_i'' + N_i'') \cdot C > 0$ for such curves C, hence $(D_i'' + N_i'')|_{S_i''}$ is not numerically trivial over general points of V_i' which implies that we can choose the S_i so that $(**)$ holds. $\qquad\square$

7. Global ACC

In this section, we show that Global ACC (Theorem 1.6) in dimension $< d$ and ACC for generalized lc thresholds (Theorem 1.5) in dimension d together imply Global ACC in dimension d. We first deal with the pairs which are generalized lc but not generalized klt. For the general case, we will use Proposition 3.4 and do induction on the number of summands in the nef part of the pair, as illustrated in the introduction. The starting point of the induction is the important result [10, Theorem 1.5] which proves the statement when the nef part is zero.

Proposition **7.1.** — *Assume Theorem* 1.6 *holds in dimension* $\leq d - 1$. *Then Theorem* 1.6 *holds in dimension d for those* $(X', B' + M')$ *which are not generalized klt.*

Proof. — *Step 1.* Extending Λ we can assume $1 \in \Lambda$. If the proposition does not hold, then there is a sequence of generalized lc but not klt polarized pairs $(X'_i, B'_i + M'_i)$ with data $X_i \xrightarrow{f_i} X'_i \to Z_i$ and $M_i = \sum \mu_{j,i} M_{j,i}$ satisfying the assumptions of 1.6 but such that the set of the coefficients of all the B'_i together with the $\mu_{j,i}$ does not satisfy ACC.

We may assume that $f_i : X_i \to X'_i$ is a log resolution of (X'_i, B'_i). Let B_i be the sum of the birational transform of B'_i and the reduced exceptional divisor of f_i. We can write

$$K_{X_i} + B_i + M_i = f_i^*\big(K_{X'_i} + B'_i + M'_i\big) + E_i$$

where $E_i \geq 0$ is exceptional$/X'_i$. We can run an LMMP$/X'_i$ on $K_{X_i} + B_i + M_i$ with scaling of some ample divisor which contracts E_i and terminates with some model (as in the proof of Lemma 4.5). Moreover, by the generalized non-klt assumption, we can choose f_i so that there is a prime divisor S_i on X_i which is a component of $\lfloor B_i \rfloor$ but not a component of E_i, hence it is not contracted by the LMMP. Replacing X'_i with the model given by the LMMP allows us to assume that (X'_i, B'_i) is **Q**-factorial dlt and that we have a component S'_i of $\lfloor B'_i \rfloor$.

Step 2. Write $B'_i = \sum b_{k,i} B'_{k,i}$ where $B'_{k,i}$ are the distinct irreducible components of B'_i. If the set of all the coefficients $b_{k,i}$ is not finite, then we may assume that the $b_{1,i}$ form a strictly increasing sequence in which case we let $P'_i := B'_{1,i}$. On the other hand, if the set of all the coefficients $b_{k,i}$ is finite, then the set of all the $\mu_{j,i}$ is not finite hence we could assume that the $\mu_{1,i}$ form a strictly increasing sequence in which case we let $P'_i := M'_{1,i}$. In either case we can run an LMMP on

$$K_{X'_i} + B'_i + M'_i - \epsilon P'_i \equiv -\epsilon P'_i$$

for some small $\epsilon > 0$ which ends with a Mori fibre space, by Lemma 4.4(1). The generalized lc (and non-klt) property of $(X'_i, B'_i + M'_i)$ is preserved by the LMMP because $K_{X'_i} + B'_i + M'_i \equiv 0$.

Step 3. We first consider the case when S'_i is not contracted by the LMMP in Step 2, for infinitely many i. Replacing the sequence we can assume this holds for every i. In this case, we replace X'_i with the Mori fibre space constructed, hence we can assume we already have a Mori fibre structure $X'_i \to T'_i$ and that P'_i is ample$/T'_i$. Let F'_i be a general fibre of $X'_i \to T'_i$. Then we can write

$$K_{F'_i} + B_{F'_i} + M_{F'_i} = \big(K_{X'_i} + B'_i + M'_i\big)\big|_{F'_i}$$

where $B_{F'_i} = B'_i|_{F'_i}$ and $M_{F'_i} = M'_i|_{F'_i}$. Moreover, since $P'_i|_{F'_i}$ is ample, the set of the coefficients of all the $B_{F'_i}$ together with the set $\{\mu_{j,i} \mid M_{j,i}|_{F_i} \not\equiv 0\}$ is not finite where F_i is the

fibre of $X_i \to T_i'$ corresponding to F_i'. So applying induction we can assume $\dim T_i' = 0$. In particular, $P_i'|_{S_i'}$ is not numerically trivial.

Now assume the LMMP of Step 2 contracts S_i' at some step, for infinitely many i. Replacing the sequence we can assume this holds for every i. Replacing X_i' we can assume S_i' is contracted by the first step of the LMMP, say $X_i' \to X_i''$. Then P_i' is ample over X_i'', hence $P_i'|_{S_i'}$ is not numerically trivial.

From now on we assume that $P_i'|_{S_i'}$ is not numerically trivial.

Step 4. Apply generalized adjunction to get

$$K_{S_i'} + B_{S_i'} + M_{S_i'} = \left(K_{X_i'} + B_i' + M_i'\right)\big|_{S_i'} \equiv 0.$$

By Proposition 4.9, the coefficients of $B_{S_i'}$ belong to a DCC set depending only on d and Λ. Moreover, $M_{S_i'}$ is the pushdown of $M|_{S_i} = \sum \mu_{j,i} M_{j,i}|_{S_i}$. Thus by induction the set of the coefficients of all the $B_{S_i'}$ together with the set $\{\mu_{j,i} \mid M_{j,i}|_{S_i} \not\equiv 0\}$ is finite. But this contradicts Lemma 4.10. □

Proposition **7.2.** *— Assume that Theorem 1.6 holds in dimension $\leq d - 1$ and that Theorem 1.5 holds in dimension d. Then Theorem 1.6 holds in dimension d.*

Proof. — Step 1. If the statement is not true, then there is a sequence of generalized lc polarized pairs $(X_i', B_i' + M_i')$ with data $X_i \xrightarrow{f_i} X_i' \to Z_i$ and $M_i = \sum \mu_{j,i} M_{j,i}$ satisfying the assumptions of 1.6 but such that the set of the coefficients of all the B_i' and all the $\mu_{j,i}$ put together satisfies DCC but not ACC. Write $B_i' = \sum b_{k,i} B_{k,i}'$ where $B_{k,i}'$ are the distinct irreducible components of B_i'.

As in Steps 1 and 2 of the proof of Proposition 7.1, we can reduce the problem to the situation in which X_i' is a **Q**-factorial klt variety with a Mori fibre space structure. Restricting to the general fibres of the fibration and applying induction we can in addition assume X_i' is Fano of Picard number one.

For each i, let $\sigma(M_i)$ be the number of the $\mu_{j,i}$. Then, by Propositions 5.1 and 5.2, we can assume that the number of the components of B_i' plus $\sigma(M_i)$ is bounded. Thus we can assume that the number of the components of B_i' and $\sigma(M_i)$ are both independent of i. We just write σ instead of $\sigma(M_i)$.

We will do induction on the number σ. By [10, Theorem 1.5], the proposition holds when $\sigma = 0$, i.e. when $M_i = 0$ for every i. So we can assume $\sigma > 0$. We may also assume that σ is minimal with respect to all sequences as above, even if Λ is extended to a larger set.

Replacing the sequence we may assume that the numbers $b_{k,i}$ and $\mu_{j,i}$ form a (not necessarily strict) increasing sequence for each k and each j, because they all belong to the DCC set Λ. By definition, $b_{k,i} \leq 1$. We show that the $\mu_{j,i}$ are also bounded from above, i.e. $\lim_i \mu_{j,i} < +\infty$ for every j: this follows from the same arguments as in Step 3 of the proof of Proposition 5.2 by considering the generalized lc threshold of $M_{j,i}'$ with respect to

$(X'_i, B'_i + M'_i - \mu_{j,i}M'_{j,i})$ if $M_{j,i} \not\equiv 0/X'_i$ for infinitely many i, or by applying boundedness of the length of extremal rays otherwise.

Step 2. By Proposition 7.1, we may assume that $(X'_i, B'_i + M'_i)$ is generalized klt for every i. In particular, $(X'_i, B'_i + (1 + \epsilon_i)M'_i)$ is generalized klt, and $K_{X'_i} + B'_i + (1 + \epsilon_i)M'_i$ is ample for some small $\epsilon_i > 0$, noting that the Picard number $\rho(X'_i) = 1$. We may assume that $f_i : X_i \to X'_i$ is a log resolution of (X'_i, B'_i) and can write

$$K_{X_i} + B_i + (1 + \epsilon_i)M_i = f_i^*\left(K_{X'_i} + B'_i + (1 + \epsilon_i)M'_i\right) + E_i$$

where B_i is the sum of the birational transform of B'_i and the reduced exceptional divisor of f_i, and $E_i \geq 0$ is exceptional$/X'_i$. So $K_{X_i} + B_i + (1 + \epsilon_i)M_i$ is big, and since the $\mu_{j,i}$ are bounded from above, we deduce that $K_{X_i} + B_i + \sum nM_{j,i}$ is also big for some fixed natural number $n \gg 1$ independent of i.

Now by Proposition 3.4, there exists a natural number m, independent of i, such that $|m(K_{X_i} + B_i + \sum nM_{j,i})|$ defines a birational map for every i. In particular,

$$H^0\left(X_i, \left\lfloor m\left(K_{X_i} + B_i + \sum nM_{j,i}\right)\right\rfloor\right) \neq 0$$

so

$$\left\lfloor m\left(K_{X_i} + B_i + \sum nM_{j,i}\right)\right\rfloor \sim m\overline{D}_i$$

for some integral divisor $m\overline{D}_i \geq 0$. The coefficients of $m\overline{D}_i$ belong to $\mathbf{N}$, a DCC set. Now let D_i be the $\mathbf{R}$-divisor so that mD_i is the sum of $m\overline{D}_i$ and the fractional part $\langle m(K_{X_i} + B_i + \sum nM_{j,i})\rangle$. Since $K_{X_i} + \sum nM_{j,i}$ is Cartier, $mD_i = m\overline{D}_i + \langle mB_i \rangle$. On the other hand, since the coefficients of B_i belong to the DCC set $\Lambda \cap [0, 1]$, the coefficients of $\langle mB_i \rangle$ belong to a DCC set as well by Lemma 2.1. Therefore, the coefficients of mD_i and hence of D_i belong to a DCC set, depending only on Λ. By extending Λ we can assume that the coefficients of D_i belong to Λ.

By construction,

$$0 \leq D_i \sim_{\mathbf{R}} K_{X_i} + B_i + \sum nM_{j,i}$$

which in turn implies that

$$0 \leq D'_i \sim_{\mathbf{R}} K_{X'_i} + B'_i + \sum nM'_{j,i}$$
$$= K_{X'_i} + B'_i + M'_i + \sum(n - \mu_{j,i})M'_{j,i} \equiv \sum(n - \mu_{j,i})M'_{j,i}.$$

Note that we can assume $n - \mu_{j,i} > 0$ for every j, i.

Step 3. By Lemma 2.2, replacing the sequence X_i' and reordering the indexes j, we may assume that $M_{j,i}' \equiv \lambda_{j,i} M_{1,i}'$ so that for each j the numbers $\lambda_{j,i}$ form a decreasing sequence. By Step 2, we get

$$D_i' \equiv \sum (n - \mu_{j,i}) M_{j,i}' \equiv \sum (n - \mu_{j,i}) \lambda_{j,i} M_{1,i}' =: \rho_i M_{1,i}'$$

where we have defined

$$\rho_i := \sum (n - \mu_{j,i}) \lambda_{j,i}.$$

For each j, the numbers $n - \mu_{j,i}$ and $\lambda_{j,i}$ form decreasing sequences hence the ρ_i also form a decreasing sequence by Lemma 2.1.

Now let $N_i := \sum_{j \geq 2} \mu_{j,i} M_{j,i}$ and let u_i be the generalized lc threshold of D_i' with respect to $(X_i', B_i' + N_i')$. Since $D_i' \equiv \rho_i M_{1,i}'$ we get

$$K_{X_i'} + B_i' + N_i' + u_i D_i' \equiv K_{X_i'} + B_i' + N_i' + u_i \rho_i M_{1,i}'.$$

Assume that $u_i \rho_i \geq \mu_{1,i}$ for every i. Let $v_i \leq u_i$ be the number so that

$$K_{X_i'} + B_i' + N_i' + v_i D_i' \equiv K_{X_i'} + B_i' + M_i' \equiv 0$$

that is, $v_i = \frac{\mu_{1,i}}{\rho_i}$. As the $\mu_{1,i}$ form an increasing sequence and the ρ_i form a decreasing sequence, the v_i form an increasing sequence. Moreover, if the $\mu_{1,i}$ form a strictly increasing sequence, then the v_i also form a strictly increasing sequence. Thus the set of the coefficients of all the $B_i' + v_i D_i'$ together with the $\{\mu_{j,i} \mid j \geq 2\}$ is a DCC set but not ACC. Now $(X_i', B_i' + v_i D_i' + N_i')$ is generalized lc with boundary part $B_i' + v_i D_i'$ and nef part N_i, and $\sigma(N_i) < \sigma$ which contradicts the minimality assumption on σ in Step 1. Therefore, from now on we may assume that $u_i \rho_i < \mu_{1,i}$ for every i.

Step 4. Fix i. Let Σ_i be the set of those elements $(\alpha, \beta) \in [0, \frac{\mu_{1,i}}{\rho_i}] \times [0, \mu_{1,i}]$ such that

$$K_{X_i'} + B_i' + N_i' + \alpha D_i' + \beta M_{1,i}' \equiv K_{X_i'} + B_i' + M_i'$$

which is equivalent to $\alpha \rho_i + \beta = \mu_{1,i}$. Note that $(0, \mu_{1,i}) \in \Sigma_i$ hence $\Sigma_i \neq \emptyset$. Now let

$$s_i = \sup \{ \alpha \mid (\alpha, \beta) \in \Sigma_i, \; (X_i', B_i' + \alpha D_i' + N_i' + \beta M_{1,i}') \text{ is generalized lc} \}$$

where the pair in the definition has boundary part $B_i' + \alpha D_i'$ and nef part $N_i + \beta M_{1,i}$. Letting $t_i = \mu_{1,i} - s_i \rho_i$ we get $(s_i, t_i) \in \Sigma_i$.

We show that s_i is actually a maximum hence in particular

$$\left(X_i', B_i' + s_i D_i' + N_i' + t_i M_{1,i}' \right)$$

is generalized lc. If not, then there is a sequence $(\alpha^l, \beta^l) \in \Sigma_i$ such that the α^l form a strictly increasing sequence approaching s_i and the β^l form a strictly decreasing sequence approaching t_i. Since

$$\left(X_i', B_i' + \alpha^l D_i' + N_i' + t_i M_{1,i}'\right)$$

is generalized lc, the generalized lc threshold of D_i' with respect to $(X_i', B_i' + N_i' + t_i M_{1,i}')$ is at least $\lim \alpha^l = s_i$ by Theorem 1.5. So

$$\left(X_i', B_i' + s_i D_i' + N_i' + t_i M_{1,i}'\right)$$

is also generalized lc. Hence s_i is indeed a maximum. Note that $s_i \le u_i$.

Step 5. Since the coefficients of D_i' belong to Λ and since u_i is the generalized lc threshold of D_i' with respect to $(X_i', B_i' + N_i')$, u_i is bounded from above by Theorem 1.5. Thus s_i is also bounded from above. So we may assume the s_i and the t_i each form an increasing or a decreasing sequence hence $s = \lim s_i$ and $t = \lim t_i$ exist. Since the $\mu_{1,i}$ form an increasing sequence and the ρ_i form a decreasing sequence, the s_i or the t_i form an increasing sequence. We will show that in fact the t_i form an increasing sequence.

Assume otherwise, that is, assume the t_i form a decreasing sequence. We can assume it is strictly decreasing. Then the s_i form a strictly increasing sequence. Since

$$\left(X_i', B_i' + s_i D_i' + N_i' + t M_{1,i}'\right)$$

is generalized lc, we may assume that

$$\left(X_i', B_i' + s D_i' + N_i' + t M_{1,i}'\right)$$

is generalized lc too, by Theorem 1.5. Now we can find $\tilde{s}_i > s_i$ such that $(\tilde{s}_i, t) \in \Sigma_i$, that is, $\tilde{s}_i \rho_i + t = \mu_{1,i}$. Since the $\mu_{1,i}$ form an increasing sequence and the ρ_i form a decreasing sequence, the $\tilde{s}_i$ form an increasing sequence. Moreover, since

$$t < t_i \le \mu_{1,i} \le \lim \mu_{1,i} \quad \text{and} \quad s(\lim \rho_i) + t = \lim(s_i \rho_i + t_i) = \lim \mu_{1,i}$$

we deduce $\lim \rho_i > 0$. Thus as

$$\lim(\tilde{s}_i \rho_i + t) = \lim \mu_{1,i},$$

we get $\lim \tilde{s}_i = \lim s_i = s$. In particular this means $s \ge \tilde{s}_i > s_i$, hence

$$\left(X_i', B_i' + \tilde{s}_i D_i' + N_i' + t M_{1,i}'\right)$$

is generalized lc which contradicts the maximality assumption of s_i in Step 4.

So we have proved that the t_i form an increasing sequence. Now by definition s_i is the generalized lc threshold of D_i' with respect to

$$\left(X_i', B_i' + N_i' + t_i M_{1,i}'\right).$$

So they form a decreasing sequence by Theorem 1.5.

Step 6. The purpose of this step is to modify B_i' so that we can assume $s = \lim s_i = 0$. Let $\tilde{t}_i$ be the number so that $s\rho_i + \tilde{t}_i = \mu_{1,i}$. As $s_i \geq s$, $\tilde{t}_i \geq t_i \geq 0$, hence $(s, \tilde{t}_i) \in \Sigma_i$. Since the $\mu_{1,i}$ (resp. ρ_i) form an increasing (resp. decreasing) sequence, the $\tilde{t}_i$ form an increasing sequence. Moreover,

$$\lim \tilde{t}_i = \lim(\mu_{1,i} - s\rho_i) = \lim(\mu_{1,i} - s_i\rho_i) = \lim t_i = t$$

which implies $\tilde{t}_i \leq t$.

We claim that

$$(*) \quad \left(X_i', B_i' + sD_i' + N_i' + \tilde{t}_i M_{1,i}'\right)$$

is generalized lc. Indeed, let c_i be the generalized lc threshold of $M_{1,i}'$ with respect to $(X_i', B_i' + sD_i' + N_i')$. Then $c_i \geq t_i$ and by Theorem 1.5, we may assume that the c_i form a decreasing sequence. Thus

$$c_i \geq \lim c_i \geq \lim t_i = t \geq \tilde{t}_i$$

and the claim follows.

Now we define the boundary $C_i := B_i + s\tilde{D}_i'$ on X_i where B_i, as in Step 2, is the sum of the birational transform of B_i' and the reduced exceptional divisor of $X_i \to X_i'$, and $\tilde{D}_i'$ is the birational transform of D_i'. Then $C_i' = B_i' + sD_i'$ and

$$\left(X_i', C_i' + N_i' + \tilde{t}_i M_{1,i}'\right)$$

is generalized lc by $(*)$, and

$$K_{X_i'} + C_i' + N_i' + \tilde{t}_i M_{1,i}' \equiv 0.$$

Moreover, the set of the coefficients of all the C_i' union the set $\{\mu_{j,i} \mid j \geq 2\} \cup \{\tilde{t}_i\}$ satisfies DCC but not ACC (note that if the $\mu_{1,i}$ form a strictly increasing sequence, then so do the $\tilde{t}_i$).

On the other hand, let $G_i := D_i + s\tilde{D}_i'$ and let $r_i := \frac{s_i - s}{1 + s}$. Then

$$0 \leq G_i \sim_{\mathbf{R}} K_{X_i} + C_i + \sum n M_{j,i}$$

and $G_i' = (1 + s)D_i'$, and

$$K_{X_i'} + C_i' + r_i G_i' + N_i' + t_i M_{1,i}' = K_{X_i'} + B_i' + s_i D_i' + N_i' + t_i M_{1,i}' \equiv 0.$$

The equality also shows

$$\left(X_i', C_i' + r_i G_i' + N_i' + t_i M_{1,i}'\right)$$

is generalized lc and that r_i is the generalized lc threshold of G_i' with respect to $(X_i', C_i' + N_i' + t_i M_{1,i}')$. Therefore extending Λ, replacing B_i with C_i, replacing $\mu_{1,i}$ with $\tilde{t}_i$, replacing D_i with G_i, and replacing s_i with r_i allow us to assume that $s = \lim s_i = 0$.

Step 7. After replacing X_i we may assume that there is a prime divisor S_i on X_i whose generalized log discrepancy with respect to the generalized lc polarized pair

$$\left(X_i', B_i' + s_i D_i' + N_i' + t_i M_{1,i}'\right)$$

is 0: this follows from our choice of s_i, t_i.

First assume that S_i is not contracted over X_i' for every i which means that S_i' is a component of $\lfloor B_i' + s_i D_i' \rfloor$. Let d_i be the coefficient of S_i' in D_i' and let p_i be the real number such that

$$K_{X_i'} + B_i' + s_i d_i S_i' + N_i' + p_i M_{1,i}' \equiv 0.$$

Obviously $p_i \leq \mu_{1,i}$, and equality holds if and only if $s_i d_i S' \equiv 0$, i.e., $s_i d_i = 0$. Since $s_i d_i S_i' \leq s_i D_i'$ and

$$K_{X_i'} + B_i' + s_i D_i' + N_i' + t_i M_{1,i}' \equiv 0$$

we have $t_i \leq p_i$. Then from $\lim s_i = 0$ and $\mu_1 := \lim \mu_{1,i} = \lim t_i$ we arrive at $\lim p_i = \mu_1$. So we may assume that the p_i form an increasing sequence approaching μ_1.

Let w_i be the generalized lc threshold of $M_{1,i}'$ with respect to

$$\left(X_i', B_i' + s_i d_i S_i' + N_i'\right).$$

Then $w_i \geq t_i$. Applying Theorem 1.5, we can assume that the w_i form a decreasing sequence. Then

$$w_i \geq \lim w_i \geq \lim t_i = \mu_1 = \lim p_i \geq p_i$$

which implies that

$$\left(X_i', B_i' + s_i d_i S_i' + N_i' + p_i M_{1,i}'\right)$$

is generalized lc with boundary part $\Delta_i' := B_i' + s_i d_i S_i'$ and nef part $R_i := N_i + p_i M_{1,i}$. The set of the coefficients of all the Δ_i' union the set $\{\mu_{j,i} \mid j \geq 2\} \cup \{p_i\}$ satisfies DCC. Therefore, by Proposition 7.1, we may assume that p_i is a constant independent of i.

Now

$$\mu_1 = \lim p_i = p_i \leq \mu_{1,i} \leq \lim \mu_{1,i} = \mu_1$$

Thus $p_i = \mu_{1,i}$, hence $s_i d_i = 0$, $\Delta_i' = B_i'$, and $R_i = M_i$. In other words, $(X_i', B_i' + M_i')$ is not generalized klt. This contradicts Proposition 7.1.

So after replacing the sequence we may assume that S_i is exceptional over X'_i for every i.

Step 8. By Lemma 4.6, there is an extremal contraction $g_i \colon X''_i \to X'_i$ extracting S''_i with X''_i being **Q**-factorial. We can assume $X_i \dashrightarrow X''_i$ is a morphism. We can write

$$K_{X''_i} + B''_i + s_i\tilde{D}''_i + N''_i + t_iM''_{1,i} = g^*_i\big(K_{X'_i} + B'_i + s_iD'_i + N'_i + t_iM'_{1,i}\big) \equiv 0$$

where B''_i is the pushdown of B_i, $\tilde{D}''_i$ is the birational transform of D'_i, $M''_{1,i}$ is the pushdown of $M_{1,i}$, and N''_i is the pushdown of N_i. Now S''_i is a component of $\lfloor B''_i \rfloor$. By Lemma 4.4(1) we can run the $-S''_i$-LMMP which terminates on some Mori fibre space $X'''_i \to T'''_i$. We may assume that $\dim T'''_i = 0$ for every i, or $\dim T'''_i > 0$ for every i. Replacing X_i we may assume $X_i \dashrightarrow X'''_i$ is a log resolution of $(X'''_i, B'''_i + s_i\tilde{D}'''_i)$.

Since $(X'_i, B'_i + M'_i)$ is generalized lc and $K_{X'_i} + B'_i + M'_i \equiv 0$, we deduce that $K_{X_i} + B_i + M_i$ is pseudo-effective. Thus $K_{X'''_i} + B'''_i + M'''_i$ is pseudo-effective too. Moreover, by construction

$$K_{X'''_i} + B'''_i + s_i\tilde{D}'''_i + N'''_i + t_iM'''_{1,i} \equiv 0.$$

So there is the largest number $q_i \in [t_i, \mu_{1,i}]$ such that

$$K_{X'''_i} + B'''_i + N'''_i + q_iM'''_{1,i} \equiv 0/T'''_i.$$

From $s = \lim s_i = 0$ we get $\lim t_i = \lim \mu_{1,i} = \mu_1$ from which we derive $\lim q_i = \mu_1$. So we may assume that the q_i form an increasing sequence approaching μ_1. Let w_i be the generalized lc threshold of $M'''_{1,i}$ with respect to the generalized lc polarized pair $(X'''_i, B'''_i + N'''_i)$. Then $w_i \geq t_i$ as

$$\big(X'''_i, B'''_i + s_i\tilde{D}'''_i + N'''_i + t_iM'''_{1,i}\big)$$

is generalized lc. Moreover, by Theorem 1.5 we can assume the w_i form a decreasing sequence, hence

$$q_i \leq \mu_{1,i} \leq \mu_1 = \lim t_i \leq \lim w_i \leq w_i.$$

So the pair $(X'''_i, B'''_i + N'''_i + q_iM'''_{1,i})$ is generalized lc. But the pair is not generalized klt because S'''_i is a component of $\lfloor B'''_i \rfloor$.

Step 9. Assume that $\dim T'''_i = 0$ for every i. Applying Proposition 7.1, we can assume that the set of the coefficients of all the B'''_i union the set $\{\mu_{j,i}|j \geq 2\} \cup \{q_i\}$ is finite. In particular, this means we can assume $q_i = \mu_{1,i} = \mu_1$ for every i, and that $\mu_{j,i} = \mu_j$ for every j, i where $\mu_j := \lim_i \mu_{j,i}$. On the other hand, assume that $\dim T'''_i > 0$ for every i. If $M'''_{1,i} \equiv 0/T'''_i$, then $q_i = \mu_{1,i}$. But if $M'''_{1,i} \not\equiv 0/T'''_i$, then by restricting to the general fibres of $X'''_i \to T'''_i$ and applying induction, we deduce that $\{q_i\}$ is finite, hence $q_i = \mu_1$ for $i \gg 1$; so we can assume $q_i = \mu_{1,i} = \mu_1$. Moreover, by restricting to the general fibres

of $X_i''' \to T_i'''$ and applying induction once more, we may assume that the set of the horizontal/T_i''' coefficients of all the B_i''' together with the set $\{\mu_{j,i} \mid M_{j,i}''' \not\equiv 0/T_i'''\}$ is finite.

The last paragraph shows that in either case $\dim T_i''' = 0$ or $\dim T_i''' > 0$, we can assume

$$(**) \quad K_{X_i'''} + B_i''' + M_i''' \equiv 0/T_i'''.$$

Let $\overline{B}_i$ be obtained from B_i by replacing the coefficient $b_{k,i}$ with $b_k := \lim_i b_{k,i}$. Let $\overline{M}_i$ be obtained from M_i by replacing $\mu_{j,i}$ with $\mu_j = \lim_i \mu_{j,i}$. Then $K_{X_i'} + \overline{B}_i' + \overline{M}_i'$ is ample because $\rho(X_i') = 1$ and because either $b_{k,i} < b_k$ for some k or $\mu_{j,i} < \mu_j$ for some j. Moreover, by Theorem 1.5, we can assume $(X_i', \overline{B}_i' + \overline{M}_i')$ is generalized lc. Thus

$$K_{X_i} + \overline{B}_i + \overline{M}_i \geq f_i^*\left(K_{X_i'} + \overline{B}_i' + \overline{M}_i'\right)$$

is big. This in turn implies that $K_{X_i'''} + \overline{B}_i''' + \overline{M}_i'''$ is big too. On the other hand, by the last paragraph, we may assume that on the general fibres F_i''' of $X_i''' \to T_i'''$ we have: $\overline{B}_i'''|_{F_i'''} = B_i'''|_{F_i'''}$ and $\overline{M}_i'''|_{F_i'''} \equiv M_i'''|_{F_i'''}$. This contradicts $(**)$. $\qquad\square$

8. Proof of main results

In this section, we prove our main results stated in the introduction.

Proof of Theorem 1.5 and Theorem 1.6. — By Proposition 6.1, Theorem 1.6 in dimension $< d$ implies Theorem 1.5 in dimension d. On the other hand, by Proposition 7.2, Theorem 1.6 in dimension $< d$ and Theorem 1.5 in dimension d imply Theorem 1.6 in dimension d. So both theorems follow inductively the case $d = 1$ being trivial. $\qquad\square$

Next we prove a result bounding pseudo-effective thresholds which will be needed for the proof of Theorem 1.3.

Theorem **8.1**. — *Let d be a natural number and Λ a DCC set of nonnegative real numbers. Then there is a real number $e \in (0, 1)$ depending only on Λ, d such that if:*

- *(X, B) is projective lc of dimension d,*
- *$M = \sum \mu_j M_j$ where M_j are nef Cartier divisors,*
- *the coefficients of B and the μ_j are in Λ, and*
- *$K_X + B + M$ is a big divisor,*

then $K_X + eB + eM$ is a big divisor.

Proof. — It suffices to show the assertion: there is an $e \in (0, 1)$ depending only on Λ, d such that $K_X + eB + eM$ is pseudo-effective; because then

$$\mathrm{vol}\left(K_X + \frac{1}{2}(e+1)(B+M)\right) = \mathrm{vol}\left(\frac{1}{2}(K_X + B + M + K_X + eB + eM)\right)$$

$$\geq \mathrm{vol}\left(\frac{1}{2}(\mathrm{K_X} + \mathrm{B} + \mathrm{M})\right) > 0$$

and hence $\mathrm{K_X} + e'\mathrm{B} + e'\mathrm{M}$ is big for $e' := \frac{1}{2}(1 + e) \in (0, 1)$.

If there is no e as in the last paragraph, then there is a sequence of pairs $(\mathrm{X}_i, \mathrm{B}_i)$ and divisors $\mathrm{M}_i = \sum \mu_{j,i}\mathrm{M}_{j,i}$ satisfying the assumptions of the theorem but such that the pseudo-effective thresholds e_i of $\mathrm{B}_i + \mathrm{M}_i$ form a strictly increasing sequence approaching 1: by definition $\mathrm{K}_{\mathrm{X}_i} + e_i\mathrm{B}_i + e_i\mathrm{M}_i$ is pseudo-effective but $\mathrm{K}_{\mathrm{X}_i} + c_i\mathrm{B}_i + c_i\mathrm{M}_i$ is not pseudo-effective for any $c_i < e_i$.

We can extend Λ and replace the $\mathrm{X}_i, \mathrm{B}_i$ so that we may assume $(\mathrm{X}_i, \mathrm{B}_i)$ is log smooth klt. By Lemma 4.4(2), we can run an LMMP on $\mathrm{K}_{\mathrm{X}_i} + e_i\mathrm{B}_i + e_i\mathrm{M}_i$ which ends with a minimal model X'_i on which $\mathrm{K}_{\mathrm{X}'_i} + e_i\mathrm{B}'_i + e_i\mathrm{M}'_i$ is semi-ample defining a contraction $\mathrm{X}'_i \to \mathrm{T}'_i$. Since $\mathrm{K}_{\mathrm{X}'_i} + \mathrm{B}'_i + \mathrm{M}'_i$ is big and $\mathrm{K}_{\mathrm{X}'_i} + e_i\mathrm{B}'_i + e_i\mathrm{M}'_i \equiv 0/\mathrm{T}'_i$, we deduce that $\mathrm{B}'_i + \mathrm{M}'_i$ is big over T'_i.

Replacing X_i we may assume that $\mathrm{X}_i \dashrightarrow \mathrm{X}'_i$ is a log resolution of $(\mathrm{X}'_i, \mathrm{B}'_i)$. Let F'_i be a general fibre of $\mathrm{X}'_i \to \mathrm{T}'_i$ and F_i the corresponding fibre of $\mathrm{X}_i \to \mathrm{T}'_i$. By restricting to F'_i we get

$$\mathrm{K}_{\mathrm{F}'_i} + e_i\mathrm{B}_{\mathrm{F}'_i} + e_i\mathrm{M}_{\mathrm{F}'_i} := \left(\mathrm{K}_{\mathrm{X}'_i} + e_i\mathrm{B}'_i + e_i\mathrm{M}'_i\right)|_{\mathrm{F}'_i} \equiv 0.$$

This contradicts Theorem 1.6 because $e_i\mathrm{B}_{\mathrm{F}'_i} + e_i\mathrm{M}_{\mathrm{F}'_i}$ is big hence nonzero for every i, so the set of the coefficients of all the $e_i\mathrm{B}_{\mathrm{F}'_i}$ union with the set $\{e_i\mu_{j,i} \mid \mathrm{M}_{j,i}|_{\mathrm{F}_i} \not\equiv 0\}$ is not finite. $\square$

Proof of Theorem 1.3. — As usual by taking a log resolution we may assume (X, B) is log smooth. By Theorem 8.1, there exist a rational number $e \in (0, 1)$ depending only on Λ, d, r such that $\mathrm{K_X} + e\mathrm{B} + e\mathrm{M}$ is big, so $\mathrm{K_X} + e\mathrm{B} + \mathrm{M}$ is also big. As in Step 2 of the proof of Proposition 3.4, there is $p \in \mathbf{N}$ depending only on e, Λ, r such that $r|p$ and for any nonzero $\lambda \in \Lambda$ we can find $\gamma \in [e\lambda, \lambda)$ such that $p\gamma$ is an integer. In particular, we can find a boundary Δ such that $e\mathrm{B} \leq \Delta \leq \mathrm{B}$, $p\Delta$ is Cartier, $\mathrm{K_X} + \Delta + \mathrm{M}$ is big, and (X, Δ) is klt. Replacing B with Δ we can then assume $\Lambda = \{\frac{i}{p} \mid 0 \leq i \leq p - 1\}$ and that (X, B) is klt.

By Proposition 3.4, there exist $l, n \in \mathbf{N}$ depending only on Λ, d, r such that $r|n$ and that $|l(\mathrm{K_X} + \mathrm{B} + n\mathrm{M})|$ defines a birational map. By replacing l with pl we can assume $p|l$. There is a resolution $\phi: \mathrm{W} \to \mathrm{X}$ such that

$$\phi^* l(\mathrm{K_X} + \mathrm{B} + n\mathrm{M}) \sim \mathrm{H} + \mathrm{G}$$

where H is big and base point free and $\mathrm{G} \geq 0$. Perhaps after replacing l with $(2d + 1)l$, we can also assume that H is potentially birational [9, Lemma 2.3.4].

Applying Theorem 8.1 once more, there exist rational numbers $s, u \in (0, 1)$ depending only on Λ, d, r such that $\mathrm{K_X} + s\mathrm{B} + u\mathrm{M}$ is big. Perhaps after replacing s, u, we

can choose a sufficiently large natural number q so that qs is integral and divisible by p, qu is integral and divisible by r,

$$s' := \frac{qs + l}{q + l + 1} < 1, \quad \text{and} \quad \frac{qu + ln}{q + l + 1} = 1.$$

Let X' be a minimal model of $K_X + sB + uM$, which exists by Lemma 4.4(2). We can assume that the induced map $\psi \colon W \dashrightarrow X'$ is a morphism. Since X' is a minimal model,

$$\phi^*(K_X + sB + uM) = \psi^*\big(K_{X'} + sB' + uM'\big) + E$$

where E is effective. Let

$$D = \psi^*\big(K_{X'} + sB' + uM'\big).$$

Since H is potentially birational, by Lemma 3.1, $qD + H$ is potentially birational and $|K_W + \lceil qD + H \rceil|$ defines a birational map. Thus

$$\Big|K_W + \big\lceil \phi^* q(K_X + sB + uM) \big\rceil + \phi^* l(K_X + B + nM)\Big|$$

also defines a birational map which in turn implies that

$$\Big|K_X + \big\lceil q(K_X + sB + uM) \big\rceil + l(K_X + B + nM)\Big|$$

defines a birational map. Hence the linear system

$$\Big|(q + l + 1)\big(K_X + s'B + M\big)\Big|$$

defines a birational map. Therefore

$$\Big|(q + l + 1)(K_X + B + M)\Big|$$

also defines a birational map.

By construction $r|qu$ and $r|ln$, so $r|(q + l + 1 = qu + ln)$. Now put $a := m(\Lambda, d, r) := q + l + 1$. Then aM is Cartier, and for any $b \in \mathbf{N}$, the linear system $|b\lfloor a(K_X + B + M) \rfloor|$ defines a birational map. But since aM is Cartier and B is effective,

$$b\big\lfloor a(K_X + B + M) \big\rfloor \le \big\lfloor ba(K_X + B + M) \big\rfloor$$

which means $|m(K_X + B + M)|$ also defines a birational map where $m = ba$. $\square$

Next we prove a result similar to 1.3 but we allow a more general nef part M. This result is not used elsewhere in this paper.

Theorem **8.2.** *— Let d be a natural number and Λ a DCC set of nonnegative real numbers. Then there is a natural number m depending only on Λ, d such that if:*

- (X, B) *is projective lc of dimension* d,
- $M = \sum \mu_j M_j$ *where* M_j *are nef Cartier divisors,*
- *the coefficients of* B *and the* μ_j *are in* Λ, *and*
- $K_X + B + M$ *is big,*

then the linear system $|\lfloor m(K_X + B) \rfloor + \sum \lfloor m\mu_j \rfloor M_j|$ *defines a birational map.*

Proof. — As usual we may assume (X, B) is log smooth. By Theorem 8.1, there exists a rational number $e \in (0, 1)$ depending only on Λ, d such that $K_X + eB + eM$ is pseudo-effective. As in the proof of 1.3, there is $p \in \mathbf{N}$ depending only on e, Λ such that we can find a boundary $\Delta \le B$ and numbers $\nu_j \in [e\mu_j, \mu_j]$ such that $p\Delta$ and pN are Cartier divisors and $K_X + \Delta + N$ is big where $N = \sum \nu_j M_j$.

Applying Theorem 1.3, there is $l \in \mathbf{N}$ depending only on p, d (hence only on Λ, d) such that $|l(K_X + \Delta + N)|$ defines a birational map and $p|l$. Replacing l by a multiple we can in addition assume that $l(K_X + \Delta + N)$ is potentially birational. Then by Lemma 3.1,

$$l(K_X + \Delta + N) + \sum \alpha_j M_j$$

is potentially birational for any $0 \le \alpha_j \in \mathbf{Z}$, and

$$\left| K_X + l(K_X + \Delta + N) + \sum \alpha_j M_j \right|$$

defines a birational map. Since $\nu_j \le \mu_j$, we can take α_j so that $l\nu_j + \alpha_j = \lfloor (l+1)\mu_j \rfloor$. Therefore

$$\left| (l+1)K_X + l\Delta + \sum \lfloor (l+1)\mu_j \rfloor M_j \right|$$

defines a birational map which in turn implies that

$$\left| \lfloor (l+1)(K_X + B) \rfloor + \sum \lfloor (l+1)\mu_j \rfloor M_j \right|$$

defines a birational map because $l\Delta \le \lfloor (l+1)B \rfloor$. Now put $m = l+1$. $\square$

Proof of Theorem 1.2. — Replacing W we can assume the Iitaka fibration $I: W \dashrightarrow X$ is a morphism, i.e. can assume $V = W$ using the notation before Theorem 1.2. Also we can assume $\kappa(W) \ge 1$ otherwise there is nothing to prove.

Let $b := b_F$ and $\beta := \beta_{\widetilde{F}}$. Let

$$N = N(\beta) = \mathrm{lcm}\{ m \in \mathbf{N} \mid \varphi(m) \le \beta \}$$

where φ denotes Euler's φ-function. Let

$$A(b, N) := \left\{ \frac{bNu - v}{bNu} \;\middle|\; u, v \in \mathbf{N}, \; v \le bN \right\}$$

which is a DCC subset of the interval $[0, 1)$.

By the results of [7] (which is summarized in [24, Lemma 1.2]), replacing W and X by high enough resolutions, we may assume that X is smooth and that there exist a boundary B on X (the discriminant part of $I : W \to X$) and a nef $\mathbf{Q}$-divisor M (the moduli part of $I : W \to X$) such that

- NbM is Cartier,
- B has simple normal crossing support with coefficients in $A(b, N)$,
- $K_X + B + M$ is big,
- we have isomorphisms

$$H^0(W, mbK_W) \cong H^0\big(X, mb(K_X + B + M)\big)$$

  for every $m \in \mathbf{N}$, and
- the rational map defined by $|mbK_W|$ is birational to the Iitaka fibration $I : W \to X$ if and only if $|mb(K_X + B + M)|$ gives rise to a birational map.

By letting $\Lambda = A(b, N)$ and $r = Nb$, and applying Theorem 1.3, there is a constant $m(\Lambda, d, r)$ depending only on Λ, d, r, (hence depending only on d, b, β) such that $|m(K_X + B + M)|$ defines a birational map for any $m \in \mathbf{N}$ divisible by $m(\Lambda, d, r)$. Now simply let $m(d, b_F, \beta_{\widetilde{F}}) = bm(\Lambda, d, r)$. $\qquad\square$

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