PUBLICATIONS MATHÉMATIQUES DE L'I.H.É.S.

HERBERT CLEMENS

The Neron model for families of intermediate jacobians acquiring "algebraic" singularities

Publications mathématiques de l'I.H.É.S., tome 58 (1983), p. 5-18

<http://www.numdam.org/item?id=PMIHES_1983__58__5_0>

© Publications mathématiques de l'I.H.É.S., 1983, tous droits réservés.

L'accès aux archives de la revue « Publications mathématiques de l'I.H.É.S. » (http://www.ihes.fr/IHES/Publications/Publications.html) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

THE NERON MODEL FOR FAMILIES OF INTERMEDIATE JACOBIANS ACQUIRING "ALGEBRAIC" SINGULARITIES

by HERBERT CLEMENS

1. Introduction

Let V be an irreducible complex projective manifold of dimension $2m-1$. Let $\{Z_s\}_{s\in S}$ be an algebraic family of algebraic $(m-1)$-cycles on V whose members Z_s are all homologically equivalent. In Appendix A of his paper "Periods of integrals on algebraic manifolds, III" ([G]; p. 165), P. Griffiths defines an analytic map, called the *Abel-Jacobi homomorphism*

$$S \to J(V). \tag{1.1}$$

Here $J(V) = \dfrac{[F^m H^{2m-1}(V; \mathbf{C})]^*}{H_{2m-1}(V; \mathbf{Z})}$ is the Jacobian variety of V, and (1.1) is defined by picking a basepoint $s_0 \in S$ and sending

$$s \mapsto \int_{Z_{s_0}}^{Z_s}. \tag{1.2}$$

Next let X be a complex projective manifold of dimension $2m$, let Δ be the unit disc, and let

$$V_t, \quad t \in \Delta, \tag{1.3}$$

be an analytic family of divisors on X which are irreducible and non-singular as long as $t \neq 0$. Suppose

$$\{Z_s\}_{s\in S_t} \tag{1.4}$$

is an algebraic family of algebraic $(m-1)$-cycles on V_t for each $t \in \Delta$. Suppose further that, for fixed t, all cycles Z_s, $s \in S_t$, are *homologous* to one another. Finally suppose that

$$\mathscr{S} = \bigcup_{t\in\Delta} (\{t\} \times S_t)$$

is a smooth analytic variety with

$$\mathscr{S} \to \Delta \tag{1.5}$$

everywhere of maximal rank. We make no assumptions about properness of (1.5) or connectivity of fibres.

Let $\mathscr{J}^* \to \Delta^* = (\Delta - \{0\})$ be the bundle of complex tori over the punctured disc whose fibre over t is $J(V_t)$. Then for every section $\tau : \Delta \to \mathscr{S}$ of (1.5) there is a commutative diagram of proper morphisms

$$\begin{array}{ccc} \mathscr{S}^* & \longrightarrow & \mathscr{J}^* \\ & \searrow \quad \swarrow & \\ & \Delta^* & \end{array} \tag{1.6}$$

defined fibrewise by (1.2) with $s_0 = \tau(t)$. The point of this paper is to complete (1.6) to a commutative diagram

$$\begin{array}{ccc} \mathscr{S} & \longrightarrow & \mathscr{J} \\ & \searrow \quad \swarrow & \\ & \Delta & \end{array} \tag{1.7}$$

where $\mathscr{J}$ is obtained from $\mathscr{J}^*$ by filling in over $t = 0$ with a commutative complex Lie group. The complex Lie group in question is the fibre over $t = 0$ of an analytic analogue of the *Neron minimal model.*

We will be able to carry out this program only after putting some severe restrictions on the family (1.3). We devote the rest of the introduction to explaining these restrictions.

Let $\mathscr{V}^* = \bigcup_{t \in \Delta} (\{t\} \times V_t)$ and

$$\mu : \mathscr{V}^* \to \Delta^*. \tag{1.8}$$

We pull-back the bundle (1.8) via the universal covering map

$$\begin{aligned} &\tilde{\Delta} \to \Delta^* \\ &u \mapsto t = e^{2\pi i u} \end{aligned}$$

to obtain $\tilde{\mu} : \tilde{\mathscr{V}} \to \tilde{\Delta}$.

Abusing notation we write

$$\tilde{\mu}^{-1}(u) = V_u = V_t = \mu^{-1}(t)$$

whenever $t = e^{2\pi i u}$. The derived bundles $R_{2m-1}\tilde{\mu}_*(\mathbf{Z})$ and $R^{2m-1}\tilde{\mu}_*(\mathbf{Z})$ are trivial and we will denote their modules of global sections, taken modulo torsion, by $H_{\mathbf{Z}}$ and $H^{\mathbf{Z}}$ respectively. Also $H^{\mathbf{C}} = H^{\mathbf{Z}} \otimes \mathbf{C}$.

The natural isomorphism

$$H_{\mathbf{Z}} \cong H_{2m-1}(V_{u+1}) = H_{2m-1}(V_t) = H_{2m-1}(V_u) \cong H_{\mathbf{Z}}$$

is not the identity map on $H_{\mathbf{Z}}$ but rather the *monodromy isomorphism*

$$T_*: H_{\mathbf{Z}} \to H_{\mathbf{Z}}.$$

Let T^* be the adjoint of T_* with respect to the natural unimodular pairing

$$H_{\mathbf{Z}} \times H^{\mathbf{Z}} \to \mathbf{Z}$$
$$(\gamma, \omega) \to \int_\gamma \omega.$$

Our first major assumption is that

(1.9) $$(T_* - I)^2 = (T^* - I)^2 = 0.$$

Let $N_* = \log T_*$, $N^* = \log T^*$. We then have a filtration $\{W_*\}$ on $H^{\mathbf{Z}}$ defined by

$$W_{2m-3}H^{\mathbf{Z}} = 0$$
$$W_{2m-2}H^{\mathbf{Z}} = (\ker N_*)^{\perp}$$
$$W_{2m-1}H^{\mathbf{Z}} = (\text{image } N_*)^{\perp}$$
$$W_{2m}H^{\mathbf{Z}} = H^{\mathbf{Z}}.$$

This filtration is called the *asymptotic weight filtration.*

Under the identification

$$H^{\mathbf{C}} = H^{2m-1}(V_u; \mathbf{C})$$

the Hodge filtration on $H^{2m-1}(V_u; \mathbf{C})$ induces a filtration F_u^* on $H^{\mathbf{C}}$. This filtration varies with $u \in \tilde{\Delta}$, but there is a well-defined filtration

(1.10) $$F_\infty^* = \lim_{t \to 0} \exp(-uN^*)F_u^*$$

on $H^{\mathbf{C}}$ called the *asymptotic Hodge filtration.*

In [S], W. Schmid shows that the array $(H^{\mathbf{Z}}, W_*, F_\infty^*)$ is a *mixed Hodge structure* such that

$$N^*: H^{\mathbf{C}} \to H^{\mathbf{C}}$$

is a morphism of mixed Hodge structures of type $(-1, -1)$. In fact in this situation which "comes from geometry", we have that

$$N^*: H^{\mathbf{Z}}/W_{2m-1} \to W_{2m-2}$$

is an isomorphism over $\mathbf{Q}$. Our second major assumption is:

(1.11) The Hodge structure of weight $2m$ on $H^{\mathbf{C}}/W_{2m-1}$ induced by (1.10) is of pure type (m, m).

Finally we make an assumption which is not essential but will simplify the exposition:

(1.12) All of $H^{2m-1}(V_t; \mathbf{C})$ is primitive cohomology.

2. Growth of normal functions

Let

$$\tau : \Delta \to \mathscr{S} \tag{2.1}$$

be a section of the fibration $\mathscr{S} \to \Delta$ considered in (1.7). Let

$$Z_t \subseteq V_t, \quad t \in \Delta,$$

be the corresponding family of $(m-1)$-cycles. By Kleiman's smoothing theorem ([K]; p. 297), we can assume that, if $t \neq 0$,

$$Z_t = Z_t' - Z_t'' \tag{2.2}$$

where Z_t' and Z_t'' are smooth and do not meet. We can resolve the family

$$\bigcup_{t \in \Delta} (\{t\} \times V_t) \subseteq \Delta \times X \tag{2.3}$$

along V_0 so that:

i) the fibre over $t = 0$ is a normal crossing variety in a smooth ambient space of dimension $2m$;

ii) the proper transform $\tilde{Z}$ of

$$\bigcup_{t \in \Delta} (\{t\} \times Z_t)$$

is smooth and meets the fibre over zero transversely.

In ([C1]; p. 245), we explicitly construct an action of the semigroup $[0, 1] \times \mathbf{R}$ on the resolved ambient space which is equivariant with the action

$$(r, \theta) . t = re^{2\pi i\theta} t$$

of $[0, 1] \times \mathbf{R}$ on Δ. It is easy to see that this action can be defined so as to respect $\tilde{Z}$ since it is constructed first locally and then pieced together via fibrations which can be constructed to be compatible with $\tilde{Z}$.

As before, let

$$u = \frac{1}{2\pi i} \log t \tag{2.4}$$

and take, for some fixed u_0, a $(2m-1)$-chain Γ_{u_0} such that

$$\partial \Gamma_{u_0} = Z_{t_0}.$$

For $(r, \theta) \in [0, 1] \times \mathbf{R}$, define

$$\Gamma_u = (r, \theta) . \Gamma_{u_0}$$

where $u = u_0 + \left(\theta + \dfrac{\log r}{2\pi i}\right)$. Then Γ_u has as its boundary the algebraic cycle Z_t with $t = e^{2\pi i u}$.

Lemma **(2.5).** — *The cycle* $(\Gamma_{u+1} - \Gamma_u) \in H_{2m-1}(V_t; \mathbf{Z})$ *has zero intersection number with any cycle which is invariant under the monodromy transformation*

$$T_*: H_{2m-1}(V_t; \mathbf{Z}) \to H_{2m-1}(V_t; \mathbf{Z})$$
$$\gamma \mapsto (1, 1).\gamma.$$

Proof. — The cycle $(\Gamma_{u+1} - \Gamma_u)$, by construction, bounds in the ambient space of the resolution of the family (2.3). Therefore, by the Local Invariant Cycle Theorem ([C¹]; p. 230), this cycle integrates to zero against any invariant element of $H^{2m-1}(V_t; \mathbf{C})$. Since the Poincaré duals of the invariant cocycles are the invariant cycles, the lemma is proved.

Next let $\omega(t)$ be a section of the canonical prolongation of

$$F^m R^{2m-1}\mu_*(\mathbf{C}),$$

where $\mu : \mathcal{V}^* \to \Delta^*$

is our family (1.8) and the canonical prolongation is as established in ([D]; pp. 91-92).

Lemma **(2.6).** — *The integrals*

$$\int_{Z_{t'}}^{Z_t} \omega(t) = \int_{\Gamma_u} \omega(t)$$

are all of the form

$$a(t) \log t + f(t)$$

where $a(t)$ *and* $f(t)$ *are holomorphic on* Δ.

Proof. — The idea of the proof is to make the integrals in question into period functions, *i.e.* integrals over *cycles*, for some two-parameter family of varieties. Let X as in (1.3) denote the ambient variety for the family $\{V_t\}$ and let

$$U_t \subseteq X, \quad t \in \Delta,$$

be an analytic family of very ample hypersurfaces such that:

i) if $t \neq 0$, U_t is smooth and meets V_t transversely,
ii) $Z_t \subseteq (U_t \cap V_t)$;
iii) Z_t is homologous to zero in U_t.

That such U_t exist is an application of the results of [K]. (See the Columbia University Thesis of Spencer Bloch, 1971, pp. 9-10.) If we choose the family U_t sufficiently amply, there will be a two-parameter family

$$W_{(t,t')}, \quad (t, t') \in \Delta \times \Delta,$$

of hypersurfaces in X such that:

i) $W_{(t,t')}$ is smooth and irreducible for $(t, t') \in \Delta^* \times \Delta^*$,
ii) $W_{(t,0)} = U_t \cup V_t$.

Next choose chains $\Sigma_u \subseteq U_t$ such that

$$\partial\Sigma_u = Z_t$$

and define $\Gamma_{(u,0)} = \Gamma_u - \Sigma_u$.

Finally because of the simple nature of the degenerations $W_{(t,t')} \to W_{(t,0)}$ we can form a continuously varying family of *cycles*

$$\Gamma_{(u,u')} \subseteq W_{t,t'}$$

such that $\lim_{t'\to 0} \Gamma_{(u,u')} = \Gamma_{(u,0)}$.

Now the assumption (1.12) that the differential $\omega(t)$ on V_t is primitive implies that there is a two-parameter family of differentials

$$\omega(t, t') \in F^m H^{2m-1}(W_{(t,t')}; \mathbf{C})$$

such that:

i) $\omega(t, 0) = \omega(t)\Big|_{V_t} + 0\Big|_{U_t}$;

ii) the family $\omega(t, t')$ extends over $\Delta \times \Delta$ to give a section of the canonical prolongation of the Hodge bundle for the two-variable degeneration (1).

In fact, let $\varphi_{(u,u')} = \Gamma_{(u,u'+1)} - \Gamma_{(u,u')}$. Then $\varphi_{(u,u'+1)} = \varphi_{(u,u')}$ and, by i),

$$\lim_{t'\to 0} \int_{\varphi_{(u,u')}} \omega(t, t') = 0.$$

Therefore the integrals

$$\int_{\Gamma_{(u,u')}} \omega(t, t') - u' \int_{\varphi_{(u,u')}} \omega(t, t')$$

are well-defined functions of the variables u and

$$t' = e^{2\pi i u'}.$$

Also since periods with respect to the canonical prolongation have at most logarithmic growth, these functions must be holomorphic along $\Delta^* \times \{0\}$, and have value

$$\int_{\Gamma_u} \omega(t)$$

at $(t, 0)$. Finally we use logarithmic growth of periods with respect to the canonical prolongation once again, this time in the t-direction. This allows us to conclude that

$$a(t) = \frac{1}{2\pi i} \int_{\Gamma_{u+1} - \Gamma_u} \omega(t)$$

is bounded at $t = 0$, since, by Lemma (2.5), it is a well-defined function of t. So

$$\int_{\Gamma_u} \omega(t) - a(t) \log t$$

is well-defined and so also bounded at $t = 0$ and the lemma is proved.

---

(1) For a more complete discussion of this point, see the Appendix.

3. The Neron model

Recall that in § 1 we defined

$$N_*: H_{\mathbf{Z}} \to H_{\mathbf{Z}}.$$

Now (image N_*) is not in general a direct summand of $H_{\mathbf{Z}}$ so we enlarge it:

$$E_{van} = \{\varphi \in H_{\mathbf{Z}} : \text{some integral multiple of } \varphi \text{ lies in (image } N_*)\} \tag{3.1}$$

is called the module of *vanishing cycles*. Also

$$E_{inv} = (\ker N_*) \tag{3.2}$$

is called the module of *invariant cycles*. E_{van} is totally isotropic for the intersection pairing on $H_{\mathbf{Z}}$, in fact, the intersection bilinear form is identically zero on

$$E_{inv} \times E_{van}.$$

So it easy to see that there is a splitting of $H_{\mathbf{Z}}$

$$H_{\mathbf{Z}} = L \oplus E \oplus E_{van} \tag{3.3}$$

such that $E \oplus E_{van} = E_{inv}$, and the intersection pairing is unimodular on $L \oplus E_{van}$ and on E, and these two symplectic modules are orthogonal. Furthermore we can adjust the definition of L so that L is totally isotropic.

Let $\{\varphi_j\}$ be a basis for E_{van} satisfying (1.12). Let $\{\delta_\ell, \varepsilon_\ell\}_{\ell=1}^{s}$ be a symplectic basis for E and let $\{\lambda_j\} \subseteq L$ be such that $\{\lambda_j, \varphi_j\}_{j=1}^{r}$ is a symplectic basis for $L \oplus E_{van}$. The Riemann relations imply that if $\{\omega_i(t), \eta_k(t)\}$ is a framing of $F^m R^{2m-1}\mu_*(\mathbf{C})$ for μ as in (1.8), then the matrix

$$\begin{bmatrix} \int_{\varphi_j} \omega_i(t) & \int_{\varepsilon_\ell} \omega_i(t) \\ \int_{\varphi_j} \eta_k(t) & \int_{\varepsilon_\ell} \eta_k(t) \end{bmatrix} \tag{3.4}$$

is invertible for each $t \neq 0$. So we can normalize the choice of the framing to make (3.4) be the identity matrix for each $t \neq 0$. (Notice that the matrix (3.4) is well-defined as a function of t since the cycles φ_j and ε_ℓ are invariant.)

Now if " * " denotes Poincaré dual, we can write any framing $\{\omega_i(t), \eta_k(t)\}$ of $F^m H^{2m-1}(V_u; \mathbf{C})$ for $t = e^{2\pi i u}$ as

$$\text{i)} \quad \sum_j \left(\int_{\varphi_j} \omega_i(t)\right) \lambda_j^* + \sum_\ell \left(\int_{\varepsilon_\ell} \omega_i(t)\right) \delta_\ell^* + \sum_j \left(\int_{\lambda_j} \omega_i(t)\right) \varphi_j^* + \sum_\ell \left(\int_{\delta_\ell} \omega_i(t)\right) \varepsilon_\ell^* \tag{3.5}$$

$$\text{ii)} \quad \sum_j \left(\int_{\varphi_j} \eta_k(t)\right) \lambda_j^* + \sum_\ell \left(\int_{\varepsilon_\ell} \eta_k(t)\right) \delta_\ell^* + \sum_j \left(\int_{\lambda_j} \eta_k(t)\right) \varphi_j^* + \sum_\ell \left(\int_{\delta_\ell} \eta_k(t)\right) \varepsilon_\ell^*.$$

The elements (3.5) therefore frame $F_u^m \subseteq H^{\mathbf{C}}$ (see (1.10)).

Now Schmid's theory and our normalization of (3.4) imply a considerable amount about the entries in (3.5). First of all we wish to compute F^m_∞ as in (1.10). To do this, suppose

(3.6) $$N_*(\lambda_j) = \Sigma m_{jj'} \varphi_{j'}.$$

Then

(3.7) $$\int_{\lambda_j} - u \sum_{j'} m_{jj'} \int_{\varphi_{j'}}$$

is a well-defined function of $t = e^{2\pi i u}$, and F^m_∞ is obtained by replacing $\int_{\lambda_j}$ in (3.5) by the operator (3.7) and taking limits as $t \to 0$.

Now our assumption in (1.11) is that

$$(F^m_\infty + W_{2m-1}) \supseteq L^* = \Sigma \mathbf{C}\lambda^*_j$$

for L as in (3.3). But we have arranged that the $\left(\int_{\varphi_j} \omega_i(t)\right) \equiv$ Kronecker δ_{ij} and $\left(\int_{\varphi_j} \eta_k(t)\right) \equiv 0$ in (3.5). So replacing $\int_{\lambda_j}$ by (3.7), all entries in (3.5 i)) *stay bounded* as $t \to 0$. Thus in particular

(3.8) $$\int_{\lambda_j} \omega_i(t) = m_{ji} u + (\text{holo. fn. of } t).$$

Also, the fact that $F^m_\infty \cap W_{2m-1}$ induces a Hodge structure on E^* implies that all entries in (3.5) ii) are bounded and therefore holomorphic functions of t at $t = 0$. So we can rewrite (3.5) as follows:

(3.9) i) $$\omega_i(t) = \lambda^*_i + \sum_j (m_{ij} u + \omega_{ij}(t))\varphi^*_j + \sum_l (\omega_{il}(t))\varepsilon^*_l$$

ii) $$\eta_k(t) = \delta^*_k + \sum_j (\eta_{kj}(t))\varphi^* + \sum_l (\eta_{kl}(t))\varepsilon^*_l$$

where all functions of t on the right-hand-side are holomorphic at $t = 0$. We were able to replace m_{ji} with m_{ij} in the above formula because

$$\begin{aligned} m_{ji} &= (\lambda_i . N_* \lambda_j) = (\lambda_i . T_* \lambda_j) = (T_*^{-1}\lambda_i . \lambda_j) \\ &= (-N_*\lambda_i . \lambda_j) = (\lambda_j . N_* \lambda_i) = m_{ij}. \end{aligned}$$

From (3.9) and the characterization of the canonical prolongation in ([Z]; p. 189), one concludes that the framing (3.9) of

(3.10) $$F^m R^{2m-1}\mu_*(\mathbf{C})$$

is in fact a framing defining the canonical prolongation of Deligne. Therefore Lemma (2.6) applies to the families of differentials $\{\omega_i(t), \eta_k(t)\}$.

Also if we use the dual basis to (3.9) to frame the dual bundle to (3.10), then the "Jacobian bundle"

$$\mathscr{J}^* \to \Delta^*$$

in (1.6) can be described as the bundle whose fibre over t is obtained by dividing the affine space $\mathbf{C}^{r+s}$ by the lattice generated by the columns of the matrix

$$\begin{bmatrix} \mathrm{I} & \mathrm{o} & (m_{ij}u + \omega_{ij}(t)) & (\omega_{i\ell}(t)) \\ \mathrm{o} & \mathrm{I} & (\gamma_{kj}(t)) & (\gamma_{k\ell}(t)) \end{bmatrix}$$

where, as always, $u = \dfrac{1}{2\pi i}\log t$. Let

(3.11) $$\mathscr{J}' \to \Delta$$

be the analytic fibration obtained by filling in over $t = 0$ with the quotient of $\mathbf{C}^{r+s}$ by the partial lattice generated by the columns of

(3.12) $$\begin{bmatrix} \mathrm{I} & \mathrm{o} & (\omega_{i\ell}(0)) \\ \mathrm{o} & \mathrm{I} & (\gamma_{k\ell}(0)) \end{bmatrix}.$$

(The fact that these columns are indeed independent over $\mathbf{R}$ follows from the fact that $[\mathrm{I}(\gamma_{k\ell}(0))]$ is the period matrix for the Hodge structure of weight $2m-1$ on $\mathrm{W}_{2m-1}/\mathrm{W}_{2m-2}$ in the asymptotic mixed Hodge structure.) The fibration (3.11) is as in ([Z]; p. 191).

Our next step is to enlarge the fibre over $t = 0$ in (3.11) as is done in the construction of the Neron model associated to the degeneration of a family of abelian varieties. The purpose is the same, namely, so that the sections of $\mathscr{J}^* \to \Delta^*$ considered in § 2

$$\int_{Z_{i'}}^{Z_i} \in \mathrm{J}(\mathrm{V}_t)$$

extend over $t = 0$.

To accomplish this we refer to (3.3) and define

(3.13) $$\hat{\mathrm{L}} = \{\lambda \in \mathrm{L} \otimes \mathbf{Q} : \lambda \text{ has } \textit{integral} \text{ intersection number with each element of } \mathrm{N}_*\mathrm{L}\}.$$

Then the group $\hat{\mathrm{L}}/\mathrm{L}$ is naturally the dual of the group $\mathrm{E}_{\mathrm{van}}/\mathrm{N}_*\mathrm{L}$ and so has order equal to $\det(m_{ij})$ where (m_{ij}) is the non-degenerate symmetric matrix in (3.6). Furthermore the natural map

(3.14) $$\mathrm{N}_* : \hat{\mathrm{L}}/\mathrm{L} \to \mathrm{E}_{\mathrm{van}}/\mathrm{N}_*\mathrm{L}$$

is an isomorphism.

Since each element $\lambda \in \hat{\mathrm{L}}$ is a section of $\mathrm{R}_{2m-1}\mu_*(\mathbf{Q})$ which is invariant modulo elements of $\mathrm{R}_{2m-1}\mu_*(\mathbf{Z})$, it gives a well-defined section

(3.15) $$\Delta^* \to \mathscr{J}^*.$$
$$t \mapsto \int_\lambda.$$

Such a section is zero if and only if $\lambda \in \mathrm{L}$. Thus we have an isomorphism of $\hat{\mathrm{L}}/\mathrm{L}$ with a group of sections (3.15). From now on we will denote this group simply as

(3.16) $$\mathscr{G}.$$

We are now ready to define the Neron model associated to the degeneration

(3.17) $$\mathscr{J}^* \to \Delta^*$$

of complex tori. We take $|\mathscr{G}|$ copies of $\mathscr{J}'$ in (3.11) and index them by the elements of $\mathscr{G}$. We identify a point x in the fibre of $\mathscr{J}'_{g_1}$ over $t \neq 0$ with a point y in the fibre of $\mathscr{J}'_{g_2}$ over the same t if and only if

(3.18) $$x - y = g_1 - g_2$$

in $J(V_t)$. The result is a smooth complex manifold

(3.19) $$\mathscr{J} \to \Delta$$

whose restriction to Δ^* is (3.17), and whose fibre J_0 over $t = 0$ fits into the exact sequence

(3.20) $$0 \to J'_0 \to J_0 \to \mathscr{G} \to 0$$

where J'_0 is the fibre of $\mathscr{J}'$ over $t = 0$.

By Lemma (2.6), if $\omega(t)$ is any section of the canonical prolongation of $F^m R^{2m-1}\mu_*(\mathbf{C})$, then

$$\int_\lambda \omega(t) - u\int_{N_*(\lambda)} \omega(t)$$

is a well-defined function of t holomorphic at $t = 0$ for each $\lambda \in \hat{L}$. Thus

(3.21) $$u\int_{N_*(\lambda)}$$

is a well-defined section of (3.19) which extends over $t = 0$. In fact (3.21) gives a section of (3.19) which passes through the same component of J_0 that $\int_\lambda$ does, that is, the component given by $g = \int_\lambda$.

So now suppose we have an analytic family of algebraic $(m-1)$-cycles

$$Z_t = Z'_t - Z''_t$$

as in (2.2). Let

$$\partial\Gamma_u = Z'_t - Z''_t$$

as before. By Lemma (2.5),

(3.22) $$\Gamma_{u+1} - \Gamma_u = \varphi \in E_{van}$$

so that there is $\lambda \in \hat{L}$ such that $N_*(\lambda) = \varphi$. By Lemma (2.6), the integrals

$$\int_{\Gamma_u} \omega(t) - u\int_\varphi \omega(t)$$

are bounded holomorphic functions of t for $\omega(t) \in \{\omega_i(t), \eta_k(t)\}$. Thus:

Theorem **(3.23).** — *The Abel-Jacobi map*

$$\Delta^* \to \mathscr{J}^*$$
$$t \mapsto \int_{Z_i''}^{Z_i'} = \int_{\Gamma_u}$$

extends over $t = 0$ *to a section of*

$$\mathscr{J} \to \Delta$$

whose value at $t = 0$ *lies in the component of* J_0 *given by*

$$g = \int_\lambda \in \mathscr{G}$$

with $N_*\lambda = \Gamma_{u+1} - \Gamma_u$.

Since all our constructions can be carried out holomorphically with respect to auxiliary parameters we conclude:

Corollary **(3.24).** — *The diagram* (1.6) *extends to a commutative diagram*

$$\begin{array}{ccc} \mathscr{S} & \longrightarrow & \mathscr{J} \\ & \searrow \quad \swarrow & \\ & \Delta & \end{array}$$

This last corollary is just the analytic analogue of the " universal property " of the Neron model in the algebraic case.

Appendix

The purpose of this appendix is to justify the assertion, made during the proof of Lemma (2.6), that the integrals

(A.1) $$\int_{\Gamma_u} \omega(t)$$

appear as coefficients, along $t' = 0$, of sections of the canonical prolongation of

(A.2) $$\{F^2H^3(W_{(t,t')})\}_{(t,t') \in \Delta^* \times \Delta^*}$$

with respect to a flat basis of

(A.3) $$\{H^3(W_{(t,t')})_{\mathbf{Z}}\}_{(t,t') \in \Delta^* \times \Delta^*}.$$

It is this fact that allows us to conclude from ([D]; pp. 91-92) that the normal functions (A.1) have at worst logarithmic growth.

Let

$$\pi : \tilde{\Delta} \times \tilde{\Delta} \to \Delta^* \times \Delta^*$$
$$(u, u') \mapsto (t, t')$$

be the universal covering map as in § 2. Using π^*, we pull the bundle (A.3) back to a trivial bundle on $\tilde{\Delta} \times \tilde{\Delta}$ whose global sections will be denoted by $K^{\mathbf{Z}}$. There are two commuting, nilpotent endomorphisms of $K^{\mathbf{Z}}$, namely

$$N = \text{logarithm of monodromy around } t = 0,$$
$$N' = \text{logarithm of monodromy around } t' = 0.$$

Notice that $W_{(t,0)} = V_t \cup U_t$, the union of two smooth manifolds meeting transversely. So $(N')^2 = 0$, in fact, the topological part of our analysis in § 3 applies to the one-variable degeneration

$$W_{(t,t')} \to W_{(t,0)}.$$

We want to apply the Cattini-Kaplan-Schmid theory of asymptotic mixed Hodge structures to the two-parameter family (A.3). This theory says, first of all, that $K^{\mathbf{Z}}$ has a mixed Hodge structure whose weight filtration is defined by the nilpotent endomorphism $N + N'$, and whose Hodge filtration is given by

$$\mathbf{(A.4)} \qquad F^* = \lim_{(t,t')\to(0,0)} \exp(-uN - u'N')\pi^*(F^*H^3(W_{(t,t')})).$$

If $H^{\mathbf{Z}}$ is as in (1.8)-(1.11), then there is a subquotient of $K^{\mathbf{Z}}$ which can be identified with $H^{\mathbf{Z}}$. Namely let

$$\mathbf{(A.5)} \qquad Gr_3^{N'} = \frac{\text{kernel}(N' : K^{\mathbf{Z}} \to K^{\mathbf{Z}})}{((\text{image } N') \otimes \mathbf{Q}) \cap K^{\mathbf{Z}}}.$$

The filtrations on $H^{\mathbf{Z}} \otimes \mathbf{C}$ induced by the weight and Hodge filtrations on $K^{\mathbf{Z}} \otimes \mathbf{C}$ define a mixed Hodge structure on (A.5) which is isomorphic to (the mixed Hodge structure on)

$$\mathbf{(A.6)} \qquad H^{\mathbf{Z}} \oplus G^{\mathbf{Z}}.$$

Here $H^{\mathbf{Z}}$ has the mixed Hodge structure in (1.8)-(1.11), and $G^{\mathbf{Z}}$ has the asymptotic mixed Hodge structure for the family $\{U_t\}_{t\in\Delta^*}$ constructed in the proof of Lemma (2.6).

Now let φ_j^* and ε_l^* be as in (3.5). These are flat, N-invariant sections of $\{H^3(V_t)_{\mathbf{Z}}\}_{t\in\Delta^*}$ and can be extended to flat sections of (A.3) which are both N and N′ invariant. We compute the limit (A.4) in two steps, first letting t' go to zero and then letting t go to zero. As in § 3, we see that after the first step, the vector space

$$F_t^2 = \lim_{t'\to 0} \exp(-u'N')\pi^*F^2H^3(W_{(t,t')})$$

can be written as the direct sum of two subspaces,

$$\mathbf{(A.7)} \qquad M_t = (F_t^2 \cap \{\Sigma\mathbf{Z}\varphi_j^* + \Sigma\mathbf{Z}\varepsilon_l^*\}^{\perp})$$

and a second subspace, which we will call

$$\mathbf{(A.8)} \qquad L_t,$$

which lies in $(\ker N') \otimes \mathbf{C}$ and gives $F^2H^3(V_t)$ in the weight three graded quotient of the asymptotic mixed Hodge structure associated to the one-parameter family

(A.9) $$W_{(t,t')} \to W_{(t,0)} = V_t \cup U_t.$$

The important point is that

$$L = \lim_{t\to 0} \exp(-uN)L_t$$

and

$$M = \lim_{t\to 0} \exp(-uN)M_t$$

is a direct-sum decomposition of F^2 (see (A.4)). This is because

i) $\dim L_t = \frac{1}{2}\dim H^3(V_t) = \dim\{\Sigma \mathbf{C}\varphi_j^* + \Sigma \mathbf{C}\varepsilon_\ell^*\} = \dim F^2(W_3(H^{\mathbf{C}}))$;

ii) $M \subseteq \{\Sigma \mathbf{C}\varphi_j^* + \Sigma \mathbf{C}\varepsilon_\ell^*\}^\perp$ and so, by § 3, $M \cap (\ker N') \otimes \mathbf{C}$ projects to zero in $F^2(W_3(H^{\mathbf{C}}))$.

Therefore there is a framing $\omega(t)$ of L_t which extends to a partial framing $\omega(t, t')$ of the canonical prolongation of (A.2). These are the differentials which occur in the proof of Lemma (2.6).

The differentials

$$\omega(t)\Big|_{V_t} + 0\Big|_{U_t}$$

framing L_t are well-defined modulo

$$F^2((\text{image } N') \otimes \mathbf{C})$$

by the fact that they are dual to the basis $\{\varphi_j\}$, $\{\varepsilon_\ell\}$ of $F^2(W_3(H^{\mathbf{C}}))^*$. More precisely, these differentials map isomorphically to a framing of L_t under the natural morphism of mixed Hodge structures

$$H^3(V_t \cup U_t) \to (\text{asymptotic mixed Hodge structure for the family (A.9)}).$$

For Γ_u as in (A.1), $\partial\Gamma_u$ is algebraic so that the integral (A.1) will occur as the coefficient of an *algebraic* basis element of

$$(\text{image } N') \otimes \mathbf{Q} \underset{\S 3}{=} \frac{H^2(U_t \cap V_t)_{\mathbf{Q}}}{\{\text{hyperplane section}\}}.$$

So if we change the framing $\omega(t)$ by an element of

$$F^2((\text{image } N') \otimes \mathbf{C}),$$

the value of the coefficient is unchanged.

REFERENCES

[C1] Clemens, H., Degeneration of Kähler manifolds, *Duke Math. Journal*, **44** (1977), 215-290.

[C2] Clemens, H., Double Solids, *Advances in Math.* (to appear).

[D] Deligne, P., Équations Différentielles à Points Singuliers Réguliers, *Springer Lecture Notes in Math.*, **163** (1970).

[G] Griffiths, P. A., Periods of integrals on algebraic manifolds, III, *Publ. Math. I.H.E.S.*, **38** (1970), 125-180.

[K] Kleiman, S., Geometry on Grassmannians and applications to splitting bundles and smoothing cycles, *Publ. Math. I.H.E.S.*, **36** (1969), 281-298.

[S] Schmid, W., Variation of Hodge structure : The singularities of the period mapping, *Inventiones math.*, **22** (1973), 211-319.

[Z] Zucker, S., Generalized intermediate Jacobians and the theorem on normal functions, *Inventiones math.*, **33** (1976), 185-222.

Department of Mathematics,
The University of Utah,
Salt Lake City, Utah 84112.

Manuscrit reçu le 12 janvier 1981.
Révisé le 5 mai 1982.