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Generic immersions of curves, knots, monodry and Gordian number

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GENERIC IMMERSIONS OF CURVES, KNOTS, MONODROMY AND GORDIAN NUMBER

by Norbert A'CAMPO

Dedicated to Rob Kirby on his 60th birthday

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1. Introduction

A divide P is a generic relative immersion of a finite number of copies of the unit interval $(I, \partial I)$ in the unit disk $(D, \partial D)$. The image of each copy of the unit interval is called a branch of the divide P. The link L(P) of a divide P is

$$L(P) := \{ (x, u) \in T(P) \mid \|(x, u)\| = 1 \} \subset S(T(\mathbf{R}^2)) = S^3,$$

where we use the following notation: For a tangent vector $(x, u) \in T(\mathbf{R}^2) (= \mathbf{R}^4)$ of $\mathbf{R}^2$ the point $x \in \mathbf{R}^2$ represents its foot and the vector $u \in T_x(\mathbf{R}^2)$ its linear part. The unit sphere $S(T(\mathbf{R}^2)) := \{ (x, u) \in T(\mathbf{R}^2) \mid \|(x, u)\| := x_1^2 + x_2^2 + u_1^2 + u_2^2 = 1 \}$ should not be confused with the tangent circle bundle of $\mathbf{R}^2$ and is homeomorphic to the 3-sphere S^3. Finally, $T(P) \subset T(D) \subset T(\mathbf{R}^2)$ is the space of tangent vectors of the divide P, where at a crossing point s by definition the space $T_s(P)$ is the union of the two 1-dimensional subspaces of $T_s(D)$, which are the tangent spaces of the local branches of P passing through s. The link L(P) is an embedding of a union of r circles in S^3, where r is the number of branches of the divide P. So, for a divide P consisting of one branch the link L(P) is a classical knot.

A divide is called connected if the image of the immersion is a connected subset of the disk. The following is the main theorem of this paper.

Theorem. — The link L(P) *of a connected divide* P *is a fibered link.*

The monodromy of the fibered link L(P) of a connected divide is given by Theorem 2 of Section 3 in terms of the combinatorics of the underlying divide. Since

it is very easy to give examples of connected divides we obtain a huge class of links, such that the complement admits a fibration over the circle and that the isotopy class of the monodromy diffeomorphism is explicitly known. The links of plane curve singularities belong to this class (see [AC1, AC2, AC3, G-Z]). In Section 5 we show that the gordian number of the link of a divide equals the number of crossing points of the divide. The figure eight knot does not belong to this class. Many knots of this class are hyperbolic, as we will see in a forthcoming paper. Theorem 2 is used in the proof of the main theorem of [AC3].

I would like to thank Michel Boileau, Yasha Eliashberg, Rainer Kaenders, Dieter Kotschick, Tom Mrowka and Bernard Perron for helpful discussions during the preparation of this paper.

2. The fibration of the link of a divide

A regular isotopy of a divide in the space of generic immersions does not change the isotopy type of its link. So, without loss of generality, we may choose a divide to be linear and orthogonal near its crossing points. For a connected divide $P \subset D$, let $f_P : D \to \mathbf{R}$ be a generic C^∞ function, such that P is its 0-level and that each region has exactly one non-degenerate maximum or minimum and that each region, which meets the boundary, has exactly one non-degenerate maximum or minimum on the intersection of the region with ∂D. Such a function exists for a connected divide and is well defined up to sign and isotopy. In particular, there are no critical points of saddle type other than the crossing points of the divide. Moreover without loss of generality, we may assume that the function f_P is quadratic and euclidean in a neighborhood of those of its critical points, that lie in the interior of D, i.e. for euclidean coordinates (X, Y) with center at a critical point c of f_P, in a neighborhood of c we have the expression $f_P(X, Y) = f_P(c) + XY$, if c is a saddle point, $f_P(X, Y) = f_P(c) - X^2 - Y^2$, if c is a local maximum, or $f_P(X, Y) = f_P(c) + X^2 + Y^2$, if c is a local minimum. Further, we may also assume, that the function f_P is linear in a neighborhood of every relative critical point on ∂D, i.e. at a critical point $c \in \partial D$ of the function f_P we have the expression $f_P(c + h) = f_P(c) + \langle h, c \rangle$ or $f_P(c + h) = f_P(c) - \langle h, c \rangle$, where we denote by $\langle\ ,\ \rangle$ the scalar product of $\mathbf{R}^2$. Let $\chi : D \to [0, 1]$ be a positive C^∞ function which equals zero outside of small neighborhoods where f_P is quadratic and equals 1 in some smaller neighborhood U of the critical points of f_P. Moreover, we choose the function χ to be rotational symmetric around each critical point, i.e. we assume that locally near each critical point the function χ depends only on the distance to the critical point. For $\eta \in \mathbf{R}, \eta > 0$ let $\theta_{P,\eta} : S^3 \to \mathbf{C}$ be given by

$$\theta_{P,\eta}(x, u) := f_P(x) + i\,\eta\, df_P(x)(u) - \frac{1}{2}\eta^2\chi(x)H_{f_P}(x)(u, u).$$

Observe that the Hessian H_{f_P} is locally constant in the neighborhood of the critical points of f_P where f_P is euclidean. Let $\pi_{P,\eta} : S^3 \setminus L(P) \to S^1$ be defined by

$$\pi_{P,\eta}(x, u) := \theta_{P,\eta}(x, u)/|\theta_{P,\eta}(x, u)|.$$

Theorem 1. — *Let* P *be a connected divide. For* $\eta > 0$ *and sufficiently small, the map* $\pi_P := \pi_{P,\eta}$ *is a fibration of the complement of* L(P) *over* S^1.

Proof. — There exists a regular product tubular neighborhood N of L(P), such that the map $\pi_{P,\eta}$ for any $1 \geqslant \eta > 0$ is on $N \setminus L(P)$ a fibration over S^1, for which near L(P) the fibers look like the pages of a book near its back. It is crucial to observe that in the intersection of the link L(P) with the support of the function

$$(x, u) \in S^3 \mapsto \chi(x) \in \mathbf{R},$$

the kernel of the Hessian of $\theta_{P,\eta}$ and the kernel of the differential of the map

$$(x, u) \in S^3 \mapsto f_P(x) \in \mathbf{R}$$

coincide. For any $\eta > 0$, the map $\pi_{P,\eta}$ is regular at each point of $U' := \{ (x, u) \in S^3 \mid x \in U \}$. There exists $\eta_0 > 0$ such that for any η, $0 < \eta < \eta_0$, the map $\pi_{P,\eta}$ is regular on $S^3 \setminus (N \cup U')$. Hence, due to the quadratic scaling, for η sufficiently small the map $\pi_{P,\eta}$ is a submersion, so since already a fibration near L(P), it is a fibration by a theorem of Ehresmann. □

3. The monodromy diffeomorphism

Let P be a connected divide and let $\pi_P : S^3 \setminus L(P) \to S^1$ be its fibration of Theorem 1. We will show how to read off geometrically the fibers $\pi_P^{-1}(\pm 1)$. Two diffeomorphisms S''_i, S''_{-i} between the fibers $\pi_P^{-1}(\pm 1)$ modified by half Dehn twists will after a suitable composition give the monodromy. For our construction we orient the disk D by one of the possible orientations, which we think of as an orthogonal complex structure $J : T(D) \to T(D)$. We start out with a description of the fiber $F_1 := \pi_P^{-1}(1)$ and at the same time of the fiber $F_{-1} := \pi_P^{-1}(-1)$. Put

$$P_+ := \{ x \in D \setminus \partial D \mid f_P(x) > 0, df_P(x) \neq 0 \}.$$

The level curves of f_P define a oriented foliation F_+ on P_+, where a tangent vector u to a level of f_P at $x \in P_+$ is oriented if $df_P(x)(Ju) > 0$. Put

$$P_{+,+} := \{ (x, u) \in S^3 \mid x \in P_+, \ u \in T(F_+) \}$$

and

$$P_{+,-} := \{ (x, u) \in S^3 \mid x \in P_+, \ u \in T(F_-) \},$$

where F_- is the foliation with the opposite orientation. Put

$$F_M := \{ (x, u) \in S^3 \mid x = M \}$$

for a maximum M, and

$$F_m := \{ (x, u) \in S^3 \mid x = m \}$$

for a minimum m of f_P. Put

$$F_{s,+} := \{ (x, u) \in S^3 \mid x = s, \ H_{f_P}(x)(u, u) < 0 \}$$

and

$$F_{s,-} := \{ (x, u) \in S^3 \mid x = s, \ H_{f_P}(x)(u, u) > 0 \}$$

for a crossing point s of P, which is also a saddle point of f_P. Observe that the angle in between $u, v \in F_m$ or $u, v \in F_M$ is a natural distance function on F_m or F_M, which allows us to identify F_m and F_M with a circle. Finally, put

$$\partial D_+ := \{ x \in \partial D \mid f_P(x) > 0 \}.$$

Let $p_{\mathbf{R}} : S^3 \to D$ be the projection $(x, u) \mapsto x$. The projection $p_{\mathbf{R}}$ maps each of the sets $P_{+,+}$ and $P_{+,-}$ homeomorphically onto P_+. The sets F_m or F_M are homeomorphic to S^1, if M or m is a maximum or minimum of f_P respectively, and the sets $F_{s,\pm}$ are homeomorphic to a disjoint union of two open intervals if s is a crossing point of P. The set ∂D_+ is homeomorphic to a disjoint union of open intervals. We have that F_1 and F_{-1} are disjoint unions of these sets:

$$F_1 = P_{+,+} \cup P_{+,-} \cup \partial D_+ \cup \bigcup_{s \in P} F_{s,+} \cup \bigcup_{M \in P_+} F_M,$$

and accordingly, with the obvious changes of signs:

$$F_{-1} = P_{-,+} \cup P_{-,-} \cup \partial D_- \cup \bigcup_{s \in P} F_{s,-} \cup \bigcup_{m \in P_-} F_m.$$

In fact, for $(x, u) \in P_{+,+} \cup P_{+,-}$ we have $\theta_P(x, u) \in \mathbf{R}_{>0}$ since

$$\theta_P(x, u) := f_P(x) + i\eta \ df_P(x)(u) - \frac{1}{2}\eta^2 \chi(x) H_{f_P}(x)(u, u),$$

where $f_P(x) > 0$, $df_P(x)(u) = 0$, $\chi(x) H_{f_P}(u, u) \leqslant 0$. Hence, $P_{+,+} \cup P_{+,-}$ is an open and dense subset in F_1. Forming the closure of $P_{+,+} \cup P_{+,-}$ in F_1 leads to the following combinatorial description of the above decomposition. First, we add to the open surface F_1 its boundary and get

$$\overline{F}_1 := F_1 \cup L(P)$$

Let R be a connected component of P_+. The inverse image $p_R^{-1}(R) \cap \overline{F}_1$ in $\overline{F}_1$ consists of two disjoint open cells or cylinders $R_+ \subset P_{+,+}$ and $R_- \subset P_{+,-}$ which are in fact subsets of F_1. The closure of R_+ in $\overline{F}_1$ is a surface $\overline{R}_+$ with boundary and corners. The set F_M is a common boundary component without corners of $\overline{R}_+$ and $\overline{R}_-$ if M is a maximum in R. If there is no maximum in R the closures $\overline{R}_+$ and $\overline{R}_-$ meet along the component of ∂D_+ which lies in the closure of R. Let S, R be connected components of P_+ such that the closures of R and S have a crossing point s in common. The closures of R_+ and S_- in F_1 meet along one of the components of $F_{s,+}$ and the closures of R_- and S_+ in F_1 meet along the other component of $F_{s,+}$. The closure of $F_{s,+} \cap \overline{R}_+$ in $\overline{R}_+$ intersects L(P) in 2 corners, that are also corners of the closure of $F_{s,+} \cap \overline{S}_-$ in $\overline{S}_-$ (see Fig. 1). Notice that the foliation F_+ on P_+ does not lift to a foliation, which extends to an oriented foliation on F_1.

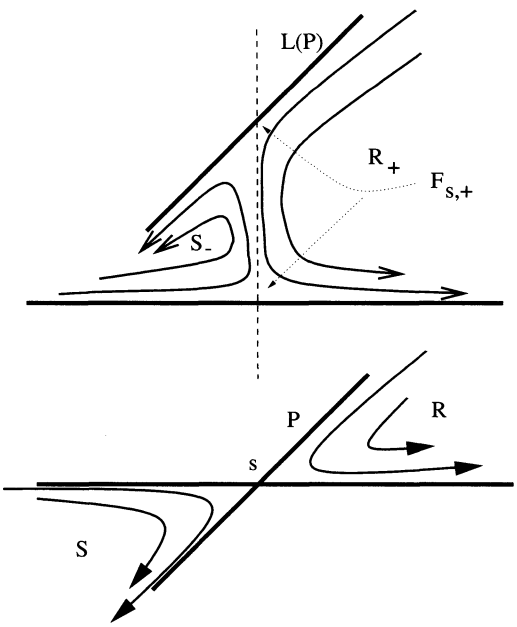

FIG. 1. – Gluing of the lifts of R with + and S with - foliation to F_1

Now we will work out the fibers $F_i := \pi_P^{-1}(i)$ and $F_{-i} := \pi_P^{-1}(-i)$. First observe that F_i and F_{-i} are projected to a subset of $P \cup \mathrm{supp}(\chi)$ by p_R. Put

$$F_{i,P} := \{ (x, u) \in S^3 \mid x \in P,\ \chi(x) = 0,\ df_P(x)(u) > 0 \}.$$

For a crossing point c of P we put

$$F_{i,c} := \{ (x, u) \in S^3 \mid \chi(x) > 0,\ df_P(x)(u) > 0,$$
$$f_P(x) - \tfrac{1}{2}\eta^2\chi(x)H_{f_P}(c)(u, u) = 0 \}.$$

In order to get nice sets it is necessary to choose a nice bump function χ. The set $F_{i,P} \cup F_{i,c}$ is an open and dense subset in F_i and forming its closure in $\overline{F}_i := F_i \cup L(P)$ leads to a combinatorial description of F_i.

Our next goal is the description of the monodromy diffeomorphism. We will use the integral curves of the distribution J(kernel(df_P)), which pass through the crossing points of the divide P. In a connected component R of $D \setminus P$, those integral curves of J(kernel(df_P)) meet at the critical point of f_P in the component R with distinct tangents or go to distinct points of ∂D.

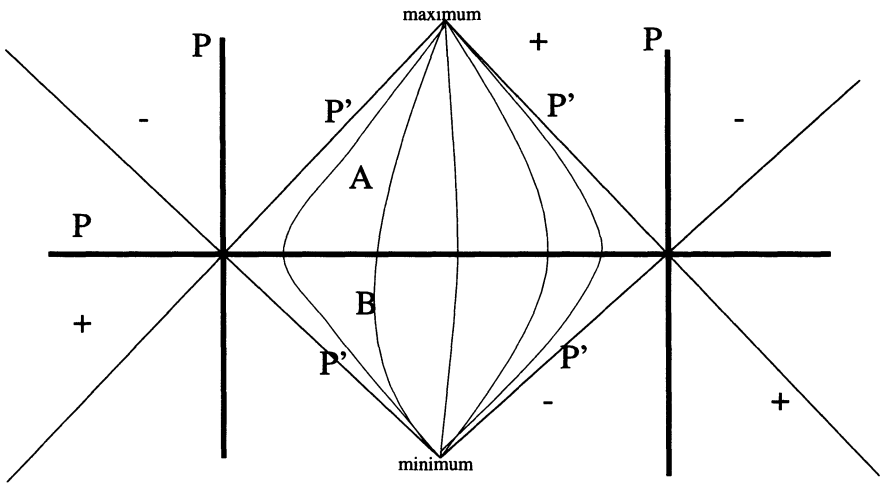

FIG. 2. – Two tiles with the $\mathcal{J}$(kernel(df_P)) foliation

We denote by P′ the union of the integral curves of J(kernel(df_P)), which pass through the crossing points of P. The complement in D of the union $P' \cup P \cup \partial D$ is a disjoint union of tiles, which are homeomorphic to open squares or triangles. We call a pair (A, B) of tiles opposite, if $A \neq B$ and the closures of A and B in D have a segment of P in common. For an opposite pair of tiles (A, B) let $A \mid B$ be the interior in D of the union of the closures of A and B in D. The set is foliated by the levels of f_P and also by the integral lines of the distribution J(kernel(df_P)). Both foliations are non-singular and meet in a J-orthogonal way (see Fig. 2).

Let R and S be the components of $D \setminus (P \cup \partial D)$, which contain A and B. Put

$$F_{1,A|B} := \{ (x, u) \in F_1 \mid x \in A \mid B \}$$

and

$$F_{-1,A|B} := \{ (x, u) \in F_{-1} \mid x \in A \mid B \}.$$

The sets $F_{1,A|B}$ and $F_{-1,A|B}$ each have two connected components:

$$F_{\pm 1,A|B} = F_{\pm 1,+,A|B} \cup F_{\pm 1,-,A|B}$$

where

$$F_{1,+,A|B} := \{ (x, u) \in F_1 \mid x \in A \mid B, \ df_P(Ju) > 0 \},$$

$$F_{-1,+,A|B} := \{ (x, u) \in F_{-1} \mid x \in A \mid B, \ df_P(Ju) > 0 \},$$

and

$$F_{1,-,A|B} := \{ (x, u) \in F_1 \mid x \in A \mid B, \ df_P(Ju) < 0 \},$$

$$F_{-1,-,A|B} := \{ (x, u) \in F_{-1} \mid x \in A \mid B, \ df_P(Ju) < 0 \}.$$

The closures of $F_{1,\pm,A|B}$ in $\overline{F}_1$ and of $F_{-1,\pm,A|B}$ in $\overline{F}_{-1}$ are polygons with 6 edges: let M, c, c' be the vertices of the triangle A; the six edges of the closure H of $F_{1,+,A|B}$ in $\overline{F}_1$ are $\{ (x, u) \in H \mid x = M \}$, $\{ (x, u) \in H \mid x \in [c, M] \}$, $\{ (x, u) \in H \mid x = c \}$, $\{ (x, u) \in H \mid x \in [c, c'] \}$, $\{ (x, u) \in H \mid x = c' \}$, $\{ (x, u) \in H \mid x \in [c', M] \}$ where $[M, c]$ and $[M, c']$ are segments included in P′ and $[c, c']$ is a segment in P.

We will define two diffeomorphisms:

$$S_{i,A|B} : F_{1,A|B} \to F_{-1,A|B}$$

and

$$S_{-i,A|B} : F_{1,A|B} \to F_{-1,A|B}.$$

To do so it is convenient to choose the function $f_P : D \to \mathbf{R}$ such that the maxima are of value 1 and the minima of value -1. Moreover, we modify the function f_P at the boundary ∂D so that along each of the integral lines of the foliation given by the distribution J(kernel(df_P)) the function f_P takes all values in an interval $[-m, m]$ with $1 \geqslant m > 0$. The latter modification of f_P is useful if the tile A or B meets ∂D. We also need the rotations $J_\theta : T(D) \to T(D)$ about the angle $\theta \in [-\pi, \pi]$. Remember $J = J_{\pi/2}$. The map S_i acts as follows: for $(x, u) \in F_1$ with $x \in A \mid B$ let $y \in A \mid B$ be the point in the opposite tile on the integral line of the distribution J(kernel(df_P)) with $f_P(x) = -f_P(y)$; now we move x to y along the integral curve $\gamma_{x,y}(t)$, $t \in [f_P(x), f_P(y)]$ which connects x and y with the parameterization $f_P(\gamma_{x,y}(t)) = t$; the vector u will be moved along the path

$$(\gamma_{x,y}(t), U_{x,y}(t)) := (\gamma_{x,y}(t), s(x, t)(J_{\theta(x,t)}(|f_P(x) \mid u_{x,y}(t)/2) + u_{x,y}(t))),$$

where $(\gamma_{x,y}(t), u_{x,y}(t)) \in S^3$ is the continuous vector field along $\gamma_{x,y}(t)$ such that $u_{x,y}(t)$ stays in the kernel of df_P and $u_{x,y}(f_P(x)) = U_{x,y}(f_P(x)) = u$, where the rotation angle

$\theta(x, t)$ at time t is given by $\theta(x, t) := \dfrac{(|\, f_P(x) \,| - t)\pi}{2 \,|\, f_P(x) \,|}$ and where the stretching factor $s(x, t) \geqslant 1$ is chosen such that $(\gamma_{x,y}(t), U_{x,y}(t)) \in S^3$ holds; define

$$S_i((x, u)) := (y, u_{x,y}(f_P(y))) = (y, U_{x,y}(f_P(y))).$$

The definition of S_{-i} is analogous, but uses rotations in the sense of $-J$. The names S_i or S_{-i} indicate that the flow lines $(\gamma_{x,y}(t), u(t))$ pass through the fiber F_i or F_{-i} respectively. The flow lines defining S_i or S_{-i} are different. However, the maps S_i and S_{-i} are equal. The system of paths $(\gamma_{x,y}(t), U_{x,y}(t)) \in S^3$ is local near the link $L(P)$, i.e. for every neighborhood N in S^3 of a point $(x', u') \in L(P)$ there exists a neighborhood M of (x', u') in S^3 such that each path $(\gamma_{x,y}(t), U_{x,y}(t))$ with $(x, u) \in F_1 \cap M$ stays in N. It will follow that the flow lines of the monodromy vector field are meridians of the link $L(P)$ in its neighborhood.

The partially defined diffeomorphisms S_i and S_{-i}

$$S_i, S_{-i} : \bigcup_{A|B} F_{1,A|B} \to \bigcup_{A|B} F_{-1,A|B}$$

are obtained by gluing the maps $S_{i,A|B} : F_{1,A|B} \to F_{-1,A|B}$ and $S_{-i,A|B} : F_{1,A|B} \to F_{-1,A|B}$ for all opposite pairs of tiles (A, B) with $A \subset P_+$. The gluing poses no problem since those unions are disjoint, but the diffeomorphisms S_i and S_{-i} do not extend continuously to F_1. We will see that the discontinuities, which are the obstruction for extending S_i and S_{-i}, can be compensated by a composition of right half Dehn twists.

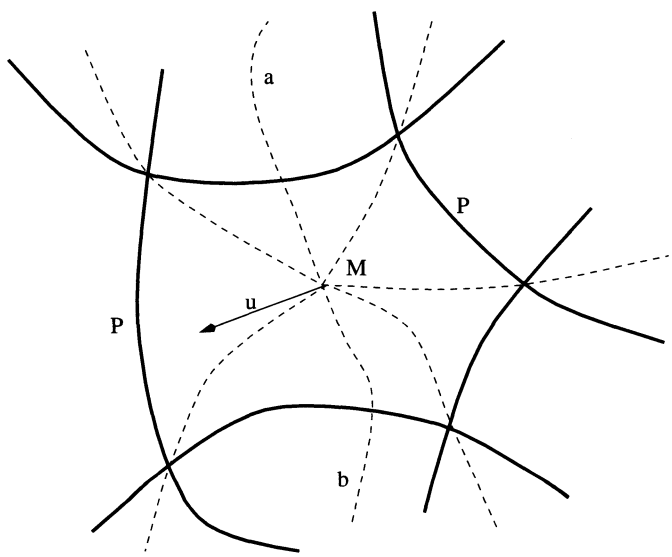

FIG. 3. – The discontinuity at F_M

At a maximum $M \in D$ of f_P each vector $(M, u) \in S^3$ belongs to F_1. Let a and b be the integral curves of $J(\text{kernel}(df_P))$ with one endpoint at M and orthogonal to u. We assume that neither a nor b passes through a crossing point of P (see Fig. 3) and that a and b belong to different pairs of opposite tiles. A continuous extension of the maps S_i or S_{-i} has to map the vector (M, u) to two vectors based at the other endpoint of a and b. Since these endpoints differ in general, a continuous extension is impossible.

In order to allow a continuous extension at the common endpoint of a and b we make a new surface F'_1 by cutting F_1 along the cycles F_M, where M runs through all the maxima of f_P and by gluing back after a rotation of angle π of each of the cycles F_M. In the analogous manner, we make the surface F'_{-1} in doing the half twist along F_m, where m runs through the minima of f_P. The subsets $F_{1,A|B}$ do not meet the support of the half twists, so they are canonically again subsets of F'_1, which we denote by $F'_{1,A|B}$. Analogously, we have subsets $F'_{-1,A|B}$ in F'_{-1}. A crucial observation is that the partially defined diffeomorphisms

$$S'_i, S'_{-i} : \bigcup_{A|B} F'_{1,A|B} \to \bigcup_{A|B} F'_{-1,A|B}$$

have less discontinuities, which are the obstruction for a continuous extension. We denote by a' and b' the arcs on F'_1, which correspond to the arcs a and b on F_1. Indeed, the continuous extension at the end points of a' and b' is now possible.

Let s be a crossing point of P and let $I_{s,+}$ be the segment of P', which passes through s and lies in P_+. The inverse image of $Z^\circ_s := p_{\mathbf{R}}^{-1} I_{c,+} \cap F_1$ is not a cycle, except if both endpoints of $I_{s,+}$ lie on ∂D. If a maximum M of f_P is an endpoint of $I_{s,+}$, the inverse image $p_{\mathbf{R}}^{-1}(M) \cap F_1$ consists of two points on F_M, which are antipodal. On the new surface F'_1 the inverse image $p_{\mathbf{R}}^{-1}(I_{s,+}) \cap F'_{-1}$ is a cycle. An extension of S'_i and S'_{-i} will be discontinuous along this cycle (see Fig. 4). We now observe that the partially defined diffeomorphisms S'_i and S'_{-i} have discontinuities along the cycle $p_{\mathbf{R}}^{-1}(I_{s,+}) \cap F'_{-1}$, which can be compensated by half twists along the inverse images $p_{\mathbf{R}}^{-1}(I_{s,-}) \cap F'_{-1}$, where s runs through the crossing points of P. Note that for a crossing point s of P the curve $Z'_{s,-1} := p_{\mathbf{R}} - 1(I_{s,-}) \cap F'_{-1}$ is in fact a simply closed curve on F'_{-1}.

For a crossing point s of the divide P we now define a simply closed curve on F_1, by putting

$$Z_s := Z^\circ_s \cup \bigcup_{M \in \partial I_{s,+}} F_{s,M},$$

where for an endpoint M of $I_{s,+}$, which is a maximum of f_P, the set $F_{s,M}$ is the simple arc of F_M, which connects the two points of $Z^\circ_s \cap F_M$ and contains an inward tangent vector of $I_{s,+}$ at M. As we already have noticed the set $Z^\circ_s \cap F_M$ has only one element if $M \in \partial D$, so we define $F_{s,M} := \emptyset$ in that case.

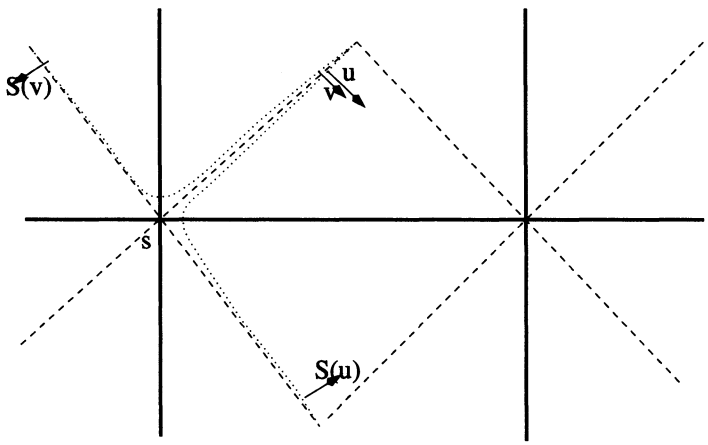

FIG. 4. – The discontinuity along Z_s

We have the inclusion $F_m \subset F_{-1}$. We now define the cycle $Z_m \subset F_1$. Define for a minimum m of f_P the region

$$B_m := \bigcup_{A|B, m \in \overline{B}} F_{-1,A|B}.$$

Let $B_{m,\varepsilon}$ be the level curve

$$B_{m,\varepsilon} := \{ (x, u) \in B_m \mid f_P(x) = -\varepsilon \}.$$

For a small ε the set

$$(S_i)^{-1}(B_{m,\varepsilon} \cap \bigcup_{A|B} F_{-1,A|B})$$

is a union of copies of an open interval and is not a cycle but nearly a cycle. The unions closes up to a cycle by adding small segments which project to the integral lines through the crossing points of P. We denote this cycle by $Z_m \subset F_1$.

We are now able to state the main theorem.

Theorem **2.** — *Let* P *in* D *be a connected divide. Let* $\pi_P : S^3 \setminus L(P) \to S^1$ *be the fibration of Theorem* 1. *The counter clockwise monodromy of the fibration* π_P *is the composition of right Dehn twists* $T := T_- \circ T_{\cdot} \circ T_+ : F_1 \to F_1$, *where* T_- *is the product of the right twists along* Z_m, m *running through the minima of* f_P, $T_{\cdot}$ *is the product of the right Dehn twists along the cycles* Z_s, s *running through the crossing points of* P, *and* T_+ *is the product of the right twists along* F_M, M *running through the maxima of* f_P.

Before giving the proof, we will define positive and negative half Dehn twists. Let X be an oriented surface and let z be a simply closed curve on X. Let X′ be the

surface obtained from the surface X by cutting X along z and by gluing back with a diffeomorphism of degree one. The surfaces X and X′ are of course diffeomorphic. A minimal positive pair of Dehn twists from X to X′ is a pair of diffeomorphisms (p, q) from X to X′ such that the following holds:

a) The composition $q^{-1} \circ p : X \to X$ is a right Dehn twist with respect to the orientation of X having the curve z as core. In addition $p(z) = q(z) = z$ holds.

b) There exists a regular bicollar neighbourhood N of z in X such that both p and q are the identity outside N.

c) For some volume form ω on N, which we think of as a symplectic structure, we have $p^*\omega = q^*(\omega) = \omega$, and the sum of the Hofer distances ([H-Z], see Chap. 5,) to the identity of the restrictions of p and q to $N \setminus z$ is minimal.

Minimal positive pairs of Dehn twists exist and are well defined up to isotopy. For a minimal positive pair (p, q) of Dehn twists, the member p is called positive or right and the member q is called negative or left.

Proof of Theorem **2.** — We need to introduce one more surface. Let F''_{-1} be the surface obtained from the surface F'_{-1} by cutting F'_{-1} along the cycles $Z'_{s,-1}$ and by gluing back after a half twist along each $Z'_{s,-1}$, s running through the crossing points of P. We still have partially defined diffeomorphisms

$$S''_i, S''_{-i} : \bigcup_{A|B} F'_{1,A|B} \to \bigcup_{A|B} F''_{-1,A|B}$$

since the cutting was done in the complement of $\bigcup_{A|B} F'$. By a direct inspection we see that the diffeomorphisms extend continuously to

$$S''_i, S''_{-i} : F'_1 \to F''_{-1}.$$

Let

$$(p_+, q_+) : F_1 \to F'_1$$
$$(p_\cdot, q_\cdot) : F''_{-1} \to F'_{-1}$$
$$(p_-, q_-) : F'_{-1} \to F_{-1}$$

be minimal positive pairs of Dehn twists. A direct inspection shows that the composition

$$(q_- \circ q_\cdot \circ S'_{-i} \circ q_+)^{-1} \circ p_- \circ p_\cdot \circ S'_i \circ p_+ : F_1 \to F_1$$

is the monodromy of the fibration π_P. This composition evaluates to

$$T_- \circ T_\cdot \circ T_+ : F_1 \to F_1. \quad \square$$

Remark **1.** — We list special properties of the monodromy of links and knots of divides. The number of Dehn twists of the above decomposition of the monodromy equals the first betti number $\mu = 2\delta - r + 1$ of the fiber, and the total number of

intersection points among the core curves of the involved Dehn twists is less then 5δ. This means that the complexity of the monodromy is bounded by a function of μ. For instance, the coefficients of the Alexander polynomial of the link of a divide are bounded by a quantity, which depends only on the degree of the Alexander polynomial. This observation suggests the following definition for the complexity C of an element of the mapping class group ϕ of a surface: the minimum of the quantity $L + I$ over all decompositions as product of Dehn twists of ϕ, where L is the number of factors and I is the number of mutual intersections of the core curves. We do not know properties of this exhaustion of the mapping class group. Notice, that the function $(\phi, \psi) \mapsto C(\psi^{-1} \circ \phi) \in \mathbf{N}$ defines a left invariant distance on the mapping class group.

A crossing point of P is "à quatre vents", if the 4 sectors of the complement of P in D meet the boundary of D. If there are no crossing points a quatre vents, then none of the core curves of the twists involved in the decomposition of the monodromy does separate the fiber, so the twists of the decomposition are all conjugated in the orientation-preserving mapping class group of the fiber.

It is easily seen that for any link of a divide the monodromy diffeomorphism and its inverse are conjugate by an orientation reversing element in the mapping class group. In our previous notation this conjugacy is given by the map $(x, u) \in F_1 \mapsto (x, -u) \in F_1$, which moreover realizes geometrically the symmetry of G. Torres [To] $t^{\mu}\chi(1/t) = (-1)^{\mu}\chi(t)$ for the Alexander polynomial $\chi(t)$ of knots.

Remark **2.** — In fact the proof of Theorem 2 shows that the fibration of the link of a connected divide P can be filled with a singular fibration in the 4-ball, which has three singular fibers with only quadratic singularities, as in the case of a divide of the singularity of a complex plane curve. The filling has only two singular fibers if the function f_P has no maxima or no minima. By this construction from a connected divide we obtain a contractible 4-dimensional piece with a Lefschetz pencil. It is sometimes possible to glue these local pieces and to get 4-manifolds with a Lefschetz pencil.

4. Examples, symplectic and contact properties

The figure eight is not the knot of a divide. The figure eight knot's complement fibers over the circle with as fiber the punctured torus and as monodromy the isotopy class of the linear diffeomorphism given by a matrix in $SL(2, \mathbf{Z})$ of trace 3. Such a matrix M is not the product of two unipotent matrices, which are conjugate in $SL(2, \mathbf{Z})$ and the matrices M and M^{-1} are not conjugate by an integral matrix of determinant -1. So according to the remarks of Section 3, the figure eight cannot be the knot of a divide. A third argument to rule out the figure eight as the knot of a divide goes as follows. The first betti number of the fiber of the figure eight knot is 2. But only two connected divides have fibers with betti number 2 and these two have monodromies with trace equal to 1 or 2.

The connected sum of two divides (D_1, P_1) and (D_2, P_2) is done by making a boundary connected sum of D_1 and D_2 such that a boundary point of P_1 matches with a boundary point of P_2. For divides with one branch we have the formula:

$$L(P_1 \# P_2) = L(P_1) \# L(P_2)$$

The Theorems 1 and 2 remain true for generic immersions of disjoint unions of intervals and circles in the 2-disk. It is also possible to start with a generic immersion of a 1-manifold I in an oriented compact connected surface with boundary S. The pair (S, I) defines a link L(S, I) in the 3-manifold $M_S := T^+(S)/\text{zip}$, where $T^+(S)$ is the space of oriented tangent directions of the surface S and where zip is the identification relation, which identifies $(x, u), (y, v) \in T^+(S)$ if and only if $x = y \in \partial S$ or if $(x, u) = (y, v)$. In order to get a fibered link, the topological pair $(R, R \cap \partial S)$ has to be contractible for each connected component R of $S \setminus I$ and moreover, the complement $S \setminus I$ has to allow a chess board coloring in positive and negative regions. The proofs do not need any modification.

A relative immersion $i : I \to D$ of a copy of $[0, 1]$ in D, such that at selftangencies the velocities are with opposite orientations, defines an embedded and oriented arc I' in S^3 by putting

$$I' := \{ (x, u) \in S^3 \mid x \in i(I), (di^{-1})_x(u) \geqslant 0 \}.$$

Let $j : I \to D$ be a relative immersion with only transversal crossings and opposite selftangencies, such that the endpoints of i and j are tangent with opposite orientations and that all tangencies of i and j are generic and have opposite orientations. The union $I' \cup J'$ is the oriented knot of the pair (i, j). A divide P defines pairs (i_P, j_P) of relative immersions with opposite orientations by taking both orientations. Those pairs (i_P, j_P) have a special 2-fold symmetry. For instance the complex conjugation realizes this 2-fold symmetry for a divide, which arises as a real deformation of a real plane curve singularity. It is interesting to observe that this symmetry acts on F_1 with as fixed point set the intersection $D \cap F_1$, which is a collection of r disjointly embedded arcs in F_1. The quotient of $\overline{F_1}$ by the symmetry is an orbifold surface with exactly $2r$ boundary $\frac{\pi}{2}$-singularities. Any link of singularity of a plane curve can be obtained as the link of a divide (see [AC3]). It is an interesting problem to characterize links of singularities among links of divides.

We finish this section with some symplectic and contact properties. The link of a divide is transversal to the standard contact structure in the 3-sphere. This can be seen explicitly by the following computation, where we use the multiplication of quaternions. Let P be a divide in the unit disk. We assume that the part of P, which lies in the collar of ∂D with inner radius $\frac{1}{\sqrt{2}}$, consists of radial line segments. We think of the branches of P as parametrized curves $\gamma(t) = (a(t), b(t))$, $-A \leqslant t \leqslant A$, where the

parameter speed is adjusted so that $a^2 + b^2 + \dot{a}^2 + \dot{b}^2 = 1$. To the branch γ correspond two arcs Γ^+ and Γ^- on the sphere of quaternions of unit length:

$$\Gamma^+(t) := a(t) - \dot{a}(t)i + b(t)j + \dot{b}(t)k,$$
$$\Gamma^-(t) := a(-t) + \dot{a}(-t)i + b(t)j - \dot{b}(-t)k.$$

The left invariant speed of Γ^+ at time t is

$$v(t) := \Gamma^+(t)^{-1}\frac{d}{dt}\Gamma^+(t).$$

We have

$$v = a\dot{a} + \dot{a}\ddot{a} + b\dot{b} + \dot{b}\ddot{b} + [-a\ddot{a} + \dot{a}^2 - b\ddot{b} + \dot{b}^2]i + v_j j + v_k k.$$

The coefficient $v_0 := a\dot{a} + \dot{a}\ddot{a} + b\dot{b} + \dot{b}\ddot{b}$ vanishes, since $\frac{d}{dt}\Gamma(t)$ is perpendicular to $\Gamma(t)$, and hence we can rewrite the coefficient v_i of i in v as

$$v_i = -a\ddot{a} + \dot{a}^2 - b\ddot{b} + \dot{b}^2 = <(a + \dot{a}, b + \dot{b}) \mid (\dot{a}, \dot{b})>.$$

Outside of the collar neighborhood of ∂D we have $v_i > 0$, since $a^2 + b^2 < 1/2 < \dot{a}^2 + \dot{b}^2$. In the collar we also have $v_i > 0$ by a direct computation. Since the left invariant contact structure on the unit sphere in the skew field of the quaternions is given by the span of the tangent vectors j and k at the point 1, we conclude that Γ^+ with its orientation is in the positive sense transversal to the left invariant contact structure S^3.

For the link of a divide we now will construct a polynomial, hence a symplectic, spanning surface in the 4-ball. For $\lambda \in \mathbf{R}$, $\lambda > 0$ put

$$B_\lambda := \{ p + ui \in \mathbf{C}^2 \mid p, u \in \mathbf{R}^2, ||p||^2 + \lambda^{-2}||u|| \leqslant 1 \}.$$

We have $B_\lambda \cap \mathbf{R}^2 = D$ and B_λ is a strictly holomorphically convex domain with smooth boundary in $\mathbf{C}^2$. The map $(p, u) \mapsto ((p, u/\lambda)$ identifies ∂B_λ with the unit 3-sphere of $\mathbf{C}^2$.

Theorem **3.** — *Let* P *be a connected divide in the disc* D *with* δ *double points and* r *branches. There exist* $\lambda > 0, \eta > 0$ *and there exists a polynomial function* $F : B_\lambda \to \mathbf{C}$ *with the following properties:*

a) the function F *is real, i.e.* $F(\overline{p + ui}) = \overline{F(p + ui)}$,

b) the set $P_0 := \{ p \in D \mid F(p) = 0 \}$ *is a divide, which is* C^1 *close to the divide* P, *and hence the divides* P *and* P_0 *are combinatorially equivalent,*

c) the function F *has only non degenerate singularities, which are all real,*

d) for all $t \in \mathbf{C}$, $|t| \leqslant \eta$ *the intersection* $K_t := \{ (p + iu) \in B_\lambda \mid F(p + iu) = t \} \cap \partial B_\lambda$ *is transversal and by a small isotopy equivalent to the link* L(P),

e) for the link K_η *the surface* $\{ (p + iu) \in B_\lambda \mid F(p + iu) = \eta \}$ *is a connected smooth symplectic spanning surface of genus* $\delta - r + 1$ *in the* 4*-ball.*

Proof. — Let the divide P be given by smooth parametrized curves $\gamma_l : [-1, 1] \to \mathbf{R}^2$, $1 \leqslant l \leqslant r$. Using the Weierstrass Approximation Theorem, we can construct polynomial approximations $\gamma_{l,0} : [-1, 1] \to \mathbf{R}^2$ being C^2 close to γ_l and henceforth give a divide P_0 with the combinatorics of the divide P. We may choose $\gamma_{l,0}$ such that $\gamma_{l,0}(s) \notin D$, $|s| > 1$. Let $F : \mathbf{C}^2 \to \mathbf{C}$ be a real polynomial map such $F = 0$ is a regular equation for the union of the images of $\gamma_{l,0}$. Let S^3_λ be the sphere $S^3_\lambda := \{ p + iu \in \mathbf{C}^2 \mid ||p||^2 + \lambda^{-2}||u||^2 = 1 \}$. For a sufficiently small $\lambda > 0$ we have that the 0-level of F on S^3_λ is a model for the link L(P). For $t \in \mathbf{C}$, $t \neq 0$, and t sufficiently small, say $|t| \leqslant \eta$, the surface $X_t := \{ p + iu \in B^4_\lambda \mid F(p + iu) = t \}$ is connected and smooth of genus $\delta - r + 1$, and has a polynomial equation, hence is a symplectic surface in the 4-ball B_λ equipped with the standard symplectic structure of $\mathbf{C}^2$. The intersection $K_\eta := X_\eta \cap \partial B_\lambda$ is also a model for the link L(P) and has hence a symplectic filling with the required properties. □

Remark. — Unfortunately, it is not the case that the restriction of F to B_λ is a fibration with only quadratic singularities, such that for some $\eta > 1$ the fibers $f_0^{-1}(t)$, $t \in \mathbf{C}$, $|t| < \eta$, are transversal to the boundary of B_λ. So, we do not know, as it is the case for divides coming from plane curve singularities, if it is possible to fill in with a Picard-Lefschetz fibration, which is compatible with the contact and symplectic structure.

5. The gordian number of the link of a divide

The δ-invariant of a plane curve singularity S is the number of local double points in $\mathbf{C}^2$, that occur in the union of its branches after a small generic deformation of the parametrizations of the branches. The δ-invariant is also the dimension as $\mathbf{C}$ vector space of the quotient of the normalisation of the local ring of S by the local ring of S. The Überschneidungszahl or gordian number s(L) of a link L in S^3 is the smallest number of cutovers, see Fig. 5, by which the link can be made trivial [W].

J. Milnor proposed the term unknotting number and conjectured for plane curve singularities, that the unknotting number of the link of the singularity equals the δ-invariant of the singularity [M]. This conjecture has been proved by P. Kronheimer and T. Mrowka (see [K1, K2, K3, K-M1, K-M2, K-M3, K-M4, K-M5]). Local links of plane curve singularities are special among links of divides and the δ-invariant of

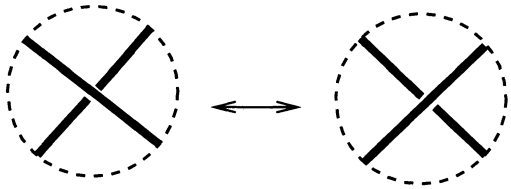

FIG. 5. – The move cutover

a plane curve singularity, which has only real branches, equals the number of double points of a divide for the singularity. The following theorem extends the computation of the gordian number to links of divides in general. The 4-ball genus of a link L in S^3 is the minimal genus of a smooth embedded oriented surface in B^4 bounded by L.

Theorem **4.** — *Let* P *be a divide with one branch. The gordian number and the* 4*-ball genus of the knot* L(P) *equal the number of double points of the divide* P.

The proof will be given at the end of this section. For the proof of Theorem 4, we can work in the ball B_λ and use the arguments of Kronheimer and Mrowka as in their work on the Thom conjecture, together with their extension to the relative case [K-M3] of a theorem of Taubes [Ta]. In the proof below we will use the global curve given by the polynomial equation $\{ F = t \}$ of Theorem 3 together with its completion in $P^2(\mathbf{C})$ and apply the affirmative answer of Kronheimer and Mrowka to the Thom conjecture [M-K2]. It is also possible to compute the Thurston-Bennequin number directly from the combinatorial data of the divide using a global displacement in the direction of the left invariant vector field given by j on S^3 and to conclude with an inequality of D. Bennequin [E] (see [G]), that the number of crossing points of the divide is a lower bound for the gordian number of its link. The point is that here luckily, in view of the inequality of Bennequin and the lemma below, the linking number of L(P) with j.L(P) is maximal among the displacements X.L(P) given by global non-vanishing vector fields X, which are tangent to the left invariant contact distribution spanned by $[k, j]$, and therefore yields the Thurston-Bennequin number.

Proof of Theorem **4.** — Let P have δ double points. Let $X \subset P^2(\mathbf{C})$ be the projective curve given by the equation $\{ F = 0 \}$ of Theorem 3. The curve X intersects ∂B_λ transversally and has δ transversal double points in B_λ. Let $Y := \{ F = t \}$ be a non-singular approximation of X. Since the genus of Y is minimal among all smooth surfaces in $P^2(\mathbf{C})$ representing [Y] by the work of Kronheimer and Mrowka on the Thom conjecture, the genus of $Y \cap B_\lambda$ is minimal among all smooth spanning surfaces in B_λ of the link $Y \cap \partial B_\lambda$ and the 4-ball genus of the link $Y \cap \partial B_\lambda$ equals δ. Since the links $Y \cap \partial B_\lambda$ and L(P) are equivalent, we conclude that the 4-ball genus of the link L(P) equals δ. At this point it follows that the gordian number of the link

L(P) equals or exceeds δ since the 4-ball genus of a link is a lower bound of its gordian number. It is consequence of the following lemma that the gordian number of L(P) equals δ. □

Lemma. — *Let* P *be a divide with* δ *double points. The gordian number of the link* L(P) *does not exceed* δ.

Proof. — We will produce an isotopy from the link L(P) to the trivial link which has exactly δ cutovers. First we need to choose a co-orientation of the branches of the divide. Next, at each point p of P, we consider the normal vector n_p in the direction of the choosen co-orientation such that for its length we have the rule $||p||^2 + ||n_p||^2 = 1$. For $\sigma \in [0, \pi/2)$ we define $L(P, \sigma)$ to be the link, possibly with transversal self-crossings,

$$L(P, \sigma) := \{ (x, \cos(\sigma)u + \sin(\sigma)n_x) \in T(D) \mid (x, u) \in L(P) \} \subset S(T(\mathbf{R}^2)) = S^3.$$

The link $L(P, \sigma)$ will have a singularity above the crossing point c of the divide P if $\sigma = \dfrac{\pi - \alpha_c}{2}$ where α_c is the angle between the two normals to P at c. The link $L(P, \sigma_0)$ is trivial if $\pi/2 > \sigma_0 > \dfrac{\pi - \alpha_c}{2}$ for all crossing points of P, since $L(P, \sigma_0)$ is spanned by the union of embedded disks $\cup_{t \in [\sigma_0, \pi/2]} L(P, t) \subset S^3$. Indeed, observe that the above formula defines a curve $L(P, \pi/2)$ which is a disjoint union of embedded arcs in S^3 and that $\cup_{t \in [\sigma_0, \pi/2]} L(P, t)$ is a disjoint union of smoothly embedded 2-disks in S^3. The family $L(P, t)$, $t \in [0, \sigma_0]$, connects the link L(P) with the trivial link and has δ cutovers.

Example. — The knot of the divide hart (Fig. 6) with 2 double points is the knot with 10 crossings 10_{145} (see the table F.1 of [Ka] page 261). From Theorem 4 it follows that the 4-ball genus and the gordian number of the knot 10_{145} are 2, which allows us to complete entries of table F.3 of [Ka]. As I learned from Toshifumi Tanaka he has determined by an other method the gordian number of the knot 10_{145} [T]. The gordian numbers of the knots 10_{139} and 10_{152} are proved to be 4 by Tomomi Kawamura [Kaw] and she deduced from $s(10_{139}) = 4$ the gordian number $s(10_{161}) = 3$.

Remark. — Let P be a divide with two branches P_1 and P_2. The homological linking number of the two oriented knots $L(P_1)$ and $L(P_2)$ equals the number of intersection points of P_1 and P_2. The minimal number of cutovers needed to separate by a smooth 2-sphere the components of the link L(P) equals also the number of intersection points of P_1 and P_2. It follows with Theorem 4 that the gordian number of the link of a divide equals the number of double points of the divide.

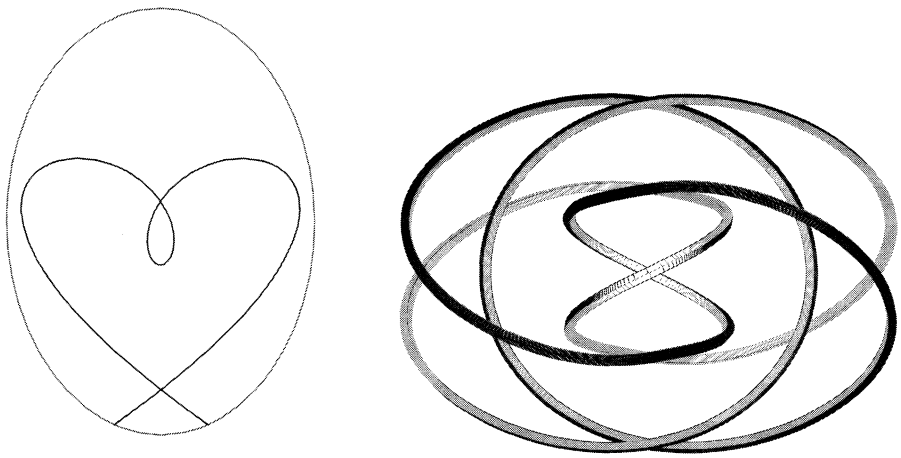

FIG. 6. – The divide hart and its knot 10_{145}

Remark. — H. Pinkham proved that for links of singularities of plane curves the gordian number does not exceed the δ-invariant of the singularity. Since the δ-invariant of a singularity is also the number of double points of a divide of a singularity of the same topological type, we have reproved the result of Pinkham.

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