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THE HOMOTOPY LIE ALGEBRA FOR FINITE COMPLEXES

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1. Introduction. — A generic question in topology asks how geometric restrictions on a topological space S are reflected in restrictions on $\pi_*(S)$. A classical example is this: which discrete groups G admit a finite CW complex as classifying space?

In this paper we shall deal with an analogous question for Lie algebras and simply connected spaces. Henceforth, and throughout this paper, we shall consider only those spaces which are simply connected and have the homotopy type of CW complexes whose rational homology is finite dimensional in each degree. Such spaces will be called *1-connected* CW *spaces of finite* $\mathbf{Q}$*-type.*

For such spaces $\pi_p(S) \otimes \mathbf{Q}$ is finite dimensional (each p), and the Whitehead product in $\pi_*(S)$, transferred to $\pi_*(\Omega S)$ by the canonical isomorphism, makes $\pi_*(\Omega S) \otimes \mathbf{Q}$ into a connected graded Lie algebra of finite type (*i.e.* finite dimensional in each degree): the *rational homotopy Lie algebra of* S. A striking result of Quillen [Q] asserts that every connected graded Lie algebra (over $\mathbf{Q}$) of finite type arises in this way.

The situation for *finite* complexes is very different, and the question referred to above, which forms the starting point of this paper, can be stated as the

Problem. — What restrictions are imposed on the rational homotopy Lie algebra of a space S, if S is a finite, 1-connected, CW complex?

We shall establish serious restrictions, both on the integers $\dim \pi_p(S) \otimes \mathbf{Q}$, and on the Lie structure.

These restrictions, moreover, turn out to hold for the much larger class of those 1-connected CW spaces of finite $\mathbf{Q}$-type whose rational Lusternik-Schnirelmann category is finite. Recall that the *Lusternik Schnirelmann category* of a space S, as normalized by Ganea [Ga], is the least integer m such that S can be covered by $m + 1$ open sets, each

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contractible in S. The *rational* LS *category*, denoted by $\mathrm{cat}_0(S)$, is the LS category of the localization $S_{\mathbf{Q}}$—cf. for instance [LS]. The inequality

$$\mathrm{cat}_0(S) \leq \sup\{n \mid H^n(S; \mathbf{Q}) \neq 0\}$$

implies that finite CW complexes have finite rational category.

Rational category has the following " hereditary property "—cf. [FH; Theorem (5.1)]: If $\varphi : S \to T$ is a continuous map of spaces such that

$$\varphi_{\#} : \pi_*(S) \otimes \mathbf{Q} \to \pi_*(T) \otimes \mathbf{Q}$$

is injective, then

$$\mathrm{cat}_0(S) \leq \mathrm{cat}_0(T).$$

In particular all the Postnikov fibres of a finite CW complex have finite rational category (but usually not finite dimensional rational homology). It is this fact (which plays an important role in the proofs to come) that has led us to weaken our hypothesis from " finite CW complex " to " finite rational category ".

We wish, then, to deal with the

Problem. — What restrictions are imposed on the rational homotopy Lie algebra of a 1-connected CW space of finite $\mathbf{Q}$-type, whose rational LS category is finite?

And our first result is the

Theorem (1.1). — *Let* S *be a* 1-*connected* CW *space of finite* $\mathbf{Q}$-*type and finite rational category. Then either*

(i) *For some* N, $\pi_p(S) \otimes \mathbf{Q} = 0$ *for* $p > N$ (*i.e.* $\pi_*(S) \otimes \mathbf{Q}$ *is finite dimensional*), *or*

(ii) *For some* k *and* C $(C > 1)$ *there is an infinite increasing sequence of odd integers*, q, *such that* $q_i \leq k^i$ *and*

$$\dim \pi_{q_i}(S) \otimes \mathbf{Q} \geq C^i.$$

We call spaces in the subclass (i) *rationally elliptic* and those in the subclass (ii) *rationally hyperbolic.*

The " generic " space (or manifold) is rationally hyperbolic, and the results of this paper deal with the homotopy Lie algebras of such spaces. But it seems worthwhile to briefly recall some of the known properties of rationally elliptic spaces, if only to illustrate how exceptional is this subclass.

Indeed for rationally elliptic spaces, their rational cohomology algebra is necessarily finite dimensional [FH] and satisfies Poincaré duality [H_1]; the integers $\dim \pi_p(S) \otimes \mathbf{Q}$ determine the degree n of the fundamental class [H_1], and the sum of the Betti numbers is $\leq 2^n$ [H_2]; all the possibilities for the integers $\dim \pi_p(S) \otimes \mathbf{Q}$ are completely classified [Fr-H]; the Euler Poincaré characteristic is necessarily non-negative [H_1].

As for rationally hyperbolic spaces, we have available the very beautiful

Conjecture (Avramov-Felix). — *Suppose* S *is a* 1*-connected* CW *space of finite* $\mathbf{Q}$*-type such that*

$$\mathrm{cat}_0(S) \text{ is finite and } \dim \pi_*(S) \otimes \mathbf{Q} \text{ is infinite.}$$

Then the rational homotopy Lie algebra contains a free sub Lie algebra on two homogeneous generators.

As a first step towards this conjecture we establish

Theorem (**1.2**). — *Suppose* S *is a* 1*-connected* CW *space of finite* $\mathbf{Q}$*-type, such that* $\mathrm{cat}_0(S)$ *is finite and* $\dim \pi_*(S) \otimes \mathbf{Q}$ *is infinite. Then the rational homotopy Lie algebra is not solvable.*

Corollary. — *If* $\mathrm{cat}_0(S)$ *and* $\mathrm{cocat}_0(S)$ [Ga] *are both finite then* S *has finite dimensional rational homotopy and finite dimensional rational homology.*

The main step in deriving Theorem (1.2) is a result which guarantees a plethora of nonzero Whitehead products for a rationally hyperbolic space. Recall that a linear transformation $\varphi : V \to V$ is *locally nilpotent* if for each $v \in V$ we can find an $n = n(v)$ such that $\varphi^n(v) = 0$.

Theorem (**1.3**). — *Suppose* S *is a* 1*-connected* CW *space of finite* $\mathbf{Q}$*-type such that*

$$\mathrm{cat}_0(S) \leq m \text{ and } \dim \pi_*(\Omega S) \otimes \mathbf{Q} \text{ is infinite.}$$

Assume $\alpha_1, \ldots, \alpha_m \in \pi_{2k}(\Omega S) \otimes \mathbf{Q}$ *(some k) are linearly independent elements such that each* $\mathrm{ad}\, \alpha_i$ *is locally nilpotent.*

Then the sub Lie algebra generated by the α_i *is infinite dimensional.*

Theorems (1.1) and (1.3) are in fact special cases of somewhat more technical results; these are stated in § 2. First, however, we should like to point out analogues for these topological theorems in two other, apparently very different categories: the category of graded Lie algebras, and that of local noetherian rings. We begin with Lie algebras.

Let $L = \sum_{p>0} L_p$ be a connected graded Lie algebra (over $\mathbf{Q}$) with universal enveloping algebra UL. A representation of L in a graded vector space $V = \sum_{p \geq 0} V_p$ makes V into a UL module in which elements of degree p act by linear transformations of degree p. The minimum length of a resolution of V by free graded UL modules is called the homological dimension of V; the supremum of the homological dimensions as V ranges over all graded UL-modules satisfying $V_i = 0$, $i < 0$, is called the *global dimension of* UL.

Now recall [Q] that L determines a chain coalgebra $(\Lambda sL, \partial)$ in which ΛsL is the free graded cocommutative coalgebra on the suspension sL. We may regard ΛsL as the space of " graded symmetric " tensors on sL; then ∂ is homogeneous of tensor

degree -1. It is well known [L] that glb dim UL is the largest m such that $(\Lambda sL, \partial)$ has nontrivial homology in tensor degree m.

On the other hand if L has finite type the cochain algebra dual to $(\Lambda sL, \partial)$ is the minimal model of a $\mathbf{Q}$-local space S, uniquely determined up to homotopy type ([BG], [S]), and called the *π-formal space* associated with L (cf. § 3 for minimal models). As we shall see in § 3, it is well known that

(i) $L = \pi_*(\Omega S) \otimes \mathbf{Q}$ (graded Lie algebras) and $sL = \pi_*(S) \otimes \mathbf{Q}$.
(ii) $\operatorname{glb} \dim UL = \operatorname{cat}_0(S)$.
(iii) The linear map $L \cong sL \to H(\Lambda sL, \partial)$ is just the Hurewicz homomorphism $h : \pi_*(\Omega S) \otimes \mathbf{Q} \cong \pi_*(S) \otimes \mathbf{Q} \to H_*(S; \mathbf{Q})$.

We thus obtain the

Theorem (**1.4**). — *Let* $L = \sum_{p>0} L_p$ *be a graded Lie algebra of finite type. Then theorems* (1.1)-(1.3), *as well as the theorems of* § 2, *when translated by the dictionary above, hold for* L.

The other setting in which graded Lie algebras appear is local algebra, and here the analogy with topology, while less precise, is striking and deep. (Although this analogy was known to Quillen in the mid sixties, it is the recent article of Roos [R] which has drawn attention to its importance.)

In this setting we do not offer theorems, but shall propose the appropriate translation of the results above as conjectures. Here the dictionary runs as follows:

Suppose R is a commutative local noetherian ring with residue field $\mathbf{k}$. Then $\operatorname{Ext}_R(\mathbf{k}, \mathbf{k})$ is the universal enveloping algebra of a graded Lie algebra L. We associate:

(i) $L \leftrightarrow \pi_*(\Omega S) \otimes \mathbf{Q}$; $sL \leftrightarrow \pi_*(S) \otimes \mathbf{Q}$.
(ii) $\dim_{\mathbf{k}}(I/I^2) - \operatorname{depth} R \leftrightarrow \operatorname{cat}_0(S)$. (I the maximal ideal in R.)
(iii) R not a complete intersection $\leftrightarrow \dim \pi_*(\Omega S) \otimes \mathbf{Q} = \infty$.

Notice that in this dictionary the dichotomy between rationally elliptic and hyperbolic spaces corresponds to the well known dichotomy between complete intersections and everything else.

In an earlier version of this paper we made the

Conjecture (**1.5**). — *With the dictionary above theorems* (1.1), (1.2), (1.3) *hold for local noetherian rings.*

Now Avramov [A_1], [A_2] has shown that if R is not a complete intersection then L is not abelian. Since our original preprint he has established Theorem (1.1) and, together with Halperin, showed that if R is not a complete intersection then L is not nilpotent.

The full conjecture can be established when $\operatorname{char} \mathbf{k} = 0$ and R is the localization of an evenly graded connected noetherian commutative $\mathbf{k}$-algebra H, by applying the techniques of [F-H] and of this paper to the " bigraded model " of H—cf. [F-T] where this is carried out for the case of Theorem (1.1). (An easy modification of a result of Gulliksen [Gu] shows that the Koszul complex of H has category $\leq \dim_{\mathbf{k}}(I/I^2) - \operatorname{depth} R$. If $I^{m+1} = 0$ one could also replace $\operatorname{cat}_0(S)$ by m.)

The main tool in this paper is Sullivan's theory of minimal models which we recall briefly for the convenience of the reader in § 3. This is preceded in § 2 by the statements of the main results for topological spaces in full generality. In particular, Theorem (1.1) is a special case of Theorem (2.1) while Theorem (1.3) is a special case of Theorem (2.2). (For completeness we restate Theorem (1.2) as Theorem (2.6).) The proof of theorem (2.1) (ii) is in § 4, the key lemmas relating LS category and Whitehead products are in § 5 and are followed by the proofs of the other theorems in § 6, 7 and 8.

2. Statement of results. — In the following, S denotes a 1-connected CW space of finite $\mathbf{Q}$-type. We denote the Hurewicz homomorphism, combined with the canonical isomorphism

$$\pi_*(\Omega S) \otimes \mathbf{Q} \cong \pi_*(S) \otimes \mathbf{Q}$$

by

$$h : \pi_*(\Omega S) \otimes \mathbf{Q} \to H_*(S; \mathbf{Q}).$$

Theorem **(2.1)**. — *Suppose* $\operatorname{cat}_0(S) \leq m$ *and* $\dim \pi_*(S) \otimes \mathbf{Q}$ *is infinite. Then*

(i) *There is an infinite arithmetic progression* $p_k = r + dk$ *(d even) such that* $\pi_{p_k}(S) \otimes \mathbf{Q} \neq 0$ *for all* $k \geq 0$.

(ii) *There is a constant* $C > 1$ *and an infinite increasing sequence of positive integers* q_i *such that for* $i \geq 1$, $q_i = \ell_i q_{i-1} - 1$ *for some integer* $\ell_i \in [2, m+1]$ *and such that*

$$\dim \pi_{q_i}(S) \otimes \mathbf{Q} \geq C^{q_i}, \quad i \geq 0.$$

Theorem **(2.2)**. — *Suppose* $\operatorname{cat}_0(S) \leq m$ *and* $\dim \pi_*(S) \otimes \mathbf{Q}$ *is infinite. Assume that*

$$\alpha_1, \ldots, \alpha_m \in \pi_*(\Omega S) \otimes \mathbf{Q}$$

are m linearly independent elements whose degrees are even and lie in an interval of the form $[k, 2k-2]$. *If the* $\operatorname{ad} \alpha_i$ *are locally nilpotent then the* α_i *generate an infinite dimensional sub Lie algebra.*

Theorem **(2.3)**. — *Suppose* $\operatorname{cat}_0(S) \leq m$ *and* $\dim \pi_*(S) \otimes \mathbf{Q}$ *is infinite. Suppose*

$$a_1, \ldots, \alpha_m \in \pi_*(\Omega S) \otimes \mathbf{Q}$$

are homogeneous elements of even degrees such that $h(\alpha_1), \ldots, h(\alpha_m)$ *are linearly independent homology classes. If the* $\operatorname{ad} \alpha_i$ *are locally nilpotent then the* α_i *generate an infinite dimensional sub Lie algebra.*

As an immediate corollary of Theorems (2.2) and (2.3) we have

Theorem **(2.4)**. — *Suppose* $\mathrm{cat}_0(S) \leq m$ *and* $\dim \pi_*(S) \otimes \mathbf{Q}$ *is infinite, and denote by* $Z(S)$ *the centre of the homotopy Lie algebra* $\pi_*(\Omega S) \otimes \mathbf{Q}$. *Then*

(i) $$\sum_{n=k}^{2k-1} \dim Z_{2n} < m$$

and (ii) $$\dim h(Z) < m.$$

Theorems (2.2) and (2.3) give no information as to how low the least degree of a nonzero Whitehead product must be. More precise information about the location of nonzero Whitehead products is contained in

Theorem **(2.5)**. — *Suppose* $\mathrm{cat}_0(S) \leq m$ *and* $\dim \pi_*(S) \otimes \mathbf{Q}$ *is infinite. If* n *is an integer such that* $\dim \pi_{2n}(\Omega S) \otimes \mathbf{Q} \geq m + 1$, *then for some* $p \in [4n, (4n+1)m]$, $\pi_p(\Omega S) \otimes \mathbf{Q}$ *contains a nonzero Whitehead product.*

Finally we have

Theorem **(2.6)**. — *If* $\mathrm{cat}_0(S)$ *is finite and* $\dim \pi_*(S) \otimes \mathbf{Q}$ *is infinite then the rational homotopy Lie algebra is not solvable.*

3. Minimal models and Lie algebras. — With a 1-connected CW space, S, of finite $\mathbf{Q}$-type is associated its *minimal model* $(\Lambda X, d)$; $(\Lambda X, d)$ is a graded differential algebra over $\mathbf{Q}$ with

(3.1) $$x = \sum_{p \geq 2} X^p \quad \text{and} \quad \dim X^p < \infty.$$

(3.2) $$\Lambda X = \text{exterior algebra } (X^{\mathrm{odd}}) \otimes \text{symmetric algebra } (X^{\mathrm{even}}).$$

(3.3) $$d \text{ is homogeneous of degree } 1 \text{ and } \operatorname{Im} d \subset (\Lambda X)^+ . (\Lambda X)^+.$$

The minimal model is determined (up to isomorphism of graded differential algebras) by S, and every g.d.a. $(\Lambda X, d)$ satisfying (2.1)-(2.3) arises as the minimal model of a space ([S], [BG], [H_3], [L]).

If $(\Lambda X, d)$ is a minimal model we denote by $(\Lambda X)^p$ the subspace of degree p. Because of (3.2), however, a second *(wedge) gradation* $\Lambda X = \sum_{p \geq 0} \Lambda^p X$ is defined: $\Lambda^p X$ is the linear span of elements of the form $x_1 \wedge \ldots \wedge x_p$, $x_i \in X$. Let $\zeta : \Lambda^+ X \to X$ denote the projection with kernel $\Lambda^{\geq 2} X$; then $\zeta \circ d = 0$ (by (3.3)) and so ζ induces a linear map $\zeta^* : H^+(\Lambda X) \to X$.

Again by (3.3) we can write d as the sum of derivations d_i $(i \geq 2)$ with $d_i : X \to \Lambda^i X$. In particular $(d_2)^2 = 0$. We define a graded space $L = \sum_{p \geq 1} L_p$ by $L_p = \mathrm{Hom}(X^{p+1}; \mathbf{Q})$ and a bilinear map $[\ ,\ ] : L_p \times L_q \to L_{p+q}$ by (cf. [F] for precise signs):

(3.4) $$\langle x; [\alpha, \beta] \rangle = \pm \langle d_2 x; \alpha, \beta \rangle \quad \alpha \in L_p, \quad \beta \in L_q, \quad x \in X^{p+q+1}.$$

The equation $d_2^2 = 0$ is equivalent to the Jacobi identity [Q] and so L becomes a graded Lie algebra. Further, put $H_*(\Lambda X) = \mathrm{Hom}(H(\Lambda X, d); \mathbf{Q})$; then ζ^* dualizes to a linear map of degree 1

$$h: L \to H_*(\Lambda X). \tag{3.5}$$

Finally, observe that for each m, $\Lambda^{>m}X$ is stable under d, and so a quotient g.d.a. $(\Lambda X/\Lambda^{>m}X, \bar{d})$ is defined. It follows from Sullivan's theory [S], [H_3] that there is a commutative diagram of g.d.a. morphisms (of course everything depends on m)

$$\begin{tikzcd} \Lambda X \arrow[r, "i"] \arrow[dr, "\mathrm{proj}"'] & \Lambda X \otimes \Lambda Y \arrow[d, "\eta", "\simeq"'] \\ & \Lambda X/\Lambda^{>m}X \end{tikzcd}$$

in which $\Lambda X \otimes \Lambda Y = \Lambda(X \oplus Y)$ satisfies (3.1) and (3.2) and η induces a cohomology isomorphism. (It can be shown that $\Lambda X \otimes \Lambda Y$ also satisfies (3.3), but we shall not need this.)

Following [F-H] we shall say that $(\Lambda X, d)$ *has category* $\leq m$ ($\mathrm{cat}_0(\Lambda X, d) \leq m$) if there is a g.d.a. morphism $r: \Lambda X \otimes \Lambda Y \to \Lambda X$ such that ri is the identity.

Now suppose $(\Lambda X, d)$ is the model of a space S. There are then standard isomorphisms (the first multiplicative)—[S], [BG]—

$$H^*(\Lambda X, d) \cong H^*(S; \mathbf{Q}) \quad \text{and} \quad X \cong \mathrm{Hom}_{\mathbf{Z}}(\pi_*(S); \mathbf{Q}).$$

The second can be interpreted as an isomorphism $L \cong \pi_*(\Omega S) \otimes \mathbf{Q}$, and a theorem of Andrews and Arkowitz [A-A] shows this to be a Lie algebra isomorphism. Moreover the map h (3.5) is identified with the composite

$$\pi_*(\Omega S) \otimes \mathbf{Q} \xrightarrow{\cong} \pi_*(S) \otimes \mathbf{Q} \xrightarrow{\text{Hurewicz}} H_*(S; \mathbf{Q}).$$

Finally ([F-H]; Theorem VIII), $\mathrm{cat}_0(S) = \mathrm{cat}_0(\Lambda X, d)$.

Next, suppose E is any graded Lie algebra as described in the introduction. Then the cochain algebra on E (dual to the chain coalgebra described there) is in fact a minimal model of the form $(\Lambda X, d_2)$. By definition the Lie algebra L which results is just E again. Next notice that $H(\Lambda X, d_2)$ inherits a second gradation from the wedge gradation of ΛX. If UE has global dimension m then, as we observed in the introduction, $H(\Lambda X, d_2)$ is zero in wedge degrees greater than m. Choose a graded complement I for the space of cocycles in $\Lambda^m X$. Then $J = I + \sum_{j>m} \Lambda^j X$ is an acyclic ideal. The projection $\psi: \Lambda X \xrightarrow{\cong} \Lambda X/J$ factors over the projection $\Lambda X \to \Lambda X/\Lambda^{>m}X$ to yield a retraction $\Lambda X \otimes \Lambda Y \to \Lambda X$. It follows that $\mathrm{cat}_0(\Lambda X, d_2) \leq m$. On the other hand a non-trivial cohomology class of wedge degree $> m$ would clearly prevent a retraction and we conclude that

$$\mathrm{glb\ dim\ UE} = \mathrm{cat}_0(\Lambda X, d_2).$$

4. The integers $\dim x^p$. — In view of the preceding section, Theorem (2.1) (ii) is a consequence of

Theorem **(4.1)**. — *Let* $(\Lambda X, d)$ *satisfy* (3.1)-(3.3) *and suppose* $cat_0(\Lambda X, d) \leq m$ *and* $\dim X = \infty$. *There is then an infinite sequence* $q_0, q_1, \ldots$ *with* (*for* $i \geq 1$) $q_i = \ell_i q_{i-1} - 1$, *some integer* $\ell_i \in [2, m+1]$, *and there is a constant* $C > 1$ *such that*

$$\dim X^{q_i} \geq C^{q_i}, \quad i \geq 0.$$

To prove this theorem we write $X^{[k,\ell]} = \sum_{p=k}^{\ell} X^p$. We first establish

Lemma **(4.2)**. — *With the hypotheses of* (4.1), *the integers* $k(n) = \dim X^{[n, 2n-2]}$ *are unbounded.*

Lemma **(4.3)**. — *With the hypotheses of* (4.1) *the integers* $\dim X^p$ *are unbounded.*

Proof of Lemma (4.2). — Define an *ℓ-widget* (strictly for the purpose of this proof!) to be a sequence of nonzero homogeneous elements $x_1, \ldots, x_\ell \in X$ such that

(i) each x_i has odd degree,
(ii) $\deg x_i > \deg x_1 + \ldots + \deg x_{i-1}$ $(i > 1)$, and
(iii) there is a g.d.a. morphism

$$\psi : (\Lambda X, d) \to (\Lambda(x_1, \ldots, x_\ell), 0)$$

with $\psi x_i = x_i$.

It then follows from ([F-H]; Theorem (5.1)) that $\ell \leq m = \mathrm{cat}_0(\Lambda X, d)$ for any ℓ-widget. We shall prove the lemma by showing that the three hypotheses: $\mathrm{cat}_0(\Lambda X, d) \leq m$, $\dim X = \infty$ and $k(n) \leq K$ for all n imply the existence of ℓ-widgets for all ℓ.

Indeed assume $k(n) \leq K$ in addition to the hypotheses of (4.1). Since $\mathrm{cat}_0(\Lambda X, d) < \infty$ it is immediate that $X^{\mathrm{odd}} \neq 0$ and the existence of 1-widgets follows. Suppose by induction that $(\Lambda X, d)$ always has ℓ-widgets.

If $\sigma_1 = x_1, \ldots, x_\ell$ is such an ℓ-widget put $q = \sum_1^{\ell} \deg x_i$ and consider the quotient model $(\Lambda X^{>q}, \bar{d})$ obtained by dividing by $X^{\leq q}$. By [F-H; Theorem (5.1)] this model also satisfies our three hypotheses and so by the induction hypothesis admits an ℓ-widget $\sigma_2 = x_{\ell+1}, \ldots, x_{2\ell}$. Clearly the composite $\Lambda X \to \Lambda X^{>q} \to \Lambda(x_{\ell+1}, \ldots, x_{2\ell})$ exhibits σ_2 as an ℓ-widget for ΛX.

Continuing in this way we find a sequence of $K + 1$ ℓ-widgets

$$x_1, \ldots, x_\ell;\ x_{\ell+1}, \ldots, x_{2\ell};\ \ldots;\ x_{K\ell+1}, \ldots, x_{(K+1)\ell}$$

with $\deg x_{s\ell+1} > \sum_{i=1}^{\ell} \deg x_{(s-1)\ell+i}$.

Now put $N = \sum_{i=1}^{\ell} \deg x_{K\ell+i}$. The quotient model $\Lambda X^{>N}$ having finite category, we can find a nonzero generator x of odd degree $p > N$. The $K + 1$ intervals

$$I_s = [p + \deg x_{s\ell+1} - 1, p + \sum_{i=1}^{\ell} \deg x_{s\ell+i} - 1], \quad 0 \leq s \leq K$$

are disjoint and they are all contained in $[p, 2p-2]$. Since $k(p) \leq K$ it follows that for one of the intervals I_s,

$$X^q = 0, \quad q \in I_s.$$

Fix such an s, and write $y_i = x_{s\ell+i}$, $1 \leq i \leq \ell$. We will show that $y_1, \ldots, y_\ell, x$ is an $\ell + 1$ widget for ΛX. Certainly the condition on the degrees is satisfied, so we have only to construct the morphism ψ.

Restrict the projection $(\Lambda X, d) \to (\Lambda(y_1, \ldots, y_\ell), 0)$ to $\Lambda(X^{<p})$ and then form the g.d.a. morphism.

$$\Lambda X = \Lambda X^{<p} \otimes \Lambda X^{\geq p} \to \Lambda(y_1, \ldots, y_\ell) \bigotimes_{\Lambda X^{<p}} \Lambda X = \Lambda(y_1, \ldots, y_\ell) \otimes \Lambda X^{\geq p}.$$

This gives a surjective map of minimal models

$$(\Lambda X, d) \to (\Lambda(y_1, \ldots, y_\ell) \otimes \Lambda X^{\geq p}, d')$$

in which $d' y_i = 0$.

Write $X^p = (x) \oplus Y^p$ and note that because $p > \Sigma \deg y_i$, $d'(X^p) = 0$. We can thus project Y^p to zero, arriving at a morphism

$$(\Lambda X, d) \to (\Lambda(y_1, \ldots, y_\ell, x) \otimes \Lambda X^{>p}, d'').$$

It follows now from the condition $X^q = 0$, $q \in I_s$ that the ideal generated by $X^{>p}$ is d''-stable. Divide by this ideal to achieve the desired morphism, and complete the proof.

Proof of Lemma (4.3). — Fix an integer N, and in the quotient model $(\Lambda X^{\geq N}, \bar{d})$ write

$$\bar{d}x = \alpha(x) + \beta(x), \quad x \in X,$$

where $\alpha(x) \in \Lambda X^{[N, 2N-2]}$ and $\beta(x)$ is in the ideal generated by $X^{>2N-2}$. Since $\mathrm{cat}_0(\Lambda X^{\geq N}, \bar{d}) \leq m$ by [F-H; Theorem (5.1)] and since $\bar{d}(X^{[N, 2N-2]}) = 0$ it follows that $\Lambda^{m+1} X^{[N, 2N-2]} \subset \mathrm{Im}\, \bar{d}$. This implies that

$$\alpha(X^{\geq N}) . \Lambda X^{[N, 2N-2]} \supset \Lambda^{m+1} X^{[N, 2N-2]}. \tag{4.4}$$

Recall the integers $k(n)$ of Lemma (4.2). If $\alpha_i(x)$ is the component of $\alpha(x)$ in $\Lambda^i X$ we have for degree reasons that $\alpha_i(X^p) = 0$ unless

$$p \in [iN - 1, i(2N-2) - 1] \subset [iN - 1, 2(iN - 1) - 2].$$

Thus $\dim \alpha_i(X^{\geq N}) \leq k(iN - 1)$. From (4.4) it follows that

$$\sum_{i=2}^{m+1} k(iN-1).k(N)^{m+1-i} \geq \binom{k(N)}{m+1}. \tag{4.5}$$

Put $k = \sup\{k(iN-1) \mid 2 \leq i \leq m+1\}$. Choose N so that

$$k(N) > [2^{m+1}(m+1)!\, m^2]^3$$

—possible by lemma (4.2). Then (4.5) yields

$$m.k.k(N)^{m-1} \geq \frac{k(N)^{m+1}}{2^{m+1}(m+1)!},$$

whence
$$k \geq \frac{k(N)^2}{2^{m+1}m(m+1)!} \geq k(N)^{5/3} > [2^{m+1}(m+1)!\, m^2]^3.$$

We can thus iterate the process to find a sequence $N_0 < N_1 < \ldots$ of integers with $(j \geq 1)$ $N_j = s_j N_{j-1} - 1$, some integer $s_j \in [2, m+1]$ and

$$k(N_{j+1}) \geq k(N_j)^{5/3} > k(N)^{5/3}, \quad j \geq 1.$$

Then
$$\frac{k(N_{j+1})}{N_{j+1}} \geq \frac{k(N_{j+1})}{(m+1)N_j} \geq \frac{k(N_j)^{2/3}}{m+1}\frac{k(N_j)}{N_j} \geq \frac{k(N)^{2/3}}{m+1}\frac{k(N_j)}{N_j} \geq 2\frac{k(N_j)}{N_j}.$$

Hence $\lim_{j\to\infty} \frac{k(N_j)}{N_j} = \infty$. Since $k(N_j) = \sum_{p=N_j}^{2N_j-2} \dim X^p$, this implies the lemma.

Proof of Theorem (3.1). — This follows from Lemma (4.3) as in Step III in the proof of [F-H; Theorem (8.1)]. We recall the argument for the convenience of the reader.

Put $a = \left(\frac{1}{2(m+1)}\right)^{m+1}$ and choose q so that $N = \dim X^q$ satisfies $Ne > 1$. The argument at the start of the proof of Lemma (4.3), applied to $(\Lambda X^q \otimes \Lambda X^{>q}, \bar{d})$, yields a degree 1 linear map $\alpha : X^{>q} \to \Lambda^{\geq 2} X^q$ such that $\alpha(X^{>q}).\Lambda X^q \supset \Lambda^{m+1} X^q$.

In particular $\alpha(X^p) = 0$ unless $p = \ell q - 1$, $\ell \geq 2$, and hence

$$\Lambda^{m+1} X^q = \sum_{\ell=2}^{m+1} \Lambda^{m+1-\ell} X^q . \alpha(X^{\ell q - 1}).$$

It follows that

$$\left(\frac{N}{2(m+1)}\right)^{m+1} \leq \binom{N}{m+1} \leq \dim \Lambda^{m+1} X^q$$
$$\leq \sum_{\ell=2}^{m+1} N^{m+1-\ell} \dim X^{\ell q - 1}.$$

This implies that for some $\ell \in [2, m+1]$, $\dim X^{\ell q - 1} \geq aN^\ell > N$.

Iterating this procedure yields a sequence of integers $q = q_0, q_1, \ldots$ with $q_i = \ell_i q_{i-1} - 1$ and ℓ_i an integer in $[2, m+1]$ such that

$$\dim X^{q_i} \geq a(\dim X^{q_{i-1}})^{\ell_i}.$$

It follows that

$$\dim X^{q_i} \geq a^{(1+\ell_i+\ell_{i-1}\cdot\ell_i+\ldots+\ell_2\cdot\ldots\cdot\ell_i)} N^{\ell_1\ell_2\ldots\ell_i}.$$

Since each $\ell_j \geq 2$, $(1+\ell_i+\ldots+\ell_2\cdot\ldots\cdot\ell_i)/\ell_1\ell_2\ldots\ell_i \leq \frac{1}{2^i}+\ldots+\frac{1}{2} < 1$ and so

$$\dim X^{q_i} \geq (aN)^{\ell_1\ell_2\ldots\ell_i} \geq [(aN)^{1/q}]^{q_i}, \text{ each } i.$$

To finish the proof, put $C = (aN)^{1/q}$.

Corollary **(4.6)**. — *Under the hypotheses of* (4.1), *with* $a = \left(\frac{1}{2(m+1)}\right)^{m+1}$,

$$\limsup_{q\to\infty} (\dim X^q)^{1/q} \geq \sup_p (a \dim X^p)^{1/p}.$$

5. The main lemmas. — Here we prove the essential two lemmas for the remaining theorems. First some notation: if $\sigma = (\sigma_1, \ldots, \sigma_m) \in \mathbf{Z}^m$ we define $\partial_i\sigma \in \mathbf{Z}^m$ by

$$(\partial_i\sigma)_j = \begin{cases} \sigma_j & j \neq i \\ \sigma_j - 1 & j = i. \end{cases}$$

Thus $\partial_i\partial_j = \partial_j\partial_i$. We also put $|\sigma| = \Sigma\sigma_i$, and write $\mathbf{Z}^m_* = \{\sigma \mid \sigma_i \geq 0, \text{each } i\}$.

Suppose now $(\Lambda X, d)$ satisfies (3.1)-(3.3) and $\mathrm{cat}_0(\Lambda X, d) \leq m$. Assume $\Phi_1, \ldots, \Phi_m, \Psi$ are homogeneous elements of $(\Lambda X)^+$ such that

$$\text{(5.1)} \qquad \begin{cases} d\Phi_i = 0 \text{ and } \Phi_i \text{ has odd degree, } 1 \leq i \leq m. \\ d(\Phi_1 \wedge \ldots \wedge \Phi_m \wedge \Psi) = 0. \end{cases}$$

From the definition of $\mathrm{cat}_0(\Lambda X, d)$ in § 3 it follows that the projection $\Lambda X \to \Lambda X/\Lambda^{>m}X$ is injective in cohomology. In particular, for some $u \in \Lambda X$

$$\text{(5.2)} \qquad du = \Phi_1 \wedge \ldots \wedge \Phi_m \wedge \Psi.$$

For $|\sigma| = 0$ put

$$\text{(5.3)} \qquad u_\sigma = \begin{cases} u & \sigma = (0, \ldots, 0) \\ 0 & \text{if some } \sigma_i < 0. \end{cases}$$

Lemma **(5.4)**. — *Suppose* $\mathrm{cat}_0(\Lambda X, d) \leq m$ *and* $\Phi_i, \Psi, u \in \Lambda X$ *are as described above. There are then elements* $z_\sigma \in \Lambda X$, $\sigma \in \mathbf{Z}^m$, *such that* $z_\sigma = 0$ *if some* $\sigma_i < 0$ *and*

$$\text{(5.5)} \qquad dz_\sigma = \begin{cases} 0 & |\sigma| = 0 \\ \sum_{i=1}^m \Phi_i \wedge (z_{\partial_i\sigma} - u_{\partial_i\sigma}) & |\sigma| = 1 \\ \sum_{i=1}^m \Phi_i \wedge z_{\partial_i\sigma} & |\sigma| > 1. \end{cases}$$

Proof. — Define a minimal model $(\Lambda X \otimes \Lambda W, D)$ as follows. Give W a basis of elements w_σ, $\sigma \in \mathbf{Z}^m_*$. Put $w_\sigma = 0$ if some $\sigma_i < 0$. Define D to coincide with d in ΛX, and by the formulae (5.5) in W (replacing z_σ by w_σ and d by D). The relations $\Phi_i \Phi_j = -\Phi_j \Phi_i$ and $\partial_i \partial_j = \partial_j \partial_i$ imply that $D^2 = 0$.

Define a g.d.a. homomorphism $\Lambda X \otimes \Lambda W \xrightarrow{\rho} \Lambda X / \Lambda^{>m} X$, extending the projection in ΛX, by $\rho w_0 = u$ and $\rho w_\sigma = 0$, $|\sigma| > 0$. (Note that $du \in \Lambda^{>m} X$ so that ρu is a cocycle in $\Lambda X / \Lambda^{>m} X$.) The theory of minimal models now implies that ρ lifts over η to a morphism $\varphi : \Lambda X \otimes \Lambda W \to \Lambda X \otimes \Lambda Y$ extending $\mathrm{id}_{\Lambda X}$, where η is defined in § 3. Since $\mathrm{cat}_0(\Lambda X, d) \leq m$ we have a retraction $r : \Lambda X \otimes \Lambda Y \to \Lambda X$; put $z_\sigma = r\varphi(w_\sigma)$:

$$\begin{tikzcd}
\Lambda X \arrow[r] \arrow[d, hook] & \Lambda X \otimes \Lambda Y \arrow[r, "r"] \arrow[d, "\eta", "\simeq"'] & \Lambda X \\
\Lambda X \otimes \Lambda W \arrow[r, "\rho"'] \arrow[ur, "\varphi"] & \Lambda X / \Lambda^{>m} X &
\end{tikzcd}$$

and the lemma follows.

Lemma **(5.6).** — *Let $\Lambda(x_1, \ldots, x_m)$ be the exterior algebra generated by linearly independent elements x_ν, and let* B *be any vector space. Suppose for some $p \geq 1$, $\Phi_\tau \in \Lambda(x_1, \ldots, x_m) \otimes B$ $(\tau \in \mathbf{Z}^m,\ |\tau| = p)$, satisfy*

(i) $\Phi_\tau = 0$ *if some* $\tau_i < 0$.

(ii) $\sum_{\nu=1}^{m} x_\nu \wedge \Phi_{\partial_\nu \sigma} = 0$, *all* $|\sigma| = p + 1$.

There are then elements $\Psi_\omega \in \Lambda(x_1, \ldots, x_m) \otimes B$, $|\omega| = p - 1$, such that $\Psi_\omega = 0$ if some $\omega_i < 0$ and

$$\Phi_\tau = \sum_{i=1}^{m} x_\nu \wedge \Psi_{\partial_\nu \tau}, \quad \textit{all } \tau.$$

Proof. — It is clearly sufficient to prove this when $B = \mathbf{Q}$. Assume this. The lemma is trivial when $m = 1$ and follows for general m by an easy induction.

6. Lie structure and finite category. — In this section we fix a minimal model $(\Lambda X, d)$, satisfying (3.1)-(3.3), with $\mathrm{cat}_0(\Lambda X, d) \leq m$. Let $L = \sum_{p \geq 1} L_p$ denote the corresponding Lie algebra, and recall the linear map $h : L \to H_*(\Lambda X)$ of degree 1, defined in (3.5). If $h(\alpha_1), \ldots, h(\alpha_\ell)$ are linearly independent we can find cocycles in ΛX of the form $x_i + \Phi_i$, $\Phi_i \in \Lambda^{\geq 2} X$, such that

$$\langle x_i + \Phi_i, h(\alpha_j) \rangle = \delta_{ij}.$$

Redefining X by replacing x_i by $x_i + \Phi_i$ we arrive at elements $x_i \in X$ with

$$\textbf{(6.1)} \qquad dx_i = 0 \quad \text{and} \quad \langle x_i, h(\alpha_j) \rangle = \delta_{ij}.$$

Theorem (2.3) follows from

Theorem **(6.2)**. — *Suppose* $\mathrm{cat}_0(\Lambda X, d) \leq m$ *and* $\dim X = \infty$. *Assume* $\alpha_1, \ldots, \alpha_m \in L$ *have even degrees and generate a finite dimensional subalgebra of* L. *If* $h(\alpha_1), \ldots, h(\alpha_m)$ *are linearly independent, then for some* $\beta \in L$ *and some* $i \in [1, m]$

$$(\mathrm{ad}\, \alpha_i)^p \beta \neq 0, \text{ all } p.$$

Proof. — Let $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_\ell$ be a basis of the subalgebra E generated by the α_j and note that each β_i has even degree.

Let $W \subset X$ be the subspace of vectors w such that $\langle w, E \rangle = 0$ and choose vectors $x_\nu, x'_\mu \in X$ such that

$$\langle x_\nu, \alpha_j \rangle = \delta_{\nu j} \quad \langle x'_\mu, \alpha_j \rangle = 0$$

and

$$\langle x_\nu, \beta_i \rangle = 0 \quad \langle x'_\mu, \beta_i \rangle = \delta_{\mu i}.$$

Then X is the direct sum of W and the subspace Y with basis x_ν, x'_μ, so that

$$\Lambda^2 X = \Lambda^2 Y \oplus (Y \otimes W) \oplus \Lambda^2 W.$$

In view of our remarks above we may suppose $dx_\nu = 0$, $1 \leq \nu \leq m$.

Denote by ad^* the representation of L in X dual to the adjoint representation. Since $[E, E] \subset E$, W is an E-module and $d_2 w \in Y \otimes W \oplus \Lambda^2 W$. Thus formula (3.4) yields

$$\textbf{(6.3)} \qquad d_2 w \pm \sum_{\nu=1}^{m} x_\nu \wedge (\mathrm{ad}\, \alpha_\nu)^* w \pm \sum_{\mu=1}^{\ell} x'_\mu \wedge (\mathrm{ad}\, \beta_\mu)^* w \in \Lambda^2 W, \quad w \in W.$$

Since α_j, β_i have even degree the x_ν, x'_μ have odd degree. Since $\dim X = \infty$ there is a generator $a \in X$, linearly independent of the x_ν. Choosing a of least degree we see that $da \in \Lambda(x_1, \ldots, x_m)$, so that $d(x_1 \wedge \ldots \wedge x_m \wedge a) = 0$. As remarked in § 5, because $\mathrm{cat}_0(\Lambda X, d) \leq m$, there is an element $u \in \Lambda X$ with $du = x_1 \wedge \ldots \wedge x_m \wedge a$.

Apply Lemma (5.4) to the elements x_ν, a, u. This yields elements $z_\sigma \in \Lambda X$ ($\sigma \in \mathbf{Z}^m$) which satisfy the following equations:

$$\textbf{(6.4)} \qquad z_\sigma = 0 \quad \text{if some } \sigma_i < 0$$

$$\textbf{(6.5)} \qquad dz_\sigma = \begin{cases} 0 & |\sigma| = 0 \\ \sum_{\nu=1}^{m} x_\nu \wedge (z_{\partial_\nu \sigma} - u_{\partial_\nu \sigma}), & |\sigma| = 1 \\ \sum_{\nu=1}^{m} x_\nu \wedge z_{\partial_\nu \sigma}, & |\sigma| \geq 2. \end{cases}$$

(Here we have put $u_0 = u$ and $u_\sigma = 0$ if $|\sigma| = 0$ and some $\sigma_i < 0$.)

The main step in the proof of the theorem is the following lemma.

Lemma **(6.6)**. — *With* x_ν, a, u, E *as above, fix integers* $q \geq 1$, *and* $j, k \geq 0$. *Suppose the* E*-module generated by each* $\beta \in L$ *has finite dimension. Then the elements* z_σ *can be chosen so that in addition to* (6.4) *and* (6.5) *they satisfy: for* $|\sigma| = q$

$$z_\sigma \in \sum_{i \geq j} [\Lambda^{k-i}(x_\nu, x'_\mu) \otimes \Lambda^i W] \oplus \Lambda^{\geq k+1} X.$$

Proof of the lemma. — The lemma holds trivially for all q, j when $k = 0$. If for some fixed k it holds for all $j \leq k + 1$ and all $q \geq 1$, then it holds for $k + 1$, $j = 0$ and all $q \geq 1$. Thus to prove the lemma we may (and do) assume it true for some fixed k, j and all $q \geq 1$, and establish it under this hypothesis for k, $j + 1$, and all $q \geq 1$.

Since each $\beta \in L$ generates a finite dimensional E-module, we can find an increasing sequence $\ldots A_p \subset A_{p+1} \subset \ldots$ of finite dimensional graded E-modules such that $L_{\leq p} \subset A_p$.

Let $U_p \subset X$ be the graded E-module of vectors vanishing on A_p. Then $U_p \supset U_{p+1}$, each U_p has finite codimension in X, and $U_p \subset X^{\geq p+2}$.

Now fix $q \geq 1$. Choose N_1 so large that $W^p = X^p$, $p \geq N_1$, and so that if $|\sigma| = q$ then $\deg z_\sigma < N_1$. Then $U_{N_1} \subset W$. Choose $N_2 > N_1$ so that

$$W^p = X^p \subset U_{N_1} \quad \text{if } p \geq N_2.$$

Finally choose $q' > q$ so that for $|\sigma| = q'$

$$\deg z_\sigma \geq \sum_{\nu=1}^{m} \deg x_\nu + \sum_{\mu=1}^{\ell} \deg x'_\mu + (k+1)N_2. \tag{6.7}$$

Our induction hypothesis allows us to suppose that

$$z_\sigma \in \sum_{i \geq j} [\Lambda^{k-i}(x_\nu, x'_\mu) \otimes \Lambda^i W] \oplus \Lambda^{\geq k+1} X, \quad |\sigma| = q'. \tag{6.8}$$

Let $I \subset \Lambda W$ be the graded ideal generated by U_{N_1} and let $C \subset \Lambda^j W$ be a graded complement for $I \cap \Lambda^j W$ in $\Lambda^j W$. Define graded spaces $V(s)$, $V'(s)$, $s = k$ or $k + 1$ by

$$V(s) = \Lambda^{\geq s+1} X \oplus \sum_{i \geq j+1} [\Lambda^{s-i}(x_\nu, x'_\mu) \otimes \Lambda^i W] \oplus [\Lambda^{s-j}(x_\nu, x'_\mu) \otimes (\Lambda^j W \cap I)]$$

and

$$V'(s) = \Lambda^{<s} X \oplus \sum_{i<j} [\Lambda^{s-i}(x_\nu, x'_\mu) \otimes \Lambda^i W] \oplus [\Lambda^{s-j}(x_\nu, x'_\mu) \otimes C].$$

Clearly $\Lambda X = V(s) \oplus V'(s)$, $s = k$ or $k + 1$. Moreover since U_{N_1} is an E-module so is $\Lambda^j W \cap I$. It follows thus from (6.3) that $d(V(k)) \subset V(k+1)$. We shall show now that for any $q_1 \in [q, q']$, we can choose the z_σ to satisfy (6.4), (6.5) and

$$z_\sigma \in V(k), \quad |\sigma| = q_1. \tag{6.9}$$

Indeed (6.8) and an easy degree argument show that (6.7) implies (6.9) when $q_1 = q'$. Suppose now that for some $q_1 > q$: the z_σ satisfy (6.4), (6.5), and (6.9) holds.

For $|\tau| = q_1 - 1$ write $z_\tau = v_\tau + v'_\tau$, $v_\tau \in V(k)$, $v'_\tau \in V'(k)$. Then $x_\nu \wedge v_\tau \in V(k+1)$, $x_\nu \wedge v'_\tau \in V'(k+1)$, while for $|\sigma| = q_1$, $dz_\sigma \in V(k+1)$. Thus (6.5) yields

$$\sum_{\nu=1}^{m} x_\nu \wedge v'_{\partial_\nu \sigma} = 0, \quad |\sigma| = q_1.$$

By hypothesis $q_1 - 1 \geq q \geq 1$. Thus Lemma (5.6) asserts the existence of elements $a_\omega \in \Lambda X$, $|\omega| = q_1 - 2$ such that $a_\omega = 0$ if some $\omega_i = 0$ and

$$v'_\tau = \sum_{\nu=1}^{m} x_\nu \wedge a_{\partial_\nu \tau}, \quad |\tau| = q_1 - 1.$$

Replace z_σ by z'_σ when $|\sigma| = q_1 - 1$ or $q_1 - 2$ as follows:

$$z'_\sigma = z_\sigma - v'_\sigma, \quad |\sigma| = q_1 - 1$$

and

$$z'_\sigma = z_\sigma + da_\sigma, \quad |\sigma| = q_1 - 2.$$

By definition (6.9) now holds for $|\sigma| = q_1 - 1$. A simple calculation shows that (6.4) and (6.5) continue to hold.

Continuing this way we see that we may assume (6.9) to hold for $|\sigma| = q$. But now $\deg z_\sigma < N_1$ if $|\sigma| = q$, while $U^p_{N_1} = 0$ if $p \leq N_1$. For degree reasons, then, z_σ can have no component in $\Lambda(x_\nu, x'_\mu) \otimes \Lambda^j W \cap I$ and it follows from (6.9) that

$$z_\sigma \in \sum_{i \geq j+1} [\Lambda^{k-i}(x_\nu, x'_\mu) \otimes \Lambda^i W] \oplus \Lambda^{\geq k+1} X, \quad |\sigma| = q.$$

The lemma follows by induction.

The proof of the theorem is now completed by the following two steps:

Lemma **(6.10)**. — *Some* $\beta \in L$ *generates an* E-*module which is an infinite dimensional vector space.*

Lemma **(6.11)**. — *Suppose* θ *represents a finite dimensional connected graded Lie algebra* E *in a graded vector space* V *and some* $v \in V$ *generates an* E-*module which is an infinite dimensional vector space. Suppose* E *is generated by elements* α_i, $1 \leq i \leq m$. *Then for some* $w \in V$ *and some* i

$$\theta(\alpha_i)^p w \neq 0, \quad \textit{all } p.$$

Proof of (6.10). — We suppose each $\beta \in L$ generates a finite dimensional E-module, and deduce a contradiction. Indeed with the notation above choose k so that $k > |z_\sigma|$ for $|\sigma| = 1$ and apply Lemma (6.6) with $q = 1$. It follows that we may assume $z_\sigma = 0$, $|\sigma| = 1$. Now (6.5) reads

$$0 = \sum_{\nu=1}^{m} x_\nu \Lambda (z_{\partial_\nu \sigma} - u_{\partial_\nu \sigma}), \quad |\sigma| = 1.$$

When σ is given by $\sigma_i = 1$ and $\sigma_j = 0$, $j \neq i$, this equation reads

$$x_i \wedge (z_0 - u) = 0.$$

Since this holds for each i we conclude

$$u - z_0 = x_1 \wedge \ldots \wedge x_m \wedge b, \quad \text{some } b \in \Lambda X.$$

Differentiating one last time, and recalling that $dz_0 = 0$, we find

$$x_1 \wedge \ldots \wedge x_m \wedge a = \pm x_1 \wedge \ldots \wedge x_m \wedge db,$$

which is impossible because $a \in X$ and $db \in \Lambda^{\geq 2}X$. The lemma follows.

Proof of (6.11). — Recall that a linear transformation $\varphi : V \to V$ is locally nilpotent if for $v \in V$ there is an integer $n(v)$ such that $\varphi^{n(v)}v = 0$. We shall show that if each $\theta(\alpha_i)$ is locally nilpotent then each $v \in V$ generates a finite dimensional E-module.

First observe, since E is finite dimensional and connected, that if $\beta, \gamma \in E$ then for some p; $0 \neq \omega = (\operatorname{ad} \beta)^p \gamma$, and $0 = (\operatorname{ad} \beta)^{p+1}\gamma$. A simple computation shows that if $v \in V$ satisfies $\theta(\beta)v = 0$ then

$$\theta(\beta)^{pk}\theta(\gamma)^k v = \lambda\theta(\omega)^k v,$$

where λ is a certain binomial coefficient. In particular, if $\theta(\gamma)$ is locally nilpotent then for some n, $\theta(\omega)^n v = 0$.

We show now that if $\theta(\beta)$ and $\theta(\gamma)$ are both locally nilpotent, then so is $\theta(\omega)$. Indeed suppose by induction that $\theta(\omega)^{n(v)}v = 0$ if $v \in \ker \theta(\beta)^q$. If $v \in \ker \theta(\beta)^{q+1}$ we can find k so that $\theta(\omega)^k\theta(\beta)v = 0$. But by construction $[\beta, \omega] = (\operatorname{ad} \beta)^{p+1}\gamma = 0$ and so $\theta(\beta)$ commutes with $\theta(\omega)$. It follows that $\theta(\beta)\theta(\omega)^k v = 0$. The argument just given then implies that for some ℓ, $\theta(\omega)^\ell\theta(\omega)^k v = 0$. Thus $\theta(\omega)$ is locally nilpotent.

In particular we may construct an element α in the centre of E (because E is finite dimensional and connected) of the form

$$\alpha = (\operatorname{ad} \alpha_{i_1})^{p_1} \circ \ldots \circ (\operatorname{ad} \alpha_{i_s})^{p_s}\alpha_1 \neq 0,$$

and such that for each j

$$0 = (\operatorname{ad} \alpha_{i_j})^{p_j+1} \circ (\operatorname{ad} \alpha_{i_{j+1}})^{p_{j+1}} \circ \ldots \circ (\operatorname{ad} \alpha_{i_s})^{p_s}\alpha_1.$$

The remarks above, applied to each $(\operatorname{ad} \alpha_{i_j})^{p_j} \circ \ldots \circ (\operatorname{ad} \alpha_{i_s})^{p_s}\alpha_1$ in turn, imply that $\theta(\alpha)$ is locally nilpotent.

We complete the proof by induction on dim E. If $\theta : E \to \operatorname{End} V$ is not injective we replace E by $\theta(E)$ and conclude by induction on dim E. Hence we may suppose θ injective. Consider the exact sequences of E-modules

$$0 \to \ker \theta(\alpha)^p \to \ker \theta(\alpha)^{p+1} \to \ker \theta(\alpha)^{p+1}/\ker \theta(\alpha)^p \to 0.$$

By our induction assumption (since α acts trivially in the right hand vector space) if $v \in \ker \theta(\alpha)^{p+1}$ then the E-module, $E(v)$, it generates has finite dimensional image in $\ker \theta(\alpha)^{p+1}/\ker \theta(\alpha)^p$. Thus for some n

$$E(v)_j \subset \ker \theta(\alpha)^p, \quad j \geq n.$$

Now the E-module $E(v)_{\geq n}$ is generated by the finite dimensional vector space $\sum_{j=n}^{n+s} E(v)_j$, $s = \max \deg \alpha_i$. Thus by induction on p this time, $\dim E(v)_{\geq n} < \infty$ and so $\dim E(v) < \infty$. Thus $\dim E(v) < \infty$ for all $v \in \bigcup_p \ker \theta(\alpha)^p$. But since $\theta(\alpha)$ is locally nilpotent $V = \bigcup_p \ker \theta(\alpha)^p$; hence the claim.

7. The remaining theorems, except for Theorem 2.6. — The next theorem is almost a corollary of Theorem (6.2); on the other hand, it clearly gives Theorem (2.2).

Theorem **(7.1).** — *Let* $(\Lambda X, d)$ *satisfy* (3.1)-(3.3) *and assume* $\mathrm{cat}_0(\Lambda X, d) \leq m$ *and* $\dim X = \infty$. *Suppose* $\alpha_1, \ldots, \alpha_m$ *are homogeneous elements in the associated Lie algebra,* L, *whose degrees are even and lie in an interval of the form* $[k, 2k-2]$. *If the* α_i *generate a finite dimensional subalgebra then for some* $\beta \in L$ *and some* $i \in [1, m]$

$$(\mathrm{ad}\, \alpha_i)^p \beta \neq 0 \text{ all } p.$$

Proof. — Let $(\Lambda X^{>k}, \bar{d})$ be the quotient model obtained by putting $X^p = 0$, $p \leq k$. Its associated Lie algebra (say $\bar{L}$) is a subalgebra of L; the inclusion is dual to the projection $X \to X^{>k}$, so clearly $\bar{L} = L_{\geq k}$. In particular $\alpha_1, \ldots, \alpha_m \in \bar{L}$.

On the other hand, evidently $\bar{d}(X^p) = 0$, $k+1 \leq p \leq 2k$. It follows easily that $h : \bar{L}_p \to H_{p+1}(\Lambda X^{>k})$ is injective for $p \in [k, 2k-2]$. Finally, by [F-H; Theorem (5.1)], $\mathrm{cat}_0(\Lambda X^{>k}, \bar{d}) \leq m$. We can now apply Theorem (5.2) to $\bar{L}$.

Remark **(7.2).** — Theorem (2.1) (i) and Theorem (2.4) (i) are corollaries of Theorem (2.2). Theorem (2.4) (ii) is a corollary of Theorem (2.3). It only remains to prove Theorem (2.5). It is a consequence of

Theorem **(7.3).** — *Suppose* $(\Lambda X, d)$ *satisfies* (3.1)-(3.3), $\mathrm{cat}_0(\Lambda X, d) \leq m$, *and* $\dim X = \infty$. *Assume* $x \in X^{2k+1}$, $x_1, \ldots, x_m \in X^{2n+1}$ *are linearly independent of degrees such that* $k \leq 2n$, *and let* ℓ *be the lesser of* k *and* n.

Then in the associated Lie algebra L *the derived algebra* [L, L] *is nonzero in degree* p *for some* $p \in [2\ell + 2n, 2n(m-1) + (2n+1)m + 2k]$.

Proof. — Choose a graded subspace $Y \subset X^{\leq 2n+1}$ as follows:

(a) if $k < n$: $Y^p = X^p$ unless $p = 2k+1$ or $p = 2n+1$, $Y^{2k+1} \oplus (x) = X^{2k+1}$, $Y^{2n+1} = 0$,

(b) if $k = n$: $Y^p = X^p$, $p < 2n+1$ and $Y^{2n+1} = 0$,

(c) if $k > n$: $Y^p = X^p$, $p < 2n+1$, $Y^{2n+1} \oplus (x_1, \ldots, x_m) = X^{2n+1}$.

Then Y generates a d-stable ideal I. Dividing by I produces a new minimal model which still has category $\leq m$ ([F-H; Theorem (5.1)]). It is thus sufficient to prove the theorem for this new model; *i.e.* we may assume that $Y = 0$. We then have the

following additional facts: $x_1, \ldots, x_m$ are linearly independent cocycles linearly independent of x, and $dx \in \Lambda(x_1, \ldots, x_m)$.

As in the proof of Theorem (6.2) we can write $x_1 \wedge \ldots \wedge x_m \wedge x = du$ and apply Lemma (5.4) to the elements x_i, x and u. This yields elements $z_\sigma \in \Lambda X$ such that

$$z_\sigma = 0 \quad \text{if some } \sigma_i < 0 \tag{7.4}$$

and

$$dz_\sigma = \begin{cases} 0, & |\sigma| = 0 \\ \sum_{i=1}^{m} x_i \wedge (z_{\partial_i \sigma} - u_{\partial_i \sigma}), & |\sigma| = 1 \\ \sum_{i=1}^{m} x_i \wedge z_{\partial_i \sigma}, & |\sigma| > 1. \end{cases} \tag{7.5}$$

(As in the proof of (6.2) $u_0 = u$ and $u_\sigma = 0$, otherwise.) A simple computation shows that

$$\deg z_\sigma = 2n|\sigma| + (2n+1)m + 2k.$$

Now suppose in L that $[\alpha, \beta] = 0$ whenever

$$\deg \alpha + \deg \beta \in [2\ell + 2n, 2n(m-1) + (2n+1)m + 2k].$$

We shall deduce from this a contradiction. Because of the way we have constructed ΛX, $(\Lambda^2 X)^p = 0$ in degrees $< 2\ell + 2n + 2$. It follows from (3.4) that $[\alpha, \beta] = 0$ if $\deg \alpha + \deg \beta < 2\ell + 2n$, and so

$$[\alpha, \beta] = 0, \quad \deg \alpha + \deg \beta \leq 2n(m-1) + (2n+1)m + 2k.$$

We can thus apply (3.4) to get

$$d_2 = 0 \text{ in } (\Lambda X)^{\leq N}, \quad N = 2n(m-1) + (2n+1)m + 2k.$$

In particular $d_2 z_\sigma = 0$, $|\sigma| \leq m - 1$.

We shall show now by induction on i that we can arrange that

$$z_\sigma \in \Lambda^{\geq i+1} X, \quad |\sigma| = m - 1 - i, \quad 0 \leq i \leq m - 2.$$

Indeed this is trivial when $i = 0$. Suppose it holds for some $i < m - 2$. For $|\sigma| = m - 1 - i$, $d_2 z_\sigma = 0$ and so

$$dz_\sigma \in \Lambda^{\geq i+3} X.$$

For every $|\tau| = m - i - 2$ we write $z_\tau = v_\tau + w_\tau$, $v_\tau \in \Lambda^{\geq i+2} X$, $w_\tau \in \Lambda^{< i+2} X$. For $|\sigma| = m - 1 - i$, $dz_\sigma = \Sigma x_\nu \wedge v_{\partial_\nu \sigma} + x_\nu \wedge w_{\partial_\nu \sigma}$ with $|\partial_\nu \sigma| = m - i - 2$. Since $dz_\sigma \in \Lambda^{\geq i+3} X$ we have $0 = \sum_{\nu=1}^{m} x_\nu \wedge w_{\partial_\nu \sigma}$ and $dz_\sigma = \sum_{\nu=1}^{m} x_\nu \wedge v_{\partial_\nu \sigma}$.

We can thus apply Lemma (5.6) to find elements a_ω, $|\omega| = m - i - 3$, such that $a_\omega = 0$ if some $\omega_i = 0$ and, for $|\tau| = m - i - 2$

$$w_\tau = \sum_{\nu=1}^{m} x_\nu \wedge a_{\partial_\nu \tau}.$$

Modify the z_τ by replacing z_τ by $v_\tau = z_\tau - w_\tau$ if $|\tau| = m - i - 2$, and by replacing z_ω by $z_\omega + a_\omega$ if $|\omega| = m - i - 3$. Leave the other z_σ's untouched. A simple calculation shows the new z_σ's satisfy (7.4) and (7.5) and that $z_\sigma \in \Lambda^{\geq i+2}X$ if $|\sigma| = m - 2 - i$.

We may thus suppose $z_\sigma \in \Lambda^{\geq m-1}X$ when $|\sigma| = 1$. Write $u - z_0 = \Phi + \Psi$, $\Phi \in \Lambda^{\leq m-1}X$, $\Psi \in \Lambda^{\geq m}X$. Using (7.5) we deduce that (since $dz_\sigma \in \Lambda^{\geq m+1}X$)

$$x_i \wedge \Phi = 0, \qquad 1 \leq i \leq m,$$

whence $\Phi = 0$. It follows that

$$x_1 \wedge \ldots \wedge x_m \wedge x = du = d\Psi \in \Lambda^{\geq m+2}X,$$

again because (for degree reasons) $d_2\Psi = 0$. This is the desired contradiction.

8. Proof of Theorem 2.6. — The main step in the proof of the theorem is the

Proposition **(8.1).** — *Assume* $\mathrm{cat}_0(S) < \infty$ *and* $\alpha \in \pi_{2n}(\Omega S) \otimes \mathbf{Q}$, $\beta \in \pi_{2\ell-1}(\Omega S) \otimes \mathbf{Q}$ *satisfy*

$$(\mathrm{ad}\,\alpha)^p \beta \neq 0, \text{ all } p.$$

Then ad α, *restricted to* $\pi_{\mathrm{even}}(\Omega S) \otimes \mathbf{Q}$, *is not a nilpotent transformation.*

Assuming the proposition for the moment, we deduce the theorem as follows. Let $L = \sum_k \pi_{2k}(\Omega S) \otimes \mathbf{Q}$, and let $L^{(s)}$ denote the s-th derived algebra of $L : L^{(s+1)} = [L^{(s)}, L^{(s)}]$. We show by induction on s that: For any N there is an integer $k \geq N$ such that

$$\textbf{(8.2)}_s \qquad \sum_{i=k}^{2k-2} \dim L_i^{(s)} \geq m.$$

This certainly establishes Theorem (2.6).

Now $(8.2)_0$ is true by Theorem (2.1). Suppose it holds for some s. Fix N and choose m linearly independent homogeneous elements

$$\alpha_1, \ldots, \alpha_m \in \sum_{i=k}^{2k-2} L_i^{(s)}, \qquad \text{some } k \geq N.$$

If the α_i generate an infinite dimensional Lie algebra there will, in particular, be an infinite sequence $i_1, i_2, i_3, \ldots$ such that

$$\gamma_\nu = [\alpha_{i_\nu}[\alpha_{i_{\nu-1}}[\ldots[\alpha_{i_2}, \alpha_{i_1}] \ldots] \neq 0, \text{ each } \nu.$$

For $\nu \geq 2$ each γ_ν is in $L^{(s+1)}$. For sufficiently large r we have

$$\deg \gamma_{r+m} \leq 2 \deg \gamma_{r+1} - 2$$

and so $(8.2)_{s+1}$ follows in this case.

On the other hand, if the α_i generate a finite dimensional subalgebra, Theorem (2.2) implies that for some β and some i

$$(\mathrm{ad}\,\alpha_i)^p \beta \neq 0, \text{ all } p.$$

We may clearly take $\deg \beta > N$. If $\deg \beta$ is even then $[\alpha_i, \beta] \in L^{(s)}$ and so $(\operatorname{ad} \alpha_i)^p \beta \in L^{(s+1)}$, $p \geq 2$. By choosing p large we may arrange that

$$\deg(\operatorname{ad} \alpha_i)^{p+m}\beta \leq 2 \deg(\operatorname{ad} \alpha_i)^{p+1}\beta - 2,$$

and so establish $(8.2)_{s+1}$.

Finally, suppose $\deg \beta$ is odd. By Proposition (8.1), there are then elements $\gamma \in L$ of arbitrarily high degree, such that $(\operatorname{ad} \alpha_i)^m \gamma \neq 0$. Formula $(8.2)_{s+1}$ follows, and with it Theorem (2.6).

It remains to give the

Proof of (8.1). — We suppose $\operatorname{ad} \alpha$ nilpotent of order k in $\pi_{\text{even}}(\Omega S) \otimes \mathbf{Q}$ and deduce a contradiction. By replacing β by a suitable $(\operatorname{ad} \alpha)^p \beta$ we may arrange that $\ell > knm(m+1)!$. Denote the minimal model of S by $(\Lambda X, D)$. By projecting onto a quotient model we may, without affecting our hypotheses, arrange that $(\Lambda X, D)$ has the form

$$\Lambda X = \Lambda(x) \otimes \Lambda Y, \qquad Y = y_0 \oplus Y^{>2\ell},$$

where x is dual to α, and y_0 is dual to β. Thus for $\Phi \in \Lambda Y$ we have

$$D\Phi = d\Phi + x \wedge \theta\Phi;$$

here d is the induced differential in the quotient model ΛY. In particular $\theta = \theta_1 + \theta_2 + \dots$ is the sum of derivations each of degree $-2n$ with $\theta_i : Y \to \Lambda^i Y$. By formula (3.4), θ_1 is dual to $\operatorname{ad} \alpha$.

The hypothesis $(\operatorname{ad} \alpha)^p \beta \neq 0$, all p, can thus be restated as: $y_0 \in \bigcap_p \operatorname{Im}(\theta_1)^p$. If $\theta_1^p y = y_0$ then

$$\theta^p y - \theta_1^p y \in (\Lambda^{\geq 2} Y)^{2\ell} = 0$$

and so also $y_0 \in \bigcap_p (\operatorname{Im} \theta^p)$.

Now for any q

$$\dots \geq \dim[(\Lambda Y)^q \cap \operatorname{Im}(\theta^p)] \geq \dim[(\Lambda Y)^q \cap \operatorname{Im}(\theta^{p+1})] \geq \dots,$$

and, since $\dim(\Lambda Y)^q < \infty$, it follows that for some p_q;

$$(\Lambda Y)^q \cap \operatorname{Im} \theta^{p_q} = (\Lambda Y)^q \cap (\bigcap_p \operatorname{Im} \theta^p).$$

This implies that $\theta : \Lambda Y^{q+2n} \cap (\bigcap_p \operatorname{Im} \theta^p) \to (\Lambda Y)^q \cap (\bigcap_p \operatorname{Im} \theta^p)$ is surjective. In particular we can find an infinite sequence $y_0, y_1, \dots$ such that $\theta y_i = y_{i-1}$. Evidently none of the y_i are in $\Lambda^{\geq 2} Y$ and so (by a new choice of Y) we may suppose all the y_i belong to Y. Note that $\deg y_i = 2(\ell + in)$.

Denote by $U \subset Y$ the span of the y_i. For any $\Phi \in \Lambda Y$

$$\Phi \wedge y_i = \theta\Big(\sum_{k=0}^{\infty} (-1)^k \theta^k \Phi \wedge y_{i+k+1}\Big). \tag{8.3}$$

It follows that $\theta : U.\Lambda Y \to U.\Lambda Y$ is surjective. Thus a straight-forward argument by increasing induction on degree permits us to write $\Lambda Y = \Lambda U \otimes \Lambda Z$, with ΛZ stable under θ. We may suppose (by a new choice of Y) that $Y = U \oplus Z$.

Recall that $\theta = \Sigma\theta_i$ with $\theta_i(Y) \subset \Lambda^i Y$. Define linear maps $\varphi_i : Z \to \Lambda^i U$, $i \geq 2$, of degree 1 by the equations

$$dz - \sum_i \varphi_i z \in Z.\Lambda Y, \qquad z \in Z.$$

We shall show that

$$\varphi_i\theta_1 = \theta_1\varphi_i, \qquad i \geq 2. \tag{8.4}$$

Indeed, put $\varphi = \Sigma\varphi_i$. Then since $\theta_j z \in \Lambda^{\geq 2}Z$, $j \geq 2$, we have $d\theta_j z \in Z.\Lambda Y$, $j \geq 2$ and so

$$d\theta z - \psi\theta_1 z \in Z.\Lambda Y.$$

On the other hand, θ preserves the ideal $Z.\Lambda Y$ and so

$$\theta dz - \theta\varphi z \in Z.\Lambda Y.$$

Since $\theta d = d\theta$ and since θ reduces to θ_1 in ΛU, (8.4) follows.

The hypothesis that ad α is nilpotent of order k in $\pi_{\text{even}}(\Omega S) \otimes \mathbf{Q}$ (adopted at the start of the proof) implies that $\theta_1^k = 0$ in Z^{odd}. In view of (8.4) we may conclude that

$$\varphi_i(Z^{\text{odd}}) \subset \ker \theta_1^k, \qquad i \geq 2. \tag{8.5}$$

Put $s = km(m+1)!$, and denote by $\rho : \Lambda U \to \Lambda(y_0, \ldots, y_s)$ the projection whose kernel is generated by the y_j with $j > s$. Then

$$\begin{aligned} d[(\Lambda Y)^{\text{odd}}] &\cap \Lambda^{m+1}(y_0, \ldots, y_s) \\ &\subset [\rho\varphi(Z^{\text{odd}}).\Lambda U] \cap \Lambda^{m+1}(y_0, \ldots, y_s) \\ &\subset \{\sum_i [\rho\varphi_i(Z^{\text{odd}}).\Lambda U]\} \cap \Lambda^{m+1}(y_0, \ldots, y_s) \\ &\subset \sum_{i=2}^{m+1} \rho\varphi_i(Z^{\text{odd}}).\Lambda^{m+1-i}(y_0, \ldots, y_s). \end{aligned}$$

In Lemma (8.6) below we use (8.5) to show that $\dim \rho\varphi_i(Z^{\text{odd}}) \leq k(s+1)^{i-1}$, $i \geq 2$. Thus the inclusions above give

$$\dim\{d((\Lambda Y)^{\text{odd}}) \cap \Lambda^{m+1}(y_0, \ldots, y_s)\} \leq km(s+1)^m.$$

On the other hand, by hypothesis $\ell > knm(m+1)! = ns$, and so

$$\deg y_s = \deg y_0 + 2ns \leq \deg y_0 + 2(\ell - 1) = 4\ell - 2 = 2\deg y_0 - 2.$$

Since $(\Lambda^{\geq 2}Y)^{\geq 4\ell - 2} = 0$, we find $dy_j = 0$, $0 \leq j \leq s$. Because the quotient model $(\Lambda Y, d)$ also satisfies $\text{cat}_0(\Lambda Y, d) \leq m$, we have

$$\Lambda^{m+1}(y_0, \ldots, y_s) \subset d((\Lambda Y)^{\text{odd}})$$

and hence

$$\binom{m+s+1}{m+1} = \dim \Lambda^{m+1}(y_0, \ldots, y_s) \leq km(s+1)^m.$$

This implies that

$$\frac{(s+1)^{m+1}}{(m+1)!} \leq km(s+1)^m,$$

whence $km(m+1)! + 1 = (s+1) \leq km(m+1)!$, an obvious contradiction. The proof is complete.

Lemma **(8.6)**. — *With the notation and hypotheses of Proposition* (8.1) *and its proof*:

$$\dim \rho\varphi_i(Z^{\mathrm{odd}}) \leq k(s+1)^{i-1}, \quad i \geq 2.$$

Proof. — Fix i and denote by γ the restriction of θ_1 to $\Lambda^i U$. In view of (8.5) it suffices to prove that

$$\dim \rho(\ker \gamma^k) \leq k(s+1)^{i-1}, \quad i \geq 2.$$

For this, linearly order the monomials in ΛU, putting $y_0^{k_0} y_1^{k_1} \ldots < y_0^{\ell_0} y_1^{\ell_1} \ldots$ if for some $\nu : k_j = \ell_j$ $(j < \nu)$ and $k_\nu < \ell_\nu$.

If g is a linear combination of monomials, each greater than a given monomial f, we write $g > f$.

Next, if $f = y_0^{k_0} y_1^{k_1} \ldots$ is a monomial we write

$$\partial_\nu f = \begin{cases} \dfrac{1}{k_\nu} \dfrac{\partial f}{\partial y_\nu} & \text{if } k_\nu > 0 \\ 0 & \text{if } k_\nu = 0. \end{cases}$$

Thus if $k_\nu > 0$, $(\partial_\nu f) y_\nu = f$.

Consider the set $\mathscr{S}$ of monomials $f \in \Lambda^i U$ of the form $f = y_0^{k_0} \cdot y_0^{k_1} \cdot \ldots \cdot y_t^{k_t}$ with $k_\nu \geq 0$ and $k_t \geq 2$. For $f \in \mathscr{S}$ put

$$\Omega_f = -\sum_{j=0}^{\infty} (-\theta_1)^j (\partial_t \gamma f) \cdot y_{t+j+1}.$$

Then

(8.7) $\qquad f < \Omega_f,$

and

(8.8) $\qquad \Omega_f \in$ ideal generated by the y_ν, $\quad \nu > t.$

Recall, moreover, that $\theta = \theta_1$ in ΛU. Thus by (8.3)

(8.9) $\qquad f + \Omega_f \in \ker \gamma.$

Because of (8.7) the elements $f + \Omega_f$ are linearly independent ($f \in \mathscr{S}$). We show now they are a basis for $\ker \gamma$. Indeed if $\Phi = g + \Psi \in \ker \gamma$ with g a monomial and $g < \Psi$, we shall show that $g \in \mathscr{S}$. In fact write $g = y_0^{k_0} \cdot \ldots \cdot y_t^{k_t}$ with $k_t \geq 1$. If $k_t = 1$ it is easy to see that

$$\gamma\Psi > y_0^{k_0} \cdot \ldots \cdot y_{t-1}^{k_{t-1}+1}$$

while
$$\gamma g = y_0^{k_0} \cdot \ldots \cdot y_{t-1}^{k_{t-1}+1} + \Psi'$$

with $\Psi' > y_0^{k_0} \cdot \ldots \cdot y_{t-1}^{k_{t-1}+1}$. These two facts would contradict $\gamma(g + \Psi) = 0$ and so we conclude $k_t \geq 2$; *i.e.* $g \in \mathscr{S}$.

Thus $\Phi - (g + \Omega_g) \in \ker \gamma$ can be written as a linear combination of monomials $> g$; proceeding in this way and using the fact that $\ker \gamma$ is finite dimensional in each degree, we see that $\ker \gamma$ is spanned by the vectors $f + \Omega_f$, $f \in \mathscr{S}$.

It follows from (8.3) that γ is surjective. Thus for $f \in \mathscr{S}$ we can find $f_0, f_1, \ldots \in \Lambda^i U$ such that

$$f_0 = f + \Omega_f \quad \text{and} \quad \gamma f_j = f_{j-1}, \quad j \geq 1.$$

Clearly the elements f_j, $0 \leq j < k$, $f \in \mathscr{S}$ form a basis of $\ker \gamma^k$.

Now if f is in the ideal generated by the y_ν, $\nu > s$, then so is $f_0 = f + \Omega_f$, by (8.8). In this case by (8.3) we can choose $f_1, f_2, \ldots$ so that they too are in the ideal generated by the y_ν, $\nu > s$. Thus for these f we have $\rho f_j = 0$, $j \geq 0$.

It follows that $\dim \rho(\ker \gamma^k) \leq kq$, where q is the number of monomials $f \in \mathscr{S}$ such that $f \in \Lambda^i(y_0, \ldots, y_s)$. There is a one to one correspondence between such monomials and monomials $g \in \Lambda^{i-1}(y_0, \ldots, y_s)$ given by (if $g = y_0^{\ell_0} \ldots y_t^{\ell_t}$, $t \geq 1$)

$$g \to y_0^{\ell_0} \cdot \ldots \cdot y_t^{\ell_t + 1}.$$

Thus $q = \dim \Lambda^{i-1}(y_0, \ldots, y_s)$ and so

$$\dim \rho(\ker \gamma^k) \leq k \dim \Lambda^{i-1}(y_0, \ldots, y_s) \leq k(s+1)^{i-1},$$

q.e.d.

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