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ON TOPOLOGICAL TITS BUILDINGS AND THEIR CLASSIFICATION

by KEITH BURNS [1] and RALF SPATZIER [2]

Abstract

We define topological Tits buildings. If a topological building Δ satisfies some technical conditions and is irreducible, compact, locally connected and satisfies the topological equivalent of the Moufang property, then it is a building canonically associated with a Lie group. If Δ satisfies all the other conditions and has rank $\geqslant 3$, it must be topologically Moufang.

INTRODUCTION

We introduce the notion of a *topological Tits building*. Roughly speaking, this is a Tits building Δ with a topology which makes the incidence relation closed. We will always assume that the building is spherical, i.e. the number of chambers in an apartment is finite. If the building is a projective space our definition agrees with the usual notion of a topological projective space. For technical convenience we will usually consider buildings where the topology is given by a metric.

We investigate topological buildings via their automorphism groups. To be precise, given a topological building Δ we let its *topological automorphism group* be the group of all homeomorphic (combinatorial) automorphisms of Δ. A basic tool for this paper is the

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Theorem **(2.1)**. — *If* Δ *is an irreducible compact metric building of rank at least* 2 *then its topological automorphism group is locally compact in the compact open topology.*

This generalises a theorem of Salzmann for projective planes [Sa 2].

If G is a connected semisimple Lie group of the noncompact type then the set of all parabolic subgroups of G can be given a building structure (cf. 1.2). The topology inherited from G makes this a topological building. We call it the *classical (topological) building* $\Delta(G)$ attached to G. The component of the identity in the topological automorphism group of $\Delta(G)$ is G.

We characterize the classical buildings by intrinsic properties. Most important is the topological analogue of the *Moufang property*: it assures that Δ has sufficiently many topological automorphisms (Definition 3.1). Such a condition is necessary since there are topological projective planes with very nice topological properties but few or even no topological automorphisms [Sa 1, Introduction].

Main Theorem. — *Let* Δ *be an infinite, irreducible, locally connected, compact, metric, topologically Moufang building of rank at least* 2. *Then* Δ *is classical. More precisely, let* G *be the topological automorphism group of* Δ *and* G^0 *its connected component of the identity. Then* G^0 *is a simple noncompact real Lie group without center and* Δ *is isomorphic to* $\Delta(G^0)$ *as a topological building.*

In the combinatorial theory, buildings of rank 2 are quite different from higher rank buildings because irreducible buildings of rank at least 3 are automatically Moufang. The same is true in the topological theory:

Theorem **(5.1, 5.2)**. — *An irreducible compact metric building of rank at least* 3 *is topologically Moufang. Hence, if* Δ *is also locally connected and infinite, then* Δ *is classical.*

These theorems generalise results of Kolmogorov and Salzmann. Using coordinate methods, Kolomogorov showed in [K] that a connected compact projective n-space, for $n \geqslant 3$, is a projective space over the real or complex numbers or quaternions. Salzmann [Sa 2] proves that a flag transitive compact connected topological projective plane is a plane over the real, complex, quaternion or Cayley numbers. There is an analogous conjecture describing combinatorial Moufang buildings [T2] which has been proved in some of the most difficult special cases. The version of the Moufang property considered in this conjecture is stronger than the direct combinatorial analogue of our topological Moufang property. Tits has pointed out to us that the results of [T3] show that there is no hope of classifying combinatorial Tits buildings satisfying this weaker Moufang condition.

Let us describe one application of the above theory which also was our main motivation to do this work:

Consider a complete Riemannian manifold M of bounded nonpositive sectional curvature and finite volume. For any geodesic c, let rank c be the dimension of the space of parallel Jacobi fields along c. Let rank M be the minimum of the ranks of all geodesics. In [BS] we attach to M a topological building $\Delta(M)$. It is constructed using the Weyl chambers of M introduced in [BBS]. Its rank equals rank M. Also $\Delta(M)$ is compact, locally connected and topologically Moufang. Moreover $\Delta(M)$ is irreducible if and only if M is locally irreducible as a Riemannian manifold. Combining this with our Main Theorem above and a result of Gromov [BGS] we obtain the

Theorem. — Let M *be a complete Riemannian manifold of bounded nonpositive sectional curvature and finite volume. If* M *is irreducible and has rank at least* 2, *then* M *is locally symmetric.*

This theorem was proved by W. Ballmann in [B] using a completely different argument. For more details of our argument we refer to [BS].

We now give a brief outline of the present paper. In Section 1 we discuss the basic notions and elementary properties of topological buildings. For the combinatorial theory we refer to the first three chapters of [T1].

In Section 2 we consider irreducible compact metric buildings of rank at least 2. We prove that their automorphism groups are locally compact. First we consider rank 2 buildings, since they are much simpler combinatorially than those of higher rank: the apartments are just partitions of a circle into intervals. The general claim follows by considering the stars of faces of codimension 2, as these are buildings of rank 2.

In Sections 3 and 4 we prove the Main Theorem in the following three steps.

Step 1 (Section 3): The topological automorphism group G is a Lie group. We show this by applying Gleason and Yamabe's theorem on small subgroups.

Step 2 (Section 3): The connected component of the identity G^0 of G is a non-compact simple Lie group and the stabiliser of a chamber in G^0 is a parabolic subgroup P.

In both these steps we analyse the orbit of a normal subgroup N of G^0. To illustrate the basic idea, suppose N is normal in G. Let C be a chamber and G_C its stabiliser in G. For simplicity, suppose further that some chamber D opposite C is in N.C. The Moufang condition guarantees that $G_C.D$ contains all chambers opposite C. As N is normal in G, $N.C = N.D \supset G_C.D$. It follows that N.C contains all chambers. This shows in particular that G does not contain any small normal subgroups, since their orbits would also be small. However, we need to work with G^0 rather than G and have to elaborate this basic argument considerably.

Step 3 (Section 4): We analyse the BN-pair given by the stabilizers in G^0 of a chamber $C \in \Delta$ and an apartment containing C. In the building $\widetilde{\Delta}$ of this BN-pair we realize Δ as the subcomplex of all faces of certain prescribed types. In an irreducible building such a subcomplex is a building of rank 2 or more only if $\tilde{\Delta} = \Delta$. This approach to Step 3, which supersedes an earlier more complicated version, was suggested to us by J. Tits.

We conclude the paper in Section 5 by showing that irreducible topological buildings of rank at least 3 are topologically Moufang. This result was inspired by its combinatorial equivalent [T3, 3.5].

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0. Preliminaries

This section introduces some notations and combinatorial properties of Tits buildings that will be used later. Throughout this paper, k is the rank and d the diameter of a (spherical) Tits building Δ. For $0 \leqslant r \leqslant k$, let Δ_r be the set of elements of Δ with r vertices. Thus Δ_1 and Δ_k are the sets Vert Δ and Cham Δ of vertices and chambers respectively of Δ. Often we will call elements of Δ_{k-1} *hyperfaces* of Δ. Recall that an apartment Σ of Δ is a Coxeter complex [T1, 3.7]. We will usually call a root in Σ a *half-apartment* and the wall of a root an *equator* of Σ (cf. [T1, 1.12]).

We define the length of a gallery and the distance dist(A, B) between two elements A and B of Δ as in [T1, 1.3]. Note that dist(A, B) is one less than the number of chambers in a minimal gallery from A to B.

0.1. *Definition.* — If $A \in \Delta$, $\mathrm{Opp}(A) = \{ B \in \Delta : B \text{ is opposite } A \}$.

It is not difficult to establish the following criterion for two hyperfaces to be opposite.

0.2. *Lemma.* — *Let* $A, A' \in \Delta_{k-1}$. *Then* $\mathrm{dist}(A, A') \leqslant d - 1$ *with equality if and only if* A *and* A' *are opposite.* □

Recall the notion of type [T1, 2.5].

0.3. *Definition.* — The *type* of a gallery $(C_0, C_1, \ldots, C_m)$ is the sequence

$$(\mathrm{typ}(C_0 \cap C_1), \mathrm{typ}(C_1 \cap C_2), \ldots, \mathrm{typ}(C_{m-1} \cap C_m)).$$

It is easy to prove the following.

0.4. *Lemma.* — *Suppose* $\mathscr{G} = (C_0, C_1, \ldots, C_m)$ *is a minimal gallery and* $\mathscr{G}' = (C'_0, C'_1, \ldots, C'_m)$ *is a gallery with the same type as* $\mathscr{G}$. *Then* $\mathscr{G}'$ *is also minimal.* □

Finally recall [T1, 3.19] that if $A, B \in \Delta$ there is a unique maximal element of the full convex hull [T1, 1.5, 3.18] of A and B containing A. This element is called the *projection of* B *onto* A and is denoted by $\text{proj}_A B$.

Rank 2

In Section 2, we will make extensive use of some special combinatorial properties of rank 2 buildings. For the rest of this section, we assume that Δ has rank 2 and diameter d. It follows from Lemma 0.4 that a gallery with d or fewer chambers is minimal if and only if it does not stammer.

0.5. *Observation.* — *If $x, y \in \text{Vert}\,\Delta$ are distinct and non-opposite, there is a unique minimal gallery with initial vertex x and final vertex y, which we denote by $[x, y]$.* □

0.6. *Corollary.* — *A closed gallery $\mathcal{G}$ (i.e. a gallery with the same initial and final vertices) that has fewer than $2d$ chambers must stammer. Each chamber of $\mathcal{G}$ must occur at least twice in $\mathcal{G}$.* □

0.7. *Definition.* — Two vertices x, y of Δ are *almost opposite* if $\text{dist}(x, y) = d - 2$.

0.8. *Lemma.* — *If $x, y \in \text{Vert}\,\Delta$ have the same type, there is $z \in \text{Vert}\,\Delta$ almost opposite both x and y.*

Proof. — By [T1, 3.30] there is a vertex z' opposite both x and y. Let z be the other vertex of a chamber containing z'. □

0.9. *Lemma.* — *If x and y are distinct vertices of Δ with the same type, they are joined by a non-stammering gallery with $2d - 2$ chambers.*

Proof. — Note firstly that any gallery joining x and y has an even number of chambers. Since x and y lie in an apartment, they are joined by a non-stammering gallery with at least d and at most $2d - 2$ chambers. Thus it suffices to show that if x and y are joined by a non-stammering gallery $\mathcal{G}$ with ℓ chambers, where $d \leqslant \ell \leqslant 2d - 4$, then they are also joined by a non-stammering gallery with $\ell + 2$ chambers. Let z be the first vertex of $\mathcal{G}$ that is almost opposite x. Choose chambers $A \in \text{Star}\, x$ and $B \in \text{Star}\, z$ that are not in $\mathcal{G}$. Let a (resp. b) be the vertex of A (resp. B) that is not x (resp. z). Then a and b are almost opposite and $(A, [a, b], B, [z, y])$ is our desired gallery. □

Finally it is convenient to modify Tits' definition of distance in the rank 2 case.

0.10. *Definition.* — If $x, y \in \text{Vert}\,\Delta$, $D(x, y) = \text{dist}(x, y) + 1$ if $x \neq y$ and $D(x, y) = 0$ if $x = y$.

Thus $D(x, y)$ is the number of chambers in a gallery stretched between x and y.

1. Topological Buildings

Let Δ be a Tits building of rank k. Fix an ordering of the k types of vertex in Δ. Henceforth we identify Δ_r with a subset of $(\operatorname{Vert}\Delta)^r$ by identifying $A \in \Delta_r$ with the sequence $(x_1, \ldots, x_r)$ such that $\{x_1, \ldots, x_r\}$ is the set of vertices of A and $\operatorname{typ} x_1 < \ldots < \operatorname{typ} x_r$.

1.1. *Definition.* — A *topological Tits building* is a Tits building Δ with a Hausdorff topology on the set $\operatorname{Vert}\Delta = \Delta_1$ of all vertices such that Δ_r is closed in the product topology on $(\operatorname{Vert}\Delta)^r$ for $0 \leqslant r \leqslant \operatorname{rank}\Delta$. We give Δ_r the topology induced from $(\operatorname{Vert}\Delta)^r$. We say Δ is *compact*, *connected*, *locally connected*, *finite*, or *infinite* if $\operatorname{Cham}\Delta = \Delta_{\operatorname{rank}\Delta}$ has the appropriate property. The *topological automorphism group* of Δ is the group $\operatorname{Auttop}(\Delta)$ formed by all (combinatorial) automorphisms of Δ whose restrictions to each Δ_r, $0 \leqslant r \leqslant \operatorname{rank}\Delta$, are homeomorphisms.

Topological buildings arise naturally from Lie groups.

1.2. *Example.* — We define the *classical buildings*. Let G be a connected real semisimple Lie group and let $\Delta(G)$ denote the set of all parabolic subgroups of G. If A is a maximal $\mathbf{R}$-split torus of G, let Σ_A denote the set of all parabolic subgroups of G containing A. We call Σ_A an *apartment* and denote the collection of all apartments Σ_A by $\mathscr{A}$. If $C_1, C_2 \in \Delta(G)$, call C_1 a *face* of C_2 and write $C_1 \leqslant C_2$ if $C_2 \supset C_1$. This partial order on $\Delta(G)$, together with $\mathscr{A}$, makes $\Delta(G)$ into a Tits building [T1, 5.2].

The chambers of $\Delta(G)$ are the minimal parabolic subgroups of G. The group G acts on $\Delta(G)$ by conjugation. Since any two minimal parabolic subgroups are conjugate, G is transitive on $\operatorname{Cham}\Delta(G)$. Therefore G is also transitive on the set of vertices of a given type, and thus induces topologies on these sets. We topologize the set of vertices of $\Delta(G)$ by the sum of these topologies. Clearly $\Delta(G)$ is compact and locally connected. Moreover G is the component of the identity in $\operatorname{Auttop}(\Delta(G))$.

We mention two other simple examples.

1.3. *Example.* — A finite Tits building with the discrete topology is a compact topological Tits building.

1.4. *Example.* — The star of an element of a compact topological Tits building is also a compact topological Tits building.

Henceforth in this paper, Δ will be a compact topological Tits building with rank k and diameter d. We begin by observing some topological properties of the combinatorial structure. Note firstly that each of the spaces Δ_r is compact. For if $\{A_\alpha\} \subseteq \Delta_r$ is a net, we can choose chambers $C_\alpha \supseteq A_\alpha$; and since $\{C_\alpha\}$ must accumulate, so does $\{A_\alpha\}$. It is clear that the function dist is lower semicontinuous on each of the spaces Δ_r^2, $0 \leqslant r \leqslant k$. Since $C, C' \in \operatorname{Cham}\Delta$ are opposite if and only if $\operatorname{dist}(C, C') = d$ [T1, 3.23], we see that $\{(C, C') \in (\text{Cham }\Delta)^2 : C$ is opposite $C'\}$ is open. The corresponding result for hyperfaces follows in the same way from Lemma 0.2. It follows easily that type is locally constant in Δ_{k-1}.

1.5. *Proposition. — Type is locally constant in each* Δ_r, $0 \leqslant r \leqslant k$.

Proof. — This is trivial if $r = 0$ or k, so assume $1 \leqslant r \leqslant k-1$. If the lemma is false, there are $A \in \Delta_r$ and a net $A_\alpha \to A$ such that typ A_α is constant and is not typ A. For each α choose a hyperface $B_\alpha \in$ Star A so that typ B_α is constant and hyperfaces of this type do not contain faces with the same type as A. Since Δ_r is compact, $\{B_\alpha\}$ subconverges to a hyperface $B \in$ Star A. By the remark preceding the lemma, B has the same type as each B_α, and so cannot contain A, which is absurd. □

1.6. *Lemma. — Suppose* $A, A' \in \Delta$ *are opposite. Then* Opp(A′) *is a neighbourhood of* A.

Proof. — If not, there is a net $A_\alpha \to A$ such that no A_α is opposite A′. Choose chambers $C_\alpha \in$ Star A_α. By passage to a subnet, we can assume that $C_\alpha \to C \in$ Star A. Choose $C' \in$ Cham Star A′ opposite C. Then C_α is opposite C′ for all large enough α. Also, by Proposition 1.5, A_α has the opposite type to A′ for all large enough α. It follows that, for large enough α, A_α is the face of C_α opposite A′, which is absurd. □

If T is the type of an element of Δ, there is a canonical map

$$\pi_T : \text{Cham }\Delta \to \{A \in \Delta : \text{typ } A = T\}.$$

1.7. *Proposition. — The map* π_T *is surjective, continuous and open.*

Proof. — Surjectivity is clear, since every element of Δ is contained in a chamber. Continuity follows from Proposition 1.5, since a chamber contains a unique face of type T. To prove openness, we show that if $A \in \Delta$, $C \in$ Cham Star A_α and $\{A_\alpha\}$ is a net converging to A, then we can find $C_\alpha \in$ Cham Star A_α such that $C_\alpha \to C$. Choose A′ opposite A and let $C' = \text{proj}_{A'} C$. By Lemma 1.6, we may assume that A_α is opposite A′ for all α. Choose $C_\alpha = \text{proj}_{A_\alpha} C'$. Clearly $\text{dist}(C_\alpha, C') = \text{dist}(C, C')$ for all α. That $C_\alpha \to C$ follows since C (resp. C_α) is the unique chamber of Star A (resp. Star A_α) closest to C′. □

1.8. *Corollary. — If* Δ *is locally connected, so is each* Δ_r, $0 \leqslant r \leqslant k$. □

Note that Δ_1 is not connected when rank $\Delta \geqslant 2$, even if Δ is connected. This is clear from Proposition 1.5.

1.9. *Proposition. — The set* $\{(A, A') \in \Delta_r^2 : A$ is opposite $A'\}$ *is open for* $0 \leqslant r \leqslant k$.

Proof. — Suppose $A_\alpha \to A$ and $A'_\alpha \to A'$ with A and A′ opposite each other. Choose chambers $C \in$ Star A and $C' \in$ Star A′ which are opposite. Use Proposition 1.7

to choose chambers $C_\alpha \in \operatorname{Star} A_\alpha$ and $C'_\alpha \in \operatorname{Star} A'_\alpha$ such that $C_\alpha \to C$ and $C'_\alpha \to C'$. Then, if α is large enough, C_α is opposite C'_α and $\operatorname{typ} A_\alpha = \operatorname{typ} A$ is opposite $\operatorname{typ} A'_\alpha = \operatorname{typ} A'$. Hence A_α is opposite A'_α for all large enough α. □

1.10. *Proposition.* — *Suppose the chamber* C *contains a face opposite* $A \in \Delta$. *If* $A_\alpha \to A$ *and* $C_\alpha \to C$, *then* $\operatorname{proj}_{A_\alpha} C_\alpha \to \operatorname{proj}_A C$.

Proof. — By Proposition 1.9, we can assume that each C_α contains a face opposite A_α. Hence $\operatorname{dist}(C_\alpha, \operatorname{proj}_{A_\alpha} C_\alpha) = \operatorname{dist}(C, \operatorname{proj}_A C)$ for all α. The proposition follows, since $\operatorname{proj}_A C$ is the unique chamber of Star A closest to C. □

1.11. *Corollary.* — *If* A *is opposite* A′, *then* $\operatorname{proj}_A : \operatorname{Star} A' \to \operatorname{Star} A$ *is a homeomorphism.* □

Metric Buildings

Henceforth in this paper, we will restrict attention to *metric Tits buildings*, i.e. buildings in which the topology on Vert Δ, and hence on each Δ_r, is given by a metric, denoted by ρ. The classical buildings of Example 1.2 are metric buildings.

Rank 2

The remainder of this section contains some special properties of (compact metric) buildings with rank 2 that are needed in Section 2. First we make two simple observations.

1.12. *Lemma.* — *Suppose* $x, y \in \operatorname{Vert} \Delta$ *are not opposite or identical. If* $x_n \to x$, $y_n \to y$ *and* $\operatorname{dist}(x_n, y_n) = \operatorname{dist}(x, y)$ *for all* n, *then* $[x_n, y_n] \to [x, y]$. □

1.13. *Lemma.* — *The set* $\{(x,y) \in (\operatorname{Vert} \Delta)^2 : x$ *is almost opposite* $y\}$ *is open.* □

1.14. *Lemma.* — *Suppose in addition that* Δ *is infinite and irreducible. Let* $x \in \operatorname{Vert} \Delta$. *Then no chamber of* Star x *is isolated in* Star x.

Proof. — We break the proof into four steps.

(1) *There is a chamber of* Δ *that is not isolated in the star of one of its vertices.*

Consider a fixed apartment Σ_0. Since every chamber in Δ is opposite some chamber of Σ_0 [T1, 4.2], we see that there is a chamber of Δ contained in infinitely many apartments. Hence there is a vertex x_0 of Δ whose star contains infinitely many vertices. Since Star x_0 is compact, it contains a non-isolated chamber. This proves (1).

Let S and T be the two types of vertex in Δ. Call a chamber S-*good* (T-*good*) if it is not isolated in the star of its vertex of type S (type T). Because of (1), we can assume that Δ contains an S-good chamber.

(2) *Let y be a vertex of type* T, *and suppose* Star y *contains a chamber* C *that is* S-*good. Then every chamber of* Star y *is* S-*good.*

Let $C' \in$ Cham Star y and let x' be the other vertex of C'. Choose z almost opposite y so $[y, z]$ does not contain C or C'. By Corollary 1.11, the chamber $D = \text{proj}_z\, C$ is not isolated in Star z. Similarly $C' = \text{proj}_{x'}\, D$ is S-good.

(3) *Let y and y' be vertices of type* T *that are joined by a gallery containing* 2 *chambers. Suppose every chamber of* Star y *is* S-*good. Then* Star y' *contains an* S-*good chamber.*

Suppose $[y, y'] = (D, D')$. Choose a vertex z such that $[D \cap D', z]$ contains $d - 2$ chambers and does not contain D or D' (note that $d - 2 > 0$, since Δ is irreducible). Choose $C \in \text{Star}\, y \backslash \{ D \}$ and $C' \in \text{Star}\, y' \backslash \{ D' \}$ and let x, x' be the other vertices of C, C'. Then z is opposite both x and x'. Since C is S-good, it follows from Corollary 1.11 that $E = \text{proj}_z\, C$ is not isolated in Star z. Similarly $C' = \text{proj}_{x'}\, E$ is S-good. This proves (3).

It follows from (2) and (3) that every chamber of Δ is S-good.

(4) *If Δ contains an* S-*good chamber, it also contains a* T-*good chamber.*

Let C be an S-good chamber with vertices x and y of types S and T respectively.

(i) Suppose $d = \text{diam}\, \Delta$ is odd. Then any vertex z opposite x has type T. It is clear from Corollary 1.9 that $\text{proj}_z\, C$ is T-good.

(ii) Suppose d is even. Note firstly that the stars of any two vertices of type T are homeomorphic. This is clear from Corollary 1.11, since there is always a vertex opposite both any two given vertices with the same type. Thus if Δ does not contain any T-good chamber, then the stars of all vertices of type T contain the same finite number of chambers. The map π_T of Proposition 1.7 is a covering, as can be seen from Corollary 1.11. Therefore by the compactness of Δ, $\inf\{ \rho(D, D') : D, D' \in \text{Cham}\, \Delta$ have a common face of type T and $D \neq D' \} > 0$. We now show that this is impossible.

Choose $C' \in \text{Cham Star}\, y \backslash \{ C \}$ and let x' be the other vertex of C'. Let $\mathcal{G}$ be a non-stammering gallery with $d - 2$ chambers that starts at y and does not contain C or C'. (Note again that $d - 2 > 0$, since Δ is irreducible.) The final vertex z of $\mathcal{G}$ is almost opposite both x and x'. Now suppose $C_n \to C$ in Cham Star x, and let y_n be the other vertex of C_n. By Proposition 1.7, there are galleries $\mathcal{G}_n$ with $d - 2$ chambers and initial vertex y_n such that $\mathcal{G}_n \to \mathcal{G}$. Let z_n be the final vertex of $\mathcal{G}_n$. Each z_n has type T and, by Lemma 1.13, z_n is almost opposite both x and x' for all large enough n. Let $D_n = \text{proj}_{z_n}\, x$ and $D'_n = \text{proj}_{z_n}\, x'$. Then $D_n \neq D'_n$ for any n, since their projections to y, namely C and C', are different. But $\{ D_n \}$ and $\{ D'_n \}$ both converge to the final chamber of $\mathcal{G}$ by Lemma 1.12.

This completes the proof of (4). Interchanging the roles of S and T in (2) and (3) proves that every chamber of Δ is T-good. □

1.15. *Lemma.* — *Let* Δ *be as in the previous lemma. For each* $m > 0$, *there is* $\delta_m > 0$ *such that if* $x \in \Delta_1$ *and* $C_1, \ldots, C_m \in \mathrm{Cham}\,\Delta$, *there is* $C \in \mathrm{Cham\,Star}\,x$ *with*

$$\min(\rho(C, C_1), \ldots, \rho(C, C_m)) > \delta_m.$$

Proof. — If not, there are $y \in \mathrm{Vert}\,\Delta$, $D^1, \ldots, D^m \in \mathrm{Cham\,Star}\,y$ and sequences $y_n \to y$ and $D^i_n \to D^i$, $1 \leqslant i \leqslant m$, with the following property: if $E_n \in \mathrm{Cham\,Star}\,y_n$ for each n, then

$$\min_{1 \leqslant i \leqslant m} (\rho(E_n, D^i_n)) \to 0$$

as $n \to \infty$. This is absurd. Indeed by the previous lemma, there is a chamber $E \in \mathrm{Cham\,Star}\,y \backslash \{D^1, \ldots, D^m\}$; and by Proposition 1.7 we can choose $E_n \in \mathrm{Cham\,Star}\,y_n$ such that $E_n \to E$. □

2. Local Compactness of the Automorphism Group

This section contains the proof of

2.1. *Theorem.* — *Let* Δ *be a compact irreducible metric Tits building with rank at least* 2. *Then* $G = \mathrm{Auttop}(\Delta)$ *is locally compact in the compact open topology.*

2.2. *Remark.* — If Δ has rank 1, G is the group of homeomorphisms of Vert Δ. This is not locally compact in general.

To prove the theorem we show that $G_\varepsilon = \{ \varphi \in G : \rho(A, \varphi A) \leqslant \varepsilon \text{ for all } A \in \Delta \}$ is compact for any small enough $\varepsilon > 0$. This is trivial when Δ is finite, so we assume Δ is infinite.

We consider first the case when Δ has rank 2. Let $d = \mathrm{diam}\,\Delta$. We will assume (by virtue of Lemma 1.15) that ε is so small that if $x \in \mathrm{Vert}\,\Delta$ and $C_1, \ldots, C_{100d} \in \mathrm{Cham}\,\Delta$, then there is $C \in \mathrm{Cham\,Star}\,x$ with $\rho(C, C_i) > 3\varepsilon$ for $1 \leqslant i \leqslant 100\,d$. This allows us to construct (one chamber at a time) a gallery of any reasonable length, starting from any given vertex, whose chambers are mutually 3ε-separated.

By Arzela-Ascoli it is enough to prove that G_ε is an equicontinuous family of maps. This will follow if G_ε is equicontinuous on Vert Δ. If that is not the case, there will be sequences of vertices $\{ x_n \}$ and $\{ y_n \}$ converging to a common limit and a sequence $\{ \varphi_n \} \subseteq G_\varepsilon$ such that $p_n = \varphi_n x_n$ and $q_n = \varphi_n y_n$ converge to p and q respectively with $p \neq q$. Since $\{ \varphi_n^{-1} \} \subseteq G_\varepsilon$ also, we see that it suffices to prove

2.3. *Assertion.* — *Suppose* $\{ p_n \}, \{ q_n \} \subseteq \mathrm{Vert}\,\Delta$ *and* $p_n \to p$, $q_n \to q$ *with* $p \neq q$. *Then* $\{ \psi_n p_n \}$ *and* $\{ \psi_n q_n \}$ *do not have a common accumulation point for any* $\{ \psi_n \} \subseteq G_\varepsilon$.

We first reduce this assertion to the case where $D(p_n, q_n) = D(p, q) = 2$ for all n (see Definition 0.10). Suppose that $\{ \psi_n p_n \}$ and $\{ \psi_n q_n \}$ have a common accumulation point. Since type is locally constant, $\mathrm{typ}\, \psi_n p_n = \mathrm{typ}\, \psi_n q_n$ and hence $\mathrm{typ}\, p_n = \mathrm{typ}\, q_n$

for infinitely many n. Hence $\operatorname{typ} p = \operatorname{typ} q$. By Lemma 0.9, p and q are joined by a non-stammering gallery $\mathscr{G}$ with $2d-2$ chambers. Let y be the middle vertex of $\mathscr{G}$. Since p and q are both almost opposite y, we see from Lemma 1.13 that, for all large enough n, p_n and q_n are joined by a gallery $\mathscr{G}_n$ with $2d-2$ chambers that passes through y. Moreover $\mathscr{G}_n \to \mathscr{G}$.

Since $\{\psi_n p_n\}$ and $\{\psi_n q_n\}$ have a common accumulation point, there is a subsequence of $\{\psi_n \mathscr{G}_n\}$ that converges to a closed gallery $\mathscr{H}$ with $2d-2$ chambers. By Corollary 0.6, $\mathscr{H}$ must contain two adjacent chambers that are identical. Hence we can choose adjacent chambers C_n and D_n of $\mathscr{G}_n$ such that $\{\psi_n C_n\}$ and $\{\psi_n D_n\}$ accumulate to the same chamber. We can then pass to a subsequence so that $\{\psi_{n_k} C_{n_k}\}$ and $\{\psi_{n_k} D_{n_k}\}$ have a common accumulation point and $\{C_{n_k}\}$ and $\{D_{n_k}\}$ converge to adjacent chambers C and D of $\mathscr{G}$. Since $\mathscr{G}$ does not stammer, $C \neq D$. We see that Assertion 2.3 will follow from

2.4. *Assertion.* — *If $p_n \to p$, $q_n \to q$ and $D(p_n, q_n) = D(p, q) = 2$ for all n, then $\{\psi_n p_n\}$ and $\{\psi_n q_n\}$ do not have a common accumulation point for any sequence $\{\psi_n\} \subseteq G_\varepsilon$.*

The proof of this assertion is based on

2.5. *Lemma.* — *Suppose $a_n \to a$, $b_n \to b$ and $D(a_n, b_n) = D(a, b) = 2$ for all n. Then there is a neighborhood U of a in Vert Δ such that if $\{\varphi_n\} \subseteq G_\varepsilon$ and $\{\varphi_n a_n\}$ and $\{\varphi_n b_n\}$ converge to a common limit x, then $\varphi_n u_n \to x$ for every sequence $\{u_n\} \subseteq U$.*

Proof of Assertion 2.4. — We will show below that if the assertion is false, then $\{\psi_n \mid \operatorname{Vert}\Delta\}$ has a subsequence that converges uniformly to a map $\psi : \operatorname{Vert}\Delta \to \operatorname{Vert}\Delta$ that is locally constant. This is absurd. For ψ cannot be surjective, since Vert Δ is compact and we assumed above that Δ and hence Vert Δ are infinite. But ψ must be surjective, because each $\psi_n \mid \operatorname{Vert}\Delta$ is.

We will say that a neighbourhood U of a vertex v is *good* if every subsequence $\{\psi_{n_k}\}$ for which $\{\psi_{n_k} v\}$ converges is uniformly convergent to a constant function on U. Since Δ is compact, it is easy to find ψ as above if every vertex of Δ has a good neighbourhood.

If the assertion is false, we can pass to a subsequence so that $\{\psi_n p_n\}$ and $\{\psi_n q_n\}$ converge to a common limit. Then, by Lemma 2.5, p has a neighbourhood on which $\{\psi_n\}$ converges uniformly to a constant function. This neighbourhood is good. We now show that if $v \in \operatorname{Vert}\Delta$ has a good neighbourhood U_v, then so does any $w \in \operatorname{Vert}\Delta$ with $D(v, w) = 1$. It will then follow by induction on $D(p, \cdot)$ that every vertex of Δ has a good neighbourhood.

It is clear from our choice of ε, Lemma 1.14 and Proposition 1.7 that we can find a non-stammering gallery $\mathscr{G} = (C, D, D', C')$ such that

(i) v is the initial vertex of $\mathscr{G}$ and $w = C \cap D$;
(ii) $\rho(C, D) \geqslant 3\varepsilon$;
(iii) the final vertex v' of $\mathscr{G}$ is in U_v.

Let $w' = C' \cap D'$. We show that if $\{ \psi_{n_k} w \}$ converges, then $\{ \psi_{n_k} w' \}$ converges to the same limit. If not we can assume by a further passage to a subsequence that $\{ \psi_{n_k} \mathscr{G} \}$ converges to a gallery $\mathscr{H}$ in which $\lim_{k \to \infty} \psi_{n_k} D \neq \lim_{k \to \infty} \psi_{n_k} D'$. Since each ψ_{n_k} moves chambers by at most ε, $\lim_{k \to \infty} \psi_{n_k} C \neq \lim_{k \to \infty} \psi_{n_k} D$. Also $\lim_{k \to \infty} \psi_{n_k} v = \lim_{k \to \infty} \psi_{n_k} v'$ since $v' \in U_v$. Hence $\mathscr{H}$ is a closed gallery and all four of its chambers are distinct. Since Δ is irreducible, this is impossible. Thus $\psi_{n_k} w' \to \lim_{k \to \infty} \psi_{n_k} w$ whenever $\{ \psi_{n_k} w \}$ converges.

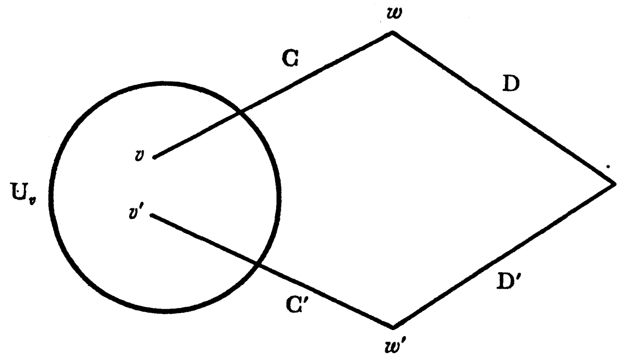

FIG. 1

Since $D(w, w') = 2$, Lemma 2.5 shows that w has a neighbourhood U_w on which $\{ \psi_{n_k} \}$ must converge uniformly to a constant if $\{ \psi_{n_k} w \}$ and $\{ \psi_{n_k} w' \}$ converge to a common limit. It follows from the previous paragraph that U_w is a good neighbourhood of w.

Proof of Lemma 2.5. — Note that the diameter d of Δ is at least 3, since Δ is irreducible. We consider two cases.

a) *d is odd.*

2.6. *Definition.* — Let a and b be vertices of an apartment Σ with $D(a, b) = 2$. Let a' be the vertex of Σ that is almost opposite both a and b, and let b' be the vertex of Σ with $D(a', b') = 2$ and $D(b, b') = d - 3$. Suppose $\Sigma = (C_1, \ldots, C_d, C'_d, \ldots, C'_1)$ where $[a, b] = (C_{(d+1)/2}, C_{(d+3)/2})$ and $[a', b'] = (C'_{(d+1)/2}, C'_{(d+3)/2})$. We say that (Σ, a, b) has the *forcing property* if each of the following sets of chambers is pairwise 3ε-separated:

(i) $C_1, \ldots, C_{(d-1)/2}, C_{(d+5)/2}, \ldots, C_d$;

(ii) $C'_1, \ldots, C'_{(d-1)/2}, C'_{(d+5)/2}, \ldots, C'_d$.

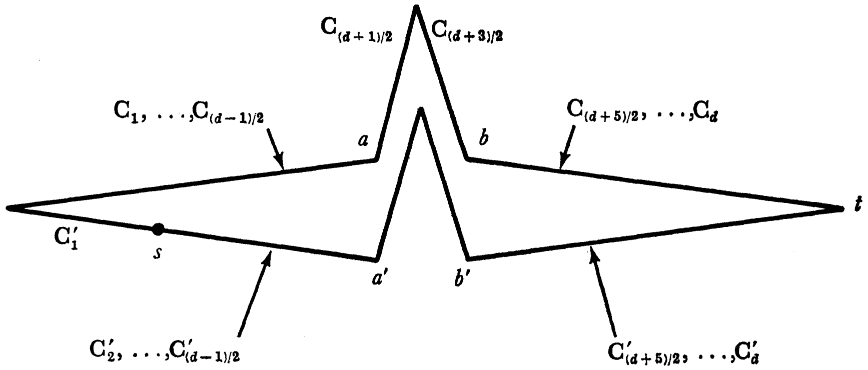

FIG. 2

Given any pair of vertices a and b with $D(a, b) = 2$, we can find an apartment Σ such that (Σ, a, b) has the forcing property. It is clear from our choice of ε that we can choose a nonstammering gallery $\mathscr{G}_0 = (C_1, \dots, C_d)$ such that $[a, b] = (C_{(d+1)/2}, C_{(d+3)/2})$ and the chambers described in (i) above are 3ε-separated. Adjoin a chamber $C'_1 \neq C_1$ to the beginning of $\mathscr{G}_0$. Then the ends s and t of the gallery $\mathscr{G}_1 = (C'_1, C_1, \dots, C_d)$ are almost opposite. Let Σ be the apartment formed by $\mathscr{G}_1$ and $[s, t]$. Since, by Lemma 1.14, C'_1 can be chosen as close to C_1 as we wish, it is clear by Lemma 1.12 that we can ensure that (Σ, a, b) has the forcing property.

The main point of the forcing property is

2.7. *Sublemma. — Suppose (Σ_n, a_n, b_n) has the forcing property for each $n \geqslant 1$. Let a'_n be the vertex of Σ_n almost opposite both a_n and b_n and let b'_n be the vertex of Σ_n with $D(a'_n, b'_n) = 2$ and $D(b'_n, b_n) = d - 3$. If $\{\varphi_n a_n\}$ and $\{\varphi_n b_n\}$ converge to a common limit x, then $\{\varphi_n a'_n\}$ and $\{\varphi_n b'_n\}$ both converge to x.*

Proof. — It suffices to prove the assertion for convergent subsequences. Therefore let us assume that $\varphi_n a'_n \to y$ and $\varphi_n b'_n \to z$. We must show that $x = y = z$. Define the chambers of Σ_n, C_{in} and C'_{in}, $1 \leqslant i \leqslant d$, analogously to C_i and C'_i in Definition 2.6. By a further passage to a subsequence, we can assume that there is a closed gallery $(D_1, D_2, \dots, D_d, D'_d, D'_{d-1}, \dots, D'_1)$ such that $\varphi_n C_{in} \to D_i$ and $\varphi_n C'_{in} \to D'_i$ for $1 \leqslant i \leqslant d$. Since $\{\varphi_n a_n\}$ and $\{\varphi_n b_n\}$ both converge to x, $D_{(d+1)/2} = D_{(d+3)/2}$ and $\mathscr{F} = (D_1, \dots, D_{(d-1)/2}, D_{(d+5)/2}, \dots, D_d)$ is a gallery. Since each φ_n moves chambers by at most ε, we see from the forcing property that any two consecutive chambers of $\mathscr{F}$ are distinct. Hence $\mathscr{F}$ is minimal. Since $(D'_1, \dots, D'_d)$ has two more chambers than $\mathscr{F}$ and the same end vertices, it cannot be a minimal gallery. Hence $D'_i = D'_{i+1}$ for some i with $1 \leqslant i \leqslant d - 1$. Again since the φ_n do not move chambers by more than ε, it is clear from the forcing property that $D'_i = D'_{i+1}$ is possible only when $i = (d+1)/2$. It follows that $y = z$ and $\mathscr{F}' = (D'_1, \dots, D'_{(d-1)/2}, D'_{(d+5)/2}, \dots, D'_d)$ is a gallery. The galleries $\mathscr{F}$ and $\mathscr{F}'$ have the same length and the same initial and final vertices. Since $\mathscr{F}$ is minimal and contains fewer than $d+1$ chambers, we see from Observation 0.5 that $\mathscr{F} = \mathscr{F}'$. Therefore $x = y = z$, since x is $D_{(d-1)/2} \cap D_{(d+5)/2}$ and $y = z$ is $D'_{(d-1)/2} \cap D'_{(d+5)/2}$. □

2.8. *Sublemma.* — *Suppose* (Σ, e, f) *has the forcing property and* e' *is the vertex of* Σ *almost opposite both* e *and* f. *Suppose* $e_n \to e$, $f_n \to f$, $e'_n \to e'$ *and* $D(e_n, f_n) = 2$ *for each* n. *Then, for any large enough* n, *there is an apartment* Σ_n *such that* (Σ_n, e_n, f_n) *has the forcing property and* e'_n *is the vertex of* Σ_n *almost opposite both* e_n *and* f_n. *Moreover* $\Sigma_n \to \Sigma$ *as* $n \to \infty$.

Proof. — By Lemma 1.13, e'_n is almost opposite both e_n and f_n for any large enough n. For such n, $\Sigma_n = ([e_n, f_n], [f_n, e'_n], [e'_n, e_n])$ is an apartment, and $\Sigma_n \to \Sigma$ by Lemma 1.12. Clearly (Σ_n, e_n, f_n) has the forcing property when it is close enough to (Σ, e, f). □

Now we prove the lemma. Construct, as described above, an apartment Σ containing a and b such that (Σ, a, b) has the forcing property. Let a' and b' be as in Definition 2.6. By Sublemma 2.8, there is, for each large enough n, an apartment Σ_n containing a_n, b_n and a' such that (Σ_n, a_n, b_n) has the forcing property. Let a'_n and b'_n be the vertices of Σ_n analogous to a' and b' in Definition 2.6. Then $a'_n = a'$ for all n and $b'_n \to b'$.

Notice from the symmetry of Definition 2.6 that (Σ, a', b') also has the forcing property. By Sublemma 2.8, there is a neighbourhood U of a and a number n_0 such that if $u \in U$ and $n \geqslant n_0$, then there is an apartment $\Sigma(u, n)$ containing a'_n, b'_n and u such that $(\Sigma(u, n), a'_n, b'_n)$ has the forcing property and u is the vertex of $\Sigma(u, n)$ almost opposite both a'_n and b'_n.

If $\{u_n\} \subseteq U$, let $\Sigma'_n = \Sigma(u_n, n)$. Sublemma 2.7 applied to (Σ_n, a_n, b_n) shows that $\varphi_n a'_n \to x$ and $\varphi_n b'_n \to x$. Applying Sublemma 2.7 to (Σ'_n, a'_n, b'_n) now shows that $\varphi_n u_n \to x$.

b) *d is even.* This case is a little more complicated. The forcing property will now apply to closed galleries with $2d+2$ chambers. The arguments are similar to those when d is odd.

2.6'. *Definition.* — Suppose $\mathscr{R}$ is a closed nonstammering gallery with $2d+2$ chambers and a, b are vertices of $\mathscr{R}$ with $D(a, b) = 2$. Let a' be the vertex of $\mathscr{R}$ opposite both a and b and let b' be the vertex of $\mathscr{R}$ such that $D(a', b') = 2$ and $D(b', b) = d - 2$. Suppose $\mathscr{R} = (C_1, \ldots, C_{d+1}, C'_{d+1}, \ldots, C'_1)$ where $[a, b] = (C_{(d/2)+1}, C_{(d/2)+2})$ and $[a', b'] = (C'_{(d/2)+1}, C'_{(d/2)+2})$. Let

$$u = C_1 \cap C_2, \quad v' = C'_d \cap C'_{d+1} \quad \text{and} \quad (C''_2, C''_3, \ldots, C''_d) = [u, v'].$$

We say that $(\mathscr{R}, a, b)$ has the *forcing property* if each of the following sets of chambers is pairwise 3ε-separated:

(i) $C_1, \ldots, C_{d/2}, C_{(d/2)+3}, \ldots, C_{d+1}$;
(ii) $C'_1, \ldots, C'_{d/2}, C'_{(d/2)+3}, \ldots, C'_{d+1}$;
(iii) $C''_2, \ldots, C''_{d/2}, C''_{(d/2)+3}, \ldots, C''_d$.

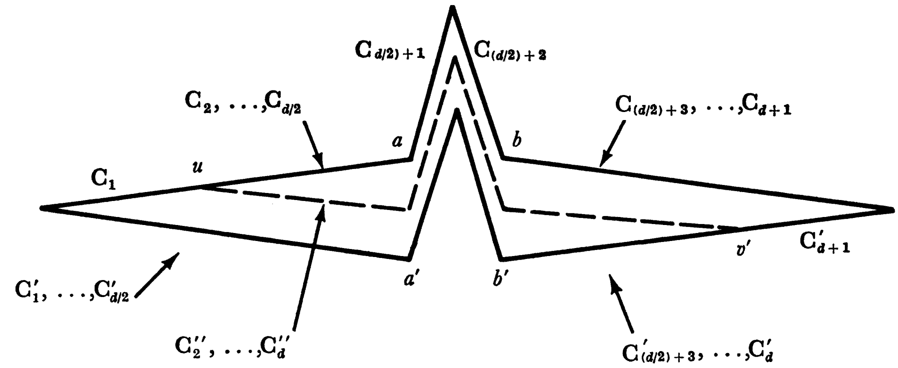

FIG. 3

Any vertices a and b with $D(a, b) = 2$ are contained in a gallery $\mathscr{R}$ such that $(\mathscr{R}, a, b)$ has the forcing property. First choose a nonstammering gallery $\mathscr{G}_0 = (C_1, \ldots, C_{d+1})$ such that $[a, b] = (C_{(d/2)+1}, C_{(d/2)+2})$ and the chambers listed in (i) above are 3ε-separated. Now adjoin $C'_1 \neq C_1$ and $C'_{d+1} \neq C_{d+1}$ to the beginning and end of $\mathscr{G}_0$ to form a gallery $\mathscr{G}_1$. If C'_1 and C'_{d+1} are close enough to C_1 and C_{d+1} respectively, the ends of $\mathscr{G}_1$ are almost opposite and we can adjoin their convex hull to $\mathscr{G}_1$ to form a closed gallery $\mathscr{R}$ such that $(\mathscr{R}, a, b)$ has the forcing property.

2.7'. *Sublemma. — Suppose $(\mathscr{R}_n, a_n, b_n)$ has the forcing property for each $n \geqslant 1$. Let a'_n be the vertex of $\mathscr{R}_n$ opposite both a_n and b_n and b'_n the vertex of $\mathscr{R}_n$ with $D(a'_n, b'_n) = 2$ and $D(b'_n, b_n) = d - 2$. Suppose $\{ \varphi_n a_n \}$ and $\{ \varphi_n b_n \}$ converge to a common limit x. Then $\{ \varphi_n a'_n \}$ and $\{ \varphi_n b'_n \}$ both converge to x.*

Proof. — It suffices to prove the assertion for convergent subsequences. Therefore let us assume that $\varphi_n a'_n \to y$ and $\varphi_n b'_n \to z$. We must show that $x = y = z$. Define the chambers $C_{in}, C'_{in}, C''_{in}$ by analogy with Definition 2.6'. By another passage to a subsequence, we assume that $C_{in} \to D_i$, $C'_{in} \to D'_i$ and $C''_{in} \to D''_i$ where $(D_1, \ldots, D_{d+1}, D'_{d+1}, \ldots, D'_1)$ is a closed gallery and $(D''_2, \ldots, D''_d)$ is a gallery with initial vertex $D_1 \cap D_2$ and final vertex $D'_d \cap D'_{d+1}$. Since $\{ \varphi_n a_n \}$ and $\{ \varphi_n b_n \}$ both converge to x, we have $D_{(d/2)+1} = D_{(d/2)+2}$.

We now use this fact and the argument from the proof of Sublemma 2.7 based on how far φ_n can move chambers. First consider the closed gallery with $2d$ chambers $(D_2, \ldots, D_d, D_{d+1}, D'_{d+1}, D''_d, D''_{d-1}, \ldots, D''_2)$. Since $D_{(d/2)+1} = D_{(d/2)+2}$, we see that $D''_{(d/2)+1} = D''_{(d/2)+2}$ and then that $D_i = D'_i$ for $2 \leqslant i \leqslant d/2$ and $(d/2) + 3 \leqslant i \leqslant d + 1$. It follows that $(D_1, \ldots, D_d, D'_d, \ldots, D'_1)$ is a closed gallery with $2d$ chambers. Since

$D_{(d/2)+1} = D_{(d/2)+2}$ we see from this gallery that $D'_{(d/2)+1} = D'_{(d/2)+2}$ and then that $D_i = D'_i$ for $1 \leqslant i \leqslant d/2$ and $(d/2) + 3 \leqslant i \leqslant d$. Hence $x = y = z$. □

2.8′. *Sublemma.* — *Suppose* $(\mathscr{R}, e, f)$ *has the forcing property and* e' *is the vertex of* $\mathscr{R}$ *opposite both* e *and* f. *Suppose* $e_n \to e$, $f_n \to f$, $e'_n \to e'$ *and* $D(e_n, f_n) = 2$ *for each* n. *Then there are galleries* $\mathscr{R}_n \to \mathscr{R}$ *such that for any large enough* n, $(\mathscr{R}_n, e_n, f_n)$ *has the forcing property and* e'_n *is the vertex of* $\mathscr{R}_n$ *opposite both* e_n *and* f_n.

Proof. — Let E (resp. F) be the chamber of $\mathscr{R}$ that contains e (resp. f) and does not belong to $[e, f]$. By Proposition 1.7, there are chambers $E_n \in \operatorname{Star} e_n$ and $F_n \in \operatorname{Star} f_n$ converging to E and F respectively. Let g_n (resp. h_n) be the other vertex of E_n (resp. F_n). If n is large enough, e'_n is almost opposite both g_n and h_n. We take

$$\mathscr{R}_n = (E_n, [e_n, f_n], F_n, [h_n, e'_n], [e'_n, g_n]). \quad \square$$

Lemma 2.5 now follows from the two sublemmas in the same way as it did when d was odd. □

This completes the proof that G_ε is compact for any small enough $\varepsilon > 0$ in the case when Δ has rank 2. Now we consider the case when rank $\Delta \geqslant 3$.

2.9. *Assertion.* — *If* ε *is small enough, any sequence* $\{\varphi_n\} \subseteq G_\varepsilon$ *has a subsequence that converges in the compact-open topology.*

The compactness of G_ε follows easily from this assertion. Suppose $\{\psi_n\} \subseteq G_\varepsilon$. Then $\{\psi_n^{-1}\} \subseteq G_\varepsilon$ and there is a subsequence $\{\psi_{n_k}\}$ such that $\{\psi_{n_k}\}$ and $\{\psi_{n_k}^{-1}\}$ both converge in the compact-open topology. It is easy to see that $\lim_{k\to\infty} \psi_{n_k} \in G_\varepsilon$ and has inverse $\lim_{k\to\infty} \psi_{n_k}^{-1}$.

To prove Assertion 2.9, fix an apartment Σ of Δ. For $0 \leqslant i \leqslant \operatorname{diam} \Sigma$, let $\Sigma^i = \{ A \in \Delta : \text{there are } C \in \operatorname{Cham} \operatorname{Star} A,\ D \in \operatorname{Cham} \Sigma \text{ with } \operatorname{dist}(C, D) \leqslant i \}$. We will use the rank 2 case of the theorem to show that $\{\varphi_n\}$ has a subsequence that converges uniformly on Σ^1. Then we show inductively that this subsequence converges uniformly on $\Sigma^2, \ldots, \Sigma^{\operatorname{diam}\Delta} = \Delta$, and thus converges in the compact-open topology.

Call $A, B \in \Delta$ *δ-opposite* if every $A' \in \Delta_{\operatorname{rank} A}$ with $\rho(A', A) \leqslant \delta$ is opposite every $B' \in \Delta_{\operatorname{rank} B}$ with $\rho(B', B) \leqslant \delta$. Since every element of Δ is opposite some element of Σ by [T1, 4.2], it follows from the compactness of Δ that if ε is small enough, then every $A \in \Delta$ is 2ε-opposite some $B \in \Sigma$. We assume henceforth that ε has this property.

Clearly we can pass to a subsequence so that $\{\varphi_n\}$ converges on Σ. Now suppose A is a codimension 2 face of Σ whose star is irreducible. Let B be the face of Σ opposite A. Note that B is 2ε-opposite A and also has an irreducible star. By the rank 2 case of the theorem, there is $\beta > 0$ such that $H_\beta = \{ \theta \in \operatorname{Auttop}(\operatorname{Star} B) : \rho(x, \theta x) \leqslant \beta \text{ for all } x \in \operatorname{Star} B \}$ is compact. It is clear from Propositions 1.9 and 1.10 that if ε is small enough and $\psi \in G_\varepsilon$, then $\widetilde{\psi} = \operatorname{proj}_B \circ \psi \circ \operatorname{proj}_A \mid \operatorname{Star} B$ is a continuous automorphism of Star B. It follows that if ε is small enough, $\widetilde{\psi} \in H_\beta$ for every $\psi \in G_\varepsilon$.

By passing to a subsequence, we can assume that $\{ \tilde{\varphi}_n \}$ converges in the compact-open topology on H_β and thus converges uniformly on Cham Star B. Now, if $C \in$ Cham Star A, $\varphi_n C = \text{proj}_{\varphi_n A} \circ \tilde{\varphi}_n \circ \text{proj}_B C$. Since $\{ \varphi_n A \}$ converges, we see from Proposition 1.10 that $\{ \varphi_n \}$ converges uniformly on Cham Star A.

By iterating the above argument, we see that if ε is small enough we can make successive passages to a subsequence so that $\{ \varphi_n \}$ converges uniformly on Σ and Cham Star A for every codimension 2 face A of Σ whose star is irreducible. Since the Coxeter diagram for Σ is connected, every hyperface of Σ contains a face with codimension 2 in Σ whose star is irreducible. Thus $\{ \varphi_n \}$ converges uniformly on Cham Σ^1 and hence on Σ^1.

Assume now that $\{ \varphi_n \}$ converges uniformly on Σ^i. To show that $\{ \varphi_n \}$ converges uniformly on Σ^{i+1}, it suffices to prove uniform convergence on Cham Σ^{i+1}. We use the same general idea as [T1, 4.1.1].

Firstly we show that if $C \in$ Cham Σ^{i+1}, then $\{ \varphi_n C \}$ converges. Let A be a hyperface along which C is adjacent to a chamber of Σ^i and choose a face B of Σ that is 2ε-opposite A. Note that $\varphi_n A$ is opposite $\varphi_n B$ for all n and, if $\text{proj}_B C = D$, then $\varphi_n C = \text{proj}_{\varphi_n A}(\varphi_n D)$. Since A and D are both in Σ^i, $\{ \varphi_n A \}$ and $\{ \varphi_n D \}$ converge. It follows from Proposition 1.10 that $\{ \varphi_n C \}$ converges.

Secondly we show that $\{ \varphi_n \}$ converges uniformly on Cham Σ^{i+1}. Suppose $\{ C_n \} \subseteq$ Cham Σ^{i+1} is a convergent sequence. Note that $C = \lim_{n \to \infty} C_n$ is a chamber of Σ^{i+1}, since Σ^{i+1} is closed. We show that $\varphi_n C_n \to \lim_{n \to \infty} \varphi_n C$. For each n, let A_n be a hyperface of C_n which is in Σ^i. By passing to a subsequence, we can assume that $\{ A_n \}$ converges to a hyperface A of C, which is in Σ^i, since Σ^i is closed. We can also assume that $\rho(A_n, A) < \varepsilon$ for all n. Choose a face B of Σ that is 2ε-opposite A. Then $\varphi_n A_n$ is opposite $\varphi_n B$ for all n, and

$$\varphi_n C_n = \text{proj}_{\varphi_n A_n}(\varphi_n D_n)$$

where $D_n = \text{proj}_B C_n$. Clearly $D_n \in \Sigma^i$ for all n, and $D_n \to D = \text{proj}_B C$ by Proposition 1.10. Since $\{ \varphi_n \}$ converges uniformly on Σ^i, $\varphi_n A_n \to \lim_{n \to \infty} \varphi_n A = A'$ and $\varphi_n D_n \to \lim_{n \to \infty} \varphi_n D = D'$. It follows from Proposition 1.10 that

$$\lim_{n \to \infty} \varphi_n C_n = \text{proj}_{A'} D' = \lim_{n \to \infty} \varphi_n C$$

as required.

Thus $\{ \varphi_n \}$ converges uniformly on Cham Σ^{i+1} and hence on Σ^{i+1}. This completes the proof of Assertion 2.9. □

3. Topological Moufang Buildings And Their Automorphism Groups

We define a topological analogue of a Moufang building. This means that there are many topological automorphisms. Most of this section is devoted to proving the following:

Let Δ be an irreducible, compact, metric, locally connected, topologically Moufang building of rank at least 2. *Then the topological automorphism group* G *of* Δ *is a finite extension of a connected, simple, noncompact Lie group. Furthermore the stabiliser of a chamber in* Δ *in the connected component of the identity* G^0 *of* G *is a parabolic subgroup of* G^0.

If Δ is finite dimensional instead of Moufang, we also show that G is a Lie group. In fact this is much easier. However, we need the machinery of the Moufang case and the Moufang condition itself to show the other properties of G. Our tools are Gleason and Yamabe's famous theorem on small subgroups and Furstenberg's characterization of parabolic subgroups in terms of proximal actions.

Let Δ be an irreducible topological building of rank at least 2. We will always denote its topological automorphism group by G. Furthermore, if $A \subset \Delta$ is a half-apartment, we let U_A be the group of all $g \in G$ that fix all the chambers in A.

3.1. *Definition.* — A subgroup $H \subset G$ is called *Moufang* (*for* Δ) if for any half-apartment A the group $H \cap U_a$ acts transitively on all the apartments containing A. We call Δ a *topologically Moufang building* if G itself is Moufang.

Since we will only deal with topologically Moufang buildings in this paper, we will often refer to them simply as Moufang buildings. Note that our Moufang condition is slightly weaker than the combinatorial one [T1, Addendum], since an element of U_A does not have to fix all the stars of all hyperfaces in $A \backslash \partial A$.

3.2. *Lemma.* — *If* H *is Moufang for* Δ *and* Σ, Σ' *are two apartments, then there is* $h \in H$ *such that* $h(\Sigma) = \Sigma'$ *and* h *fixes every chamber of* $\Sigma \cap \Sigma'$.

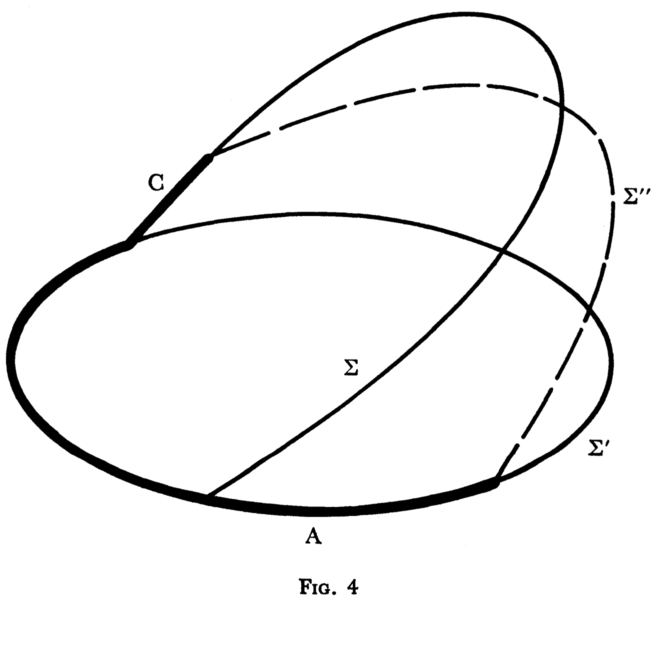

Fig. 4

Proof. — Recall that the number L of chambers in a half-apartment of Δ does not depend on the half-apartment in question. Let ℓ be the number of chambers in $\Sigma \cap \Sigma'$. We argue by a descending induction on ℓ. By [T1, 2.19] either $\Sigma = \Sigma'$ or $\Sigma \cap \Sigma'$ is an intersection of half-apartments. Hence if $\ell > L$, then $\Sigma = \Sigma'$ and the lemma is obvious. If $\ell = L$ then $\Sigma \cap \Sigma'$ is a half-apartment. Since H is Moufang, the lemma follows. Suppose $0 < \ell < L$. Then there is a chamber C in $\Sigma \setminus \Sigma'$ such that $C \cap \Sigma \cap \Sigma'$ is a hyperface of C. Let A be a half-apartment of Σ' such that $A \supset \Sigma \cap \Sigma'$ but $C \notin A$. By [T1, 3.27] the convex hull of A and C is an apartment Σ''. Since $\Sigma \cap \Sigma'' \supset (\Sigma \cap \Sigma') \cup \{ C \}$ there is $h_1 \in H$ such that $h_1 \Sigma = \Sigma''$ and h_1 fixes every element of $\Sigma \cap \Sigma''$. Since $\Sigma' \cap \Sigma'' \subset A$, there is $h_2 \in H$ such that $h_2 \Sigma'' = \Sigma'$ and h_2 fixes all elements of A. Clearly $h_2 \circ h_1(\Sigma) = \Sigma'$ and $h_2 \circ h_1$ fixes all elements of $\Sigma \cap \Sigma'$.

Finally suppose $\ell = 0$. Let D be a chamber in Σ'. By [T1, 4.2], D is opposite a chamber C in Σ. Let Σ'' be the apartment containing C and D. By the above, there is $h_1 \in H$ with $h_1 \Sigma = \Sigma''$ and $h_2 \in H$ with $h_2 \Sigma'' = \Sigma'$. Then $h = h_2 \circ h_1$ maps Σ to Σ'. □

3.3. *Corollary.* — *If* H *is Moufang on* Δ, *then* H *is transitive on* Cham Δ.

Proof. — By [T1, 3.31] there is a chamber E opposite any two given chambers C and D. Let Σ, Σ' be the apartments through C, E and D, E respectively. By Lemma 3.2, there is an $h \in H$ that maps Σ to Σ' and fixes E. Clearly $h(C) = D$. □

3.4. *Corollary.* — *Let* T *be the type of a minimal gallery. For all* $C \in \text{Cham}\,\Delta$, *the stabilizer* H_C *of* C *in* H *is transitive on* $T(C) = \{ D \in \text{Cham}\,\Delta : D$ can be joined to C by a gallery of type T $\}$.

Proof. — This is obvious, since any apartment that contains C contains a unique chamber in T(C). □

3.5. *Lemma.* — *Let* Δ *be a locally connected, infinite, irreducible, compact, metric building. Then the star of any hyperface of* Δ *is connected. Moreover* Δ *itself is connected.*

Proof. — (1) Suppose Δ has rank 2 and let $x \in \text{Vert}\,\Delta$. We show that Star x is connected. Let $C_1, C_2 \in \text{Cham Star}\,x$, and let y_i be the other vertex of C_i, $i = 1, 2$. By Lemma 1.14, there is $D_i \in \text{Cham Star}\,y_i \setminus \{ C_i \}$ as close to C_i as we wish. Let x_i be the other vertex of D_i. Note that x_1 and x_2 are close to x. For each i construct a minimal gallery $\mathcal{G}_i$ starting from x_i and ending at a vertex z_i almost opposite x_i. We can assume, by Lemma 1.13, that z_1 and z_2 are almost opposite x. The space $\mathcal{O}$ of vertices almost opposite x is open (Lemma 1.13) and hence locally connected. Thus if D_i is close enough to C_i and $\mathcal{G}_1$ close enough to $\mathcal{G}_2$, then z_1 and z_2 will lie in the same component of $\mathcal{O}$. Since $C_i = \text{proj}_x z_i$ and $\text{proj}_x \mid \mathcal{O}$ is continuous, it follows that C_1 and C_2 are in the same component of Cham Star x.

(2) Suppose rank $\Delta > 2$. We show first that if a chamber C has a hyperface h whose star is not discrete, then the star of any hyperface of C is connected.

Let x be the vertex of C that is not in h. Let y be a vertex in h adjacent to x in the Coxeter diagram. Let f be the codimension 2 face in C that contains neither x nor y. By [T1, 3.12], Star f is irreducible. Clearly Star f is not discrete. Let h' be the hyperface of C that misses y. Then Star h' is the star of x in Star f. From (1) we see that Star h' is connected. In particular, Star h' is not discrete. Since the Coxeter diagram for Δ is connected, the claim follows.

Since Δ is infinite there is a least one such chamber C. Since we can connect any hyperface to C by a gallery, it follows from the above that the star of any hyperface is connected, and also that Δ itself is connected. □

3.6. *Lemma.* — *Let* T *be the type of a minimal gallery and let* $C \in \text{Cham}\,\Delta$. *Define* T(C) *as in Corollary* 3.4. *Then* T(C) *is locally connected.*

Proof. — Let $D_0 \in T(C)$ and let $\mathscr{G}_0$ be the gallery of type T from C to D_0. Extend $\mathscr{G}_0$ to a minimal gallery $\widetilde{\mathscr{G}}_0$ from C to some chamber E_0 opposite C. Let U be a connected neighborhood of E_0 in Opp C. If $E \in U$, let $\Sigma(E)$ be the apartment determined by C and E. Let D(E) be the unique chamber of $\Sigma(E)$ in T(C). Clearly the map $E \to D(E)$ is continuous and $V = \{ D(E) : E \in U \}$ is a connected neighborhood of D_0 in T(C). □

The following sequence of technical lemmata will lead up to the proof that G is a Lie group (Theorem 3.12). Unless otherwise stated we will assume henceforth that Δ is infinite, irreducible, compact, metric, locally connected, topologically Moufang and has rank at least 2. Let G^0 be the component of the identity of G. If $C \in \text{Cham}\,\Delta$, set $P_C = G_C \cap G^0$, where G_C is the stabilizer of C in G. Note that G^0 is type-preserving.

3.7. *Lemma.* — *The action of* G *on* Cham Δ *is open and* Cham Δ *is a topological homogeneous space of* G.

Proof. — Since Cham Δ is compact metric, it is second countable. Hence G is second countable by the definition of the compact-open topology. Therefore G is separable. If $G' \subset G$ is any open subgroup, then G/G' is countable. By [MZ, 2.3.1], G has an open subgroup G' such that G'/G^0 is compact. By [MZ, 2.13] and Corollary 3.3, Cham Δ is homeomorphic to G/G_C where G_C is the stabiliser of a chamber $C \in \text{Cham}\,\Delta$. Clearly the action of G is open. □

3.8. *Lemma.* — *The action of* G^0 *on* Cham Δ *is transitive.*

Proof. — Let $C \in \text{Cham}\,\Delta$. We first show that $G^0.C \supset \text{Opp}\,C$.

By Lemmata 3.5 and 3.7, Cham Δ is a connected homogeneous space of G. By [Bou, III, § 4, no. 6, cor. 3], $G^0.C$ is dense in Cham Δ. Hence there is $g_0 \in G^0$ such that $g_0 C \in \text{Opp}\,C$. Let $E \in \text{Opp}\,C$. By Corollary 3.4 there is $g \in G_C$ such that $E = gg_0 C$. Now $E = gg_0 g^{-1} gC = gg_0 g^{-1} C \in G^0.C$ since G^0 is normal in G. Therefore $G^0.C \supset \text{Opp}\,C$.

Let $C' \in \mathrm{Cham}\,\Delta$. By [T1, 3.30] there is a chamber C'' opposite both C and C'. Then $C' \in \mathrm{Opp}\,C'' \subset G^0.C'' = G^0.C$ since $C'' \in \mathrm{Opp}\,C \subset G^0.C$. □

3.9. *Lemma. — Let* N *be a normal subgroup of* G^0 *and* T *the type of a minimal gallery. Then* $\overline{N.C} \cap T(C)$ *is open and closed in* $T(C)$ *for all* $C \in \mathrm{Cham}\,\Delta$.

Proof. — Clearly $\overline{N.C} \cap T(C)$ is closed in $T(C)$. Let $D \in \overline{N.C} \cap T(C)$. By Lemma 3.6 there is a connected open neighborhood U of D in T(C). By Corollary 3.4, U is a connected subset of a homogeneous space of G_C. By [Bou, III, § 4, no. 6, cor. 3], $(G_C)^0.D$ and hence $P_C.D$ are dense in U. Since N is normal in G^0 we have $P_C.D \subset \overline{N.C}$. Hence $\overline{N.C} \supset U$. □

3.10. *Lemma. — If* N *is a normal subgroup of* G^0 *and* $\mathcal{O}$ *is an* N*-orbit in* $\mathrm{Cham}\,\Delta$, *then* $\overline{\mathcal{O}}$ *contains the convex hull of* $\mathcal{O}$.

Proof. — We need the following fact.

If Y *is a connected, locally connected, Hausdorff topological space and* $p \in Y$, *then* p *lies in the closure of every connected component of* $Y \backslash \{p\}$.

Let $C \in \mathrm{Cham}\,\Delta$, $D \in \overline{N.C}$ and let $\mathcal{G} = (C, C_1, \ldots, C_{\ell-1}, D)$ be a minimal gallery from C to D. By [T1, 2.23] it suffices to prove that $\mathcal{G} \subset \overline{N.C}$. Let T be the type of $\mathcal{G}$ and let $h = C_{\ell-1} \cap D$. Observe that $T(C) \cap \mathrm{Star}\,h = \mathrm{Star}\,h \backslash \{C_{\ell-1}\}$. Let Z be the connected component of D in this set. By Lemma 3.9, $Z \subset \overline{N.C}$. By Lemma 3.5 and the above fact, $C_{\ell-1} \in \overline{Z} \subset \overline{N.C}$. By an induction we see that $\mathcal{G} \subset \overline{N.C}$. □

3.11. *Lemma. — The connected component of the identity* G^0 *of* G *is a Lie group.*

Proof. — By a theorem of Gleason and Yamabe [Gle, Ya, Theorem 4] it suffices to prove that there is a neighborhood of the identity that does not contain any nontrivial normal subgroup of G^0.

Let $C \in \mathrm{Cham}\,\Delta$. Let U be a compact neighborhood of the identity in G^0 such that U.C does not contain the star of any hyperface of C. Suppose U contains a nontrivial normal subgroup N of G^0. Since U is compact, we may assume that N is compact.

Suppose N fixes C. Since N is normal in G^0, N fixes every chamber in $G^0.C$. By Lemma 3.8, N fixes all elements of $\mathrm{Cham}\,\Delta$. Then N is trivial.

Hence $N.C \neq \{C\}$. As N.C is compact, it is convex by Lemma 3.10. Hence there is a hyperface h of C and $C \neq D \in \mathrm{Cham}\,\Delta$ such that $D \in N.C \cap \mathrm{Star}\,h$. By Lemma 3.9, $N.C = N.D$ intersects in open subsets with both $\mathrm{Star}\,h \backslash \{C\}$ and $\mathrm{Star}\,h \backslash \{D\}$. Hence $N.C \cap \mathrm{Star}\,h$ is open and clearly also closed. By Lemma 3.5, $N.C \supset \mathrm{Star}\,h$. This contradicts the choice of U. □

3.12. *Theorem.* — *If* Δ *is an irreducible, locally connected, compact, metric, topologically Moufang building of rank at least* 2, *then its topological automorphism group* G *is a Lie group.*

Proof. — Clearly we may also assume that Δ is infinite. Let $C \in \mathrm{Cham}\,\Delta$. By Lemma 3.8, $\mathrm{Cham}\,\Delta = G^0/P_C$. Hence $\mathrm{Cham}\,\Delta$ is a manifold by Lemma 3.11. If $g \in G$, there are $g_1 \in G^0$ and $g_2 \in G_C$ such that $g = g_1 g_2$. For any $g_0 \in G^0$, $gg_0 C = g_1 g_2 g_0 C = g_1(g_2 g_0 g_2^{-1}) g_2 C = g_1(g_2 g_0 g_2^{-1}) C$. Hence g acts on $\mathrm{Cham}\,\Delta$ via the automorphism $x \mapsto g_2 x g_2^{-1}$ and the translation $g_1 \in G^0$. As all continuous automorphisms of Lie groups are smooth, it is clear that g is a C^1-diffeomorphism of $\mathrm{Cham}\,\Delta$. By [MZ, V, Thm. 2], G is a Lie group. □

Let us point out a direct generalization of a theorem on projective planes [Sa 2]. It does not require that Δ be topologically Moufang.

3.13. *Theorem.* — *If* Δ *is an irreducible, finite dimensional, compact, connected building and its automorphism group* G *acts transitively on* $\mathrm{Cham}\,\Delta$, *then* G *is a Lie group.*

Proof. — By [MZ, p. 238], G has an open subgroup $H \supset G^0$ that is a projective limit of Lie groups. More precisely, we may assume that H satisfies condition A of [MZ, p. 237]. Since Δ is connected, H acts transitively on $\mathrm{Cham}\,\Delta$. By [MZ, 6.3 Corollary] H and therefore G are Lie groups. □

For the remainder of this section recall our assumption that unless otherwise stated Δ is an infinite, irreducible, compact, metric, locally connected, topologically Moufang building of rank at least 2. First a sequence of lemmata will prove that G is a finite extension of a simple Lie group.

3.14. *Lemma.* — *Suppose a Lie group* H *acts transitively on a connected, locally compact, Hausdorff space* M. *Then the connected component of the identity* H^0 *of* H *acts transitively on* M.

Proof. — By [MZ, 2.13], M is a homogeneous space of H. Since H^0 is open in H, all H^0-orbits are open. Hence they are also closed. □

3.15. *Lemma.* — *The group* G^0 *is Moufang for* Δ.

Proof. — Let A be a half-apartment. Let $C \in A$ be a chamber intersecting ∂A in a hyperface. Let f be the opposite hyperface in A. Let D be the unique chamber in $\mathrm{Star}\,f \cap A$. Note that the set of all apartments Σ that contain A is in 1 — 1 correspondence with $\mathrm{Star}\,f\backslash\{D\}$. Hence it suffices to prove that $U_A \cap G^0$ is transitive on $\mathrm{Star}\,f\backslash\{D\}$. Since G is Moufang for Δ, $\mathrm{Star}\,f$ is an orbit of G_f and hence a compact manifold. It follows by Lemma 3.5 that $\mathrm{Star}\,f\backslash\{D\}$ is a connected manifold. Since G is Moufang for Δ, $\mathrm{Star}\,f\backslash\{D\}$ is an orbit of U_A. By Lemma 3.14, $\mathrm{Star}\,f\backslash\{D\}$ is an orbit of the connected component U_A^0 which is contained in G^0. □

3.16. *Lemma.* — *Any nontrivial normal subgroup* N *of* G^0 *acts transitively on* Cham Δ.

Proof. — (1) *Let h be a hyperface of a chamber* C. *Suppose* $C \neq D \in \text{Star } h \cap N.C$. *Then* Star $h \subset N.C$.

By Lemma 3.15 and Corollary 3.4, we have Star $h \setminus \{C\} \subset P_C.D$. Since N is normal in G^0, we get Star $h \setminus \{C\} \subset P_C.D \subset N.C$.

(2) *Let* $C \in \text{Cham } \Delta$. *Then* N.C *is convex.*

Let $D \in N.C$ and let $\mathcal{G} = (C, C_1, \ldots, C_{\ell-1}, D)$ be a minimal gallery from C to D. Let T be the type of $\mathcal{G}$. By Lemma 3.15 and Corollary 3.4, P_C is transitive on T(C). Also $\text{Star}(C_{\ell-1} \cap D) \setminus \{C_{\ell-1}\} \subset T(C)$. Let $E \in \text{Star}(C_{\ell-1} \cap D) \setminus \{C_{\ell-1}, D\}$. Then $E \in P_C.D$. Hence $E \in N.C = N.D$, since N is normal in G^0. By (1) applied to D and E, $\text{Star}(C_{\ell-1} \cap D) \subset N.C$. Hence $C_{\ell-1} \in N.C$. By an induction, $\mathcal{G} \subset N.C$. By [T1, 2.23], N.C is convex.

(3) *We prove that* N.C *is not contained in* Star x *for any vertex* x.

Suppose $N.C \subset \text{Star } x$ for some vertex x. Then N fixes x. Lemma 3.8 shows that G^0 is transitive on the set V of all vertices of the same type as x. Since N is normal in G^0, it follows that N fixes V elementwise. Since Δ is irreducible, the convex hull of V is all of Δ. Hence N fixes Δ elementwise. Therefore N is trivial, in contradiction to the hypothesis.

(4) *Let* $C \in \text{Cham } \Delta$ *and let h be any hyperface of* C. *Then* Star $h \subset N.C$.

Let x be the vertex opposite h in C. By (2) and (3) (applied to this x) there is a hyperface h' of the same type as h whose star contains two chambers D and E in N.C. By (1), Star $h' \subset N.D = N.C$. Note that $h' \in N.h$ since $D \in N.C$. Hence Star $h \subset N.C$.

(5) *Finally we show that* $N.C = \text{Cham } \Delta$.

Let $D \in \text{Cham } \Delta$ and let $(C, C_1, \ldots, C_{\ell-1}, D)$ be a gallery from C to D. By (4) we have $C_{i+1} \in N.C_i$ and hence $D \in N.C$. □

3.17. *Lemma.* — *The component of the identity* G^0 *of* G *is semisimple without center.*

Proof. — Let N be the nilradical of G^0 and Z(N) and $Z(G^0)$ the centers of N and G^0 respectively. Set $Z = \overline{Z(N).Z(G^0)}$. By structure theory, it suffices to show that $N = \{1\}$ and $Z(G^0) = \{1\}$. Suppose the contrary. Then $Z \neq \{1\}$. Note that Z is normal in G^0, since conjugation by any $g \in G^0$ induces an automorphism, which leaves N and therefore Z invariant. By Lemma 3.16, Z acts transitively on Cham Δ. Let C and D be distinct chambers with a common face f. Then there is a $z \in Z$ such that $zC = D$. Hence $zf = f$. Since Z acts transitively on Cham Δ, it also acts transitively on the set of faces of type f. Since z commutes with Z, z fixes all faces of type f. Since Δ is irreducible, the convex hull of the faces of type f is all of Δ. Hence $z = 1$ in contradiction to $C \neq D$. □

3.18. *Theorem.* — *If* Δ *is an irreducible, compact, metric, locally connected, topologically Moufang building of rank at least* 2, *then its topological automorphism group* G *is a finite extension of its connected component of the identity* G^0. *Furthermore* G^0 *is a simple Lie group.*

Remark. — This theorem applies to both finite and infinite buildings. Since the finite case is trivial, we continue in the proof with our standing assumption that Δ is infinite.

Proof. — We first prove that G^0 is simple. Suppose the contrary. Since G^0 is semisimple and does not have center, $G^0 = G_1 \times G_2$ for some nontrivial subgroups G_1 and G_2 of G^0. By Lemma 3.16, both G_1 and G_2 are transitive on Cham Δ. Let C and D be two chambers in the star of some vertex x. Then $D = g_1 C$ for some $g_1 \in G_1$. Hence $g_1 x = x$. Since G_2 commutes with g_1 and is transitive on Cham Δ, g_1 fixes all vertices of the same type as x. Since Δ is irreducible, $g_1 = 1$ and hence $C = D$. This is a contradiction.

Now suppose $g \in G$ commutes with G^0. We will show that $g = 1$.

Suppose there is a chamber C with $gC \neq C$. Let $\mathcal{G} = (C, C_1, \ldots, C_{\ell-1}, gC)$ be a minimal gallery from C to gC. Set $f = C_{\ell-1} \cap gC$. Then

$$\mathrm{Star}\, f \backslash \{ C_{\ell-1} \} \subset P_C . gC = g P_C . C = \{ gC \}.$$

This is a contradiction.

It follows that G embeds into the automorphism group Aut G^0 of G^0. Since G^0 is simple with trivial center, it is well known that Aut G^0/Int G^0 is finite, where Int G^0 denotes the group of inner automorphisms of G^0 [Mu, § 1, Corollary 2]. Since $G^0 = \mathrm{Int}\, G^0$, G is a finite extension of G^0. □

Finally we investigate the stabiliser of a chamber. We recall some generalities on proximal actions [G].

Let a group H act on a topological space X. We call $x, y \in X$ *proximal* if there exists a net $\{ h_i \} \subset H$ such that $\lim h_i x = \lim h_i y$. We call the action *proximal* if all pairs of elements $x, y \in X$ are proximal.

3.19. *Lemma.* — *The action of* G^0 *on* Cham Δ *is proximal. Hence* G^0 *is noncompact.*

Proof. — Let C, D $\in$ Cham Δ be arbitrary. We show by an induction on dist(C, D) that $C \in \overline{P_C . D}$. Let $(C, C_1, \ldots, C_{\ell-1}, D)$ be a minimal gallery from C to D. By the inductive hypothesis we may assume that $C \in \overline{P_C . C_{\ell-1}}$. Let $h = C_{\ell-1} \cap D$. Then Star $h \backslash \{ C_{\ell-1} \} \subset P_C . D$ by Lemma 3.15 and Corollary 3.4. Hence $C_{\ell-1} \in \overline{P_C . D}$ and therefore $C \in \overline{P_C . D}$. □

Call an action of a group H on a topological space X *projective* if there is a representation of H into PGL(m, **R**) and a continuous injection of X into $\mathbf{P}^{m-1}(\mathbf{R})$ such that the action of H on X is the restriction of the action of PGL(m, **R**) on $\mathbf{P}^{m-1}(\mathbf{R})$.

3.20. *Lemma.* — *Let* $\mathfrak{p}$ *be the Lie algebra of* $P = P_C$ *for some chamber* C. *Then* P *is the normaliser of* $\mathfrak{p}$ *and the action of* G^0 *on* Cham Δ *is projective.*

Proof. — Let $g \in G^0$ normalise $\mathfrak{p}$ under the adjoint action. Then g normalises P^0. Hence P^0 fixes $D = gC$. Let T be the type of a minimal gallery from C to D. Then P^0 is transitive on T(C) by Lemma 3.15 and Corollary 3.4. This implies that C = D. Hence $g \in P$ and P contains the normaliser of $\mathfrak{p}$ in G^0. That P normalises $\mathfrak{p}$ is a general fact.

Let $k = \dim \mathfrak{p}$. It is standard that G^0/P embeds into the projectivised k-th exterior product $\mathbf{P}(\Lambda^k \mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of G. Moreover G^0 acts on $\mathbf{P}(\Lambda^k \mathfrak{g})$ via the adjoint action. □

3.21. *Theorem.* — *The stabiliser* P *in* G^0 *of a chamber is a parabolic subgroup of* G^0.

Proof. — It follows from Lemmata 3.19 and 3.20 that Cham $\Delta = G^0/P$ is a projective proximal G^0-space. As G^0 is transitive on Cham Δ by Lemma 3.15 and Corollary 3.3 we see that G^0 is transitive on Cham Δ. By Proposition 4.3 of [F], Cham Δ is a G^0-equivariant image of the boundary of G^0. The stabiliser of a point in the boundary of G^0 is a minimal parabolic subgroup of G^0. Hence P contains a minimal parabolic subgroup and thus P is a parabolic subgroup. □

We will see in the next section that P is in fact minimal.

4. Classification

Consider an infinite, irreducible, locally connected, compact, metric, topologically Moufang building Δ of rank at least 2. We know from the last section that G^0, the connected component of the identity of the topological automorphism group G, is a noncompact simple Lie group. As explained in the Introduction, the set of parabolic subgroups of G^0 forms a topological building $\widetilde{\Delta}$. In this section we will prove the Main Theorem of the Introduction, namely that Δ and $\widetilde{\Delta}$ are isomorphic as topological buildings.

4.1. *Lemma.* — *Let* $\mathcal{F}$ *be the set of pairs* (C, Σ) *consisting of an apartment* Σ *in* Δ *and a chamber* $C \in \Sigma$. *Then* G^0 *acts transitively on* $\mathcal{F}$. *Furthermore* G^0 *is a group of special automorphisms of* Δ.

Proof. — Let (C, Σ) and (C', Σ') be in $\mathcal{F}$. By [T1, 3.31] there is a chamber D opposite both C and C'. Let Σ_1 and Σ_2 be the apartments determined by D, C and D, C' respectively. By Lemmata 3.2 and 3.15 there are $g_i \in G^0$, $i = 1, 2, 3$, such that $g_1(C, \Sigma) = (C, \Sigma_1)$, $g_2(D, \Sigma_1) = (D, \Sigma_2)$ and $g_3(C', \Sigma_2) = (C', \Sigma')$. Moreover $g_2(C) = C'$. Hence G^0 is transitive on $\mathcal{F}$.

By Proposition 1.5 every connected component of Vert Δ is contained in the vertices of a given type. Thus G^0 preserves type. □

We need two purely combinatorial propositions which are due to J. Tits.

4.2. *Proposition.* — *Let* Δ *be a building and* $\mathscr{F}$ *as in Lemma* 4.1. *If* L *is a group of special automorphisms of* Δ *which acts transitively on* $\mathscr{F}$, *then* Δ *is isomorphic with the set* $\widetilde{\Delta}$ *of all subgroups of* L *conjugate to a subgroup containing the stabiliser of a given chamber, ordered by the inverse of the inclusion relation.*

Proof. — Fix $(C, \Sigma) \in \mathscr{F}$. Let B (respectively N) be the stabiliser of C (respectively Σ). By [T1, 3.11], (B, N) is a saturated BN-pair in L whose Weyl group W is the Weyl group of Σ. By [T1, 3.2.6], $\widetilde{\Delta}$ is a building. By the proof of [T1, 3.11], the distinguished generating set Ψ of W determined by (B, N) [T1, 3.2.1] is given by the reflections of Σ in the hyperfaces of C. Recall [T1, 3.2.2] that any subgroup $B' \supset B$ of L is generated by B and a subset $\Theta \subset \Psi$. Thus B' is the stabiliser of a face of C. Therefore the map sending $A \in \Delta$ to its stabiliser is an isomorphism of Δ with $\widetilde{\Delta}$ mapping apartments to apartments. □

4.3. *Proposition.* — *Let* Δ *be a building, let* I *be the set of types of vertices (identified with the set of vertices of the Coxeter graph), and let* J *be a subset of* I. *Assume that all entries in the Coxeter matrix are finite. Then the set* Δ' *of all faces of* Δ *whose type is contained in* J *is a building if and only if every connected component of* I *(in the Coxeter graph) either is entirely contained in* J *or has at most one element in* J.

Remark. — The assumption that the entries in the Coxeter matrix are finite is necessary. The proposition fails if the Coxeter matrix of Δ is $\begin{pmatrix} 1 & 3 & \infty \\ 3 & 1 & 3 \\ \infty & 3 & 1 \end{pmatrix}$. Such a Tits building is constructed in [MT].

We need three lemmas before we prove this proposition. The first is a special case of Lemma 3 of [T3]; we reprove it for the convenience of the reader.

4.4. *Lemma.* — *If* Σ *is an irreducible Coxeter complex with rank at least* 2, *then every root* Φ *contains a chamber that does not intersect* $\partial\Phi$.

Proof. — Use induction on rank Σ. The lemma is obvious when rank $\Sigma = 2$. If rank $\Sigma > 2$, there is a vertex t whose star is contained in Φ. Let

$$S = \{ E \in \operatorname{Star} t : E \cap \partial\Phi \neq \emptyset \}.$$

Then S is convex, since $S = \{ E \in \operatorname{Star} t : E \cap \bar{\Phi} \neq \emptyset \}$, where $\bar{\Phi}$ is the opposite root of Φ. Moreover $S \neq \operatorname{Star} t$, for otherwise Σ would be reducible. Hence S lies in a root of Star t and the inductive hypothesis shows that Star t contains a chamber that is not in S. □

4.5. *Definition.* — If A is an element of a chamber complex, Star′ A will denote the chamber complex formed by the faces complementary to A of the elements of Star A, together with the inclusion relation induced from Star A.

4.6. *Lemma.* — *Let p, q and r be the vertices of a chamber in a Coxeter complex of rank* 3, *and let Φ be a root with $q, r \in \partial\Phi$. Assume* Star r *is irreducible. Then q is the only vertex of its type in $\partial\Phi \cap$* Star′ p.

Proof. — Since $\partial\Phi \cap$ Star′ p is convex and contains a chamber in Star′ p, it can contain two vertices of type q only if it contains three consecutive vertices q, r^* and q^* of Star′ p. But then q, p and q^* would be three consecutive vertices of Star′ r^* with q and q^* both in the root wall $\partial\Phi \cap$ Star′ r^* of Star′ r^*. This would imply reducibility of Star′ r^* and hence of Star r. □

4.7. *Lemma.* — *Let Σ be an irreducible Coxeter complex with rank at least* 3. *Let i and j be distinct types of vertex in Σ. Assume that the ij-th entry m_{ij} of the Coxeter matrix is finite. Let Γ be the graph formed by the elements of Σ whose types are contained in $\{i, j\}$. Then the maximum distance of two vertices in Γ is greater than m_{ij}.*

Proof. — First assume that $m_{ij} = 2$. If the lemma were false, each vertex of type i would be adjacent in Γ to every vertex of type j. Choose a pair, Φ and $\overline{\Phi}$, of opposite roots in Σ. By Lemma 4.4, there are chambers $C \in \Phi$ and $\overline{C} \in \overline{\Phi}$ with $C \cap \Phi = \emptyset = \overline{C} \cap \partial\Phi$. The vertex of type i in C is not adjacent in Γ to the vertex of type j in $\overline{C}$ because any path joining them contains a vertex of $\partial\Phi$.

Now assume that $m_{ij} \geqslant 3$. By renaming i and j if necessary, we may assume that j is adjacent in the Coxeter graph to $k \notin \{i, j\}$. Choose $A \in \Sigma$ with type complementary to $\{i, j, k\}$. Let Γ_A be the graph formed by the elements of Star′ A with type contained in $\{i, j\}$. If p and q are vertices of Γ_A, their distance in Γ_A is the same as in Γ. For it is clear that $\mathrm{dist}_{\Gamma_A}(p, q) \geqslant \mathrm{dist}_{\Gamma}(p, q)$. On the other hand, since Star A is convex and contains a chamber, it is the image of an idempotent type-preserving morphism φ of Σ, see [T1, 2.19, 2.20]. The image under φ of a path joining p and q in Γ is a path of the same length joining them in Γ_A.

Henceforth we work in Star′ A. Fix a vertex c of type k. Since Star c is convex and contains a chamber, the argument used above shows that the distance in Γ_A between two vertices of Star′ c is realized by a path in Star′ c. Choose $a \in$ Star′ c with type i. Since m_{ij} is finite there is a vertex $\bar{a}$ opposite a in Star′ c. Clearly $\mathrm{dist}_{\Gamma_A}(a, \bar{a}) = m_{ij}$. Since Star′ a is irreducible, it contains at least three vertices of type j. Exactly two of these, b and b' say, lie in Star′ c. Thus there is a vertex b'' of type j such that $b'' \in$ Star′ a and $b'' \notin$ Star′ c.

We show that $\mathrm{dist}_{\Gamma_A}(b'', \bar{a}) > m_{ij}$. Note that Star′ A is irreducible, since j is adjacent in the Coxeter graph to both i and k. By Lemma 4.4, there is a root Φ of Star′ A such that $a \cup c \subseteq \Phi$ and $(a \cup c) \cap \partial\Phi = \emptyset$. It is clear that $b'', \bar{a} \in \Phi$ and $\mathrm{dist}_{\Gamma_A}(b'', \bar{a})$

is realized by a path in $\Gamma_A \cap \Phi$. Since Star c is irreducible, Lemma 4.6 shows that b'' does not lie in either of the root walls in Star′ A defined by $b \cup c$ and $b' \cup c$. It follows that b'' and $\bar{a}$ lie on opposite sides of each of the $m_{ij} - 1$ root-walls that pass through c and do not contain a and $\bar{a}$. Any path from b'' to $\bar{a}$ in $\Gamma_A \cap \Phi$ contains at least one vertex from each of these root-walls. The only vertex of Φ that two of these root-walls can have in common is c, and $c \notin \Gamma_A$. Hence $\mathrm{dist}_{\Gamma_A}(b'', \bar{a}) \geqslant m_{ij}$. Since b'' is adjacent in Γ_A to a, it follows that $\mathrm{dist}_{\Gamma_A}(b'', \bar{a}) \geqslant m_{ij} + 1$. □

Proof of Proposition 4.3. — Clearly we can assume that Δ is irreducible.

(i) It is clear that Δ' is a building if $\mathrm{card}\, J = 1$ or $J = I$.

(ii) Assume $2 = \mathrm{card}\, J < \mathrm{card}\, I$. Let Σ be an apartment of Δ. Since Σ is the image of an idempotent type-preserving morphism of Δ [T1, 3.3], the distance in Δ' between two vertices of $\Delta' \cap \Sigma$ is realized by a path in Σ. It follows from Lemma 4.7 that $\mathrm{diam}\, \Delta' > m_{ij}$. On the other hand, if $A \in \Delta$ has type $I \setminus J$, Star′ A is a non-stammering closed gallery in Δ' with $2m_{ij}$ chambers, and so $\mathrm{diam}\, \Delta' \leqslant m_{ij}$ by Corollary 0.6. Thus Δ' cannot be a Tits building.

(iii) Assume $3 \leqslant \mathrm{card}\, J < \mathrm{card}\, I$. Choose $J' \subset J$ with $\mathrm{card}\, J' = 2$. Fix $b \in \Delta$ with type $J \setminus J'$. If Δ' is a Tits building, so is $\mathrm{Star}'_{\Delta'} B$. But this would contradict (ii), since $\mathrm{Star}'_{\Delta'} B = \{ E \in \mathrm{Star}'_{\Delta} B : \mathrm{type}\, E \subseteq J' \}$. □

4.8. *Proof of the Main Theorem.* — Let Δ, G, G^0 and $\Delta(G^0)$ be as in the Main Theorem. Let P be the stabiliser in G^0 of a chamber of Δ. By Lemma 4.1, Proposition 4.2 and Theorem 3.2.1, Δ is isomorphic with the building of all parabolic subgroups of G^0 containing a conjugate of P. Since G^0 is simple by Theorem 3.18, the Coxeter graph of $\Delta(G^0)$ is connected. Applying Proposition 4.3 to $\Delta(G^0)$ with $J = \{ P' \subset G^0 \mid P'$ is a maximal parabolic subgroup, $P' \supset P \}$, shows that either $\Delta = \Delta(G^0)$ or P is a maximal parabolic subgroup. Since $\mathrm{rank}\, \Delta \geqslant 2$, $\Delta = \Delta(G^0)$. □

5. Topological Buildings of Rank Greater Than 2

We show that irreducible topological buildings of rank greater than 2 are topologically Moufang. This is the topological analogue of Satz 1 of [T3].

5.1. *Proposition.* — *If Δ is an irreducible, compact, metric building of rank at least* 3, *then Δ is topologically Moufang.*

Proof. — Let H be a half-apartment contained in two apartments Σ and Σ'. Let $\alpha : \Sigma \to \Sigma'$ be an isomorphism that fixes every element of H. By Lemma 4.4 there is a chamber $C \in H$ that does not intersect ∂H. Let $E_i(C) = \{ D : D \cap C$ has codimension at most $i \}$. By [T1, 4.16, 4.1.1] there is a unique isomorphism $\beta : \Delta \to \Delta$ that extends α and is the identity on $E_2(C)$. We have to show that β is continuous. First we use the technique of [T1, 4.1.1] to show that β is continuous on the stars of all hyperfaces. We show by induction on dist(C', C) that $\beta \mid E_1(C')$ is continuous for all $C' \in$ Cham Δ. This is clear when dist(C', C) = 0, so assume that dist(C, C') > 0 and $\beta \mid E_1(C')$ is continuous. Let $C'' \in$ Cham Δ be adjacent to C' with dist(C, C'') = dist(C, C') − 1. By [T1, 4.2] there is $E \in \Sigma$ opposite $C' \cap C''$. By [T1, 3.31] there is a chamber $D \in$ Star E opposite both C' and C''. Let B' be a hyperface of C'. Denote by B the face of D opposite B' and let B'' be the face of C'' opposite B. By the inductive hypothesis, $\beta \mid$ Star B'' is continuous. Since β commutes with the projections from Star B'' to Star B and from Star B to Star B' respectively, it is clear that $\beta \mid$ Star B' is continuous.

Finally we show that β is continuous by an inductive argument similar to the proof of Assertion 2.9. For $0 \leqslant i \leqslant$ diam Δ, let $\Sigma^i = \{ X \in \Delta :$ there are $Y \in$ Cham Star X and $Z \in$ Cham Σ with dist(Y, Z) $\leqslant i\}$. We have shown above that β is continuous on Σ^1. We now show that β is continuous on Σ^{i+1}, assuming it is continuous on Σ^i. It suffices to prove that β is continuous on Cham Σ^{i+1}. If not, there are $C \in$ Cham Σ^{i+1} and $\{ C_n \} \subseteq$ Cham Σ^{i+1} such that $\lim_{n\to\infty} C_n = C$ and $\lim_{n\to\infty} \beta C_n \neq \beta C$. For each n, let A_n be a hyperface of C_n that is in Σ^i. By passage to a subsequence, we can assume that $\{ A_n \}$ converges to a hyperface A of C. Note that $A \in \Sigma^i$, since Σ^i is closed, and hence $\beta A_n \to \beta A$. By [T1, 4.2], there is a hyperface A' of Σ opposite A. By Proposition 1.9, we can assume that each A_n is opposite A'. Let $C'_n = \mathrm{proj}_{A'} C_n$ and $C' = \mathrm{proj}_{A'} C$. Then $C'_n \to C'$ by Proposition 1.10. Since $C'_n, C' \in \Sigma^1$, $\beta C'_n \to \beta C'$. Now

$$\beta C_n = \beta(\mathrm{proj}_{A_n} C'_n) = \mathrm{proj}_{\beta A_n}(\beta C'_n)$$

since β is a combinatorial morphism of Δ. Moreover $\beta A'$ is opposite βA and each βA_n. It follows from Proposition 1.10 that $\beta C_n \to \mathrm{proj}_A C' = C$, contrary to the choice of $\{ C_n \}$ and C. □

5.2. *Corollary.* — *An infinite, irreducible, locally connected, compact, metric building of rank at least* 3 *is classical.* □

These results constitute the last theorem of the Introduction.

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