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On the differential equations satisfied by period matrices

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ON THE DIFFERENTIAL EQUATIONS SATISFIED BY PERIOD MATRICES

by NICHOLAS M. KATZ

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INTRODUCTION

Picard-Fuchs Equations.

Let X be a non-singular projective curve of genus g, defined over a field K of characteristic zero. Recall that a meromorphic differential is said to be of the second kind if all of its residues are zero, that exact differentials are all of the second kind, and that the quotient space is of dimension $2g$ over K. (When $K = \mathbf{C}$ we may take $\gamma_1, \ldots, \gamma_{2g}$ a basis of $H_1(X, \mathbf{C})$, and map this quotient isomorphically to $\mathbf{C}^{2g}$ by $\omega \mapsto \left(\int_{\gamma_1} \omega, \ldots, \int_{\gamma_{2g}} \omega\right)$; thus the quotient is dual to $H_1(X, \mathbf{C})$, and so identified with $H^1(X, \mathbf{C})$.) Thus the K-space of differentials of the second kind modulo exact differentials provides a cohomology group which is defined over K, and defined in a purely algebraic manner [6 *a*].

Suppose X depends on certain parameters — i.e. that the field K admits non-zero derivations. Heuristically, the periods $\int_{\gamma_i}\omega$ are functions of the parameters, and hence are susceptible to differentiation with respect to the parameters. It was first observed by Fuchs that, given ω, the $2g$ periods $\int_{\gamma_i}\omega$ are all solutions of a linear differential equation of degree $2g$ which has coefficients in K, the Picard-Fuchs equation.

This comes about as follows. Let x be a non-constant function, so that the function field of X is a finite extension of $K(x)$; every derivation D of K may be extended to $K(x)$ by requiring that $D(x)=0$, and thus to a derivation of the function field of X. We call this derivation D_x to indicate the dependence on x. Similarly, $\frac{d}{dx}$ is the derivation of $K(x)$ which annihilates K and has value 1 at x; its extension to all functions is also denoted $\frac{d}{dx}$. It is easily seen that D_x and $\frac{d}{dx}$ commute, since their commutator annihilates both K and x. Finally, D_x acts on differentials by $D_x(fdg)=D_x(f)dg+fd(D_x(g))$, or, more simply, by $D_x(fdx)=D_x(f)dx$. Because $D_x(dg)=d(D_xg)$, D_x preserves exactness.

The formula $\mathrm{res}_p(D_x(\omega))=D(\mathrm{res}_p(\omega))$ insures that D_x preserves the differentials of the second kind. By passage to quotients, D_x acts as a derivation of the differentials of the second kind, modulo exact differentials. On the quotient space, the action of D_x is independent of x, because, if y is another non-constant function, $(D_y-D_x)(fdx)=d(fD_y(x))$.

Thus, the quotient space has a canonical structure of module over the algebra of derivations of the base field K. Hence, for each ω of the second kind, the differentials $\omega, D_x\omega, D_x^2(\omega), \ldots, D_x^{2g}(\omega)$, must be K-dependent modulo the exact differentials, i.e., there are $a_0, a_1, \ldots, a_{2g}\in K$ and a function g with $\sum_{i=0}^{2g} a_i D_x^i(\omega)=dg$. Integrating over the homology class γ_j gives the equation $\sum_{i=0}^{2g} a_i D_x^i \int_{\gamma_j}\omega=\int_{\gamma_j}dg=0$. Equivalently, let $\omega_1, \ldots, \omega_{2g}$ be a K-basis of differentials of the second kind modulo exact differentials; each derivation D of K gives rise to the system of equations $D\omega_i=\sum_{j=1}^{2g} a_{ji}\omega_j$, (modulo exact differentials), with $a_{ji}\in K$.

The situation for non-singular X of higher dimension is more involved. For a good notion of cohomology over K we must turn to the hypercohomology of the complex $\Omega_X^{\bullet}$ of sheaves of germs of holomorphic algebraic forms [4]. It should be remarked here that the analogue of Leray's theorem allows the hypercohomology to be obtained as the total homology of the bicomplex $(C^p(\Omega^q, \mathfrak{U}))$, where $\mathfrak{U}=\{U_i\}$ is any covering of X by affine open sets.

Let us compute the one-dimensional hypercohomology group when X is a curve. We may take the covering $\mathfrak{U}=\{U_1, U_2\}$ to consists of the complements of two disjoint finite sets of points. The cocycles in $C^0(\Omega^1, \mathfrak{U})\oplus C^1(\Omega^0, \mathfrak{U})$ are the triples, $(\omega_1, \omega_2, f_{12})$, where each ω_i is a 1-form regular on U_i, f_{12} is a function regular on $U_1 \cap U_2$, and $\omega_1 - \omega_2 + df_{12} = 0$. Thus ω_1 and ω_2 are differentials of the second kind, whose difference is an exact differential. The coboundaries are those triples $(df_1, df_2, f_1 - f_2)$ where each f_i is a function regular on U_i. The mapping $(\omega_1, \omega_2, f_{12}) \mapsto \omega_1$ establishes the desired identification of this hypercohomology group with the differentials of the second kind modulo exact differentials.

We continue with this example, and introduce the action of a derivation D of K. Select functions, x and y, so that at every point $\mathfrak{p} \in X$ either $x - x(\mathfrak{p})$ or $y - y(\mathfrak{p})$ is a uniformizing parameter. Let U_1 be those $\mathfrak{p}$ where $x - x(\mathfrak{p})$ is a uniformizing parameter, and take, for U_2, the analogous set for y. It follows [6 *a*] that D_x and D_y, respectively, are stable on the functions regular on U_1 and U_2, respectively, and that both are stable on functions regular on $U_1 \cap U_2$. Define $\lambda : \Omega^1 \to \Omega^0$, by $\lambda(f dx) = f D_y(x)$. The mapping on 1-cochains of the bicomplex, given by $(\omega_1, \omega_2, f_{12}) \mapsto (D_x \omega_1, D_y \omega_2, \lambda\omega_2 + D_x f_{12})$, preserves cocycles and coboundaries, and so induces a mapping on the one-dimensional hypercohomology. This mapping gives the action of D on differentials of the second kind, modulo exact differentials.

In the higher-dimensional case, analogous formulae will, under restrictive hypotheses (see (1.6)), endow the bicomplex $(C^p(\Omega^q, \mathfrak{U}))$ with the structure of module over the algebra of derivations of K, and thus allow the differentiation of cohomology classes. This construction is presumably a special case of Grothendieck's " Gauss-Manin connection " [1], but in any case our restrictive hypotheses are satisfied by principal affine open subsets of non-singular hypersurfaces.

The differentials of the second kind in higher dimensions no longer give the cohomology, as they did for curves. Indeed, a closed meromorphic differential ω on a projective non-singular X is said to be of the second kind if there is an affine open set U, on which ω is holomorphic, such that the cohomology class on U determined by ω lies in the image of the restriction mapping $H^*(X) \to H^*(U)$. This mapping is seldom an injection (except for $H^1(X) \to H^1(U)$), although for sufficiently nice U, one can determine the kernel (1.11).

Because differentiation of cohomology commutes, whenever it is defined, with the restriction $H^*(X) \to H^*(U)$, the image of $H^*(X)$ in $H^*(U)$ will be stable under differentiation, thus giving rise to a Picard-Fuchs equation for the subspace of $H^*(U)$ spanned by differentials of the second kind.

Parameters and the Zeta Function.

Recall that the zeta function of a variety, V, defined over GF(q), the field of q elements, is the power series, $\exp\left(\sum_{s \geq 1} \frac{N_s \tau^s}{s}\right)$, where N_s is the number of points on V

(1) (Added in proof.) It is. A general algebraic construction of the Gauss-Manin connection is given by T. Oda and the author in « On the differentiation of De Rham cohomology classes with respect to parameters », to appear in *Kyoto Journal of Maths.*

whose coordinates lie in $GF(q^s)$. This function, which we will write $Z(\tau, V/GF(q))$, is known (Dwork [1]) to be a rational function of τ, with rational integral coefficients.

Consider now a " variable " variety, V_Γ, defined over $GF(q)[\Gamma]$. For a special $\Gamma_0 \in GF(q^d)$ the special variety V_{Γ_0} is defined over $GF(q^d)$, and thus we may speak of the zeta function $Z(\tau, V_{\Gamma_0}/GF(q^d))$. We ask for its dependence upon Γ_0.

Let us begin by studying the elliptic curve E_λ given by $y^2 = x(x-1)(x-\lambda)$ (here $p \neq 2$). Let $H(\lambda) = (-1)^{\frac{p-1}{2}} \sum_{j=0}^{\frac{p-1}{2}} \binom{-\frac{1}{2}}{j}^2 \lambda^j$, the Hasse invariant, and denote by $N_s(\lambda_0)$ the number of points (including the point at infinity) on E_{λ_0} with coordinates in $GF(p^s)$, where it is understood that $\lambda_0 \in GF(p^s)$. In this case, it is well known that

$$H(\lambda_0)H(\lambda_0^p)\dots H(\lambda_0^{p^{s-1}}) \equiv 1 - N_s(\lambda_0) \pmod{p}$$

and hence

$$Z(\tau, E_{\lambda_0}/GF(p^s)) \equiv \frac{1 - H(\lambda_0)H(\lambda_0^p)\dots H(\lambda_0^{p^{s-1}})\tau}{1-\tau} \pmod{p}$$

More precisely

$$Z(\tau, E_{\lambda_0}/GF(p^s)) = \frac{(1-\omega(\lambda_0)\tau)(1-\omega(\lambda_0)^{-1}p^s\tau)}{(1-\tau)(1-p^s\tau)}.$$

where $\omega(\lambda_0) \equiv H(\lambda_0)\dots H(\lambda_0^{p^{s-1}}) \pmod{p}$. Clearly, when $H(\lambda_0) \not\equiv 0 \pmod{p}$, one reciprocal zero, $\omega(\lambda_0)$, of $Z(\tau, E_{\lambda_0}/GF(p^s))$, is distinguished by being a p-adic unit; it has been determined analytically [2 *a*].

Consider the hypergeometric series, $F(1/2, 1/2, 1, t) = \sum_{j \geq 0} \binom{-1/2}{j}^2 t^j$, as function of the p-adic variable t; it is convergent in $|t| < 1$ (here $p \neq 2$). The function $U(t) = F(1/2, 1/2, 1, t)/F(1/2, 1/2, 1, t^p)$ extends analytically to the region $|t| \leq 1$, $|H(t)| = 1$ (we now regard H as a p-adic function). Let t_0 be the Teichmüller representative of λ_0, and suppose $|H(t_0)| = 1$; then the previously distinguished $\omega(\lambda_0)$ is given by $\omega(\lambda_0) = (-1)^{\frac{p^s-1}{2}} \prod_{j=0}^{s-1} U(t_0^{p^j})$. One might say that the hypergeometric series analytically determines the zeta function of the family E_λ.

Finally, we remark that $F\left(\frac{1}{2}, \frac{1}{2}, 1, t\right)$ is annihilated by the differential operator, $4t(1-t)\frac{d^2}{dt^2} + 4(1-2t)\frac{d}{dt} - 1$, corresponding to the Picard-Fuchs equation,

$$\left(4\lambda(1-\lambda)\frac{d^2}{d\lambda^2} + 4(1-2\lambda)\frac{d}{d\lambda} - 1\right)\left(\frac{dx}{y}\right) = -d\left(\frac{y}{(x-\lambda)^2}\right).$$

Dwork's deformation theory generalizes these results to " good " families of hypersurfaces. Consider a one-parameter family, X_Γ, of hypersurfaces in $\mathbf{P}^{n+1}(\Omega)$, where Ω is the completion of the algebraic closure of $\mathbf{Q}_p$. We envision a defining form $F(X, \Gamma)$, whose coefficients are polynomials in Γ with Ω-integral coefficients, and we suppose these coefficients to reduce (mod p) to elements of $GF(q)$, thus defining a family over $GF(q)[\Gamma]$.

The specialized hypersurface, X_{Γ_0}, corresponding to the special value $\Gamma_0 \in \Omega$, is said to be non-singular, and in general position, if the forms $X_i \frac{\partial}{\partial X_i} F(X, \Gamma_0)$, $1 \leq i \leq n+2$, have no common zero. By elimination theory, there is a polynomial $R(\Gamma)$, with Ω-integral coefficients, such that the specialized X_{Γ_0} is non-singular, and in general position, precisely when Γ_0 is not a zero of $R(\Gamma)$. Applying this result in characteristic p, we find that, for ord $\Gamma_0 \geq 0$, the specialized X_{Γ_0} and its reduction (mod p) will both be non-singular, and in general position, so long as Γ_0 lies in the region $|\Gamma_0| \leq 1$, $|R(\Gamma_0)| = 1$; it is in this region that the theory of Dwork applies.

This theory associates to each Γ_0 with $|\Gamma_0| \leq 1$, $|R(\Gamma_0)| = 1$, a finite dimensional Ω-space $W^S(\Gamma_0)$, and a mapping, $\alpha(\Gamma_0) : W^S(\Gamma_0) \to W^S(\Gamma_0^q)$. For $|\Gamma_0 - \Gamma_1| < 1$, there is given a canonical isomorphism, $C(\Gamma_0, \Gamma_1) : W^S(\Gamma_0) \to W^S(\Gamma_1)$, which gives a commutative diagram

$$\begin{array}{ccc} W^S(\Gamma_0) & \xrightarrow{\alpha(\Gamma_0)} & W^S(\Gamma_0^q) \\ \Big\downarrow{\scriptstyle C(\Gamma_0,\Gamma_1)} & & \Big\downarrow{\scriptstyle C(\Gamma_0^q,\Gamma_1^q)} \\ W^S(\Gamma_1) & \xrightarrow{\alpha(\Gamma_1)} & W^S(\Gamma_1^q) \end{array}$$

Observe that when $\Gamma_0^{q^s} = \Gamma_0$ and $|R(\Gamma_0)| = 1$, the reduction (mod p) of X_{Γ_0} is defined over $GF(q^s)$, and the composition, $\alpha(\Gamma_0^{q^{s-1}}) \dots \alpha(\Gamma_0^q)\alpha(\Gamma_0)$, is an endomorphism of $W^s(\Gamma_0)$. Dwork related the number $N_s(\Gamma_0)$ of $GF(q^s)$-rational points on the reduction (mod p) of X_{Γ_0} to the trace of this composition:

$$N_s(\Gamma_0) = \frac{1 - q^{s(n+1)}}{1 - q^s} + \frac{(-1)^n \operatorname{trace} \alpha(\Gamma_0^{q^{s-1}}) \dots \alpha(\Gamma_0^q)\alpha(\Gamma_0)}{q^s}.$$

Further, we may choose simultaneously bases for the $W^S(\Gamma_0)$, in terms of which the matrix coefficients of $\alpha(\Gamma)$ are analytic functions on the region $|\Gamma| \leq 1$, $|R(\Gamma)| = 1$. (Recall that a function on such a region is analytic (in the sense of Krasner) if it is the uniform limit of rational functions regular on that domain.) The matrix of $C(0, \Gamma)$, on the other hand, has entries which are merely convergent power series for $|\Gamma| < 1$. However for $|\Gamma| < 1$, the relation

$$\alpha(\Gamma) = C(0, \Gamma^q)\alpha(0)C(\Gamma, 0)$$

holds identically in Γ. It follows, from the analyticity of the matrix entries of $\alpha(\Gamma)$, that the matrix entries of $C(0, \Gamma^q)\alpha(0)C(0, \Gamma)$, which apparently exist only for $|\Gamma| < 1$, are, in fact, analytic throughout the region $|\Gamma|\leq 1$, $|R(\Gamma)|=1$. Further, the uniqueness theorem for analytic functions shows that the relation

$$\alpha(\Gamma^{q^s})\ldots\alpha(\Gamma^q)\alpha(\Gamma)=C(0, \Gamma^{q^s})\alpha(0)C(0, \Gamma)^{-1}$$

is valid for $|\Gamma|\leq 1$, $|R(\Gamma)|=1$.

Thus, the matrix of $C(0, \Gamma)$ provides the analytic continuation of the zeta function from $\Gamma=0$ to the region $|\Gamma|\leq 1$, $|R(\Gamma)|=1$, in much the same way as the hypergeometric series analytically determines the zeta function of elliptic curves in the family E_λ. The analogy goes even deeper, for the matrix of $C(0, \Gamma)$ satisfies a differential equation. This arises as follows.

The generic space, $W^S(\Gamma)\otimes\Omega(\Gamma)$, is, in a natural way, a module for $\dfrac{\partial}{\partial\Gamma}$, by means of an action $\mathfrak{S}_\Gamma$, which arises formally as the twisting

$$\mathfrak{S}_\Gamma=C(0, \Gamma)\,.\,\frac{\partial}{\partial\Gamma}\,.\,C(\Gamma, 0)$$

where $\dfrac{\partial}{\partial\Gamma}$ operates in $W^S(0)\otimes\Omega(\Gamma)$ through the second factor. Because $\dfrac{\partial}{\partial\Gamma}$ annihilates the Ω-space $W^S(0)$, $\mathfrak{S}_\Gamma$ annihilates the Ω-space $C(0, \Gamma)(W^S(0))$. Let us write W_0 and W_Γ for the column vectors whose components are, respectively, the Ω-basis for $W^S(0)$ and the $\Omega(\Gamma)$-basis of $W^S(\Gamma)$, by means of which our matrix representation of $C(0, \Gamma)$ is given. As $\mathfrak{S}_\Gamma$ acts on the $\Omega(\Gamma)$-space, $W^S(\Gamma)\otimes\Omega(\Gamma)$, we may write

$$\mathfrak{S}_\Gamma(W_\Gamma)=B(\Gamma)\,.\,W_\Gamma$$

where $B(\Gamma)$ is a matrix of rational functions. To avoid confusion between matrices and mappings, let $C(\Gamma)$ denote the matrix of the mapping $C(0, \Gamma)$; then $C(0, \Gamma)(W_0)=C(\Gamma)\,.\,W_\Gamma$, and we have the equation

$$\begin{aligned} 0=\mathfrak{S}_\Gamma(C(\Gamma)\,.\,W_\Gamma)&=\frac{\partial C(\Gamma)}{\partial\Gamma}\,.\,W_\Gamma+C(\Gamma)\mathfrak{S}_\Gamma\,.\,W_\Gamma\\ &=\left(\frac{\partial C(\Gamma)}{\partial\Gamma}+C(\Gamma)B(\Gamma)\right).\,W_\Gamma \end{aligned}$$

whence $\dfrac{\partial C(\Gamma)}{\partial\Gamma}=-C(\Gamma)B(\Gamma)$.

In [3] Dwork computed the Picard-Fuchs equation for the family of elliptic curves, $X^3+Y^3+Z^3-3\Gamma XYZ$, and found it to be (in suitable bases)

$$\frac{\partial P(\Gamma)}{\partial\Gamma}=B(\Gamma)P(\Gamma)$$

where $P(\Gamma)$ is the period matrix of differentials of the second kind modulo exact differentials.

It should be remarked that the result is of an algebraic nature: the generic space $W^S(\Gamma)$, the operation $\mathfrak{S}_\Gamma$, and the Picard-Fuchs equations are all defined over $\Omega(\Gamma)$.

The present paper grew out of attemps to identify the differential equation for $C(0, \Gamma)$ with a Picard-Fuchs equation, in greater generality. This is achieved by

Theorem **(1.19)**. — *Let* X *be a projective hypersurface in* $\mathbf{P}^{n+1}$, *defined over a field*, K, *of characteristic zero, which is nonsingular and in general position. Denote by* X^0 *the open subset where no coordinate vanishes, and by* $H^n(X, K)$ *and* $H^n(X^0, K)$ *the n-dimensional cohomology groups of* X *and* X^0 *respectively. We regard* $H^n(X, K)$ *and* $H^n(X^0, K)$ *as modules over the algebra of derivations of* K, *as explained earlier, while the* K-*space* W^S, *associated to* X *by Dwork's theory, admits the derivations of* K *by means of the* $\mathfrak{S}$ *operation. There is an isomorphism* Θ *of* W^S *with the image of* $H^n(X, K)$ *in* $H^n(X^0, K)$, *which is an isomorphism of modules over the algebra of derivations of* K. *Furthermore, the kernel of* $H^n(X, K) \to H^n(X^0, K)$ *is reduced to zero for n odd, and is one-dimensional for n even* (*and* $n>0$).

We should point out that the matrix $C(\Gamma)$ is the transposed inverse of the matrix denoted by the same symbol in Dwork [3, p. 262], which arises by passing from the " dual space " at 0 to the generic dual space at Γ; thus that matrix is the transpose of the (suitably normalized) period matrix. (Also the B matrix is the transpose of its analogue there.)

Explicit Computations.

To explain how Theorem (1.19) comes about, it is necessary to examine the spaces of Dwork in some detail. We fix integers $n \geq 0$ and $d>0$, and define $\mathscr{L}$ to be the K-linear span of monomials $Z^{w_0}X_1^{w_1}\dots X_{n+2}^{w_{n+2}}$ with $w_i \in \mathbf{Z}$, $dw_0 = \sum_i w_i \geq 0$. $\mathscr{L}$ contains certain subspaces of interest to us:

$$\begin{array}{lll} \mathscr{L}^0, & \text{where} & w_0 \geq 1 \\ \mathscr{L}^+, & \text{where all} & w_i \geq 0 \\ \mathscr{L}^{0,+}, & \text{is} & \mathscr{L}^0 \cap \mathscr{L}^+ \\ \mathscr{L}^S, & \text{where all} & w_i \geq 1. \end{array}$$

We fix a non-zero constant, π, and corresponding to each defining form F, of degree d in $X_1, \dots, X_{n+2}$, we define twisted operators on $\mathscr{L}$

$$D_{X_i} = \exp(-\pi ZF) . X_i \frac{\partial}{\partial X_i} . \exp(\pi ZF) = X_i \frac{\partial}{\partial X_i} + \pi Z X_i \frac{\partial F}{\partial X_i}$$

$$D_Z = \exp(-\pi ZF) . Z \frac{\partial}{\partial Z} . \exp(\pi ZF) = Z \frac{\partial}{\partial Z} + \pi ZF.$$

This construction is rational over any field K, which contains π and the coefficients of the form F. For each derivation D of K, we define the twisted derivation $\mathfrak{S}_D$ of $\mathscr{L}$ as K-space by

$$\mathfrak{S}_D = \exp(-\pi ZF) . D . \exp(\pi ZF) = D + \pi ZF^D.$$

The fundamental problem is to construct a generic mapping from $\mathscr{L}^0$ to $H^n(X^\emptyset)$. Corresponding to each element ZX^w is the differential form, regular on $X^\emptyset$, given, in local coordinates, by

$$\frac{X^w}{X_{n+1}\dfrac{\partial F}{\partial X_{n+1}}}\frac{d(X_1/X_{n+2})}{X_1/X_{n+2}}\wedge\ldots\wedge\frac{d(X_n/X_{n+2})}{X_n/X_{n+2}}.$$

For the generic F, given by $\sum_{|w|=d} A_w X^w$ with the A_w independent, $\mathfrak{S}_{A_w}$ is given by $\dfrac{\partial}{\partial A_w}+\pi ZX^w$, so that $\mathscr{L}^0$, considered as module for the derivations of K, is spanned by its elements of Z-degree one. Thus, generically, there is at most one homomorphism from $\mathscr{L}^0$ to $H^n(X^\emptyset)$ which is given as above on elements of Z-degree one and which respects the derivations of K.

That this map exists, and that it annihilates $D_Z\mathscr{L}+\sum_i D_{X_i}\mathscr{L}^0$, is almost the content of Theorem 1. We regard $H^n(X^\emptyset)$ as coming from $\sum_{p+q=n} C^p(\Omega^q)$, for a suitable covering; in Theorem 1 we examine $C^0(\Omega^n)$, and in Lemma (1.8) we turn to $C^1(\Omega^{n-1})$.

In this way we obtain a surjection

$$\Theta : \mathscr{L}^0/(\sum_i D_{X_i}\mathscr{L}^0+D_Z\mathscr{L}) \to H^n(X^\emptyset)$$

with $\Theta(\mathscr{L}^S)$ lying in the image of $H^n(X)$ in $H^n(X^\emptyset)$.

We then compute the Betti numbers of $X^\emptyset$ and $\mathbf{P}^{n+1}_{n+2}-X^\emptyset$ ($\mathbf{P}^{n+1}_{n+2}$ is the open subset of $\mathbf{P}^{n+1}$ with all $X_i \neq 0$), making strong use of the assumption of general position and the explicit formulas of Hirzebruch [5]. Then we construct an isomorphism, suggested by [9],

$$\mathscr{R} : \mathscr{L}^0/(\sum_i D_{X_i}\mathscr{L}^0) \to H^{n+1}(\mathbf{P}^{n+1}_{n+2}-X^\emptyset)$$

by defining

$$\mathscr{R}(Z^{w_0}X^w)=\frac{(w_0-1)!}{(-\pi)^{w_0-1}}\frac{X^w}{F^{w_0}}\frac{d(X_1/X_{n+2})}{X_1/X_{n+2}}\wedge\ldots\wedge\frac{d(X_{n+1}/X_{n+2})}{X_{n+1}/X_{n+2}}.$$

It is clear that $\mathscr{R}(\mathscr{L}^S)$ is the image of $H^{n+1}(\mathbf{P}^{n+1}-X)$ in $H^{n+1}(\mathbf{P}^{n+1}_{n+2}-X^\emptyset)$.

The regularity condition, that the $X_i\dfrac{\partial F}{\partial X_i}$ have no common zero, insures that $\mathscr{L}^0\subset\sum_i D_{X_i}\mathscr{L}^0+$ elements of Z-degree one. We explicity compute in local coordinates $x_i=X_i/X_{n+2}$, $f=\dfrac{F}{X^d_{n+2}}$

$$\mathscr{R}(ZX^w)=\frac{x^w}{f}\frac{dx_1}{x_1}\wedge\ldots\wedge\frac{dx_{n+1}}{x_{n+1}}=\frac{x^w}{x_{n+1}\dfrac{\partial f}{\partial x_{n+1}}}\frac{dx_1}{x_1}\wedge\ldots\wedge\frac{dx_n}{x_n}\wedge\frac{df}{f}$$

and hence

$$\Theta(ZX^w)=\text{Residue}(\mathscr{R}(ZX^w)).$$

The final result is best expressed by a commutative diagram

$$\begin{tikzcd}
 & & \mathrm{H}^{n+1}(\mathbf{P}^{n+1}-\mathrm{X}) \arrow[r, "\text{residue}"] \arrow[d, "\text{restriction}"] & \mathrm{H}^{n}(\mathrm{X}) \arrow[d, "\text{restriction}"] & \\
0 \arrow[r] & \mathrm{H}^{n+1}(\mathbf{P}^{n+1}_{n+2}) \arrow[r, "\text{restriction}"] & \mathrm{H}^{n+1}(\mathbf{P}^{n+1}_{n+2}-\mathrm{X}^{\emptyset}) \arrow[r, "\text{residue}"] & \mathrm{H}^{n}(\mathrm{X}^{\emptyset}) \arrow[r] & 0 \\
0 \arrow[r] & (\mathrm{ZF}) \arrow[r] \arrow[u, "\wr"] & \mathscr{L}^{0}/(\sum_i \mathrm{D}_{\mathrm{X}_i}\mathscr{L}^{0}) \arrow[r] \arrow[u, "\wr\mathscr{R}"] & \mathscr{L}^{0}/(\mathrm{D}_{\mathrm{Z}}\mathscr{L}+\sum_i \mathrm{D}_{\mathrm{X}_i}\mathscr{L}^{0}) \arrow[r] \arrow[u, "\wr\Theta"] & 0 \\
0 \arrow[r] & (\mathrm{ZF}) \arrow[r] \arrow[u, "\wr"] & \mathscr{L}^{0,+}/(\sum_i \mathrm{D}_{\mathrm{X}_i}\mathscr{L}^{0,+}) \arrow[r] \arrow[u, "\wr"] & \mathrm{W}=\mathscr{L}^{0,+}/(\sum_i \mathrm{D}_{\mathrm{X}_i}\mathscr{L}^{0,+}+\mathrm{D}_{\mathrm{Z}}\mathscr{L}^{+}) \arrow[r] \arrow[u, "\wr"] & 0 \\
 & & \mathscr{L}^{\mathrm{S}}/(\mathscr{L}^{\mathrm{S}}\cap\sum_i \mathrm{D}_{\mathrm{X}_i}\mathscr{L}^{0,+}) \arrow[r] \arrow[u, phantom, "\cup"] & \mathrm{W}^{\mathrm{S}}=\mathscr{L}^{\mathrm{S}}/(\mathscr{L}^{\mathrm{S}}\cap(\mathrm{D}_{\mathrm{Z}}\mathscr{L}^{+}+\sum_i \mathrm{D}_{\mathrm{X}_i}\mathscr{L}^{0,+})) \arrow[u, phantom, "\cup"] &
\end{tikzcd}$$

The P-adic computation of the Zeta Function.

Let $\overline{\mathrm{F}}(\mathrm{X})=\overline{\mathrm{F}}(\mathrm{X}_1,\ldots,\mathrm{X}_{n+2})$ be a form of degree d over $\mathrm{GF}(q)$. Denote by N_s^* the number of zeros of $\overline{\mathrm{F}}$ in projective space rational over $\mathrm{GF}(q^s)$, all of whose coordinates are non-zero; this N_s^* is easily expressed as a character sum. Take a nontrivial character $\chi_s:\mathrm{GF}(q^s)^+\to\Omega^*$ of the additive group of $\mathrm{GF}(q^s)$ with "p-adic values".

For each $x=(x_1,\ldots,x_{n+2})$ rational over $\mathrm{GF}(q^s)$,

$$\sum_{z\in\mathrm{GF}(q^s)}\chi_s(z\overline{\mathrm{F}}(x))=\begin{cases} q^s, & \text{if } \overline{\mathrm{F}}(x)=0\\ 0, & \text{if } \overline{\mathrm{F}}(x)\neq 0.\end{cases}$$

Hence

$$(q^s-1)\mathrm{N}_s^*=\frac{1}{q^s}\{(q^s-1)^{n+2}+\Sigma\chi_s(z\overline{\mathrm{F}}(x))\}$$

where the sum is taken over $z, x_1, \ldots, x_{n+2}\in\mathrm{GF}(q^s)^*$.

It remains to construct χ_s. Fix a non-trivial character χ_0 of $\mathrm{GF}(p)$, and put $\chi_s=\chi_0\cdot\mathrm{tr}$ where "tr" denotes the trace mapping $\mathrm{tr}:\mathrm{GF}(q^s)\to\mathrm{GF}(p)$. Explicitly, for $b\in\mathrm{GF}(q^s)$, $\chi_s(b)=\zeta^{\mathrm{tr}(b)}$, where ζ is a fixed p-th root of unity, and $\mathrm{tr}(b)=\sum_v b^{p^v}$, the sum taken over $v=0,\ldots,s\log_p(q)-1$.

With this in mind, we fix an element π of Ω, with $\pi^{p-1}=-p$, and define a power series $\theta_0(\mathrm{Z})$ by setting $\theta_0(\mathrm{Z})=\exp(\pi\mathrm{Z}-\pi\mathrm{Z}^p)$.

Then [1]

1) $\theta_0(\mathrm{Z})$ has p-integral coefficients, and converges for $\mathrm{ord}(\mathrm{Z})>\frac{1-p}{p^2}$.

2) $\theta_0(1)=\zeta$, a primitive p-th root of unity.

3) for $\alpha\in\Omega$ with $\alpha^{p^v}=\alpha$, the formal identity,

$$\prod_{j=0}^{v-1}\theta_0(\alpha^{p^j}Z)=\theta_0(Z)^{\alpha+\alpha^p+\ldots+\alpha^{p^{v-1}}}$$

specializes at $Z=1$ to

$$\prod_{j=0}^{v-1}\theta_0(\alpha^{p^j})=\zeta^{\alpha+\alpha^p+\ldots+\alpha^{p^{v-1}}}.$$

Observe that the left-hand side is the value of $\theta_0(Z)\theta_0(Z^p)\ldots\theta_0(Z^{p^{v-1}})=\exp(\pi Z-\pi Z^{p^v})$ at $Z=\alpha$.

Thus, if $\beta=\beta^{q^s}$ is the Teichmüller representative of $b\in GF(q^s)$, we have $\chi_s(b)=$ the value at $Z=\beta$ of $\exp(\pi Z-\pi Z^{q^s})$. Hence, if we suppose $F(X)=F(X_1, \ldots, X_{n+2})$ to have coefficients which are (q^s-1)—st roots of unity in Ω, and if $\alpha, \beta_1, \ldots, \beta_{n+2}$ are (q^s-1)—st roots of unity,

$$\chi_s(\overline{\alpha}\overline{F}(\overline{\beta}))=\begin{cases}\text{the value at } Z=\alpha,\ X=\beta \text{ of}\\ \exp(\pi ZF(X)-\pi Z^{q^s}F(X^{q^s}))\end{cases}$$

where $\beta=(\beta_1, \ldots, \beta_{n+2})$, and $\alpha\mapsto\overline{\alpha}$ is reduction $(\operatorname{mod} p)$. Thus,

$$(q^s-1)N_s^*=\frac{1}{q^s}((q^s-1)^{n+2}+\Sigma\exp(\pi ZF(X)-\pi Z^{q^s}F(X^{q^s}))),$$

where, in the sum, $Z, X_1, \ldots, X_{n+2}$ vary independently over the (q^s-1)—st roots of unity in Ω. We next express this sum as a trace.

Denote by $L(0+)$ the space of power series $\Sigma A_w Z^{w_0}X^w$ which satisfy

1) $dw_0=|w|=\sum_{i=1}^{n+2}w_i,\ w_0\geq 0.$

2) For some constants $b>0$, and c, $\operatorname{ord} A_w\geq bw_0+c$.

The endomorphism ψ_q of $L(0+)$ is defined by

$$\psi_q(Z^{w_0}X^w)=\begin{cases}Z^{v_0}X^v, & \text{if each } w_i=qv_i\\ 0, & \text{if not.}\end{cases}$$

For each element $H\in L(0+)$, we write $\psi_q.H$ for the endomorphism of $L(0+)$ given by

$$\eta\mapsto\psi_q(H\eta).$$

This operator is " of trace class ", and [1, 9]

$$(q-1)^{n+3}\operatorname{tr}(\psi_q.H)=\Sigma H(Z, X)$$

where $Z, X_1, \ldots, X_{n+2}$ are independently summed over the $(q-1)$—st roots of unity in Ω.

In particular, this trace formula may be applied to $\alpha=\psi_q.H_q$, where $H_q=\exp(\pi ZF(X)-\pi Z^qF(X^q))$. It is immediate that the s-th iterate α^s of α is nothing other than $\psi_{q^s}.H_{q^s}$. Hence, combining the above formulae, we have

$$q^sN_s^*=(q^s-1)^{n+1}+(q^s-1)^{n+2}\operatorname{tr}(\alpha^s).$$

This is the connection of α with the zeta function. It is convenient to consider a one-sided inverse β to α, given by

$$\beta = \exp(\pi Z^q F(X^q) - \pi Z F(X)) \,.\, \Phi,$$

where $\Phi(Z^{w_0} X^w) = Z^{qw_0} X^{qw}$.

Connections with Formal Cohomology.

We wish to relate the operators α and β to certain operators arising in the formal cohomology theory of Washnitzer and Monsky.

We begin with a special affine variety over $k = \mathrm{GF}(q)$; this is, by definition, an algebra $\bar{A} = k[x, \tau]/\bar{I}$, where $\bar{I} = \left(\bar{f}(x),\ 1 - \tau \bar{g}(x) \dfrac{\partial \bar{f}}{\partial x_{n+1}}\right)$: here $x = (x_1, \ldots, x_{n+1})$. Fix a complete, discrete valuation ring R, of characteristic zero, whose residue field is k, and denote by $R[x, \tau]^+$ the subalgebra of $R[[x, \tau]]$ satisfying a certain growth condition. Take liftings, f and g, of $\bar{f}$ and $\bar{g}$, to $R[x]$, and define $A^+ = R[x, \tau]^+/I$, where $I = \left(f,\ 1 - \tau g \dfrac{\partial f}{\partial x_{n+1}}\right)$. This algebra is independent of choice of liftings, up to non-canonical isomorphism. Fixing these choices for a moment, every map $\varphi : \bar{A} \to \bar{B}$ of special affines may be lifted to a $\varphi^+ : A^+ \to B^+$. Passing to continuous R-differentials, the induced map, $\varphi^+ : \Omega(A^+) \otimes \mathbf{Q} \to \Omega(B^+) \otimes \mathbf{Q}$, is determined, up to homotopy, by φ. In this way, the deRham cohomology (i.e., the homology of the complex $\Omega(A^+) \otimes \mathbf{Q}$) becomes functorial in $\bar{A}$.

In particular, A^+ admits an endomorphism Fr, which lifts the q-th power mapping. Fr is an injection and A^+ is a finite module over $\mathrm{Fr}(A^+)$. Define an additive mapping, $\psi : A^+ \otimes \mathbf{Q} \to A^+ \otimes \mathbf{Q}$, by requiring that the composition, $q^n \mathrm{Fr} \circ \psi$, be the trace mapping from A^+ to $\mathrm{Fr}(A^+)$.

We first consider a special affine subset of an irreducible projective hypersurface, $F = 0$. Namely, let $f(x_1, \ldots, x_{n+1}) = F(\ldots, X_i/X_{n+2}, \ldots)$, and consider the locus $f(x) = 0$, $1 - \tau x_1, \ldots, x_{n+1} \dfrac{\partial f}{\partial x_{n+1}} = 0$, with « plus » algebra A^+. The mapping Θ of theorem 1 extends by " continuity " to a map of $L^0(0+)$ to $A^+ \otimes \mathbf{Q}$, where $L^0(0+)$ is the subspace without constant term, and our result is the commutativity of the diagram

$$\begin{tikzcd} L^0(0+) \arrow[r, "q\beta"] \arrow[d, "\Theta"] & L^0(0+) \arrow[d, "\Theta"] \\ A^+ \otimes \mathbf{Q} \arrow[r, "\mathrm{Fr}"] & A^+ \otimes \mathbf{Q} \end{tikzcd}$$

where Fr is that lifting of the q-th power map with $\mathrm{Fr}(x_i) = x_i^q$ for $i = 1, \ldots, n+1$.

Somewhat more straightforward is the case of the " complement ", i.e. the algebra $R[x, \tau]/(1-\tau x_1, \ldots, x_{n+1}f)$. Again, we select the Fr with $\mathrm{Fr}(x_i)=x_i^q$ for $i=1, \ldots, n+1$.

Our main results are the commutativity of the diagrams

$$\begin{array}{ccc} L^0(0+) & \xrightarrow{\alpha} & L^0(0+) \\ \downarrow{\scriptstyle \mathscr{R}} & & \downarrow{\scriptstyle \mathscr{R}} \\ A^+\otimes\mathbf{Q} & \xrightarrow{q\psi} & A^+\otimes\mathbf{Q} \end{array} \qquad \begin{array}{ccc} L^0(0+) & \xrightarrow{q\beta} & L^0(0+) \\ \downarrow{\scriptstyle \mathscr{R}} & & \downarrow{\scriptstyle \mathscr{R}} \\ A^+\otimes\mathbf{Q} & \xrightarrow{\mathrm{Fr}} & A^+\otimes\mathbf{Q} \end{array}$$

Here the mapping $\mathscr{R}$ is the one explained earlier.

The image of $\mathscr{R}$ in $A^+\otimes\mathbf{Q}$ consists of those functions regular on the larger open set $\{x\in\mathbf{P}^{n+1}, F(x)\neq 0\}$, while, on the level of differential forms, the image of $\mathscr{R}$ in $\Omega^{n+1}(A^+)\otimes\mathbf{Q}$ consists of those forms, meromorphic on $\{x\in\mathbf{P}^{n+1}, F(x)\neq 0\}$, whose only singularities are, at worst, first order poles along the coordinate axes.

Working with the form $X_1 \ldots X_{n+2}F(X)$ would allow a surjection,

$$\mathscr{R} : L^0(0+) \to A^+\otimes\mathbf{Q},$$

but at the cost that the differential operators D_{X_i} for this form are difficult to analyze, even under the most favorable hypotheses on F. Difficulties of this sort prevent the direct application of Dwork's work to prove the finite dimensionality of any « plus » cohomology groups.

However application of (2.15) to the form $X_1 \ldots X_{n+2}F(X)$ is easily seen to imply the trace formula obtained by Reich and Monsky for the mapping ψ of $A^+\otimes\mathbf{Q}$, namely

$$N=(q-1)^{n+1}\mathrm{tr}(\psi)$$

where N is the number of points (x), rational over $GF(q)$, where $x_1 \ldots x_{n+1}f(x)\neq 0$.

I wish to thank my teacher, B. Dwork, for so very much, and to acknowledge many helpful discussions with G. Washnitzer.

ALGEBRAIC THEORY

Notations. — We work over a field K of characteristic zero, and fix an element $\pi\in K^*$. Let $F(X_1, \ldots, X_{n+2})$ be a homogeneous form of degree d over K, defining a non-singular hypersurface X. Denote by $X^\varnothing$ the open subset where no X_i vanishes, by $U(b, i)$, $b\neq i$, the open subset where $X_i\dfrac{\partial F}{\partial X_b}\neq 0$, and by $U^\varnothing(b, i)$ the intersection $U(b, i)\cap X^\varnothing$.

Any derivation D of K extends to a derivation of each coordinate ring $\Omega^0(U(b, i))$ by requiring $D(X_j/X_i)=0$ for $i\neq j$, $j\neq b$ over $U(b, i)$; when there is no ambiguity we will denote this derivation also by D.

We define $\mathscr{L}$ to be the K-span of the monomials $Z^a X_1^{b_1} \ldots X_{n+2}^{b_{n+2}}$ (hereafter written $Z^a X^b$) for which $da = \sum_i b_i$, $a \geq 0$, $\mathscr{L}^0$ the subspace " divisible by Z ", i.e. with $a \geq 1$. (Observe that the b_i may be negative.) Define operators $D_{X_i} = X_i \frac{\partial}{\partial X_i} + \pi Z X_i \frac{\partial F}{\partial X_i}$, $D_Z = Z \frac{\partial}{\partial Z} + \pi Z F$.

Any derivation D of K extends to a derivation $\mathfrak{S}_D$ of the K-space $\mathscr{L}$ (or $\mathscr{L}^0$) by setting $\mathfrak{S}_D = D + \pi Z F^D$, where D acts only on coefficients and F^D is the result of applying D to the coefficients of F. Formally $\mathfrak{S}_D = \exp(-\pi Z F) D \exp(\pi Z F)$, while $D_{X_i} = \exp(-\pi Z F) X_i \frac{\partial}{\partial X_i} \exp(\pi Z F)$, whence each $\mathfrak{S}_D$ commutes with all D_{X_i} and D_Z, and $\mathfrak{S}_D$ commutes with $\mathfrak{S}_{D'}$ if D commutes with D'.

Theorem (1). — *There exists for every nonsingular* F *and for each* (b, i), $b \neq i$, *a unique* K-*linear mapping* $\Theta : \mathscr{L}^0 \to \Omega^0(U^0(b, i))$ *satisfying*:

a) Θ *is compatible with specializing the coefficients of the form* F;

b) $\Theta . \mathfrak{S}_D = D . \Theta$ *for every derivation* D *of* K;

c) $\Theta(Z X^w) = X^w / X_b \frac{\partial F}{\partial X_b}$;

d) $\Theta D_{X_j} = x_j \frac{\partial}{\partial x_j} \Theta$ *for* $j \neq b, i$, *where* $x_j = X_j / X_i$ *and* $\frac{\partial x_k}{\partial x_j} = \delta_{jk}$ *if* $j \neq b, i$ *and* $k \neq b, i$;

e) $\Theta(D_Z \mathscr{L}) = 0$, $\Theta(D_{X_b} \mathscr{L}^0) = 0$.

Proof. — We begin by constructing Θ for the generic form $F = \sum_w A_w X^w$ where the A_w are independent variables. To fix ideas we work over $U^0(n+1, n+2)$ and first content ourselves with verifying *b)* for the special derivations $\frac{\partial}{\partial A_w}$ (we write $\mathfrak{S}_{A_w}$ for $\mathfrak{S}_{\partial/\partial A_w}$). Uniqueness follows from *b)* and *c)*, since $\mathfrak{S}_{A_w}(Z^a X^v) = \pi Z^{a+1} X^{v+w}$, so that every monomial of $\mathscr{L}^0$ is obtained from one of Z-degree 1 by successive iteration of any $\mathfrak{S}_{A_w}$; clearly for a fixed A_w, that expression is unique.

As to existence, the last remark shows that for each A_w, there is a mapping Θ_{A_w} satisfying *b)* and *c)* for A_w. We first show that Θ_{A_w} is independent of A_w. We have $\pi^a Z^{a+1} X^b = \mathfrak{S}^a_{A_w}(Z X^{b-aw}) = \mathfrak{S}^a_{A_v}(Z X^{b-av})$, so we must show, for $f = F / X^d_{n+2}$,

$$\left(\frac{\partial}{\partial A_w}\right)^a \left(\frac{\left(\frac{\partial f}{\partial A_v}\right)^a x^{b-aw-av}}{x_{n+1} \frac{\partial f}{\partial x_{n+1}}} \right) = \left(\frac{\partial}{\partial A_v}\right)^a \left(\frac{\left(\frac{\partial f}{\partial A_w}\right)^a x^{b-aw-av}}{x_{n+1} \frac{\partial f}{\partial x_{n+1}}} \right),$$

i.e. that for every monomial x^c in the x_i and x_i^{-1} we have

$$\left(\frac{\partial}{\partial A_w}\right)^a \left(\left(\frac{\partial f}{\partial A_v}\right)^a \frac{x^c}{\frac{\partial f}{\partial x_{n+1}}} \right) = \left(\frac{\partial}{\partial A_v}\right)^a \left(\left(\frac{\partial f}{\partial A_w}\right)^a \frac{x^c}{\frac{\partial f}{\partial x_{n+1}}} \right).$$

For $a=1$ this is

$$\frac{\partial}{\partial A_w}\left(-x^c\frac{\partial x_{n+1}}{\partial A_v}\right)=\frac{\partial}{\partial A_v}\left(-x^c\frac{\partial x_{n+1}}{\partial A_w}\right)=\frac{c_{n+1}}{x_{n+1}}x^c\frac{\partial x_{n+1}}{\partial A_v}\frac{\partial x_{n+1}}{\partial A_w}-x^c\frac{\partial^2(x_{n+1})}{\partial A_w\partial A_v}.$$

Now by induction

$$\left(\frac{\partial}{\partial A_w}\right)^{a+1}\left(\frac{\left(\frac{\partial f}{\partial A_v}\right)^{a+1}x^c}{\frac{\partial f}{\partial x_{n+1}}}\right)=\frac{\partial}{\partial A_v}\left(\frac{\partial}{\partial A_w}\right)^a\left(\frac{\frac{\partial f}{\partial A_w}\left(\frac{\partial f}{\partial A_v}\right)^a x^c}{\frac{\partial f}{\partial x_{n+1}}}\right)=\left(\frac{\partial}{\partial A_v}\right)^{a+1}\left(\frac{\left(\frac{\partial f}{\partial A_w}\right)^{a+1}x^c}{\frac{\partial f}{\partial x_{n+1}}}\right).$$

Let Γ denote the coefficient of X_{n+2}^d in F, i.e. the constant term of f. As $\mathfrak{S}_\Gamma$ commutes with the D_{X_i}, and $\frac{\partial}{\partial\Gamma}$ commutes with the $x_i\frac{\partial}{\partial x_i}$ for $i=1, \ldots, n$, we are reduced to showing *d)* for a monomial ZX^b of Z-degree 1, say for D_{X_1}.

$$\begin{aligned}\Theta(D_{X_1}(ZX^b))&=\Theta\left(b_1ZX^b+\pi Z^2X^bX_1\frac{\partial F}{\partial X_1}\right)\\&=\Theta(b_1ZX^b)+\frac{\partial}{\partial\Gamma}\Theta\left(X_1Z\frac{\partial F}{\partial X_1}X^bX_{n+2}^{-d}\right)\\&=-e_1x^e\frac{\partial x_{n+1}}{\partial\Gamma}-\frac{\partial}{\partial\Gamma}\left(x_1x^e\frac{\partial x_{n+1}}{\partial x_1}\right)\qquad\text{(we write } x^e=x^bx_{n+1}^{-1})\\&=-e_1x^e\frac{\partial x_{n+1}}{\partial\Gamma}-x^ex_1\frac{\partial}{\partial x_1}\left(\frac{\partial x_{n+1}}{\partial\Gamma}\right)-\frac{\partial x_{n+1}}{\partial x_1}\frac{\partial}{\partial\Gamma}(x_1x^e)\\&=-x_1\frac{\partial}{\partial x_1}\left(x^e\frac{\partial x_{n+1}}{\partial\Gamma}\right)=x_1\frac{\partial}{\partial x_1}\Theta(ZX^b).\end{aligned}$$

As for *e)* we first use *b)* to reduce to showing $\Theta(D_Z(X^b))=0$, but this is $\Theta(\pi ZFX^b)=\dfrac{fx^b}{x_{n+1}\frac{\partial f}{\partial x_{n+1}}}=0$. Similarly it suffices to compute

$$\Theta\,.\,D_{X_{n+1}}(ZX^b)=\Theta(b_{n+1}ZX^b+\mathfrak{S}_\Gamma(ZX^bX_{n+1}\frac{\partial F}{\partial X_{n+1}}X_{n+2}^{-d}))=\frac{b_{n+1}x^b}{x_{n+1}\frac{\partial f}{\partial x_{n+1}}}+\frac{\partial}{\partial\Gamma}(x^b)=0.$$

We now regard this generic definition as providing formulas for Θ in terms of the coefficients of the defining form.

Clearly it remains only to demonstrate that *b)* holds for *all* forms F. Consider $F(X)+\lambda X_{n+2}^d$ over $K(\lambda)$ where λ is transcendental over K. Extend D by $D(\lambda)=0$ and $\frac{\partial}{\partial\lambda}$ by $\frac{\partial K}{\partial\lambda}=0$, whence D and $\frac{\partial}{\partial\lambda}$ commute, as do $\mathfrak{S}_D$ and $\mathfrak{S}_\lambda$, whence

$\Theta\mathfrak{S}_D(Z^{w_0}X^w)=\Theta\mathfrak{S}_\lambda^{w_0-1}\mathfrak{S}_D\left(\frac{ZX^w}{\pi^{w_0-1}}X_{n+2}^{-(w_0-1)d}\right)=\left(\frac{1}{\pi}\frac{\partial}{\partial\lambda}\right)^{w_0-1}\Theta.\mathfrak{S}_D(ZX^wX_{n+2}^{-(w_0-1)d})$, while $D\Theta(Z^{w_0}X^w)=\left(\frac{1}{\pi}\frac{\partial}{\partial\lambda}\right)^{w_0-1}.D.\Theta(ZX^wX_{n+2}^{-(w_0-1)d})$; hence we are reduced to $w_0=1$.

$$\Theta.\mathfrak{S}_D(ZX^w)=\Theta\mathfrak{S}_\lambda(ZF^DX^wX_{n+2}^{-d})=\frac{\partial}{\partial\lambda}\left(\frac{f^Dx^w}{x_{n+1}\,\partial f/\partial x_{n+1}}\right)=\frac{-\partial}{\partial\lambda}\left(\frac{x^w}{x_{n+1}}D(x_{n+1})\right),$$

while $D\Theta(ZX^w)=D\left(\frac{x^w}{x_{n+1}}\frac{1}{\dfrac{\partial f}{\partial x_{n+1}}}\right)=-D\left(\frac{x^w}{x_{n+1}}\frac{\partial x_{n+1}}{\partial\lambda}\right)$, and as D and $\frac{\partial}{\partial\lambda}$ commute, both sides are $\frac{-x^w}{x_{n+1}}D\left(\frac{\partial x_{n+1}}{\partial\lambda}\right)-\frac{(w_{n+1}-1)x^w}{x_{n+1}^2}\frac{\partial x_{n+1}}{\partial\lambda}D(x_{n+1})$. Q.E.D.

Corollary (**1.1**). — *Over each* $U^0(b,i)$, Θ *has a natural extension to a mapping from the Koszul complex on* $\mathcal{L}^0/(D_Z\mathcal{L}+D_{X_b}\mathcal{L}^0)$ *with operators the* D_{X_j}, $j\neq i, b$ *to the de Rham complex* $\Omega(U^0(b,i))$ *of differentials regular on* $U^0(b,i)$.

Proof. — We define the Koszul complex and give the proof in the following section.

Koszul and de Rham complexes.

Let $\varphi_1,\ldots,\varphi_n$ be commuting endomorphisms of a vector space V/K. Write $S=\{1,\ldots,n\}$, $\wedge S$ the exterior algebra of the free K-space with basis the elements of S. On $\mathrm{Hom}(\wedge S, V)$ the Koszul boundary may be defined by

$$\delta\omega(\tau)=\sum_{i\in S}\varphi_i(\omega(\tau\wedge i))$$

where $\tau\in\wedge S$, $\omega\in\mathrm{Hom}(\wedge S,V)$.

Define $*$ on $\wedge S$ by linearity and the requirement that for a monomial $\tau\in\wedge^k S$, $*\tau$ is the monomial of $\wedge^{n-k}S$ with $\tau\wedge *\tau=1\wedge 2\wedge\ldots\wedge n$, and let $*$ act on $\mathrm{Hom}(\wedge S, V)$ by defining $*\omega(*\tau)=\omega(\tau)$.

Let L/K be a function field in n variables with separating transcendence basis $x_1,\ldots,x_n$. The monomials in the dx_i/x_i form a basis for $\Omega_K(L)$ as L-space, which is thus isomorphic $(dx_i/x_i\mapsto i)$ with $\wedge S\otimes_K L$, i.e. with $\mathrm{Hom}(\wedge S, L)$. Thus exterior differentiation d induces a coboundary on $\mathrm{Hom}(\wedge S, L)$, while the $x_i\frac{d}{dx_i}$, $i=1,\ldots,n$ provide the Koszul boundary.

Proposition (**1.2**). — $*.\delta=d.*$.

Proof. — For a monomial η and $i\in S$, $*\eta=i\wedge *(\eta\wedge i)$ so long as $\eta\wedge i\neq 0$. Hence $d(*\omega)(*\eta)=\sum_i x_i\frac{d}{dx_i}(*\omega(*(\eta\wedge i)))=\sum_i x_i\frac{d}{dx_i}(\omega(\eta\wedge i))=(\delta\omega)(\eta)=*(\delta\omega)(*\eta)$. Q.E.D.

Globalization for n-forms.

Lemma (**1.3**). — *Let* $b \neq i$, *and say* $\{1, \ldots, n+2\} = \{k_1, \ldots, k_n, b, i\}$. *Define* $\omega(b, i)$ *on* $U(b, i)$ *to be*

$$\operatorname{sgn}\begin{pmatrix} 1, \ldots, n+2 \\ k_1, \ldots, k_n, b, i \end{pmatrix} \frac{X_i^{n+1} d(X_{k_1}/X_i) \wedge \ldots \wedge d(X_{k_n}/X_i)}{X_i X_b \dfrac{\partial F}{\partial X_b} \prod_{i=1}^{n} X_{k_i}}.$$

Then on $U(b, i) \cap U(c, j)$, $\omega(b, i) = \omega(c, j)$.

Proof. — Suppose first that $\{b, i\} = \{c, j\}$; to fix ideas take $b = n+1 = j$, $i = n+2 = c$, so we are asserting that

$$X_{n+2}^{n+1} \frac{\partial F}{\partial X_{n+2}} d\left(\frac{X_1}{X_{n+2}}\right) \wedge \ldots \wedge d\left(\frac{X_n}{X_{n+2}}\right) = -X_{n+1}^{n+1} \frac{\partial F}{\partial X_{n+1}} d\left(\frac{X_1}{X_{n+1}}\right) \wedge \ldots \wedge d\left(\frac{X_n}{X_{n+1}}\right).$$

Now $X_\beta d(X_\alpha/X_\beta) = dX_\alpha - (X_\alpha/X_\beta) dX_\beta$, so we want

$$\left(X_{n+1} \frac{\partial F}{\partial X_{n+1}} + X_{n+2} \frac{\partial F}{\partial X_{n+2}}\right) dX_1 \wedge \ldots \wedge dX_n =$$
$$= \left(\frac{\partial F}{\partial X_{n+1}} dX_{n+1} + \frac{\partial F}{\partial X_{n+2}} dX_{n+2}\right) \wedge \sum_{v=1}^{n} (-1)^{v-1} X_v dX_1 \wedge \ldots \wedge \widehat{dX_v} \wedge \ldots \wedge dX_n$$

and the right side $\left(\text{as } dF = \sum_i \frac{\partial F}{\partial X_i} dX_i = 0\right)$ is

$$\sum_{j=1}^{n} \frac{\partial F}{\partial X_j} dX_j \wedge \sum_{v=1}^{n} (-1)^v X_v dX_1 \wedge \ldots \wedge \widehat{dX_v} \wedge \ldots \wedge dX_n = -\sum_{v=1}^{n} X_v \frac{\partial F}{\partial X_v} dX_1 \wedge \ldots \wedge dX_n;$$

now apply the Euler relations.

In general, given (b, i) and (c, j), we compare both with (b, j), and hence, by the first part, we are reduced to comparing (b, i) with (c, i). To fix ideas we compare $(n, n+2)$ with $(n+1, n+2)$, and write $x_i = X_i/X_{n+2}$, $f(x_1, \ldots, x_{n+1}) = F(x_1, \ldots, x_{n+1}, 1)$, so that our assertion becomes $\dfrac{dx_1 \wedge \ldots \wedge dx_n}{\dfrac{\partial f}{dx_{n+1}}} = -\dfrac{dx_1 \wedge \ldots \wedge dx_{n-1} \wedge dx_{n+1}}{\dfrac{\partial f}{\partial x_n}}$ which follows, as $df = \sum_{i=1}^{n+1} \frac{\partial f}{\partial x_i} dx_i = 0$. Q.E.D.

Lemma (**1.4**). — *Let* L/K *be a separably generated function field in n variables with separating transcendence basis* $x_1, \ldots, x_n$; *let* D *be any derivation of* L *trivial on* K. *Then on* $\Omega_K(L)$, $D = d\lambda + \lambda d$, *where* (*for* $u_{i_1, \ldots, i_k} \in L$)

$$\lambda(u_{i_1, \ldots, i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k}) = u_{i_1, \ldots, i_k} \sum_{j=1}^{k} (-1)^{j-1} D(x_{i_j}) dx_{i_1} \wedge \ldots \wedge \widehat{dx_{i_j}} \wedge \ldots \wedge dx_{i_k}.$$

Proof. — We readily compute that for $\omega \in \Omega_K^j(L)$, $\lambda(\omega \wedge \tau) = \lambda(\omega) \wedge \tau + (-1)^j \omega \wedge \lambda(\tau)$, whence it follows easily that $(d\lambda + \lambda d)(\omega \wedge \tau) = (d\lambda + \lambda d)(\omega) \wedge \tau + \omega \wedge (d\lambda + \lambda d)(\tau)$, so that $d\lambda + \lambda d$ is a degree zero derivation of $\Omega_K(L)$ which commutes with d, hence is determined by its restriction to L, and $(d\lambda + \lambda d)(x_i) = D(x_i)$. Q.E.D.

Differentiating Cohomology Classes whith respect to Parameters.

Let V be a non-singular variety defined over a field of characteristic zero; a theorem of A. Grothendieck [4] gives an algebraic method of computing the cohomology of V as complex manifold. Namely, fix a covering $\{V_i\}$ of V by affine open sets; denote by Ω^q the sheaf of germs of regular algebraic differential forms, and by $C^p(\Omega^q, \{V_i\})$ the (alternating) p-cochains for the nerve of the covering $\{V_i\}$ with coefficients in the sheaf Ω^q. When the context is clear, we will simply write $C^p(\Omega^q)$. These $C^p(\Omega^q)$ form a double complex, with $d: C^p(\Omega^q) \to C^p(\Omega^{q+1})$ the usual exterior differentiation and $\partial: C^p(\Omega^q) \to C^{p+1}(\Omega^q)$ the nerve-coboundary; for $\omega(i_0, \ldots, i_p) \in C^p(\Omega^q)$, recall $(\partial\omega)(i_0, \ldots, i_{p+1}) = \sum_j (-1)^j \omega(i_0, \ldots, \hat{i_j}, \ldots, i_{p+1})$. Then the cohomology of V is the homology of the total complex (whose term of degree n is $\sum_{p+q=n} C^p(\Omega^q)$) under the differential which acts on $C^p(\Omega^q)$ by $\partial + (-1)^p d$, which we will write Δ. Now consider a derivation D of the field of definition of V; we suppose chosen for each V_i an extending derivation D_i at the function field of V in such a way that the coordinate ring of each intersection $V_{i_0} \cap \ldots \cap V_{i_p}$ is stable under $D_{i_0}, \ldots, D_{i_p}$. We recall (lemma (1.4)) that $D_i - D_j = d\lambda_{ij} + \lambda_{ij} d$ where

$$\lambda_{ij}(u\,dx_1 \wedge \ldots \wedge dx_r) = u \sum_l (-1)^l (D_i - D_j)(x_l)\, dx_1 \wedge \ldots \wedge \widehat{dx_l} \wedge \ldots \wedge dx_r.$$

Finally we define $\lambda: C^p(\Omega^q) \to C^{p+1}(\Omega^{q-1})$ by $(\lambda\omega)(i_0, \ldots, i_{p+1}) = \lambda_{i_0, i_1}(\omega(i_1, \ldots, i_{p+1}))$ where we have $i_0 < i_1 < \ldots < i_{p+1}$, and $D: C^p(\Omega^q) \to C^p(\Omega^q)$ by

$$(D\omega)(i_0, \ldots, i_p) = D_{i_0}(\omega(i_0, \ldots, i_p))$$

where again $i_0 < i_1 < \ldots < i_p$. Finally we will write $\widetilde{D}$ for the operator which is $D + (-1)^{p+1}\lambda$ on $C^p(\Omega^q)$; this is " differentiation with respect to a parameter ".

Lemma (**1.5**). — *The operators* $\widetilde{D} = D + (-1)^{p+1}\lambda$ *and* $\Delta = \partial + (-1)^p d$ *commute.*

Proof. — Let $\omega \in C^p(\Omega^q)$;

$$\widetilde{D}(\Delta\omega) = \widetilde{D}(\partial\omega + (-1)^p d\omega) = D\partial\omega + (-1)^p Dd\omega + (-1)^p \lambda\partial\omega - \lambda d\omega$$

while $\Delta(\widetilde{D}\omega) = \Delta(D\omega + (-1)^{p+1}\lambda\omega) = \partial D\omega + (-1)^{p+1}\partial\lambda\omega + (-1)^p dD\omega + d\lambda\omega$. Comparing components on both sides we must show that $Dd\omega = dD\omega$ in $C^p(\Omega^{q+1})$, that $D\partial\omega - \lambda d\omega = \partial D\omega + d\lambda\omega$ in $C^{p+1}(\Omega^q)$, and that $\lambda\partial\omega + \partial\lambda\omega = 0$ in $C^{p+2}(\Omega^{q-1})$. The first point is clear, as each D_i commutes with d. For the second,

$$\begin{aligned}(D\partial\omega - \partial D\omega)(i_0, \ldots, i_{p+1}) = D_{i_0}\omega(i_1, \ldots, i_{p+1}) &+ \\ + \sum_{j \geq 1} (-1)^j D_{i_0}\omega(i_0, \ldots, \hat{i_j}, \ldots, i_{p+1}) &- D_{i_1}\omega(i_1, \ldots, i_{p+1}) - \\ - \sum_{j \geq 1} (-1)^j D_{i_0}\omega(i_0, \ldots, \hat{i_j}, \ldots, i_{p+1}) &= (D_{i_0} - D_{i_1})\omega(i_1, \ldots, i_{p+1}),\end{aligned}$$

while $(d\lambda\omega + \lambda d\omega)(i_0, \ldots, i_{p+1}) = d\lambda_{i_0, i_1}\omega(i_1, \ldots, i_{p+1}) + \lambda_{i_0, i_1} d\omega(i_1, \ldots, i_{p+1})$

(where $i_0 < \ldots < i_{p+1}$). Finally

$$(\lambda\partial\omega+\partial\lambda\omega)(i_0, \ldots, i_{p+2}) = \lambda_{i_0,i_1}((\partial\omega)(i_1, \ldots, i_{p+2})) + \sum_{j\geq 0}(-1)^j(\lambda\omega)(i_0, \ldots, \hat{i_j}, \ldots, i_{p+2}) =$$

$$= \lambda_{i_0,i_1}\omega(i_1, \ldots, i_{p+2}) + \lambda_{i_0,i_1}\sum_{j\geq 2}(-1)^{j-1}\omega(i_1, \ldots, \hat{i_j}, \ldots, i_{p+2}) +$$

$$+\lambda_{i_1,i_2}\omega(i_2, \ldots, i_{p+2}) - \lambda_{i_0,i_2}\omega(i_2, \ldots, i_{p+2}) + \sum_{j\geq 2}(-1)^j\lambda_{i_0,i_1}\omega(i_1, \ldots, \hat{i_j}, \ldots, i_{p+2}),$$

and it is clear from the definition that $\lambda_{i_0,i_2} = \lambda_{i_0,i_1} + \lambda_{i_1,i_2}$. Q.E.D.

Lemma (**1.6**). — *Suppose for every derivation* D *of the field of definition we have chosen the various extensions* D_i *over the* V_i *in such a way that for derivations* D *and* D′ *we have* $[D, D']_i = [D_i, D'_i]$. *Suppose further that for each pair* D, D′ *and each* i, j, *the ratio* $(D_i - D_j)(h)/(D'_i - D'_j)(h)$ *is independent of the choice of function* h *(whenever it is defined) and of the choice of* (i, j). *Then the assignment* $D \mapsto \tilde{D}$ *is a Lie homomorphism from the ring of derivations of the constants to the ring of additive endomorphisms of the total complex of* $C^p(\Omega^q, \{V_i\})$. *In particular, if* D *and* D′ *commute, then* $\tilde{D}$ *and* $\tilde{D}'$ *commute.*

Proof. — Let $\omega \in C^p(\Omega^q)$; we readily compute $[\tilde{D}, \tilde{D}']\omega = \eta + \tau + \gamma$ with $\eta \in C^p(\Omega^q)$, $\tau \in C^{p+1}(\Omega^{q-1})$ and $\gamma \in C^{p+2}(\Omega^{q-2})$; here $\eta(i_0, \ldots, i_p) = [D_{i_0}, D'_{i_0}]\omega(i_0, \ldots, i_p)$, $\tau(i_0, \ldots, i_{p+1}) = (-1)^{p+1}(-\lambda'_{i_0,i_1}D_{i_1} - D'_{i_0}\lambda_{i_0,i_1} + \lambda_{i_0,i_1}D'_{i_1} + D_{i_0}\lambda'_{i_0,i_1})\omega(i_1, \ldots, i_{p+1})$, and $\gamma(i_0, \ldots, i_{p+2}) = (-\lambda_{i_0,i_1}\lambda'_{i_1,i_2} + \lambda'_{i_0,i_1}\lambda_{i_1,i_2})(\omega(i_2, \ldots, i_{p+2}))$. We first show $\gamma = 0$; write $\omega(i_2, \ldots, i_{p+2})$ as a sum of terms of the form $u\,dh_1 \wedge \ldots \wedge dh_r$ where $h_1, \ldots, h_r$ are functions; it is sufficient if $\lambda'_{i_0,i_1}(dh_a)\lambda_{i_1,i_2}(dh_b) = \lambda_{i_0,i_1}(dh_a)\lambda'_{i_1,i_2}(dh_b)$ for every a and b, and as $\lambda'_{i_0,i_1}(dh_a) = (D'_{i_0} - D'_{i_1})(h_a)$ this is insured by the hypotheses.

Turning to η, we begin by showing $\lambda''_{i_0,i_1} = \lambda_{i_0,i_1}D'_{i_1} + D_{i_0}\lambda'_{i_0,i_1} - \lambda'_{i_0,i_1}D_{i_1} - D'_{i_0}\lambda_{i_0,i_1}$ is a derivation of degree -1 of differentials. As the λ and the D are derivations of degrees -1 and 0 respectively, it follows that $D_{i_0}\lambda'_{i_0,i_1} - \lambda'_{i_0,i_1}D_{i_0}$ and $\lambda_{i_0,i_1}D'_{i_1} - D'_{i_1}\lambda_{i_0,i_1}$ are derivations of degree -1, whence it suffices that

$$\lambda'_{i_0,i_1}(D_{i_0} - D_{i_1}) - (D'_{i_0} - D'_{i_1})\lambda_{i_0,i_1} = \lambda'_{i_0,i_1}\lambda_{i_0,i_1}d - d\lambda'_{i_0,i_1}\lambda_{i_0,i_1}$$

be a derivation. Here d is a derivation of degree 1, and so it suffices for $\lambda'_{i_0,i_1}\lambda_{i_0,i_1}$ to be a derivation. However $\lambda'_{i_0,i_1}\lambda_{i_0,i_1} + \lambda_{i_0,i_1}\lambda'_{i_0,i_1}$ is a derivation, and hence it suffices if $\lambda'_{i_0,i_1}\lambda_{i_0,i_1} = \lambda_{i_0,i_1}\lambda'_{i_0,i_1}$, which is verified just as in the last paragraph. Finally the operator λ''_{i_0,i_1} does enjoy $d\lambda''_{i_0,i_1} + \lambda''_{i_0,i_1}d = [D, D']_{i_0} - [D, D']_{i_1}$, as an immediate computation shows, and hence λ''_{i_0,i_1} has the proper effect on 1-forms, and thus on all forms. Q.E.D.

Application. — The hypotheses are satisfied by a non-singular hypersurface of equation $F = 0$, if we take the covering from the $U(b, i)$. We will write functions in homogenous coordinates P/Q, and use P^D to denote the result of applying D only to the coefficients of P; then we readily compute

$$D_{(b,i)}(P/Q) = \frac{QP^D - PQ^D}{Q^2} + X_i\frac{\partial}{\partial X_b}(P/Q)\left(-F^D/X_i\frac{\partial F}{\partial X_b}\right)$$

and hence the ratio $(D_i - D_j)(h)/(D'_i - D'_j)(h) = F^D/F^{D'}$.

Globalization of the Θ Mapping in Middle Dimension.

We begin with globalization to X^0, where matters are greatly simplified by taking the covering $U^0(b, n+2)$, $b=1, \ldots, n+1$. The proof of theorem 1, together with our recent definitions, gives the following

Theorem (**1.7**). — *There is a unique mapping* $\Theta : \mathscr{L}^0 \to \sum_{p+q=n} C^p(\Omega^q)$ *which is a homomorphism of Lie modules over the ring of derivations of the field of definition, which is compatible with specializing the defining equation, and which assigns to a monomial* ZX^w *the element of* $C^0(\Omega^n)$ *which is* $X^w \omega(b, n+2)$ *over* $U^0(b, n+2)$.

In this way the elements of $\mathscr{L}^0$ of Z-degree one correspond with the algebraic n-forms regular on X^0. As these are closed, it follows that the image of $\mathscr{L}^0$ in $\sum_{p+q=n} C^p(\Omega^q)$ lies among the cycles (of the total complex), and hence by passage to quotients we have a map $\Theta : \mathscr{L}^0 \to H^n(X^0)$. Further, the aforementioned theorem of Grothendieck [4] states that on the non-singular affine variety X^0, every cohomology class is realized by a regular algebraic form, whence $\mathscr{L}^0$ maps onto $H^n(X^0)$.

Lemma (**1.8**). — *The kernel of* $\Theta : \mathscr{L}^0 \to H^n(X^0)$ *contains* $D_Z \mathscr{L} + \sum_i D_{X_i} \mathscr{L}^0$.

Proof. — The assertion for D_Z admits the same proof as given in theorem 1. To fix ideas consider $D_{X_1}\mathscr{L}^0$; it suffices to show the kernel contains $D_{X_1}(ZX^w)$ by our general reduction procedure. To this end consider the element η of $C^0(\Omega^{n-1})$ whose value over

$U^0(b, n+2)$ is 0 for $b=1$, $\dfrac{(-1)^{n+1-b}x^w}{x_b \dfrac{\partial f}{\partial x_b}} \dfrac{dx_2}{x_2} \wedge \ldots \wedge \widehat{\dfrac{dx_b}{x_b}} \wedge \ldots \wedge \dfrac{dx_{n+1}}{x_{n+1}}$ for $U^0(b, n+2)$, $b>1$.

Part *d)* of theorem 1 assures us that $\Theta(D_{X_1}(ZX^w))$ and $\Delta\eta$ (Δ = the total coboundary) agree in their components in $C^0(\Omega^n)$; it remains only to consider the components in $C^1(\Omega^{n-1})$. Consider $\Theta(D_{X_1}(ZX^w)) = \Theta(w_1 ZX^w) + \Theta\left(\pi Z^2 X^w X_1 \dfrac{\partial F}{\partial X_1}\right)$; we compute by passing to the equation $F + \Gamma X_{n+2}^d = 0$; then

$$\Theta(D_{X_1}(ZX^w)) = \Theta(w_1 ZX^w) + \frac{\widetilde{\partial}}{\partial \Gamma}\Theta\left(ZX^w X_1 \frac{\partial F}{\partial X_1} X_{n+2}^{-d}\right),$$

and the component in $C^1(\Omega^{n-1})$ assigns to $U(b, n+2) \cap U(c, n+2)$ (where $b<c$) the form

$$(-1)^1 \lambda_{b,c} (-1)^{n+1-c} \frac{x^w x_1 \dfrac{\partial f}{\partial x_1}}{x_c \dfrac{\partial f}{\partial x_c}} \frac{\partial x_1}{x_1} \wedge \ldots \wedge \widehat{\frac{dx_c}{x_c}} \wedge \ldots \wedge \frac{dx_{n+1}}{x_{n+1}} =$$

$$= \frac{x^w x_1 \dfrac{\partial f}{\partial x_1}}{x_c \dfrac{\partial f}{\partial x_c}} (-1)^{n-c}(-1)^{b-1} \left(\frac{-1}{x_b \dfrac{\partial f}{\partial x_b}}\right) \frac{dx_1}{x_1} \wedge \ldots \wedge \widehat{\frac{dx_b}{x_b}} \wedge \ldots \wedge \widehat{\frac{dx_c}{x_c}} \wedge \ldots \wedge \frac{dx_{n+1}}{x_{n+1}}.$$

In case $b=1$, this is $(-1)^{n+1-c}\dfrac{x^w}{x_c\dfrac{\partial f}{\partial x_c}}\dfrac{dx_2}{x_2}\wedge\ldots\wedge\widehat{\dfrac{dx_c}{x_c}}\wedge\ldots\wedge\dfrac{dx_{n+1}}{x_{n+1}}$, as asserted. Similarly if $1<b<c$, this is

$$\frac{(-1)^{n-c-b-1}x^w}{x_b x_c\dfrac{\partial f}{\partial x_b}\dfrac{\partial f}{\partial x_c}}\left(\frac{\partial f}{\partial x_b}dx_b+\frac{\partial f}{\partial x_c}dx_c\right)\wedge\frac{dx_2}{x_2}\wedge\ldots\wedge\widehat{\frac{dx_b}{x_b}}\wedge\ldots\wedge\frac{dx_{n+1}}{x_{n+1}}=$$

$$=\frac{(-1)^{n-c-1}x^w}{x_c\dfrac{\partial f}{\partial x_c}}\frac{dx_2}{x_2}\wedge\ldots\wedge\widehat{\frac{dx_c}{x_c}}\wedge\ldots\wedge\frac{dx_{n+1}}{x_{n+1}}+\frac{(-1)^{n-b}x^w}{x_b\dfrac{\partial f}{\partial x_b}}\frac{dx_2}{x_2}\wedge\ldots\wedge\widehat{\frac{dx_b}{x_b}}\wedge\ldots\wedge\frac{dx_{n+1}}{x_{n+1}}$$

again as asserted.

Corollary **(1.9)**. — Θ *maps* $\mathscr{L}^0/(\mathrm{D_Z}\mathscr{L}+\sum_i \mathrm{D}_{\mathrm{X}_i}\mathscr{L}^0)$ *onto* $\mathrm{H}^n(\mathrm{X}^\theta)$.

Finally we require a definition. A form F is said to be *regular* if the forms $\mathrm{X}_i\dfrac{\partial \mathrm{F}}{\partial \mathrm{X}_i}$ have no common zero; i.e. the locus of F is non-singular, as are all its intersections with the coordinate axes.

The Cohomology of Regular Hypersurfaces.

Fix in $\mathbf{P}^{n+1}$ a system of homogeneous coordinates $(\mathrm{X}_1, \ldots, \mathrm{X}_{n+2})$, and a positive integer d. X^n denotes the locus in $\mathbf{P}^{n+1}$ of a regular form of degree d, X_i^n the open subset where the first i coordinates are all non-zero, and $\mathbf{P}_{n+2}^{n+1}$ the open subset of $\mathbf{P}^{n+1}$ where no coordinate vanishes. H^j denotes the j-th singular cohomology group, $\mathrm{B}^j=\dim \mathrm{H}^j$; $\mathrm{H}^j_{(c)}$ is the j-th group with compact supports, $\mathrm{B}^j_{(c)}$ its dimension.

Theorem **(1.10)**. — $$\mathrm{B}^{n-l}(\mathrm{X}^n_{n+2})=\begin{cases}0 & \text{if } l<0\\ d^{n+1}+n & \text{if } l=0\\ \binom{n+1}{l+1} & \text{if } l>0\end{cases}$$

$$\mathrm{B}^{n+1-l}(\mathbf{P}^{n+1}_{n+2}-\mathrm{X}^n_{n+2})=\begin{cases}0 & \text{if } l<0\\ d^{n+1}+n+1 & \text{if } l=0\\ \binom{n+2}{l+1} & \text{if } l>0.\end{cases}$$

Proof. — First notice $\mathrm{X}_1^n=\mathrm{X}_0^n-\mathrm{X}_0^{n-1}$, whence the exact sequence

$$\mathrm{H}^{q-1}_{(c)}(\mathrm{X}_0^{n-1}) \to \mathrm{H}^q_{(c)}(\mathrm{X}_1^n) \to \mathrm{H}^q_{(c)}(\mathrm{X}_0^n) \xrightarrow{\rho_q} \mathrm{H}^q_{(c)}(\mathrm{X}_0^{n-1})$$

and by the Lefschetz theorem [6, p. 91] ρ_q is an isomorphism for $q\leq n-2$, an injection for $q=n-1$, and a surjection for $q\geq n$. As $\mathrm{B}^q_{(c)}(\mathrm{X}_0^n)=\mathrm{B}^q_{(c)}(\mathbf{P}^n)$ for $q\neq n$ by the same theorem, $\mathrm{B}^q_{(c)}(\mathrm{X}_1^n)=0$ if $q<n$ or $n<q<2n$, while $\mathrm{B}^{2n}_{(c)}(\mathrm{X}_1^n)=1$. Similarly for $1\leq i\leq n+2$ we have $\mathrm{X}_i^n=\mathrm{X}_{i-1}^n-\mathrm{X}_{i-1}^{n-1}$, whence $\mathrm{B}^q_{(c)}(\mathrm{X}_i^n)\leq \mathrm{B}^{q-1}_{(c)}(\mathrm{X}_{i-1}^{n-1})+\mathrm{B}^q_{(c)}(\mathrm{X}_{i-1}^n)$, which already shows $B^q_{(c)}(X^n_i)=0$ if $i\geq 1$ and $q<n$. Iterating our last inequality, we have for $l\geq 1$ that

$$B^{n+l}_{(c)}(X^n_{n+2})\leq \sum_{j=0}^{n+1}\binom{n+1}{j}B^{n-j+l}(X^{n-j}_1)=\binom{n+1}{n-l}=\binom{n+1}{l+1}.$$

To reverse the inequality we turn to the exact sequence of $(\mathbf{P}^{n+1}_{n+2}, X^n_{n+2})$,

$$H^{q-1}_{(c)}(X^n_{n+2})\to H^q_{(c)}(\mathbf{P}^{n+1}_{n+2}-X^n_{n+2})\to H^q_{(c)}(\mathbf{P}^{n+1}_{n+2}).$$

Now $\mathbf{P}^{n+1}_{n+2}$ is just the $(n+1)$-fold product of the non-zero complex numbers, so that $B^{n+1+l}_{(c)}(\mathbf{P}^{n+1}_{n+2})=\binom{n+1}{l}$ for $l\geq 0$, 0 for $l<0$. This already shows that $B^q_{(c)}(\mathbf{P}^{n+1}_{n+2}-X^n_{n+2})=0$ if $q\leq n$, while for $l\geq 1$ we have

$$B^{n+1+l}_{(c)}(\mathbf{P}^{n+1}_{n+2}-X^n_{n+2})\leq\binom{n+1}{l+1}+\binom{n+1}{l}=\binom{n+2}{l+1}=\binom{n+2}{n+1-l}.$$

By Poincaré duality, the left side is $B^{n+1-l}(\mathbf{P}^{n+1}_{n+2}-X^n_{n+2})$. In $\mathbf{P}^{n+1}_{n+2}-X^n_{n+2}$, taking $x_i=X_i/X_{n+2}, f=F(x_1, \ldots, x_{n+1}, 1)$, we claim the $n+2$ cohomology classes $\frac{dx_1}{x_1}\wedge\ldots\wedge\frac{dx_{n+1}}{x_{n+1}}$ and $\frac{df}{f}\wedge\frac{dx_1}{x_1}\wedge\ldots\wedge\widehat{\frac{dx_i}{x_i}}\wedge\ldots\wedge\frac{dx_{n+1}}{x_{n+1}}$, $1\leq i\leq n+1$ are linearly independent (hence the $\binom{n+2}{j}$ monomials of degree j in $\frac{df}{f}, \frac{dx_1}{x_1}, \ldots, \frac{dx_{n+1}}{x_{n+1}}$ will give linearly independent cohomology classes). Observe

$$\frac{df}{f}\wedge\frac{dx_1}{x_1}\wedge\ldots\wedge\widehat{\frac{dx_i}{x_i}}\wedge\ldots\wedge\frac{dx_{n+1}}{x_{n+1}}=\pm\frac{X_i\frac{\partial F}{\partial X_i}}{F}\frac{dx_1}{x_1}\wedge\ldots\wedge\frac{dx_{n+1}}{x_{n+1}}$$

and so by the Euler relation we may assume a relation

$$\sum_{i=1}^{n+2}\frac{\frac{a_i}{d}X_i\frac{\partial F}{\partial X_i}}{F}\frac{dx_1}{x_1}\wedge\ldots\wedge\frac{dx_{n+1}}{x_{n+1}}\sim 0.$$

We claim $a_j=0$; let $y_i=X_i/X_j$ if $i<j$, $y_i=X_{i+1}/X_j$ if $i\geq j$; as

$$\frac{dy_1}{y_1}\wedge\ldots\wedge\frac{dy_{n+1}}{y_{n+1}}=\pm\frac{dx_1}{x_1}\wedge\ldots\wedge\frac{dx_{n+1}}{x_{n+1}},$$

it suffices to show $a_{n+2}=0$. We rewrite our relations

$$a_{n+2}\frac{dx_1}{x_1}\wedge\ldots\wedge\frac{dx_{n+1}}{x_{n+1}}\sim\sum_{i=1}^{n+1}\left(\frac{a_i-a_{n+1}}{d}\right)\frac{df}{f}\wedge\frac{dx_1}{x_1}\wedge\ldots\wedge\widehat{\frac{dx_i}{x_i}}\wedge\ldots\wedge\frac{dx_{n+1}}{x_{n+1}}.$$

The regularity of F insures that every term X^d_i occurs with non-zero coefficient b_i, so that when $x_1, \ldots, x_{n+1}$ are all small, $f(x)$ is close to b_{n+2}, hence $\mathbf{P}^{n+1}_{n+2}-X^n_{n+2}$ contains a region

$$\{0<|x_1|<\varepsilon\}\times\ldots\times\{0<|x_{n+1}|<\varepsilon\}$$

where ε is sufficiently small so that $\log f$ is a "single-valued" holomorphic function in that region. Integrating our relation over $\gamma=\{|x_1|=|x_2|=\ldots=|x_{n+1}|=\varepsilon/2\}$ we get

$$\frac{a_{n+2}}{(2\pi i)^{n+1}}=\sum_{i=1}^{n+1}\frac{a_i-a_{n+2}}{d}\int_\gamma d\left(\log f\frac{dx_1}{x_1}\wedge\ldots\wedge\widehat{\frac{dx_i}{x_i}}\wedge\ldots\wedge\frac{dx_{n+1}}{x_{n+1}}\right)$$

and the right side vanishes.

Hence for $l\geq 1$, $B_{(c)}^{n+l}(X_{n+2}^n)=\binom{n+1}{l+1}$, $B_{(c)}^{n+1+l}(\mathbf{P}_{n+2}^{n+1}-X_{n+2}^n)=\binom{n+2}{l+1}$, and for $l<0$ everything is zero, so we must look at Euler characteristics. We readily compute $\chi(\mathbf{P}_{n+2}^{n+1}-X_{n+2}^n)=(-1)^{n+1}(B_{(c)}^{n+1}(\mathbf{P}_{n+2}^{n+1}-X_{n+2}^n)-n-1)$, $\chi(X_{n+2}^n)=(-1)^n(B_{(c)}^n(X_{n+2}^n)-n)$, while $\chi(\mathbf{P}_{n+2}^{n+1})=\chi(\mathbf{C}^*)^{n+1}=0$, so $\chi(X_{n+2}^n)=-\chi(\mathbf{P}_{n+2}^{n+1}-X_{n+2}^n)$, whence it suffices to show $\chi(X_{n+2}^n)=(-1)^n d^{n+1}$. But $\chi(X_i^n)=\chi(X_{i-1}^n)-\chi(X_{i-1}^{n-1})$ for $1\leq i\leq n+2$. Upon iteration $\chi(X_{n+2}^n)=\sum_{j\geq 0}(-1)^{n-j}\binom{n+2}{n-j}\chi(X_0^j)$, whence $\sum_{n\geq 0}\chi(X_{n+2}^n)Z^n=\sum_{j\geq 0}\chi(X_0^j)\left(\frac{Z}{1+Z}\right)^j$. But $\sum_j\chi(X_0^j)Z^j=\frac{d}{(1-Z)^2(1+(d-1)Z)}$ [5, p. 465]. Q.E.D.

Corollary (**1.11**). — *For* $i\geq 1$ *and every* q *we have a short exact sequence*

$$0\to H^q(X_i^n)\to H^q(X_{i+1}^n)\to H^{q-1}(X_i^{n-1})\to 0$$

and thus for $i\geq 1$, $0\to H^{n-2}(X_0^{n-1})\to H^n(X_0^n)\to H^n(X_i^n)$.

Proof. — The second assertion for $i=1$ is part of the Lefschetz theorem, and follows for $i>1$ from the first assertion. The first sequence is certainly exact without the end zeros, and our computation showed the alternating sum of the dimensions to be zero.

The Cohomology of the Complement.

Define a mapping $\mathscr{R}:\mathscr{L}^0\to\Omega^0(\mathbf{P}_{n+2}^{n+1}-X_{n+2}^n)$ by $\mathscr{R}(Z^aX^b)=\frac{(a-1)!}{(-\pi)^{a-1}}\frac{X^b}{F^a}$. It is easily verified that $\mathscr{R}.D_{X_i}=X_i\frac{\partial}{\partial X_i}.\mathscr{R}$, and that $\mathscr{R}(D_Z(Z^aX^b))=a\left(1-\frac{F}{F}\right)\mathscr{R}(Z^aX^b)$, so that $D_Z\mathscr{L}^0$ is precisely the kernel. Write $x_i=X_i/X_{n+2}$ and take monomials in the $\frac{dx_i}{x_i}$ as a free basis for $\Omega(\mathbf{P}_{n+2}^{n+1}-X_{n+2}^n)$ over $\Omega^0(\mathbf{P}_{n+2}^{n+1}-X_{n+2}^n)$. We obtain

Theorem (**1.12**). — *The mapping* $\mathscr{R}$, *together with the* $*$*-operation provides, an isomorphism of the de Rham complex on* $\mathbf{P}_{n+2}^{n+1}-X_{n+2}^n$ *with the Koszul complex on* $\mathscr{L}^0/D_Z\mathscr{L}^0$ *with operators* $D_{X_1},\ldots,D_{X_{n+1}}$; *in particular an isomorphism of* $H^{n+1}(\mathbf{P}_{n+2}^{n+1}-X_{n+2}^n)$ *with* $\mathscr{L}^0/(D_Z\mathscr{L}^0+\sum_i D_{X_i}\mathscr{L}^0)$.

Corollary (**1.13**). — *In the case of a regular* F, $\dim\mathscr{L}^0/(\sum_i D_{X_i}\mathscr{L}^0)=d^{n+1}+n+1$.

Corollary (**1.14**). — *For regular* F, $\dim\mathscr{L}^0/(D_Z\mathscr{L}+\sum D_{X_i}\mathscr{L}^0)=d^{n+1}+n$.

Proof. — As $D_Z = \frac{1}{d}\sum_i D_{X_i}$, $D_Z\mathscr{L} \subset \sum_i D_{X_i}\mathscr{L}^0 + D_Z$ (elements of Z-degree 0). If X^b has Z-degree zero, but $X^b \neq 1$, say $b_1 \neq 0$, then

$$D_Z(X^b) = D_{X_1}\left(D_Z\left(\frac{1}{b_1}X^b\right)\right) - D_Z\left(\frac{1}{b_1}\pi Z X_1 \frac{\partial F}{\partial X_1} X^b\right) \quad \text{so} \quad D_Z\mathscr{L} \subset \sum_i D_{X_i}\mathscr{L}^0 + (ZF).$$

But $\mathscr{R}(ZF) = \frac{dx_1}{x_1}\wedge \ldots \wedge \frac{dx_{n+1}}{x_{n+1}}$, a non-zero cohomology class, whence $ZF \notin \sum_i D_{X_i}\mathscr{L}^0$, so the drop in dimension is precisely one. Q.E.D.

Remark. — Define $\mathscr{L}^+ = \mathscr{L} \cap K[Z, X_1, \ldots, X_{n+2}]$, $\mathscr{L}^{0,+} = \mathscr{L}^0 \cap \mathscr{L}^+$. It is clear that $\mathscr{L}^0 \subset \mathscr{L}^{0,+} + \sum_i D_{X_i}\mathscr{L}^0$, whence $\mathscr{L}^0/(D_Z\mathscr{L} + \sum_i D_{X_i}\mathscr{L}^0)$ is a quotient of

$$\mathscr{L}^{0,+}/(D_Z\mathscr{L}^+ + \sum_i D_{X_i}\mathscr{L}^{0,+}).$$

In the regular case, the latter space has dimension $\leq d^{n+1}+n$, as dim $\mathscr{L}^+/\sum_i D_{X_i}\mathscr{L}^+$ is d^{1+n} [2, p. 55]. Similarly, dim $\mathscr{L}^{0,+}/\sum_i D_{X_i}\mathscr{L}^{0,+} \leq d^{1+n}+n+1$ in the regular case.

Corollary **(1.15)**. — *$\mathscr{R}$ establishes an isomorphism of $\mathscr{L}^{0,+}/\sum_i D_{X_i}\mathscr{L}^{0,+}$ with $H^{n+1}(\mathbf{P}^{n+1}_{n+2} - X^0)$, which maps $W^S = \mathscr{L}^S/(\mathscr{L}^S \cap \sum_i D_{X_i}\mathscr{L}^{0,+})$ isomorphically onto the image of $H^{n+1}(\mathbf{P}^{n+1} - X)$ in $H^{n+1}(\mathbf{P}^{n+1}_{n+2} - X^0)$.*

We write W for $\mathscr{L}^{0,+}/(D_Z\mathscr{L}^+ + \sum_i D_{X_i}\mathscr{L}^{0,+})$.

Theorem **(1.16)**. — *Let F be a regular form. The mapping $\Theta : W \to H^n(X^0)$ is an isomorphism of modules over the ring of derivations of the field of definition.*

Differentials of the " second kind ".

Consider now the complete variety X, and the open cover of all the $U(b, i)$. Writing $\mathscr{L}^S$ for the subspace of $\mathscr{L}^+$ divisible by all the variables, it is easily seen, following the proof of theorem (1.7), that there is a unique map $\Theta : \mathscr{L}^S \to \sum_{p+q=n} C^p(\Omega^q)$ with the proper effect on elements of Z-degree one, which is a homomorphism of Lie modules, and whose image lies among the " cycles " of the total complex. Thus, letting W^S denote the image of $\mathscr{L}^S$ in W, we have that the image of W^S in $H^n(X^0)$ under Θ lies in the image of $H^n(X)$ in $H^n(X^0)$.

Finally we recall that a closed algebraic differential form X is " of the second kind " if for some affine open set U on which it is regular, the cohomology class it determines on U is the restriction to U of a cohomology class on X.

Theorem **(1.17)**. — *Let F be a regular form. The isomorphism Θ of W with $H^n(X^0)$ maps W^S isomorphically onto the image of $H^n(X)$ in $H^n(X^0)$, i.e. onto the space of n-forms of the second kind holomorphic on X^0, modulo exact such.*

For any derivation D of the field of definition of F, the equations of deformation of Dwork (the action of $\mathfrak{S}_D$ on W^S) are identified with the Picard-Fuchs equations on X^0 (the action of $\widetilde{D}$ on the image of $H^n(X)$ on $H^n(X^0)$).

Proof. — By (1.11), the image of $H^n(X)$ in $H^n(X^0)$ has dimension $B^n(X)-B^n(\mathbf{P}^{n-1})$, and explicit formulas ([2, p. 54] and [5, p. 455]), show this is the dimension of $\mathscr{L}^S/(\mathscr{L}^S\cap\sum_i D_{X_i}\mathscr{L}^+)$ for $n\geq 1$. It is enough, then, to show $\mathscr{L}^S\cap\sum_i D_{X_i}\mathscr{L}^+=\mathscr{L}^S\cap\sum_i D_{X_i}\mathscr{L}^{0,+}$, or, that if $\sum_i a_i Z X_i\dfrac{\partial F}{\partial X_i}\in\mathscr{L}^S+\sum_i D_{X_i}\mathscr{L}^{0,+}$, then all $a_i=0$. To show $a_{n+2}=0$, we follow (1.10) and apply $\mathscr{R}$

$$\sum_i\frac{a_iX_i\dfrac{\partial F}{\partial X_i}}{F}\frac{dx_1}{x_1}\wedge\ldots\wedge\frac{dx_{n+1}}{x_{n+1}}\sim\text{the restriction of a class on } \mathbf{P}^{n+1}-X.$$

The cycle γ in the proof of (1.10) is homologous to zero in $\mathbf{P}^{n+1}-X$. Integrating over γ thus annihilates the right hand side while the left hand side gives $\dfrac{a_{n+2}}{(2\pi i)^{n+1}}=0$. Q.E.D.

Residues [8 *a*].

Let B be a nonsingular subvariety, of codimension one, of a nonsingular variety A, in characteristic zero. The exact sequence of cohomology with compact supports

$$H^{j-1}_{(c)}(A)\to H^{j-1}_{(c)}(B)\to H^{j}_{(c)}(A-B)\to H^{j}_{(c)}(A)\to H^{j}_{(c)}(B)$$

gives by duality an exact sequence of de Rham cohomology

$$H^{l-2}(B)\to H^{l}(A)\to H^{l}(A-B)\to H^{l-1}(B)\to H^{l+1}(A)$$

The map $H^l(A-B)\to H^{l-1}(B)$ is the residue map. When B is given by an equation $g=0$, the residue map, roughly speaking, extracts the coefficient of $\dfrac{dg}{g}$ on the level of differential forms.

Theorem **(1.18).** — $\Theta=$*residue* $\mathscr{R}$, *so that we have a commutative diagram,*

$$\begin{tikzcd}
0 \arrow[r] & H^{n+1}(\mathbf{P}^{n+1}_{n+2}) \arrow[r] & H^{n+1}(\mathbf{P}^{n+1}_{n+2}-X^0) \arrow[r, "\text{residue}"] & H^n(X^0) \arrow[r] & 0 \\
0 \arrow[r] & (ZF) \arrow[r] \arrow[u, "\mathscr{R}", "\wr"'] & \mathscr{L}^0/(\sum_i D_{X_i}\mathscr{L}^0) \arrow[r] \arrow[u, "\mathscr{R}", "\wr"'] & \mathscr{L}^0/(D_Z\mathscr{L}+\sum_i D_{X_i}\mathscr{L}^0) \arrow[r] \arrow[u, "\Theta", "\wr"'] & 0 \\
0 \arrow[r] & (ZF) \arrow[r] \arrow[u, "\wr"'] & \mathscr{L}^{0,+}/(\sum_i D_{X_i}\mathscr{L}^{0,+}) \arrow[r] \arrow[u, "\wr"'] & \mathscr{L}^{0,+}/(\sum_i D_{X_i}\mathscr{L}^{0,+}+D_Z\mathscr{L}^+) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Proof. — By assumption the forms $X_i\dfrac{\partial F}{\partial X_i}$ have no common zero, and hence every form of sufficiently high degree lies in the ideal they generate. Momentarily let us write $\mathscr{L}(n)$ for the subspace of $\mathscr{L}$ of elements whose Z-degree is n; thus $\mathscr{L}(n)=\mathscr{L}(n-1)+\sum_i ZX_i\frac{\partial F}{\partial X_i}\mathscr{L}(n-1)$; because $D_{X_i}=X_i\frac{\partial}{\partial X_i}+\pi ZX_i\frac{\partial F}{\partial X_i}$, it follows that $\mathscr{L}(n)=\mathscr{L}(n-1)+\sum_i D_{X_i}\mathscr{L}(n-1)$ and, in particular, $\mathscr{L}(1)$ spans $\mathscr{L}^0/(\sum_i D_{X_i}\mathscr{L}^0)$. For $ZX^w\in\mathscr{L}(1)$, taking local coordinates $x_i=X_i/X_{n+2}$,

$$\mathscr{R}(ZX^w)=\frac{x^w}{f}\frac{dx_1}{x_1}\wedge\ldots\wedge\frac{dx_{n+1}}{x_{n+1}}=\frac{x^w}{x_{n+1}\frac{\partial f}{\partial x_{n+1}}}\frac{dx_1}{x_1}\wedge\ldots\wedge\frac{dx_n}{x_n}\wedge\frac{df}{f}$$

Thus, residue $(\mathscr{R}(ZX^w))=\frac{x^w}{x_{n+1}\frac{\partial f}{\partial x_{n+1}}}\frac{dx_1}{x_1}\wedge\ldots\wedge\frac{dx_{n+1}}{x_{n+1}}=\Theta(ZX^w)$. Q.E.D.

Remark. — This provides an independent proof that $\Theta(W^S)$ lies in the image of $H^n(X)$ in $H^n(X^0)$, as follows from (1.15) and the commutative diagram

$$\begin{array}{ccc} H^{n+1}(\mathbf{P}^{n+1}-X) & \xrightarrow{\text{residue}} & H^n(X) \\ \Big\downarrow{\scriptstyle\text{restriction}} & & \Big\downarrow{\scriptstyle\text{restriction}} \\ H^{n+1}(\mathbf{P}^{n+1}_{n+2}-X^0) & \xrightarrow{\text{residue}} & H^n(X^0) \end{array}$$

ANALYTIC THEORY

Let $\mathcal{O}$ denote the ring of integers in a fixed finite extension of $\mathbf{Q}_p$ containing π, where now $\pi^{p-1}=-p$, with residue class field $k=GF(q)$. The valuation is normalized by $\operatorname{ord}(p)=1$. A special affine k-algebra is one of the form $A=k[x,\tau]/\bar{I}$, where x means $(x_1,\ldots,x_{n+1})$, and $\bar{I}=(\bar{f}(x), 1-\tau\bar{g}(x))$ with $\bar{g}(x)$ divisible by $\frac{\partial\bar{f}(x)}{\partial x_{n+1}}$. Fixing f and $g\in\mathcal{O}[x]$ lifting $\bar{f}$ and $\bar{g}$, we define $A^*=\mathcal{O}[x,\tau]/I$, $I=(f(x), 1-\tau g(x))$, a special affine lifting of A, and we define $A^\infty=\varprojlim A^*/p^nA^*$. Finally define A^+ as the subalgebra of A^∞ consisting of series $\sum_{n\geq 0}a_nh_n(x,\tau)$, $a_n\in\mathcal{O}$, $h_n\in A^*$ with $\operatorname{ord}a_n\geq n$ and $\frac{\text{degree } h_n}{n}$ bounded from above.

We recall without proof the basic property of these algebras [6, 8, 10].

Proposition (**2**). — *Let* A, B *be any special affine k-algebras,* A^* *and* B^* *any special affine liftings over* $\mathcal{O}$. *Given any homomorphism* $\varphi: A\to B$ *there exists* $\varphi^\infty: A^\infty\to B^\infty$

a homomorphism of $\mathcal{O}$-algebras lifting φ; φ^∞ is determined by specifying $\varphi^\infty(x_1), \dots, \varphi^\infty(x_n)$, subject only to the condition that $\varphi^\infty(x_i)$ lifts $\varphi(x_i)$; further if the $\varphi^\infty(x_i)$ lie in B^+, then φ^∞ maps A^+ to B^+.

Lemma (**2.1**). — *Suppose f depends linearly on x_1; let D_1 be the derivation of A^* with $D_1(x_i)=\delta_{1i}$ for $i=1, \dots, n$. Then $D_1^n A^* \subset n! A^*$.*

Proof. — Let $A_0^* = \mathcal{O}[x_0, x, \tau]/I_0$, $I_0 = (f(x)+x_0, 1-\tau g(x))$ and define D_0 and D_1, derivations of A_0^*, by $D_i(x_j)=\delta_{ij}$ for $i=0, 1$, $0 \leq j \leq n$; define $C_i = \{x \in A_0^* \mid D_i^n(x) \in n!\, A_0^* \text{ if } n \geq 1\}$. Leibniz' rule $D_i^n(xy) = \sum_{j=0}^{n} \binom{n}{j} D_i^j(x) D_i^{n-j}(y)$ shows that C_i is a subring, and that if $xy=1$, $x \in C_i \Rightarrow y \in C_i$. As C_i contains the subring generated by $x_0, \dots, x_n$, it suffices to show each $x_{n+1}^a \Big/ \dfrac{\partial f}{\partial x_{n+1}} \in C_i$; but recalling the relation

$$D_1^n\left(\frac{\rho(x)}{\dfrac{\partial f}{\partial x_{n+1}}}\right) = D_0^n\left(\frac{\left(\dfrac{\partial f}{\partial x_1}\right)^n \rho(x)}{\dfrac{\partial f}{\partial x_{n+1}}}\right)$$

for $\rho(x) \in \mathcal{O}[x_2, \dots, x_{n+1}]$ we see that $C_0 = A_0^* \Rightarrow C_1 = A_0^*$, whence it suffices to show $x_{n+1} \in C_0$.

Lemma (**2.2**). — *Let R be a torsion free domain and let $f(Y) \in R[Y]$ with $f'(Y) \neq 0$. Write $A = R[Y, \tau]/(1-\tau f'(Y))$ and define D to be the R-derivation of A with $D(f)=1$. Then $D^n A \subset n!\, A$.*

Proof. — We may suppose $f(0)=0$, and write $f(Y)=g(Y)+f'(0)Y$ where $Y^2 \mid g(Y)$. It thus suffices to consider the polynomial $h(Y)=g(Y)+uY$ over the ring $R[u]$ where u is a new indeterminate (then specialize $u \to f'(0)$). However $R[u, u^{-1}][[Y]] = R[u, u^{-1}][[h]]$ by the Inverse Function Theorem, so we can write $Y = \sum_{j \geq 1} a_j h^j$. Then $D^n(Y) = n! \sum_{j \geq 0} b_j h^j$, $b_j \in R[u, u^{-1}]$, hence $D^n(Y) = n! \sum_{j \geq 0} C_j Y^j$, $C_j \in R[u, u^{-1}]$, so $D^n(Y) \in n!\, R[u, u^{-1}][[Y]]$. Clearly we may write $D^n(Y) = \dfrac{p[u, Y]}{h'(Y)^{2n-1}}$ with $p[u, Y] \in R[u, Y]$. Multiplying by $h'(Y)^{2n-1}$, we have $p[u, Y] \in R[u, Y] \cap n!\, R[u, u^{-1}][[Y]]$. Q.E.D.

Corollary (**2.3**). — *There exists a constant k, depending only on the degrees and number of variables in f and g such that for $p(x, \tau) \in \mathcal{O}[x, \tau]$,* degree $D_1^n p(x, \tau) \leq$ degree $p_1(x, \tau) + nk$, *and $D_1^n p(x, \tau)$ has all coefficients in $n!\, \mathcal{O}$.*

The Analytic Θ Mapping.

Denote by $L^0(0+)$ the space of series $\sum_w A_w Z^{w_0} X_1^{w_1} \dots X_{n+2}^{w_{n+2}}$ with $dw_0 = \sum_1^{n+2} w_i$, $w_0 \geq 1$, all $w_i \geq 0$, such that for some constants $b > 0$ and c, ord $A_w \geq bw_0 + c$; the A_w are taken from $\mathcal{O} \otimes \mathbf{Q}$. Consider a form F with coefficients in $\mathcal{O}$, f the affinization

$X_{n+2}=1$. The Θ mapping for F is easily computed; pass to $F+\Gamma X^d_{n+2}$, so $\mathfrak{S}_\Gamma=\frac{\partial}{\partial\Gamma}+\pi Z X^d_{n+2}$ whence $\Theta(Z^{w_0}X^w)=\frac{1}{\pi^{w_0-1}}\left(\frac{\partial}{\partial\Gamma}\right)^{w_0-1}\left(\frac{x^w}{x_{n+1}\frac{\partial f}{\partial x_{n+1}}}\right)\Bigg|_{\Gamma=0}$, hence $\Theta(Z^{w_0}X^w)$ is a polynomial in $x_1, \dots, x_{n+1}, 1/x_{n+1}\frac{\partial f}{\partial x_{n+1}}$ all of whose coefficients have ordinal $\geq \operatorname{ord}(w_0-1)!-\operatorname{ord}\pi^{w_0-1}\geq\frac{-\log w_0}{\log p}$. Hence we have

Theorem **(2.4).** — *Θ naturally extends to a mapping* $\Theta : L^0(0+)\to A^+\otimes\mathbf{Q}$ *where* $A^*=\mathcal{O}[x,\tau]/I$, $I=\left(f(x),\ 1-\tau x_1\dots x_{n+1}\frac{\partial f}{\partial x_{n+1}}\right)$.

Deformations. Explicit Construction.

Let $f,\ h\in\mathcal{O}[x]$, $f_\Gamma=f+\Gamma h$, $A^*_\Gamma=\mathcal{O}[x,\Gamma,\tau]/I$, $I=\left(f_\Gamma,\ 1-x_{n+1}\frac{\partial f_\Gamma}{\partial x_{n+1}}\right)$ and define $A^\infty_\Gamma=\varprojlim A^*_\Gamma/(p,\Gamma)^n A^*_\Gamma$. Our previous estimates show that, defining $\frac{\partial}{\partial\Gamma}$ on A^∞_Γ by $\frac{\partial x_i}{\partial\Gamma}=0$, $i=1,\dots,n$, the series $\sum_{n\geq 0}\frac{(-\Gamma)^n}{n!}\left(\frac{\partial}{\partial\Gamma}\right)^n$ converges to an endomorphism $\hat{D}$ of A^∞_Γ (a homomorphism by Leibniz rule). Clearly $\hat{D}\equiv$ identity $\operatorname{mod}(\Gamma)$, $\hat{D}(\Gamma)=0$ so $\hat{D}(\Gamma A^\infty_\Gamma)=0$, and $\hat{D}(x)=x$ if $\frac{\partial x}{\partial\Gamma}=0$. Hence $\hat{D}$ induces $\hat{D}_{(0,\Gamma)}: A^\infty_\Gamma/\Gamma A^\infty_\Gamma\to A^\infty_\Gamma$. Write, for $\operatorname{ord}\mu>0$, $A^*_\mu=A^*_\Gamma/(\Gamma-\mu)$; composing with $\hat{D}_{(0,\Gamma}$ the specialization $\Gamma\to\mu$ then provides $\hat{D}_{(0,\mu)}: A^\infty_0\to A^\infty_\mu$, a map reducing to the identity mod $\mathfrak{p}$ and fixing $x_1,\dots,x_n$. Here $\mathfrak{p}$ is the maximal ideal of $\mathcal{O}$.

Lemma **(2.5).** — *Suppose* F *and* H *are forms of degree d, and let Θ_μ for* $\operatorname{ord}\mu>0$ *denote the Θ map for* $F+\mu H$. *Then we have the commutative diagram*

$$\begin{tikzcd}
L^0(0+) \arrow[r, "\exp(-\pi\mu H)"] \arrow[d, "\Theta_0"] & L^0(0+) \arrow[d, "\Theta_\mu"] \\
A_0^+\otimes\mathbf{Q} \arrow[r, "\hat{D}_{(0,\mu)}"] & A_\mu^+\otimes\mathbf{Q}
\end{tikzcd}$$

The Frobenius (Diagonal Case).

Denote by β the endomorphism of $L^0(0+)$ given by

$$\beta(Z^{w_0}X^w)=\exp(\pi Z^q F(X^q)-\pi Z F(X))Z^{qw_0}X^{qw},$$

and by Fr the endomorphism of A^* where $A^* = \mathcal{O}[x, \tau]/I$, $I = \left(f(x), 1 - \tau x_1 \dots x_{n+1} \frac{\partial f}{\partial x_{n+1}}\right)$ which reduces to the q-th power mapping mod $\mathfrak{p}$, and maps $x_i \mapsto x_i^q$ for $i = 1, \dots, n$. We will prove

Theorem (**2.12**). — *For* F *irreducible* mod $\mathfrak{p}$, $q\Theta \circ \beta = \mathrm{Fr} \circ \Theta$.

We begin with a special case.

Theorem (**2.6**). — *Let* $F(X_1, \dots, X_{n+2})$ *be a form of degree e such that upon affinization by* X_{n+2} *we have* $f(x_1, \dots, x_{n+1}) = g(x_1, \dots, x_n) - x_{n+1}^d$ *where d is prime to p. Then over* $U^{\varnothing}(n+1, n+2)$ *we have a commutative diagram*

$$\begin{array}{ccc} L^0(0+) & \xrightarrow{q\beta} & L^0(0+) \\ \Big\downarrow{\scriptstyle\Theta} & & \Big\downarrow{\scriptstyle\Theta} \\ A^+ \otimes \mathbf{Q} & \xrightarrow{\mathrm{Fr}} & A^+ \otimes \mathbf{Q} \end{array}$$

Proof. — Write $\Delta(x) = g(x^q) - g(x)^q$; then

$$\mathrm{Fr}(x_{n+1})^d = g(x^q) = g(x)^q + \Delta(x) = x_{n+1}^{qd}(1 + \Delta(x) x_{n+1}^{-qd})$$

whence $\mathrm{Fr}(x_{n+1}) = x_{n+1}^q \sum_m \binom{1/d}{m} \frac{\Delta(x)^m}{x_{n+1}^{mqd}}$. Let $B^* = \mathcal{O}[x, \tau]/I$ where

$$I = (g(x)^q - x_{n+1}^{qd}, 1 - \tau x_1 \dots x_{n+1}),$$

this is not special affine, but clearly there is a restriction mapping $\mathrm{res} : B^+ \to A^+$, and our formula shows that we may interpret Fr as factoring $A^+ \xrightarrow{\mathrm{Fr}} B^+ \xrightarrow{\mathrm{res}} A^+$.

Now we return to the Θ mapping over $U^{\varnothing}(n+1, n+2)$ for $G(X) - X_{n+1}^d X_{n+2}^{e-d}$. We introduce the family $G(X) - X_{n+1}^d X_{n+2}^{e-d} + \Gamma X_{n+2}^e$, and we then have

$$\Theta(Z^{w_0} X^w) = \Theta_\Gamma(\pi^{1-w_0} \mathfrak{S}_\Gamma^{w_0 - 1}(Z X^w X_{n+2}^{e(1-w_0)}))\Big|_{\Gamma=0} = \frac{1}{\pi^{w_0-1}} \frac{\partial^{w_0-1}}{\partial \Gamma^{w_0-1}} \left(\frac{x^w}{-d x_{n+1}^d}\right)\Big|_{\Gamma=0},$$

write $x^w = x_1^{w_1} \dots x_{n+1}^{w_{n+1}}$, so $\Theta(Z^{w_0} X^w) = \frac{x_1^{w_1} \dots x_n^{w_n}}{-d\pi^{w_0-1}} \frac{\partial^{w_0-1}}{\partial \Gamma^{w_0-1}} (x_{n+1}^{w_{n+1}-d})\Big|_{\Gamma=0}$; but the relation $x_{n+1}^d = g(x_1, \dots, x_n) + \Gamma$, $\frac{\partial x_i}{\partial \Gamma} = 0$ if $i \leq n$, shows

$$\frac{\partial^{w_0-1}}{\partial \Gamma^{w_0-1}} (x_{n+1}^{w_{n+1}-d}) = (w_0 - 1)! \begin{pmatrix} \frac{w_{n+1}}{d} - 1 \\ w_0 - 1 \end{pmatrix} x_{n+1}^{w_{n+1} - w_0 d},$$

so that finally

$$\Theta(Z^{w_0} X^w) = \frac{-(w_0-1)!}{d\pi^{w_0-1}} \begin{pmatrix} \frac{w_{n+1}}{d} - 1 \\ w_0 - 1 \end{pmatrix} x_1^{w_1} \dots x_n^{w_n} x_{n+1}^{w_{n+1} - w_0 d}.$$

We note that if we replace d by qd and w_{n+1} by qw_{n+1} then our answer is " essentially " integral; so define $L^{(q)}(o+)$ to be the space of series $\sum_w A_w Z^{w_0} X^w$ with the usual growth condition but with homogeneity condition $qew_0 = \sum_{i=1}^{n+2} w_i$, $w_0 \geqslant 1$, and such that each w_{n+1} is divisible by q. Then let Θ_q mean the Θ mapping for $G(X^q) - X_{n+1}^{qd} X_{n+2}^{q(e-d)}$, over $U^\varnothing(n+1, n+2)$; we have computed that $\Theta_{(q)}$ maps $L^{(q)}(o+)$ to $B^+ \otimes \mathbf{Q}$.

Finally define $\Phi^{(1)}$ and $\Phi^{(2)}$ by $\Phi^{(1)}(Z^{w_0}X^w) = Z^{w_0}X^{qw}$, $\Phi^{(2)}(Z^{w_0}X^w) = Z^{qw_0}X^w$, define

$$H = \exp(\pi Z X_{n+1}^d X_{n+2}^{e-d} - \pi Z^q X_{n+1}^{qd} X_{n+2}^{q(e-d)} + \pi Z^q G(X)^q - \pi Z G(X)) \in L^0(o+)$$

and
$$G = \exp(\pi Z G(X^q) - \pi Z G(X)^q) \in L^{(q)}(o+), \qquad \text{as } \frac{\partial G}{\partial X_{n+1}} = 0.$$

Clearly now $\beta(Z^{w_0}X^w) = H \circ \Phi^{(2)} \circ G \circ \Phi^{(1)}$, and so it suffices if both the following diagrams are commutative.

$$\begin{tikzcd} L^0(o+) \arrow[r, "qG\circ\Phi^{(1)}"] \arrow[d, "\Theta"] & L^{(q)}(o+) \arrow[d, "\Theta_q"] \\ A^+\otimes\mathbf{Q} \arrow[r, "\mathrm{Fr}"] & B^+\otimes\mathbf{Q} \end{tikzcd} \qquad \begin{tikzcd} L^{(q)}(o+) \arrow[r, "H\circ\Phi^{(2)}"] \arrow[d, "\Theta_q"] & L^0(o+) \arrow[d, "\Theta"] \\ B^+\otimes\mathbf{Q} \arrow[r, "\mathrm{res}"] & A^+\otimes\mathbf{Q} \end{tikzcd}$$

The first diagram. — Take $\eta = Z^{w_0}X^w$; then

$$\mathrm{Fr}\,\Theta(\eta) = \frac{-(w_0-1)!}{d\pi^{w_0-1}} \begin{pmatrix} \frac{w_{n+1}}{d} - 1 \\ w_0 - 1 \end{pmatrix} \frac{x_1^{qw_1} \dots x_{n+1}^{qw_{n+1}}}{x_{n+1}^{qdw_0}} \sum_m \begin{pmatrix} \frac{w_{n+1}}{d} - w_0 \\ m \end{pmatrix} \frac{\Delta(x)^m}{x_{n+1}^{qmd}} =$$

$$= \frac{-x_1^{qw_1} \dots x_{n+1}^{qw_{n+1}}}{d\pi^{w_0-1} x_{n+1}^{qdw_0}} \sum_m \frac{(m+w_0-1)!}{m!} \begin{pmatrix} (w_{n+1}/d) - 1 \\ m + w_0 - 1 \end{pmatrix} \frac{\Delta(x)^m}{x_{n+1}^{qmd}}.$$

The other way, $q\Theta_{(q)} G\Phi^{(1)}(\eta) = q\Theta_{(q)}\left(\sum_m \frac{\pi^m}{m!}(ZG(X^q) - ZG(X)^q)^m Z^{w_0}X^{qw}\right)$. The expression under the Σ is a sum of monomials whose Z-degree is $m + w_0$, and whose X_{n+1}-degree is qw_{n+1}; we have

$$q\Theta_q G\Phi^{(1)}(\eta) = q \sum_m \frac{\pi^m}{-qd\pi^{m+w_0-1} m!}(m+w_0-1)! \begin{pmatrix} (w_{n+1}/d) - 1 \\ m + w_0 - 1 \end{pmatrix} \frac{x^{qw}(\Delta(x))^m}{x_{n+1}^{qmd+qdw_0}}.$$

The second diagram. — Say $L^{(q)}(o+) \ni \eta = Z^{w_0} X_1^{w_1} \dots X_n^{w_n} X_{n+1}^{qc} X_{n+2}^{w_{n+2}}$; to compute $\Theta(H\Phi^{(2)}\eta)$, first define constants A_n and B_n by

(2.7) $$\sum_n A_n X^n = \exp(\pi X^q - \pi X), \qquad \sum_n B_n X^n = \exp(\pi X - \pi X^q).$$

Then $\Theta(\mathrm{H}\Phi^{(2)}\eta)=\Theta(\sum_{n,m}\mathrm{B}_n\mathrm{A}_m(\mathrm{ZX}_{n+1}^d\mathrm{X}_{n+2}^{e-d})^n(\mathrm{ZG(X)})^m\mathrm{Z}^{qw_0}\mathrm{X}_1^{w_1}\ldots\mathrm{X}_n^{w_n}\mathrm{X}_{n+1}^{qc}\mathrm{X}_{n+2}^{w_{n+2}})$. For fixed n, m the expression under the Σ is a sum of monomials of Z-degree $n+m+qw_0$, whose X_{n+1}-degree is $nd+qc$, so that

$$\Theta(\mathrm{H}\Phi^{(2)}(\eta))=\sum_{n,m}\frac{-\mathrm{B}_n\mathrm{A}_m(n+m+qw_0-1)!}{d\pi^{n+m+qw_0-1}}\binom{(qc/d)+n-1}{n+m+qw_0-1}\frac{x_{n+1}^{nd}g^m x_1^{w_1}\ldots x_n^{w_n}x_{n+1}^{qc}}{x_{n+1}^{d(n+m+qw_0)}}$$

but $g=x_{n+1}^d$, so

$$\Theta(\mathrm{H}\Phi^{(2)}(\eta))=\sum_{n,m}\frac{-\mathrm{B}_n\mathrm{A}_m(n+m+qw_0-1)!}{d\pi^{n+m+qw_0-1}}\binom{(qc/d)+n-1}{n+m+qw_0-1}\frac{x_1^{w_1}\ldots x_n^{w_n}x_{n+1}^{qc}}{x_{n+1}^{qdw_0}}.$$

Finally $\Theta_{(q)}(\eta)=\dfrac{(w_0-1)!}{-qd\pi^{w_0-1}}\dbinom{(c/d)-1}{w_0-1}\dfrac{x_1^{w_1}\ldots x_n^{w_n}x_{n+1}^{qc}}{x_{n+1}^{qdw_0}}$, whence we are reduced to showing

$$\frac{(w_0-1)!}{q\pi^{w_0}}\binom{(c/d)-1}{w_0-1}=\frac{1}{\pi^{qw_0}}\sum_{n,m}\frac{\mathrm{B}_n}{\pi^n}\frac{\mathrm{A}_m}{\pi^m}(n+m+qw_0-1)!\binom{(qc/d)+n-1}{n+m+qw_0-1}.$$

Lemma (**2.8**). — *Let a be a strictly positive integer, and b a rational p-adic integer. Then*

$$\frac{(a-1)!}{q\pi^{a(1-q)}}\binom{b-1}{a-1}=\sum_{n,m}\frac{\mathrm{B}_n}{\pi^n}\frac{\mathrm{A}_m}{\pi^m}(qa+n+m-1)!\binom{qb+n-1}{qa+n+m-1}.$$

Proof. — First $\mathrm{ord}(\mathrm{A}_j)\geq j\left(\frac{p-1}{pq}\right)$, $\mathrm{ord}(\mathrm{B}_j)\geq j\left(\frac{p-1}{pq}\right)$; for fixed a, both sides represent continuous functions of b on the rational p-adic integers, and so we may assume that b is a positive rational integer. Both sides vanish unless $b\geq a$, and in that case the right side becomes $\sum_n\frac{\mathrm{B}_n}{\pi^n}(qb+n-1)!\sum_m\frac{\mathrm{A}_m}{\pi^m}\frac{1}{(q(b-a)-m)!}$. The second factor is the coefficient of $\mathrm{X}^{q(b-a)}$ in $\exp(\mathrm{X})\sum_m\frac{\mathrm{A}_m\mathrm{X}^m}{\pi^m}=\exp(\mathrm{X})\exp\left(\frac{\mathrm{X}^q}{\pi^{q-1}}-\mathrm{X}\right)=\exp\left(\frac{\mathrm{X}^q}{\pi^{q-1}}\right)$, so the right side is $\frac{1}{(b-a)!\,\pi^{(q-1)(b-a)}}\sum_n\frac{\mathrm{B}_n}{\pi^n}(qb+n-1)!$ and we are reduced to showing

$$\sum_n\frac{\mathrm{B}_n}{\pi^n}(qb+n-1)!=\frac{\pi^{(q-1)b}(b-1)!}{q}.$$

Define

$$f(\mathrm{X})=\mathrm{X}^{qb-1}\exp\left(\mathrm{X}-\frac{\mathrm{X}^q}{\pi^{q-1}}\right)=\sum_n\frac{\mathrm{B}_n}{\pi^n}\mathrm{X}^{qb+n-1},$$

convergent for $\mathrm{ord}\,\mathrm{X}>\frac{1}{p-1}-\frac{p-1}{pq}$, and set $g(\mathrm{X})=\sum_{n\geq 0}\frac{d^n}{d\mathrm{X}^n}(f(\mathrm{X}))$, easily seen to converge for $\mathrm{ord}\,\mathrm{X}>\frac{1}{p-1}-\frac{p-1}{pq}$, and satisfy $g-\frac{dg}{d\mathrm{X}}=f(\mathrm{X})$. As the only solution of the homogenous equation is a constant multiple of $\exp(\mathrm{X})$, converging only for $\operatorname{ord} X > \frac{1}{p-1}$, it follows that $g(X)$ is the unique power series solution of $g - \frac{dg}{dX} = X^{qb-1}\exp\left(X - \frac{X^q}{\pi^{q-1}}\right)$ which converges for $\operatorname{ord} X > \frac{1}{p-1} - \frac{p-1}{pq}$.

In fact there is a solution of the form $H(X^q)\exp\left(X - \frac{X^q}{\pi^{q-1}}\right)$. We must have $H(X^q) - \left(1 - \frac{qX^{q-1}}{\pi^{q-1}}\right)H(X^q) - \frac{d}{dX}H(X^q) = X^{qb-1}$, or, multiplying by X,

$$\frac{q}{\pi^{q-1}}X^q H(X^q) - X\frac{d}{dX}H(X^q) = X^{qb},$$

which is equivalent with $\frac{q}{\pi^{q-1}}xH(X) - qX\frac{d}{dX}H(X) = X^b$; so

$$H(X) = \left(1 - \pi^{q-1}\frac{d}{dX}\right)^{-1}\left(\frac{\pi^{q-1}}{q}X^{b-1}\right) = \sum_{j=0}^{b-1}\left(\pi^{q-1}\frac{d}{dX}\right)^j\left(\frac{\pi^{q-1}}{q}X^{b-1}\right).$$

In particular $H(0) = \frac{\pi^{(q-1)b}}{q}(b-1)!$, and by uniqueness $g(X) = H(X^q)\exp\left(X - \frac{X^q}{\pi^{q-1}}\right)$, so $g(0) = \frac{\pi^{(q-1)b}}{q}(b-1)!$, and $g(0) = \sum_n \frac{B_n}{\pi^n}(n+qb-1)!$. Q.E.D.

We pass to the general case by a sort of analytic continuation.

Lemma **(2.9)**. — *Let* A^* *be an* $\mathcal{O}$ *algebra with* $A^*/\mathfrak{p}A^*$ *a domain, and* p *not a zero divisor. Suppose* $x \in A^*$, $x \notin \mathfrak{p}A^*$, $xA^*/\mathfrak{p}A^* \neq A^*/\mathfrak{p}A^*$, *and that for every* $a \in \mathfrak{p}$, p *is not a zero divisor in* $A^*/(x-a)A^*$. *Then* $\bigcap_{a\in\mathfrak{p}}(x-a)A^\infty = 0$.

Proof. — We proceed in short steps.

(1) $\quad x$ is not a zero divisor in $A^*/\mathfrak{p}^r A^*$.

Proof. — For $r=1$, $A^*/\mathfrak{p}A^*$ is a domain, and $x \notin \mathfrak{p}A^*$. Using induction if $xy \in \mathfrak{p}^{r+1}A^*$ then $(y) = \mathfrak{p}^r z$, whence $xz \in \mathfrak{p}A^*$, whence $z \in \mathfrak{p}A^*$.

(2) $\quad (\mathfrak{p}^r A^*/\mathfrak{p}^{r+1}A^*) \cap (x^n A^*/\mathfrak{p}^{r+1}A^*) = x^n \mathfrak{p}^r A^*/\mathfrak{p}^{r+1}A^*$.

Proof. — Let $a, b \in A^*$ with $x^n a \equiv \mathfrak{p}^r b \bmod \mathfrak{p}^{r+1}$; then $(a) = \mathfrak{p}^r d$, so $(x^n a) = x^n \mathfrak{p}^r d$.

(3) $\quad \bigcap_{n\geq 0}(x^n A^*/\mathfrak{p}^j A^*) = 0$.

Proof. — For $j=1$, $xA^*/\mathfrak{p}A^*$ is a proper ideal in a domain. Now let $y \in A^*$, with $y \bmod \mathfrak{p}^j \in \bigcap_{n\geq 0}(x^n A^*/\mathfrak{p}^j A^*)$; by induction we may suppose $y \equiv 0 \bmod \mathfrak{p}^{j-1}$, and so by (2) $y \bmod \mathfrak{p}^j \in \bigcap_{n\geq 0}(x^n \mathfrak{p}^{j-1}A^*/\mathfrak{p}^j A^*) \sim \bigcap_{n\geq 0}(x^n A^*/\mathfrak{p}A^*) = 0$.

(4) $\quad \bigcap_{a\in\mathfrak{p}}(x-a)A^\infty = 0$.

Proof. — Let $y \in \bigcap_{a\in\mathfrak{p}}(x-a)A^\infty$; let $a_1, \ldots, a_n$ be distinct elements of $\mathfrak{p}$. Write $y = (x-a_1)\xi(a_1)$ and reduce $\bmod(x-a_2)$; then $(a_2-a_1)\xi(a_1) \equiv 0 \bmod(x-a_2)$ whence $\xi(a_1) = (x-a_2)\xi(a_1, a_2)$ as p, and hence $a_2 - a_1$, is not a zero divisor $\bmod(x-a_2)$.

Continuing we get $y=(x-a_1)\dots(x-a_n)\xi(a_1, \dots, a_n)$. In particular taking all $a_i \in \mathfrak{p}^r$ shows $y \bmod \mathfrak{p}^r$ lies in $x^n A^*/\mathfrak{p}^r A^*$. As n was arbitrary we conclude by (3) that $y \equiv 0 \bmod \mathfrak{p}^r A^*$; as r was arbitrary we conclude that $y=0$ by definition of A^∞. Q.E.D.

The Pseudo-Frobenius.

Let $F(X, \Gamma)$ be a form with coefficients in $\mathcal{O}[\Gamma]$. Define $L^0_\Gamma(0+)$ to be the space of series $\sum_{w,n} A_{w,n} Z^{w_0} X^w \Gamma^n$ with the usual growth condition (i.e. $w_i \geqslant 0$, $\operatorname{ord} A_{w,n} \geq \alpha(w_0+n+\beta)$, $\alpha>0$) and homogeneity $dw_0 = \sum_{i=1}^{n+2} w_i$, $w_0 \geqslant 1$. Define an endomorphism $\beta(\Gamma)$ by

$$\beta(\Gamma)(Z^{w_0}X^w\Gamma^n) = \exp(\pi Z^q F(X^q, \Gamma^q) - \pi Z F(X, \Gamma)) Z^{qw_0} X^{qw} \Gamma^n.$$

Define

$$A^*_\Gamma = \mathcal{O}[\Gamma, x, \tau]/I, \quad \text{where} \quad I = \left(f(x, \Gamma),\ 1 - \tau x_1 \dots x_{n+1} \frac{\partial f(x, \Gamma)}{\partial x_{n+1}} \right)$$

and

$$A^*_{\Gamma^q} = \mathcal{O}[\Gamma, x, \tau]/I_q, \qquad I_q = \left(f(x, \Gamma^q),\ 1 - \tau x_1 \dots x_{n+1} \frac{\partial f(x, \Gamma^q)}{\partial x_{n+1}} \right)$$

and for $a \in \mathfrak{p}$, $A^*_a = A^*_\Gamma/(\Gamma - a)A^*_\Gamma$. We note that $f(x, \Gamma)^q \equiv f(x^q, \Gamma^q) \bmod \mathfrak{p}$ and conclude the existence of a unique homomorphism $\widetilde{F}_\Gamma : A^+_{\Gamma^q} \to A^+_\Gamma$ with $\widetilde{F}_\Gamma(\Gamma) = \Gamma$, $\widetilde{F}_\Gamma(x_i) = x_i^q$ for $i = 1, \dots, n$. (Here the $+$ is taken with Γ as a space variable.)

We denote by Θ_Γ and Θ_{Γ^q} the Θ mappings from $L^0_\Gamma(0+)$ to A^+_Γ and $A^+_{\Gamma^q}$ respectively, and for $a \in \mathfrak{p}$, Θ_a is the Θ mapping from $L^0(0+)$ to A^+_a.

Lemma (**2.10**). — *Suppose* $\Theta_0 \circ q\beta(0) = F_0 \circ \Theta_0$, *and that* $\bigcap_{a \in \mathfrak{p}} (\Gamma - a) A^\infty_\Gamma = 0$. *Then we have a commutative diagram*

$$\begin{tikzcd}
L^0_\Gamma(0+) \arrow[r, "q\beta(\Gamma)"] \arrow[d, "\Theta_{\Gamma^q}"] & L^0_\Gamma(0+) \arrow[d, "\Theta_\Gamma"] \\
A^+_{\Gamma^q} \otimes \mathbf{Q} \arrow[r, "\widetilde{F}_\Gamma"] & A^+_\Gamma \otimes \mathbf{Q}
\end{tikzcd}$$

Proof. — Let $\eta \in L^0_\Gamma(0+)$; multiplying η by a power of p we may suppose $\Theta_\Gamma(\beta(\Gamma)\eta) - \widetilde{F}_\Gamma(\Theta_{\Gamma^q}(\eta))$ lies in A^+_Γ, so it suffices to show it lies in $\bigcap_{a \in \mathfrak{p}} (\Gamma - a) A^+_\Gamma$. But specializing $\Gamma \to a$, $a \in \mathfrak{p}$ we have the diagram

$$\begin{tikzcd}
L^0(0+) \arrow[r, "\exp(\pi F(X, a^q) - \pi F(X, 0))"] \arrow[d, "\Theta_{a^q}"] & L^0(0+) \arrow[r, "q\beta(0)"] \arrow[d, "\Theta_0"] & L^0(0+) \arrow[r, "\exp(\pi F(X, 0) - \pi F(X, a))"] \arrow[d, "\Theta_0"] & L^0(0+) \arrow[d, "\Theta_a"] \\
A^+_{a^q} \arrow[r, "\widehat{D}(a^q, 0)"] & A^+_0 \arrow[r, "F_0"] & A^+_0 \arrow[r, "\widehat{D}(0, a)"] & A^+_a
\end{tikzcd}$$

The middle box commutes by assumption, and the end boxes arise from deformations, hence commute. Finally it is clear that $\widetilde{\mathrm{F}}_a = \hat{\mathrm{D}}_{(a,0)} \circ \mathrm{F}_0 \circ \hat{\mathrm{D}}_{(a^q,0)}$, as both sides have the same reduction mod $\mathfrak{p}$, and agree on $x_1, \ldots, x_n$.

Corollary **(2.11)**. — *Under the assumptions of the lemma, we have a commutative diagram for* $\mathrm{F}(\mathrm{X}, 1)$

$$\begin{tikzcd} \mathrm{L}^0(0+) \arrow[r, "q\beta(1)"] \arrow[d, "\Theta_1"] & \mathrm{L}^0(0+) \arrow[d, "\Theta_1"] \\ \mathrm{A}_1^+ \otimes \mathbf{Q} \arrow[r, "\mathrm{Frob.}"] & \mathrm{A}_1^+ \otimes \mathbf{Q} \end{tikzcd}$$

Theorem **(2.12)**. — *Let* $\mathrm{F}(\mathrm{X})$ *be a form of degree d whose reduction modulo* $\mathfrak{p}$ *is irreducible. Then* $q\Theta \circ \beta = \mathrm{Fr} \circ \Theta$.

Proof. — Over $\mathrm{U}^{\varnothing}(n+1, n+2)$, to fix ideas, write $f(x) = x_{n+1} - 1 + g(x)$ (we may suppose f has degree ≥ 2) and consider the family $f(x, \Gamma) = x_{n+1} - 1 + \Gamma g(x)$. As $f(x, 1)$ is irreducible mod $\mathfrak{p}$, it is not divisible by $x_{n+1} - 1$, hence neither is $g(x)$, whence $f(x, \Gamma)$ is irreducible, and remains so mod $\mathfrak{p}$. Hence A_Γ^* and $\mathrm{A}_\Gamma^*/\mathfrak{p}\mathrm{A}_\Gamma^*$ are domains; clearly $\Gamma \notin \mathfrak{p}\mathrm{A}_\Gamma^*$ and Γ generates a proper ideal mod $\mathfrak{p}$. Finally for $a \in \mathfrak{p}$, $f(x, a)$ is congruent to $x_{n+1} - 1$ mod $\mathfrak{p}$, so that p is not a zero divisor in any $\mathrm{A}_\Gamma^*/(\Gamma - a)\mathrm{A}_\Gamma^*$. Thus we may apply (2.11). Q.E.D.

ANALYTIC THEORY OF THE COMPLEMENT

Let F be a form of degree d (non-trivial mod $\mathfrak{p}$), f the affinization $\mathrm{X}_{n+2} = 1$; define $\mathrm{C}_f^* = \mathcal{O}[x, \tau]/\mathrm{I}$, $\mathrm{I} = (1 - x_1 \ldots x_{n+1} \tau f)$.

Deformations. — Here C_f^∞ and C_f^+ clearly depend only on f modulo $\mathfrak{p}$ (if $f \equiv g$ mod $\mathfrak{p}$, $g^{-1} = f^{-1}(1 + f^{-1}(g-f))^{-1} = \sum_n (g-f)^n f^{-n-1}$).

Lemma **(2.13)**. — *Suppose* $\mathrm{F} \equiv \mathrm{G}$ mod $\mathfrak{p}$; *then we have a commutative diagram*

$$\begin{tikzcd} \mathrm{L}^0(0+) \arrow[r, "\exp(\pi \mathrm{Z}(\mathrm{F}-\mathrm{G}))"] \arrow[d, "\mathcal{R}_f"] & \mathrm{L}^0(0+) \arrow[d, "\mathcal{R}_g"] \\ \mathrm{C}_f^+ \arrow[r, "\mathrm{id.}"] & \mathrm{C}_g^+ \end{tikzcd}$$

Proof.

$$\mathcal{R}_g \circ \exp(\pi Z(F-G))Z^aX^b = \mathcal{R}_g\left(\sum_{n\geq 0} \frac{\pi^n}{n!}(ZF-ZG)^n Z^a X^b\right) =$$

$$= \sum_{n\geq 0} \frac{\pi^n (n+a-1)!\,(f-g)^n x^b}{n!\,(-\pi)^{n+a-1} g^{n+a}} = \sum_{n\geq 0} \frac{(a-1)!}{(-\pi)^{a-1}} \binom{-a}{n} \left(\frac{f-g}{g}\right)^n \frac{x^b}{g^a} =$$

$$= \frac{(a-1)!}{(-\pi)^{a-1}} \left(1 + \left(\frac{f-g}{g}\right)\right)^{-a} \frac{x^b}{g^a} = \frac{(a-1!)}{(-\pi)^{a-1}} \frac{x^b}{f^a} = \mathcal{R}_f(Z^aX^b). \quad \text{Q.E.D.}$$

The Frobenius.

Write $\Delta = f(x^q) - f^q(x)$; $\mathrm{Fr}\left(\frac{1}{f}\right) = \frac{1}{f^q + \Delta} = \sum_n \frac{\Delta^n}{f^{q(1+n)}}$ and hence we may " factor " the Frobenius

$$C^+_{f(x)} \xrightarrow{x \to x^q} C^+_{f(x^q)} \xrightarrow{\text{id.}} C^+_{f(x)^q} \xrightarrow{\text{id.}} C^+_{f(x)}.$$

Similarly (following the proof of (2.6)) write

$$H = \exp(\pi Z^q F(X)^q - \pi Z F(X)), \qquad G = \exp(\pi Z F(X^q) - \pi Z F(X)^q),$$

$L^{(q)}(0+)$ the corresponding space for qd, and maps $\Phi^{(1)}$ and $\Phi^{(2)}$ by $\Phi^{(1)}(Z^aX^b) = Z^aX^{qb}$, $\Phi^{(1)}(Z^aX^b) = Z^{qa}X^b$; then β is

$$L^0(0+) \xrightarrow{\Phi^{(1)}} L^{(q)}(0+) \xrightarrow{G} L^{(q)}(0+) \xrightarrow{H.\Phi^{(2)}} L^0(0+).$$

Theorem **(2.14).** — $\mathrm{Fr} \circ \mathcal{R}_f = q\mathcal{R}_f \circ \beta$; i.e. *the following diagram commutes.*

$$\begin{tikzcd}
C_{f(x)} \arrow[r, "x \to x^q"] & C_{f(x^q)} \arrow[r, "\text{id.}"] & C_{f(x)^q} \arrow[r, "\text{id.}"] & C_{f(x)} \\
L^0(0+) \arrow[u, "\mathcal{R}_f"] \arrow[r, "\Phi^{(1)}"] & L^{(q)}(0+) \arrow[u, "\mathcal{R}_{f(x^q)}"] \arrow[r, "G"] & L^{(q)}(0+) \arrow[u, "\mathcal{R}_{f^q}"] \arrow[r, "qH\circ\Phi^{(2)}"] & L^0(0+) \arrow[u, "\mathcal{R}_f"]
\end{tikzcd}$$

Proof. — The commutativity of the left hand box is clear, and that of the middle box is the content of the last lemma. Writing $H = \sum_n A_n Z^n F^n$, $qH\Phi^{(2)} Z^aX^b = q\sum_n A_n Z^n F^n Z^{aq} X^b$, whence

$$\mathcal{R}_f(qH\Phi^{(2)}(Z^aX^b)) = q\sum A_n \frac{(n+aq-1)!\,f^n x^b}{(-\pi)^{n+aq-1} f^{n+aq}} = q\sum A_n \frac{(n+aq-1)!}{(-\pi)^{n+aq-1}} \frac{x^b}{f^{aq}},$$

while $\mathcal{R}_{f^q}(Z^aX^b) = \frac{(a-1)!}{(-\pi)^{a-1}} \frac{x^b}{f^{aq}}$. This identity may be established by the technique of (2.8) (for q odd, $A_n = (-1)^n B_n$ and the identity appears there). Q.E.D.

The ψ Mapping.

Define on C_f^+ a linear mapping ψ by setting, for $g \in C_f^+$

$$\psi(g)(x) = \frac{1}{q^{n+1}} \sum_{y^q = x} g(y).$$

Clearly $\psi(\mathrm{Fr}(g)h) = g\psi(h)$, and in particular

$$\psi\left(\frac{x^a}{f(x^q)^n}\right) = \frac{\psi(x^a)}{f(x)^n}$$

where $\psi(x^a) = 0$ unless $a_1, \ldots, a_{n+1}$ are all divisible by q, in which case $\psi(x^a) = x^{a/q}$. Again there is an evident " factorization " of ψ

$$C_f^+ \xrightarrow{\text{id.}} C^+_{f^q(x)} \xrightarrow{\text{id.}} C^+_{f(x^q)} \xrightarrow{\psi} C_f^+.$$

Recall the endomorphism α of $L(0+)$ defined by $\widetilde{\psi} \circ \exp(\pi Z F(X) - \pi Z^q F(X^q))$, where $\widetilde{\psi}(Z^a X^b) = 0$ unless a and $b_1, \ldots, b_{n+2}$ are divisible by q, in which case $\widetilde{\psi}(Z^a X^b) = Z^{a/q} Z^{b/q}$. Define $\psi^{(1)} : L(0+) \to L^{(q)}(0+)$ by $\psi^{(1)}(Z^a X^b) = Z^{a/q} X^b$ if $q \mid a$, 0 otherwise, and $\psi^{(2)} : L^{(q)}(0+) \to L^0(0+)$ by $\psi^{(2)}(Z^a X^b) = Z^a X^{b/q}$ if $q \mid b_1, \ldots, q \mid b_{n+2}$, and 0 otherwise.

Theorem **(2.15)**. — $\frac{1}{q}\mathscr{R}_f \circ \alpha = \psi \circ \mathscr{R}_f$, i.e. *the following diagram commutes.*

$$\begin{tikzcd}
C_f \arrow[r, "\text{id.}"] \arrow[d, "\mathscr{R}_f"] & C^+_{f^q(x)} \arrow[r, "\text{id.}"] \arrow[d, "\mathscr{R}_{f^q}"] & C^+_{f(x^q)} \arrow[r, "\psi"] \arrow[d, "\mathscr{R}_{f(x^q)}"] & C_f^+ \arrow[d, "\mathscr{R}_f"] \\
L^0(0+) \arrow[r, "\frac{1}{q}\psi^{(1)} \circ H^{-1}"] & L^{(q)}(0+) \arrow[r, "G^{-1}"] & L^{(q)}(0+) \arrow[r, "\psi^{(2)}"] & L^0(0+)
\end{tikzcd}$$

Proof. — Here it is clear that the right-most and central boxes commute. Write $H^{-1} = \sum_n B_n Z^n F^n$;

$$\psi^{(1)} H^{-1}(Z^a X^b) = \sum_{n+a \equiv 0(q)} B_n Z^{\frac{n+a}{q}} F^n X^b = \sum_{n \geq 1} B_{qn-a} Z^n F^{qn-a} X^b,$$

whence

$$\mathscr{R}_{f^q}(\psi^{(1)} H^{-1}(Z^a X^b)) = \sum_{n \geq 1} B_{qn-a} \frac{(n-1)!}{(-\pi)^{n-1}} \frac{f^{nq-a} x^b}{f^{qn}} = \sum_{n \geq 1} \frac{(n-1)!}{(-\pi)^{n-1}} B_{qn-a} \frac{x^b}{f^a},$$

whence we are reduced to showing $\frac{(a-1)!}{(-\pi)^{a-1}} = \frac{1}{q} \sum_{n \geq 1} \frac{(n-1)!}{(-\pi)^{n-1}} B_{nq-a}$.

Consider the space L of all those power series in one variable $\sum_{n \geq 0} a_n x^n$ with $\operatorname{ord} a_n \geq bn + c$ with $b > 0$, L^0 the subspace vanishing at $x = 0$, D the operator $x\frac{\partial}{\partial x} + \pi x$,

α the operator $\widetilde{\psi}.\exp(\pi x - \pi x^q)$. Clearly L^0/DL^0 is one-dimensional, with basis $\{x\}$ and the relation $\alpha D = qd\alpha$ shows α acts here. Further $\alpha(D(1)) = qD(\alpha(1)) = qD(1) + qD(\eta)$ where $\eta \in L^0$ and hence, on L^0/DL^0, α induces multiplication by q. Finally, $x^a \equiv \dfrac{(a-1)!}{(-\pi)^{a-1}} x \bmod DL^0$. Q.E.D.

Remark. — This technique gives another proof of (2.8) as β induces multiplication by $1/q$ on L^0/DL^0.

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