

MOSHE JARDEN

GOPAL PRASAD

Appendix on the discriminant quotient formula for global field

Publications mathématiques de l'I.H.É.S., tome 69 (1989), p. 115-117

[<http://www.numdam.org/item?id=PMIHES_1989__69__115_0>](http://www.numdam.org/item?id=PMIHES_1989__69__115_0)

© Publications mathématiques de l'I.H.É.S., 1989, tous droits réservés.

L'accès aux archives de la revue « Publications mathématiques de l'I.H.É.S. » (<http://www.ihes.fr/IHES/Publications/Publications.html>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

APPENDIX

The discriminant quotient formula for global fields

by Moshe JARDEN and Gopal PRASAD

We shall use the notation introduced in § 0, however, in the following ℓ will be an arbitrary finite separable extension of k and if k is a number field, we will now let A denote its ring of integers and B that of ℓ . If k is a global function field, let $\mathfrak{k}$ be its field of constants and I be that of ℓ ; q_k (resp. q_ℓ) is then the cardinality of $\mathfrak{k}$ (resp. I). For a place v of k (resp. w of ℓ), k_v (resp. ℓ_w) will denote the completion of k (resp. ℓ) at v (resp. w). If v is nonarchimedean and k is a number field, then A_v (resp. B_w) will denote the closure of A (resp. B) in k_v (resp. ℓ_w); A_v is the same as the ring denoted by $\mathfrak{o}_v$ earlier.

$|\cdot|_\infty$ will denote the usual absolute value on $\mathbf{Q}$, and for each rational prime p , $|\cdot|_p$ the p -adic absolute value.

For $v \in V_f$, the absolute value $|\cdot|_v$ extends to the fractional ideals of k if k is a number field and to the divisors of k if k is a function field.

A.1. In case k is a number field, let $\mathfrak{d}(A/\mathbf{Z})$, $\mathfrak{d}(B/\mathbf{Z})$ be the discriminants of $A/\mathbf{Z}$, $B/\mathbf{Z}$ respectively ([10: § 4]), and $D_k = |\mathfrak{d}(A/\mathbf{Z})|_\infty$, $D_\ell = |\mathfrak{d}(B/\mathbf{Z})|_\infty$. The *relative discriminant* $\mathfrak{d}(\ell/k)$ of ℓ/k is by definition the discriminant $\mathfrak{d}(B/A)$ of B/A ([10: § 4]), it is an ideal in A .

A.2. The group of divisors of function fields will be written multiplicatively.

Let K be a global function field. If $\alpha = \prod \alpha_v$ is the prime factorization of a divisor α of K , then its degree, to be denoted $\deg_K(\alpha)$, is defined by

$$q_K^{\deg_K(\alpha)} = \prod_v |\alpha_v|_v^{-1}, \quad (1)$$

where q_K is the cardinality of the field of constants of K . The discriminant D_K of K is by definition equal to $q_K^{2g_K-2}$, where g_K is the genus of K .

If L is a finite separable extension of K , then $\mathfrak{D}(L/K)$ will denote the *different* of L/K (see [8: Chapter IV, § 8] for the definition of the different). The *relative discriminant* $\mathfrak{d}(L/K)$ is by definition the divisor $N_{L/K}(\mathfrak{D}(L/K))$ of K .

A.3. For a place w of ℓ lying over a nonarchimedean place v of k , let $\mathfrak{d}(\ell_w/k_v)$ be the relative discriminant of ℓ_w/k_v . Then ℓ_w/k_v is unramified if and only if $\mathfrak{d}(\ell_w/k_v)$ is trivial. The v -component of the discriminant $\mathfrak{d}(\ell/k)$ is $\prod_{w|v} \mathfrak{d}(\ell_w/k_v)$ and $|\mathfrak{d}(\ell/k)|_v = \prod_{w|v} |\mathfrak{d}(\ell_w/k_v)|_v$; see [10: § 4, Proposition 5], [14: p. 463].

Theorem A. — *Let ℓ be a finite separable extension of k . Then*

$$D_\ell/D_k^{[\ell:k]} = \prod_{\mathfrak{v} \in \mathbb{V}_f} \prod_{\mathfrak{w}|\mathfrak{v}} |\mathfrak{d}(\ell_{\mathfrak{w}}/k_{\mathfrak{v}})|_{\mathfrak{v}}^{-1}. \quad (2)$$

Proof. — Number fields and function fields will be treated separately.

(i) *k is a number field.* We use the following relation for the relative discriminants of the ring of integers ([10: § 4, Proposition 7 (ii)])

$$\mathfrak{d}(B/\mathbb{Z})/\mathfrak{d}(A/\mathbb{Z})^{[\ell:k]} = N_{k/\mathbb{Q}}(\mathfrak{d}(B/A)). \quad (3)$$

Taking the absolute value of both sides of the above, we obtain

$$\begin{aligned} D_\ell/D_k^{[\ell:k]} &= |N_{k/\mathbb{Q}}(\mathfrak{d}(\ell/k))|_\infty \\ &= \prod_{\mathfrak{p}} |N_{k/\mathbb{Q}}(\mathfrak{d}(\ell/k))|_{\mathfrak{p}}^{-1} \quad \text{by the product formula (0.1)} \\ &= \prod_{\mathfrak{p}} \prod_{\mathfrak{v}|\mathfrak{p}} |\mathfrak{d}(\ell/k)|_{\mathfrak{v}}^{-1} \quad (\text{by [7: Theorem in § 11]}) \\ &= \prod_{\mathfrak{v} \in \mathbb{V}_f} \prod_{\mathfrak{w}|\mathfrak{v}} |\mathfrak{d}(\ell_{\mathfrak{w}}/k_{\mathfrak{v}})|_{\mathfrak{v}}^{-1} \quad (\text{cf. A.3}). \end{aligned}$$

(ii) *k is a function field*.* Let $k' = \ell k$. Then k' and ℓ have the same field of constants, the genus of k' equals that of k ([9: p. 132, Theorem 2]) and the different $\mathfrak{D}(k'/k)$ is trivial. Theorem 8 of [8: Chapter IV] implies then that $\mathfrak{D}(\ell/k') = \mathfrak{D}(\ell/k)$.

The Riemann-Hurwitz formula for ℓ/k' ([8: p. 106, Corollary 2]) gives

$$2g_\ell - 2 = [\ell:k'] (2g_k - 2) + \deg_\ell(\mathfrak{D}(\ell/k')). \quad (4)$$

By a result on p. 110 of [9], we have

$$[I:\mathfrak{f}] \deg_\ell(\mathfrak{D}(\ell/k)) = \deg_k(\mathfrak{d}(\ell/k)),$$

since $\mathfrak{d}(\ell/k) = N_{\ell/k}(\mathfrak{D}(\ell/k))$. Now multiplying (4) by $[I:\mathfrak{f}]$ we obtain

$$[I:\mathfrak{f}] (2g_\ell - 2) = [\ell:k] (2g_k - 2) + \deg_k(\mathfrak{d}(\ell/k)).$$

As $q_\ell = q_k^{[I:\mathfrak{f}]}$, this leads to

$$q_\ell^{2g_\ell - 2} = q_k^{(2g_k - 2)[\ell:k]} q_k^{\deg_k(\mathfrak{d}(\ell/k))}. \quad (5)$$

By (1) and the last result of A.3, $q_k^{\deg_k(\mathfrak{d}(\ell/k))} = \prod_{\mathfrak{v}} \prod_{\mathfrak{w}|\mathfrak{v}} |\mathfrak{d}(\ell_{\mathfrak{w}}/k_{\mathfrak{v}})|_{\mathfrak{v}}^{-1}$, formula (2) follows therefore from (5).

REFERENCES

- [1] A. BOREL, Some finiteness properties of adèle groups over number fields, *Publ. Math. I.H.E.S.*, **16** (1963), 5-30.
- [2] A. BOREL, *Linear Algebraic groups*, New York, W. A. Benjamin (1969).
- [3] A. BOREL and J. de SIEBENTHAL, Les sous-groupes fermés de rang maximum des groupes de Lie clos, *Comment. Math. Helv.*, **23** (1949), 200-221.

* We are indebted to W.-D. Geyer for a simplification of an earlier version of the proof in this case.

- [4] A. BOREL and G. PRASAD, Finiteness theorems for discrete subgroups of bounded covolume in semi-simple groups, *Publ. Math. I.H.E.S.*, **69** (1989), 119-171.
- [5] N. BOURBAKI, *Groupes et Algèbres de Lie*, chapitres IV, V et VI, Paris, Hermann (1968).
- [6] F. BRUHAT and J. TITS, Groupes réductifs sur un corps local, I, *Publ. Math. I.H.E.S.*, **41** (1972), 5-251; II, *ibid.*, **60** (1984), 5-184.
- [7] J. W. S. CASSELS, Global fields, in *Algebraic number theory* (ed. J. W. S. CASSELS and A. FRÖHLICH), London, Academic Press (1967), 42-84.
- [8] C. CHEVALLEY, Introduction to the theory of algebraic functions of one variable, *A.M.S. Math. Surveys*, Number VI (1951).
- [9] M. DEURING, Lectures on the theory of algebraic functions of one variable, *Springer-Verlag Lecture Notes Math.*, **314** (1973).
- [10] A. FRÖHLICH, Local Fields, in *Algebraic number theory* (ed. J. W. S. CASSELS and A. FRÖHLICH), London, Academic Press (1967), 1-41.
- [11] G. HARDER, Minkowskische Reduktionstheorie über Funktionenkörpern, *Inventiones Math.*, **7** (1969), 33-54.
- [12] G. HARDER, A Gauss-Bonnet formula for discrete arithmetically defined groups, *Ann. Sci. École Norm. Sup.*, Paris, **4** (1971), 409-455.
- [13] G. HARDER, Chevalley groups over function fields and automorphic forms, *Ann. Math.*, **100** (1974), 249-306.
- [14] H. HASSE, *Number theory*, Berlin, Springer-Verlag (1980).
- [15] H. JACQUET and R. P. LANGLANDS, Automorphic forms on $GL(2)$, *Springer-Verlag Lecture Notes Math.*, **114** (1970).
- [16] M. KNESER, Hasse principle for H^1 of simply connected groups, *Proc. A.M.S. Symp. Pure Math.*, **9** (1966), 159-163.
- [17] R. KOTTWITZ, Tamagawa numbers, *Ann. Math.*, **127** (1988), 629-646.
- [18] K. F. LAI, Tamagawa number of reductive algebraic groups, *Compos. Math.*, **41** (1980), 153-188.
- [19] R. P. LANGLANDS, The volume of the fundamental domain for some arithmetical subgroups of Chevalley groups, *Proc. A.M.S. Symp. Pure Math.*, **9** (1966), 143-148.
- [20] I. G. MACDONALD, The volume of a compact Lie group, *Inventiones Math.*, **56** (1980), 93-95.
- [21] G. A. MARGULIS, Cobounded subgroups in algebraic groups over local fields, *Functional Anal. Appl.*, **11** (1977), 45-57.
- [22] J. G. M. MARS, Les nombres de Tamagawa de certains groupes exceptionnels, *Bull. Soc. Math. France*, **94** (1966), 97-140.
- [23] J. G. M. MARS, The Tamagawa number of 2A_n , *Ann. Math.*, **89** (1969), 557-574.
- [24] J. OESTERLÉ, Nombres de Tamagawa, *Inventiones Math.*, **78** (1984), 13-88.
- [25] T. ONO, On algebraic groups and discontinuous groups, *Nagoya Math. J.*, **27** (1966), 279-322.
- [26] T. ONO, On Tamagawa numbers, *Proc. A.M.S. Symp. Pure Math.*, **9** (1966), 122-132.
- [27] G. PRASAD, Strong approximation, *Ann. Math.*, **105** (1977), 553-572.
- [28] G. PRASAD and M. S. RAGHUNATHAN, Topological central extensions of semi-simple groups over local fields, *Ann. Math.*, **119** (1984), 143-268.
- [29] J.-P. SERRE, *Lie algebras and Lie groups*, New York, W. A. Benjamin (1965).
- [30] T. A. SPRINGER, Reductive groups, *Proc. A.M.S. Symp. Pure Math.*, **33** (1979), Part I, 3-27.
- [31] R. STEINBERG, Regular elements of semi-simple algebraic groups, *Publ. Math. I.H.E.S.*, **25** (1965), 49-80.
- [32] J. TITS, Classification of algebraic semi-simple groups, *Proc. A.M.S. Symp. Pure Math.*, **9** (1966), 33-62.
- [33] J. TITS, Reductive groups over local fields, *Proc. A.M.S. Symp. Pure Math.*, **33** (1979), Part I, 29-69.
- [34] A. WEIL, *Adèles and algebraic groups*, Boston, Birkhäuser (1982).

Tata Institute of Fundamental Research
Homi Bhabha Road
Bombay 400 005
India

Manuscrit reçu le 24 juin 1988.