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THE CHAMELEON GROUPS OF RICHARD J. THOMPSON: AUTOMORPHISMS AND DYNAMICS

by MATTHEW G. BRIN

CONTENTS

This paper is dedicated to the memories of Craig C. Squier 1946-1992, and Peter A. Greenberg 1956-1993.

PART I. BACKGROUND

0. Introduction

In the late 1960's Richard Thompson introduced a family of groups in connection with studies in logic. Thompson's groups have since appeared in a variety of mathematical topics: the word problem, infinite simple groups, homotopy and shape theory, group cohomology, dynamical systems and analysis. Universal algebraic properties of these groups have been exploited in several of these connections.

We use the interaction between Thompson's groups, ordered permutation groups and dynamical systems to analyze the automorphisms of some of the groups. Along the way we discover rigidity properties of some local structures on the real line and the circle. To explain these connections, we describe one of our results.

Among other ways, Thompson's groups can be described as all bijections of the real line, the unit interval or the circle that satisfy certain local properties. If we view the circle S^1 as $\mathbf{R}/\mathbf{Z}$ (the reals mod the integers), then one of Thompson's groups T consists of all homeomorphisms f from S^1 to itself that

(1) are piecewise linear (PL),
(2) preserve orientation,
(3) use only slopes that are integral powers of 2,
(4) have their breaks (discontinuities of f') in the set $\mathbf{Z}[\frac{1}{2}]$ (all $a/2^b$ where a and b are integers), and
(5) satisfy $f(\mathbf{Z}[\frac{1}{2}]) \subseteq \mathbf{Z}[\frac{1}{2}]$.

Item (1) means that each x in S^1 has an open neighborhood in which x is (at most) the only break of f. Thus elements of T have finitely many breaks.

The homeomorphism $x \mapsto -x$ on $\mathbf{R}$ induces an involution μ on S^1 with fixed points $\mathbf{Z}$ and $\frac{1}{2} + \mathbf{Z}$ that normalizes T. We prove the outer automorphism group of T has order 2 with generator $[\mu]$. The first step uses the theory of ordered permutation groups to show that any automorphism of T is realized as a conjugation by some self homeomorphism of S^1. The second step uses dynamical systems techniques to show that such a self homeomorphism (up to composition with μ) is in T. We discuss this second step.

Properties (1)-(5) are local properties in the same way that being smooth is a local property. The group T both defines and is defined by a local structure $\mathcal{G}$ which we take to be the set of germs of elements of T. A function f is $\mathcal{G}$-compatible if all the germs of f are in $\mathcal{G}$. The group T is the set of all $\mathcal{G}$-compatible homeomorphisms. We reword our second step above to say that (up to composition with μ) a homeomorphism of S^1 that normalizes T is $\mathcal{G}$-compatible. Since T also defines $\mathcal{G}$, we reword this further to say that (up to composition with μ) a conjugation that preserves $\mathcal{G}$ is by an element that is $\mathcal{G}$-compatible. This last sentence states our rigidity property.

Once rigidity is established, we can lift the local structure to **R** and derive results about automorphisms of groups that are defined on **R**. In particular, we analyze the automorphisms of another of Thompson's groups known as F that acts either on the unit interval or **R**, depending on the representation chosen. (One representation of F is the lift to **R** of all elements of T that fix 0.)

All the groups that we work with are homeomorphism groups that consist of structure-compatible elements for various local structures on **R** or S^1. In order to apply a single rigidity result to all of these groups, we develop machinery that relates the local structures. With this machinery, we can reduce the rigidity result to a much narrower statement as follows. We formulate a notion of " generated by " that lets us regard the elements of T as generators of the structure $\mathcal{G}$. Using this notion we show that $\mathcal{G}$ is generated by the doubling map $x \mapsto 2x$ on S^1. See Section 3. Note that the doubling map is $\mathcal{G}$-compatible. With this single generator for $\mathcal{G}$ in hand, we derive rigidity from the following statement: If h is an orientation preserving homeomorphism from S^1 to itself that preserves the set $\mathbf{Z}[\frac{1}{2}]$, and if both h and h^{-1} conjugate the doubling map to functions that are $\mathcal{G}$-compatible, then h must be $\mathcal{G}$-compatible. Since this statement investigates conjugators of an expanding map, the use of techniques from dynamical systems is natural at this point. We prove that such an h is $\mathcal{G}$-compatible by studying functions that locally preserve certain structures on the union of those orbits of the doubling map that end at its fixed point.

We discuss another interesting aspect of Thompson's group T. The cyclic group of order two is also the outer automorphism group of $\mathrm{Homeo}_+(S^1)$, the full group of orientation preserving homeomorphisms of S^1. Thus T, a countable group, is imitating the behavior of a larger group. This coincidence is not isolated since both T and $\mathrm{Homeo}_+(S^1)$ are simple. (The group T is finitely presented, and was among the first examples of finitely presented, infinite, simple groups. See [7].) Also, on Page 188 of [13] are remarks that the cohomology of T imitates that of larger groups.

Having proofs that two objects share a property makes one curious about the proofs. The more identical the proofs, the more it is likely that one object is revealing secrets about the other. (The proofs that T and $\mathrm{Homeo}_+(S^1)$ are simple can be made identical until the last step where it must be shown the commutator subgroup contains all elements that fix sets with non-empty interior. The known proofs of the last step are quite different.) This idea has affected our choice of technique. The results that we use from ordered group theory are quite general. To apply them to Thompson's groups we only need properties (high levels of transitivity on a dense subset) that the groups share with larger homeomorphism groups. In our use of dynamical systems techniques, we have attempted to suppress purely algebraic properties of these groups and again use only properties that they share with larger homeomorphism groups. For example, the fact that these groups are finitely generated has been replaced in Sections 3 and 4 by the more general fact that the relevant local structures are finitely generated.

There is a rather extensive literature on Thompson's groups and their generalizations. We give some references in Section 1. However, the only properties that we need are the transitivity properties which we derive in Section 2.

We do not analyze some generalizations of Thompson's groups. For each integer $n > 1$, there is a group F_n that acts on $\mathbf{R}$ of which F_2 is F. For integers n and r with $1 \leqslant r < n$, there is a group $T_{n,r}$ that acts on the circle of which $T_{2,1}$ is T. Among the properties that distinguish these groups from one another is the fact that elements of F_n and $T_{n,r}$ all have slopes that are powers of n. We mention in Section 1 that our results do not hold when n is greater than 2.

There is also a family of groups $G_{n,r}$ that act on the Cantor set that generalize another of Thompson's groups (often referred to as $G = G_{2,1}$) that we have not mentioned above. We have no analysis of the automorphisms of any $G_{n,r}$.

The investigations that led to this paper were started while the author was visiting the Institut des Hautes Études Scientifiques. The author wishes to extend the warmest thanks to that institution for its support and hospitality.

1. Statements, history and outline

1.1. *Definitions and statements*

We wish to describe certain homeomorphisms of the real line $\mathbf{R}$ and the circle $S^1 = \mathbf{R}/\mathbf{Z}$. With this parametrization in place, we can discuss linearity, piecewise linearity and slope of functions on S^1 and the rationality of points in S^1 exactly as we would on $\mathbf{R}$.

Let $\mathbf{Z}[\frac{1}{2}]$ be the set of $a2^b$ in $\mathbf{R}$ or S^1 where a and b are in $\mathbf{Z}$. Let $PL_2(\mathbf{R})$ denote the set of homeomorphisms f from $\mathbf{R}$ to itself satisfying:

(1) f is piecewise linear;
(2) f is orientation preserving;
(3) all slopes of f are integral powers of 2;
(4) the " breaks " of f (discontinuities of f') are in a discrete subset of $\mathbf{Z}[\frac{1}{2}]$; and
(5) $f(\mathbf{Z}[\frac{1}{2}]) \subseteq \mathbf{Z}[\frac{1}{2}]$.

It is easy to verify that every $f \in PL_2(\mathbf{R})$ satisfies $f(\mathbf{Z}[\frac{1}{2}]) = \mathbf{Z}[\frac{1}{2}]$ and that $PL_2(\mathbf{R})$ is closed under composition and inversion and is thus a group. The term " break " appears in [8] where certain PL homeomorphisms groups of $\mathbf{R}$ are proven simple.

We are interested in various subgroups of $PL_2(\mathbf{R})$ and groups on the circle S^1 that are related to $PL_2(\mathbf{R})$. Given a homeomorphism h from a topological space X to itself, we define the support of f to be the set of points $x \in X$ for which $f(x) \neq x$. We define $BPL_2(\mathbf{R})$ to be the elements of $PL_2(\mathbf{R})$ whose support is a bounded subset of $\mathbf{R}$, and we refer to the elements of $BPL_2(\mathbf{R})$ as the elements of $PL_2(\mathbf{R})$ with bounded support.

We let F be those elements f of $PL_2(\mathbf{R})$ that are translations by integers near $\pm\infty$ in that there are integers i and j and a real M so that $f(x) = x + i$ for all $x > |M|$ and $f(x) = x + j$ for all $x < -|M|$. We have $BPL_2(\mathbf{R}) \subseteq F$. (A fact not needed in this

paper is that F is isomorphic to the elements of $PL_2(\mathbf{R})$ with support in the unit interval [0, 1]. The isomorphism can be achieved as a conjugation by a suitably chosen PL homeomorphism (with infinitely many breaks) from the open unit interval to the real line.)

We let T be those homeomorphisms from S^1 to itself that satisfy (1)-(5) above. We also consider elementary extensions of these groups. If we replace (2) with

(2′) f is orientation preserving or orientation reversing,

then we get groups $\widetilde{PL}_2(\mathbf{R})$ and $\widetilde{T}$ that respectively contain $PL_2(\mathbf{R})$ and T as subgroups of index 2.

At this point we have defined the largest and smallest objects that we will look at. Our results will be about all groups G for which $BPL_2(\mathbf{R}) \subseteq G \subseteq \widetilde{PL}_2(\mathbf{R})$ or $T \subseteq G \subseteq \widetilde{T}$. This includes two groups on S^1 and infinitely many on $\mathbf{R}$. Our " smallest " groups are $BPL_2(\mathbf{R})$ and T since, as we will see in Section 2, they have all the transitivity properties that we will need.

We are interested in automorphisms of the objects that we have defined. We get certain automorphisms by conjugation. Let G be a group for which $BPL_2(\mathbf{R}) \subseteq G \subseteq \widetilde{PL}_2(\mathbf{R})$. We let Homeo($\mathbf{R}$) to be the set of all self homeomorphisms of $\mathbf{R}$ and define N(G), the normalizer of G in Homeo($\mathbf{R}$), to be the set of all $h \in$ Homeo($\mathbf{R}$) for which $hGh^{-1} = G$. For G with $T \subseteq G \subseteq \widetilde{T}$, we define N(G) similarly with respect to Homeo(S^1). We have a homomorphism $\Phi : N(G) \to \mathrm{Aut}(G)$ where for an $h \in N(G)$, we have $(\Phi(h))f = hfh^{-1}$. Our main result follows.

Theorem 1. — *Let* G *be a group for which* $BPL_2(\mathbf{R}) \subseteq G \subseteq \widetilde{PL}_2(\mathbf{R})$ *or* $T \subseteq G \subseteq \widetilde{T}$. *Then*

(i) $\Phi : N(G) \to \mathrm{Aut}(G)$ *is an isomorphism,*

(ii) $N(G) \subseteq \widetilde{PL}_2(\mathbf{R})$ *or* $N(G) \subseteq \widetilde{T}$ *whichever applies,*

(iii) *the containment in* (ii) *is equality whenever* G *is one of* $BPL_2(\mathbf{R})$, $PL_2(\mathbf{R})$, $\widetilde{PL}_2(\mathbf{R})$, T, $\widetilde{T}$, *and*

(iv) *if* $N_+(F)$ *represents the index* 2 *subgroup of* N(F) *of orientation preserving elements, then there is a short exact sequence*

$$1 \to F \to N_+(F) \to T \times T \to 1.$$

Remark. — The appearance of $T \times T$ in (iv) above is easy to explain. Elements of F are translations by integers near $\pm\infty$. Thus near each of $-\infty$ and $+\infty$, elements of F commute with functions that are lifts of homeomorphism of S^1. Those homeomorphisms of S^1 that lift to $PL_2(\mathbf{R})$ are elements of T.

We give the main intermediate result. Any continuous function from S^1 to itself satisfying (1)-(5) in the definitions of $PL_2(\mathbf{R})$ and T will be called $\mathscr{E}$-compatible. Let $\nu_2(x) = 2x$. If we regard this as a function on S^1 then it is $\mathscr{E}$-compatible.

Theorem 2. — *Let* $h : S^1 \to S^1$ *be an orientation preserving homeomorphism for which* $h(\mathbf{Z}[\frac{1}{2}]) = \mathbf{Z}[\frac{1}{2}]$. *Assume that* $h\nu_2 h^{-1}$ *and* $h^{-1}\nu_2 h$ *are* $\mathscr{E}$*-compatible. Then* h *is PL.*

It turns out that h is $\mathscr{C}$-compatible, but this is not known until Theorem 1 is proven. The proof of Theorem 1 from Theorem 2 will be given in Section 5. The proof of Theorem 2 occupies Part III. A separate statement of a rigidity property is deferred to Section 5. See Theorem 5.2.

There are natural parallels of the definitions in this section to groups whose elements have slopes that are integral powers of a fixed integer n and breaks at elements of $\mathbf{Z}[\frac{1}{n}]$. It turns out that Theorem 2 fails spectacularly when n is greater than 2. This results in the failure of items (ii), (iii) and (iv) of Theorem 1 for the parallels of F, $\mathrm{BPL}_2(\mathbf{R})$, $\mathrm{PL}_2(\mathbf{R})$ and T for $n > 2$. Details will appear elsewhere.

1.2. *Outline of the paper*

There have been many facts discovered about the groups that we consider. However the only properties that we need are the transitivity properties. These are presented in Section 2.

Conclusion (i) of Theorem 1 follows from the transitivity properties and a result in the theory of ordered permutation groups due to McCleary and Rubin [19]. This also appears in Section 2. Once (i) of Theorem 1 is proven, we are reduced to the study of normalizers rather than automorphisms.

We consider groups of homeomorphisms defined on various domains. One of our tasks is to reduce the scope of the investigation by showing that all the normalizers that we wish to study are locally like certain conjugators of a very small set of maps. We build some general machinery for this in Section 3 that applies to groups of homeomorphisms that are defined by local properties. Section 4 applies this to the Thompson groups where we identify the small set of maps as the map ν_2 on the circle S^1.

In Section 5, we show that all PL normalizers can be analyzed. From the results of Section 4, this gives the implication that all normalizers (and thus automorphisms) are analyzed if a certain set of conjugators of the map ν_2 contains only PL homeomorphisms. This is done in the argument that Theorem 2 implies Theorem 1.

Part III contains the proof of Theorem 2. Here it must be shown that any homeomorphism h from S^1 to itself for which $h\nu_2 h^{-1}$ and $h^{-1}\nu_2 h$ are $\mathscr{C}$-compatible must be PL. Since ν_2 is an expanding map, we extract information about h from the fact that $h\nu_2 h^{-1}$ is $\mathscr{C}$-compatible and derive a criterion that determines when h is PL. We then do a fairly direct calculation of $h^{-1}\nu_2 h$ and derive a criterion that determines when $h^{-1}\nu_2 h$ is PL. We then show that h must be PL whenever $h^{-1}\nu_2 h$ is PL.

Part IV contains examples that justify hypotheses and phases of the arguments.

1.3. *Prior and related work*

Some unpublished work by others on the automorphisms of Thompson's groups has been available to the author. All have made the observation in Section 5 that the main problem is to prove that normalizers are PL. Also, all have made the observation not needed or mentioned in this paper that a normalizer that is PL on some open interval

must be PL. A preprint version [12] of [13] that considered the question of automorphisms gave us the idea used in Part III of studying Markov partitions for the conjugates of the maps v_2. The author is also indebted to Dennis Sullivan for suggesting that it might be useful to look at the non-linearity of the conjugates of the maps v_2. Conversations with Fernando Guzman were very helpful in simplifying the presentation in Part III.

The most extensive work on the material considered in this paper is the unpublished notes [1] of Bieri and Strebel. (A very small sampling of material from [1] is given in an appendix to [23].) The overlap between [1] and the current paper outside of Part III is large. Section 2 of the current paper gives a special case of the analysis in [1], and Sections 3, 4 and 5 generalize and repackage ideas in [1]. The theorem of [19] that we quote in Section 2 is a generalization of a theorem in [1] which applies to $\mathbf{R}$ which in turn is a generalization of a theorem in [18] which applies to more transitive actions. Most of the examples in Part IV have their equivalents in [1] but the machinery we use to build the examples is different and the properties are therefore easier to verify. Items from Part III found in [1] include a formula that gives h from hv_2h^{-1} and an invariant based on summing break values (which are defined below in Section 7). Missing from [1] is most of the analysis of Part III.

We state two results from [1] that complement the results of the current paper. Considered in [1] are groups $G(I; A, P)$ of all orientation preserving, PL self homeomorphisms of an interval I in $\mathbf{R}$ with slopes in a multiplicative subgroup P of the positive reals and breaks in a *finite* subset of an additive $\mathbf{Z}P$ module A in $\mathbf{R}$ (with action coming from multiplication in $\mathbf{R}$). Also considered is the subgroup $B(I; A, P)$ of those elements of $G(I; A, P)$ with support in some compact subset of the interior of I. It is proven in [1] that if G is a group with $B(I; A, P) \subseteq G \subseteq G(I; A, P)$ where P is not cyclic, then all automorphisms of G are realized as conjugations by PL self homeomorphisms of I. This result uses the density of such a P in the positive reals. It is also proven in [1] that automorphisms of $G(I; A, P)$ are realized by PL self homeomorphisms of I if I is unbounded and by PL self homeomorphisms of I with finitely many breaks if $I = \mathbf{R}$. This result uses the fact that restriction to an unbounded end gives a surjection onto a group of affine functions. The structures of the automorphism groups are also examined in [1].

The result, Theorem 2 3, that we use from [19] realizes automorphisms as conjugations. An earlier version of this result is found in [18], and is used there to analyze the automorphism groups of other homeomorphism groups. It is shown that the full group of PL homeomorphisms on $\mathbf{R}$ has trivial outer automorphism groups as does the group of PL homeomorphisms on $\mathbf{R}$ with finitely many breaks and also the largest group of homeomorphisms of $\mathbf{R}$ all of whose elements are (not necessarily continuously) differentiable. See [18] for more details and older references. A special case of Theorem 2.3 for Thompson's group T appears in [13]. The proof in [13] makes use of algebraic properties of T including its simplicity.

1.4. *Other literature on Thompson's groups*

The literature on Thompson's groups is very scattered. Because of this, many letters from different alphabets have been used to refer to Thompson's groups. Thompson's groups also admit a wide variety of representations, so the difficulty of recognition extends beyond the multiplicity of notation. We will not give a complete history, we will not sort out the notation, and we will not list all the representations. We will try to give enough references to enable the reader to get to the literature on the various topics in which these groups have appeared.

The papers [3] and [7] relate some of the representations, give keys to some of the notation used, and give brief histories. For relations with the word problem and for the early history of Thompson's groups see [20]. The connection to infinite simple groups is discussed in [3] and more recently in [22]. See also [23].

Thompson's group F was rediscovered (by Dydak and Minc and also independently by Freyd and Heller) in connection with questions in homotopy/shape theory. This is discussed in [10], [6] and [11]. It is in this topic that the universal algebraic properties of the groups are derived and exploited. The final paper [17] on the homotopy question is the first to investigate the homological properties of one of Thompson's groups. The paper [6] is the start of a systematic investigation of the (co)homology properties of Thompson's groups and their generalizations. More recent papers are [3], [4], [5], [13], [23] and [9]. The (co)homology of structures resembling those discussed in Section 3 are investigated in [14]. An acyclic extension of the infinite braid group by $\mathrm{BPL}_2(\mathbf{R})$ is constructed in [16].

The paper [13] considers the group T from a dynamic point of view.

In analysis, the existence of Thompson's group F demonstrates that either there is a non-amenable finitely presented group with no free subgroup on two generators or there is a finitely presented amenable group that is not elementary amenable. This is discussed in [7] where it is shown that F is not elementary amenable and in [2] and [7] where it is shown that F contains no free subgroup on two generators. The paper [7] also contains a version of Thompson's original proof that T is simple and a proof of Thurston's observations that T and F are isomorphic to groups of piecewise projective homeomorphisms on the circle and the interval respectively. The projective aspects of T are also discussed in [15].

2. Transitivity properties of Thompson's groups

Lemma **2.1.** — *Let* $x_1 < x_2 < \ldots < x_k$ *and* $y_1 < y_2 < \ldots < y_k$ *be elements of* $\mathbf{Z}[\frac{1}{2}]$. *Then there is an* $f \in \mathrm{BPL}_2(\mathbf{R})$ *so that* $f(x_i) = y_i$ *for all* i *with* $1 \leqslant i \leqslant k$.

Remark. — We refer to the property in the lemma by saying that $\mathrm{BPL}_2(\mathrm{R})$ acts order k-transitively or o-k-transitively on $\mathbf{Z}[\frac{1}{2}]$ for all k. Any G with $\mathrm{BPL}_2(\mathbf{R}) \subseteq \mathrm{G} \subseteq \widetilde{\mathrm{PL}}_2(\mathbf{R})$, is also o-$k$-transitive on $\mathbf{Z}[\frac{1}{2}]$ for all k.

Proof. — By adding an extra point from $\mathbf{Z}[\frac{1}{2}]$ to each sequence sufficiently far below x_1 and y_1 and another from $\mathbf{Z}[\frac{1}{2}]$ sufficiently far above x_k and y_k, we can assume that $x_1 = y_1$ and that $x_k = y_k$. If for each i with $1 \leqslant i < k$, we find a homeomorphism from $[x_i, x_{i+1}]$ to $[y_i, y_{i+1}]$ that satisfies the local properties required of functions in $\mathrm{PL}_2(\mathbf{R})$, then we can piece these functions together with the identity on $(-\infty, x_1] \cup [x_k, \infty)$ to get the required function. Thus it suffices to consider four points $a < b$, $c < d$ in $\mathbf{Z}[\frac{1}{2}]$ and build a homeomorphism from $[a, b]$ to $[c, d]$ that satisfies the properties of functions in $\mathrm{PL}_2(\mathbf{R})$.

Since all four points are in $\mathbf{Z}[\frac{1}{2}]$, there is an integral power of 2 that evenly divides $b - a$ and $d - c$. Thus we can divide the intervals $[a, b]$ and $[c, d]$ into subintervals of length an integral power of 2. If the number of intervals is not the same for $[a, b]$ and $[c, d]$, then we can increase the number of intervals for one of them as much as we want by repeatedly subdividing random subintervals into 2 equally spaced smaller intervals. This preserves the property that the lengths of all the intervals are a (perhaps varying) integral power of 2. Once the number of subintervals in $[a, b]$ and $[c, d]$ are the same, then a function can be built to map each interval in the subdivision of $[a, b]$ affinely to the corresponding interval in the subdivision of $[c, d]$.

The action is not k-transitive on all of $\mathbf{R}$. However, the set $\mathbf{Z}[\frac{1}{2}]$ is dense in $\mathbf{R}$, so we get close. If I and J are two closed intervals in $\mathbf{R}$, then we write $\mathrm{I} < \mathrm{J}$ if every point in I is less than every point in J. We say that a group acting on $\mathbf{R}$ is acting approximately o-k-transitively on $\mathbf{R}$ if whenever points $x_1 < x_2 < \ldots < x_k$ and closed intervals $\mathrm{I}_1 < \mathrm{I}_2 < \ldots < \mathrm{I}_k$ with non-empty interiors are given in $\mathbf{R}$ then there is an element of the group that takes each x_i into I_i.

Corollary **2.1.1.** — *Let* G *be a group with* $\mathrm{BPL}_2(\mathbf{R}) \subseteq \mathrm{G} \subseteq \widetilde{\mathrm{PL}}_2(\mathbf{R})$. *Then* G *acts approximately o-k-transitively on* $\mathbf{R}$ *for all k.*

On the circle there is the problem of not having a linear order. This is solved by referring to the counterclockwise orientation on S^1. We use the same terminology as on $\mathbf{R}$ and let context determine which is in use. We say that a group acting on S^1 is acting approximately o-k-transitively on S^1 if whenever different points $x_1, x_2, \ldots, x_k$ in S^1 arranged in counterclockwise order on S^1 in the order listed, and pairwise disjoint closed intervals with non-empty interiors $\mathrm{I}_1, \mathrm{I}_2, \ldots, \mathrm{I}_k$ in S^1 arranged in counterclockwise order on S^1 in the order listed are given, then there is an element of the group that takes each x_i into I_i. Since functions in T lift to functions in $\mathrm{PL}_2(\mathbf{R})$ and since functions in $\mathrm{PL}_2(\mathbf{R})$ can be built in pieces, we leave it as an easy exercise for the reader to verify the following.

Lemma **2.2.** *The group* T *acts approximately o-k-transitively on* S^1 *for all k.*

We can now show that the automorphisms of the groups that we consider are all realized as conjugations by homeomorphisms. We say that a homeomorphism of S^1

has bounded support if it fixes some open interval. All of the work is in the following theorem from [19].

Theorem **2.3** *(McCleary-Rubin).* — *Let* G *act on* $\mathbf{R}$ *or* S^1 *by homeomorphisms. Assume that* G *contains a non-identity element of bounded support. If* G *acts on* $\mathbf{R}$, *then assume that the action is approximately* 0*-2-transitive, and if* G *acts on* S^1, *then assume that the action is approximately* 0*-3-transitive. Then for each automorphism* α *of* G, *there is a unique self homeomorphism* h *of* $\mathbf{R}$ *or* S^1 *(whichever is appropriate) so that* $\alpha(f) = hfh^{-1}$ *for every* $f \in G$.

Proof of (i) *of Theorem* 1. — The fact that G satisfies the hypotheses of Theorem 2.3 follows from the other results in this section. The existence and uniqueness statements in Theorem 2.3 show that Φ is one to one and onto.

PART II. A REDUCTION

3. Germs, half germs, germ functions and germ generators

We give the facts that motivate this section. There are local criteria on $\mathbf{R}$ and S^1 that elements of $PL_2(\mathbf{R})$, T and related groups must satisfy. The groups $PL_2(\mathbf{R})$ and T consist of all elements that satisfy the criteria. The local criteria on $\mathbf{R}$ and S^1 are highly related. The local criteria can be stated in terms of matching a simple set of functions (affine) with sudden changes allowed on a given subset. The simple set of functions has a very small number of generators.

This section exploits the above observations and sets up machinery that will lead to two simplifications in the next section. There we show first that an analysis of normalizers on $\mathbf{R}$ will follow from an analysis of normalizers on S^1, and second that this analysis will follow from a study of conjugators of one function on S^1.

Let X be a topological space. Let $\mathscr{H}(X)$ be the set of homeomorphisms from open sets in X to open sets in X. For each x and y in X, let $\mathscr{H}(X)_{x,y}$ be the set of elements of $\mathscr{H}(X)$ that take x to y. Let $\mathscr{G}(X)_{x,y}$ be the set of germs at x of the elements of $\mathscr{H}(X)_{x,y}$. That is $\mathscr{G}(X)_{x,y}$ is the set of equivalence classes in $\mathscr{H}(X)_{x,y}$ in which two elements are related if they agree on a neighborhood of x.

We can compose germs in that elements of $\mathscr{G}(X)_{x,y}$ compose with those of $\mathscr{G}(X)_{y,z}$ to give elements of $\mathscr{G}(X)_{x,z}$. Also, each $\mathscr{G}(X)_{x,x}$ contains the germ of the identity. If we regard $\mathscr{G}(X)$ as a function defined on $X \times X$, then we have that $\mathscr{G}(X)$ is a category whose objects are the elements of X and whose morphism sets are the sets $\mathscr{G}(X)_{x,y}$. Each element in $\mathscr{G}(X)_{x,y}$ has an inverse in $\mathscr{G}(X)_{y,x}$ so that the two possible compositions give the identities of $\mathscr{G}(X)_{x,x}$ and $\mathscr{G}(X)_{y,y}$. Thus all morphisms are isomorphisms and $\mathscr{G}(X)$ is a groupoid. We will deal with substructures of this groupoid. We will have no need to consider topological groupoids.

Let $\mathscr{G}$ be a function on $X \times X$ written so that $\mathscr{G}_{x,y}$ represents the value of $\mathscr{G}$ at (x,y) and assume that for each $(x,y) \in X \times X$ we have $\mathscr{G}_{x,y} \subseteq \mathscr{G}(X)_{x,y}$. If we further

assume that $\mathcal{G}$ is a groupoid, then we call $\mathcal{G}$ a groupoid of invertible germs on X. We will usually omit the word invertible. (It is standard to arrive at a groupoid of germs from a pseudogroup — here, the set of $\mathcal{G}$-compatible maps defined below. It is customary to put extra hypotheses on the pseudogroup so that it is in turn derivable from a groupoid of germs. See for example [21].)

We say that $f: U \to X$, where U is an open subset of X, is $\mathcal{G}$-compatible at $x \in U$ if there is a g representing a class in $\mathcal{G}_{x, f(x)}$ that agrees with f on some neighborhood of x. We say that f is $\mathcal{G}$-compatible if it is $\mathcal{G}$-compatible at x for each $x \in U$. We let $F(\mathcal{G})$ be the set of all $\mathcal{G}$-compatible functions, we let $H(\mathcal{G})$ be the set of all $\mathcal{G}$-compatible homeomorphisms from X to X, and we let $O(\mathcal{G})$ be the orbits of $F(\mathcal{G})$. We have that $O(\mathcal{G})$ is just the set of equivalence classes in which two points x and y are related when $\mathcal{G}_{x,y}$ is not empty. Note that every element of $F(\mathcal{G})$ is a local homeomorphism in that every point in its domain has an open neighborhood that is carried homeomorphically onto an open set in X.

Note that we have not made enough assumptions to guarantee that $H(\mathcal{G})$ is not empty. This can be fixed by requiring that each $\mathcal{G}_{x,x}$ be non-empty. This will mean that $\mathcal{G}_{x,x}$ contains at least the germ of the identity function and this will make the identity function on X an element of $F(\mathcal{G})$ and thus $H(\mathcal{G})$. Equivalently we could require that the identity function on X be in $F(\mathcal{G})$. When $H(\mathcal{G})$ contains the identity function on X, we will say that the groupoid $\mathcal{G}$ contains the identity.

To imitate the structure of elements of $PL_2(\mathbf{R})$, we split germs in half. Let $\mathcal{G}$ be a groupoid of germs on $\mathbf{R}$. A function f taking x to y in $\mathbf{R}$, is $\mathcal{G}^{\pm}$-compatible at x if there is an $\varepsilon > 0$ so that f and some representative in $\mathcal{G}_{x,y}$ agree on $[x, x + \varepsilon)$ and f and some representative in $\mathcal{G}_{x,y}$ agree on $(x - \varepsilon, x]$. If A is a union of orbits in $O(\mathcal{G})$, then we say that a local homeomorphism $f: U \to \mathbf{R}$ is $(\mathcal{G}; A)$-compatible if f is $\mathcal{G}$-compatible for all $x \in U - A$ and if f is $\mathcal{G}^{\pm}$-compatible for each $x \in A$. The assumption that f is a local homeomorphism prevents combining an increasing germ and a decreasing germ from $\mathcal{G}_{x, f(x)}$. We denote the groupoid of germs of all $(\mathcal{G}; A)$-compatible functions by $(\mathcal{G}; A)$. The constructions in this paragraph can be compared to Paragraph 1.6 in [14].

We now consider the circle S^1 and its covering map, the projection $\mathbf{R} \to \mathbf{R}/\mathbf{Z}$. If $\mathcal{G}$ is a groupoid of germs on S^1, then we can lift $\mathcal{G}$ to a groupoid of germs on $\mathbf{R}$. One way to define this is to let $\mathcal{P}$ be the set of germs of the covering projection $\mathbf{R} \to \mathbf{R}/\mathbf{Z}$ and let $\mathcal{P}^{-1}$ be the inverses of the elements of $\mathcal{P}$. Then the lift of $\mathcal{G}$ to $\mathbf{R}$ is the set of all allowable compositions in $\mathcal{P}^{-1}\mathcal{G}\mathcal{P}$.

We can also go the other way. Let $\mathcal{G}$ be a groupoid of germs on $\mathbf{R}$, and let $s: \mathbf{R} \to \mathbf{R}$ be defined by $s(x) = x + 1$. We say that $\mathcal{G}$ is s-invariant if the germ of $s^q \alpha s^{-p}$ at $x + p$ is in $\mathcal{G}_{x+p, y+q}$ for every germ α in $\mathcal{G}_{x,y}$, every x and y in $\mathbf{R}$, and every p and q in $\mathbf{Z}$. If $\mathcal{G}$ contains the identity, then the s-invariance of $\mathcal{G}$ is equivalent to the requirement that s be in $H(\mathcal{G})$. This is seen by noting that the germ of s at x is the germ of $s^1 \alpha s^0$ at x where α is the germ of the identity at x. Assuming that $\mathcal{G}$ is s-invariant, we can now let the projection of $\mathcal{G}$ onto S^1 be the set of all allowable compositions in $\mathcal{P}\mathcal{G}\mathcal{P}^{-1}$.

We see how this relates to $PL_2(\mathbf{R})$ and T. We let

$$\mathrm{Aff}_2(\mathbf{R}) = \{ x \mapsto 2^k x + b \mid k \in \mathbf{Z}, b \in \mathbf{Z}[\tfrac{1}{2}] \}.$$

We let $\mathscr{A}_2(\mathbf{R})$ denote the groupoid of germs of $\mathrm{Aff}_2(\mathbf{R})$. Since $s(x) = x + 1$ is in $\mathrm{Aff}_2(\mathbf{R})$, we let $\mathscr{A}_2(S^1)$ be the projection of $\mathscr{A}_2(\mathbf{R})$ onto S^1. Now

(3.1) $$H(\mathscr{A}_2(\mathbf{R}); \mathbf{Z}[\tfrac{1}{2}]) = PL_2(\mathbf{R}) \quad \text{and} \quad H(\mathscr{A}_2(S^1); \mathbf{Z}[\tfrac{1}{2}]) = T.$$

We now consider how little information is needed to specify a set of germs. Let X be a topological space. Let $\mathscr{F}$ be a collection of local homeomorphisms defined on open subsets of X. Note that the set of germs of $\mathscr{F}$ is a subset of the collection of germs in $\mathscr{G}(X)$. Thus compositions of germs of $\mathscr{F}$ and inverses of germs of $\mathscr{F}$ exist in $\mathscr{G}(X)$. The collection of finite (allowable) compositions of germs from $\mathscr{F}$ and their inverses is a groupoid and is thus the smallest groupoid that contains the germs of $\mathscr{F}$. We refer to this groupoid as the groupoid of germs generated by $\mathscr{F}$. The following is clear.

Lemma **3.1.** — *Let* X *be a topological space and let* $\mathscr{F}$ *and* $\mathscr{F}_1$ *be collections of local homeomorphisms defined on open subsets of* X.

(i) *If* $\mathscr{F} \subseteq \mathscr{F}_1$, *then the groupoid of germs generated by* $\mathscr{F}$ *is contained in the groupoid of germs generated by* $\mathscr{F}_1$.

(ii) *If* $\mathscr{F}_c$ *is the set of finite compositions of elements of* $\mathscr{F}$, *or if* $\mathscr{F}_c$ *is a group and* $\mathscr{F}$ *is a set of generators of* $\mathscr{F}_c$, *then the groupoid of germs generated by* $\mathscr{F}_c$ *equals the groupoid of germs generated by* $\mathscr{F}$.

We give generators for the structures that we use.

Lemma **3.2.** — *The functions* $d(x) = 2x$ *and* $s(x) = x + 1$ *on* $\mathbf{R}$ *generate* $\mathscr{A}_2(\mathbf{R})$ *and the function* $\nu_2(x) = 2x$ *on* S^1 *generates* $\mathscr{A}_2(S^1)$.

Proof. — Note that $\mathrm{Aff}_2(\mathbf{R})$ is generated by $d(x) = 2x$ and $s(x) = x + 1$. This is seen by noting that s conjugated by powers of d gives all translations by integral powers of 2. Thus d and s generate translations by all elements of $\mathbf{Z}[\frac{1}{2}]$. Powers of d can carry 0 to 0 with slope any integral power of 2. Now the action of any element of $\mathrm{Aff}_2(\mathbf{R})$ on a neighborhood of 0 can be imitated by composing a power of d with a translation. This gives the first claim.

All germs in $\mathscr{A}_2(S^1)$ are projections of germs from $\mathscr{A}_2(\mathbf{R})$. These in turn are compositions of germs of d and s and their inverses. The projection of germs of d are germs of ν_2 and the projection of germs of s are trivial on S^1. Thus every germ in $\mathscr{A}_2(S^1)$ is a composition of germs of ν_2 and their inverses.

The next lemma needs a definition. If $\mathscr{G}$ is a groupoid of germs on a topological space X, then we say that a homeomorphism $h : X \to X$ preserves $\mathscr{G}$ under conjugation if hfh^{-1} is in $F(\mathscr{G})$ for all $f \in F(\mathscr{G})$. This implies $hH(\mathscr{G})\,h^{-1} \subseteq H(\mathscr{G})$.

Lemma **3.3** *(Consistency).* — *Let $\mathcal{F}$ be a family of local homeomorphisms on a topological space* X *and let $\mathcal{G}$ be the groupoid of germs generated by $\mathcal{F}$. Let $h : X \to X$ be a homeomorphism and assume that h conjugates each element of $\mathcal{F}$ into $F(\mathcal{G})$. Then h preserves $\mathcal{G}$ under conjugation. If in addition,* X *is one of* S^1 *or* $\mathbf{R}$, A *is a union of orbits in* $O(\mathcal{G})$, $h(A) \subseteq A$, *and h conjugates each element of $\mathcal{F}$ into* $F(\mathcal{G}; A)$, *then h preserves* $(\mathcal{G}; A)$ *under conjugation.*

Proof. — We give the argument that applies when X is $\mathbf{R}$ or S^1, and let the reader omit the irrelevant parts for the simpler case. Let f be in $F(\mathcal{G}; A)$. If $x \notin A$, then $h^{-1}(x) \notin A$ and the germ of f at $h^{-1}(x)$ is a composition of a finite number of germs or their inverses from elements of $\mathcal{F}$. If $x \in A$, then $h^{-1}(x)$ might be in A or not. If not, then the previous statement about f applies. If $h^{-1}(x) \in A$, then the two half germs of f at $h^{-1}(x)$ are compositions of corresponding half germs from $\mathcal{F}$. (Which half germ of each factor goes into each half germ of f depends on the order in which the factors are orientation reversing or preserving.) The conjugate of the composition is the composition of the conjugates. Since the conjugate of each factor is known to have its germ or two half germs in $(\mathcal{G}; A)$, we have that hfh^{-1} is in $F(\mathcal{G}; A)$.

The notation $(\mathcal{G}; A)$ refers not only to a groupoid, but also its construction. For the remainder of the section, we will not be concerned with the construction of groupoids and the notation $(\mathcal{G}; A)$ will not be needed.

The hypothesis of the next corollary is implied by a stronger condition that we will introduce later.

Corollary **3.3.1.** — *Let $\mathcal{G}$ be a groupoid of germs on a topological space* X *and assume that* $H(\mathcal{G})$ *generates $\mathcal{G}$. If $h : X \to X$ is a homeomorphism for which $hH(\mathcal{G})h^{-1} \subseteq H(\mathcal{G})$, then h preserves $\mathcal{G}$ under conjugation.*

If $\mathcal{G}$ is a groupoid of germs, then $\mathcal{G}_{x,x}$ is the set of germs at x of those elements of $F(\mathcal{G})$ that have x as a fixed point. The following standard observation will be needed in several places.

Lemma **3.4.** — *Let* X *be a topological space and let $\mathcal{G}$ be a groupoid of germs on* X *that contains the identity. Then each $\mathcal{G}_{x,x}$ is a group. If $h : X \to X$ is a homeomorphism and h and h^{-1} preserve $\mathcal{G}$ under conjugation, then the function from $\mathcal{G}_{x,x}$ to $\mathcal{G}_{h(x),h(x)}$ taking each $\alpha \in \mathcal{G}_{x,x}$ to the germ of $h\alpha h^{-1}$ at $h(x)$ is an isomorphism.*

For example, with $\mathcal{A}_2(\mathbf{R})$ as defined above, then $(\mathcal{A}_2(\mathbf{R}); \mathbf{Z}[\frac{1}{2}])_{x,x}$ is isomorphic to $\mathbf{Z} \times \mathbf{Z}$ if $x \in \mathbf{Z}[\frac{1}{2}]$, to $\mathbf{Z}$ if $x \in \mathbf{Q} - \mathbf{Z}[\frac{1}{2}]$, and to $\{1\}$ if $x \notin \mathbf{Q}$. Thus any homeomorphism $h : \mathbf{R} \to \mathbf{R}$ that normalizes $F(\mathcal{A}_2(\mathbf{R}); \mathbf{Z}[\frac{1}{2}])$ must preserve the three sets, $\mathbf{Z}[\frac{1}{2}]$, $\mathbf{Q} - \mathbf{Z}[\frac{1}{2}]$ and $\mathbf{R} - \mathbf{Q}$.

Let X be a topological space, let $\mathcal{G}$ be a groupoid of germs on X, and let H(X) be the set of self homeomorphisms of X. We say that $g \in H(X)$ is obtained from $f \in H(X)$ by $\mathcal{G}$-rearrangement if for every $x \in X$, the germ of g at x is that of $\alpha f \beta$ where α and β

are germs in $\mathcal{G}$ that vary with x. Note that this relation does not seem to be symmetric. This relation becomes more symmetric when we consider normalizers. We let $N(\mathcal{G})$ be the set of $h \in H(X)$ so that both h and h^{-1} preserve $\mathcal{G}$ under conjugation. Note that $H(\mathcal{G})$ is a normal subgroup of $N(\mathcal{G})$.

Lemma **3.5.** *(Rearrangement).* — *Let* X *be a topological space and let* $\mathcal{G}$ *be a groupoid of germs on* X *that contains the identity. If* f *and* g *in* $H(X)$ *are in the same left coset of* $H(\mathcal{G})$ *in* $H(X)$ *or the same right coset of* $H(\mathcal{G})$ *in* $H(X)$, *then each is a* $\mathcal{G}$*-rearrangement of the other. If* h *is in* $N(\mathcal{G})$, *then any* $\mathcal{G}$*-rearrangement of* h *is in* $N(\mathcal{G})$ *and elements of* $N(\mathcal{G})$ *are in the same coset of* $H(\mathcal{G})$ *in* $N(\mathcal{G})$ *if and only if they are* $\mathcal{G}$*-rearrangements of each other.*

Proof. — If $g \in fH(\mathcal{G})$, then $g = fh$ with $h \in H(\mathcal{G})$. This and a similar observation for right cosets finishes the first claim. If h is in $N(\mathcal{G})$, g is a $\mathcal{G}$-rearrangement of h and f is in $F(\mathcal{G})$, then the composition of germs

$$gfg^{-1} = (\alpha h \beta) f (\alpha' h \beta')^{-1}$$

shows that gfg^{-1} is in $F(\mathcal{G})$ as well. We know that elements of the same coset of $H(\mathcal{G})$ in $N(\mathcal{G})$ are $\mathcal{G}$-rearrangements of each other. Now if g and h in $N(\mathcal{G})$ satisfy $g = \alpha h \beta$ at each x for some germs α and β that vary with x, then $gh^{-1} = \alpha h \beta h^{-1}$ at each x, and $h \in N(\mathcal{G})$ implies that $gh^{-1} \in F(\mathcal{G})$ and thus in $H(\mathcal{G})$, and g and h are in the same coset of $H(\mathcal{G})$ in $N(\mathcal{G})$.

We borrow a property from the study of foliations. Let $\mathcal{G}$ be a groupoid of germs on $\mathbf{R}$. We say that $\mathcal{G}$ is interpolating if for every pair of germs $\alpha \in \mathcal{G}_{a,b}$ and $\beta \in \mathcal{G}_{c,d}$ with both α and β increasing, with $a < c$ and with $b < d$, there is an $f \in F(\mathcal{G})$ with α the germ of f at a and β the germ of f at c. This is condition (γ) in [21] and is called *germ-connected* in [14].

Lemma **3.6.** — *Let* $\mathcal{G}$ *be an interpolating groupoid of germs on* $\mathbf{R}$ *that contains the identity. Then for every pair of germs* $\alpha \in \mathcal{G}_{a,b}$ *and* $\beta \in \mathcal{G}_{c,d}$ *with both* α *and* β *increasing, with* $a < c$ *and with* $b < d$, *there is an* $h \in H(\mathcal{G})$ *with* α *the germ of* h *at* a *and* β *the germ of* h *at* c.

Proof. — There is an $f \in F(\mathcal{G})$ with α the germ of f at a and β the germ of f at c. For some x with $x < a$ and $x < b$ there is a $g_1 \in F(\mathcal{G})$ with α the germ of g_1 at a and with the germ of g_1 at x the germ of the identity. For some y with $y > c$ and $y > d$ there is a $g_2 \in F(\mathcal{G})$ with β the germ of g_2 at c and with the germ of g_2 at y the germ of the identity. Now let

$$h(t) = \begin{cases} t & t \leqslant x, \\ g_1(t) & x \leqslant t \leqslant a, \\ f(t) & a \leqslant t \leqslant b, \\ g_2(t) & b \leqslant t \leqslant y, \\ t & y \leqslant t. \end{cases}$$

Corollary **3.6.1.** — *If $\mathcal{G}$ is an interpolating groupoid of germs on* $\mathbf{R}$ *that contains the identity, then* $H(\mathcal{G})$ *generates* $\mathcal{G}$.

Lemma **3.7** *(Reduction).* — *Let $\mathcal{G}$ be an interpolating groupoid of germs on* $\mathbf{R}$ *that contains the identity and is s-invariant with respect to the map* $s(x) = x + 1$. *Let $\mathcal{G}'$ be the projection of $\mathcal{G}$ on* S^1. *Then the projection from* $\mathbf{R}$ *to* $S^1 = \mathbf{R}/\mathbf{Z}$ *induces an isomorphism from* $N(\mathcal{G})/H(\mathcal{G})$ *to* $N(\mathcal{G}')/H(\mathcal{G}')$.

Proof. — Let h be in $N(\mathcal{G})$. We wish to find a function g in the same coset of $H(\mathcal{G})$ in $N(\mathcal{G})$ so that g is a lift of a homeomorphism of S^1. Note that if g is increasing, this means that we want g to commute with s. However, if g is decreasing, we want g to satisfy $g = sgs$.

We know that hsh^{-1} is in $H(\mathcal{G})$, as are $shs^{-1}h^{-1}$ and $shsh^{-1}$. Also, $shs^{-1}h^{-1}$ and $shsh^{-1}$ are increasing no matter what the behavior of h is.

If h is increasing, then $h(0) < h(1)$ and we let $a = h(0)$ and $c = h(1)$. Now $(shs^{-1}h^{-1})\,(h(1)) = h(0) + 1$ and if we let $b = h(0)$ and $d = h(0) + 1$, then we have $a < c$ and $b < d$. There is an f in $H(\mathcal{G})$ whose germ at $h(0)$ is that of the identity and whose germ at $h(1)$ is that of $shs^{-1}h^{-1}$. Now fh takes 0 to $h(0)$ with the germ of h at 0 and takes 1 to $h(0) + 1$ with the germ of shs^{-1} at 1. Since the germ of (fh) at 1 is the germ of $s(fh)\,s^{-1}$ at 1, we can build a function g in $H(\mathcal{G})$ that agrees with fh on $[0, 1]$ and that commutes with s on all of $\mathbf{R}$. Specifically, g on $[i, i + 1]$ is defined to be $s^i(fh)\,s^{-i}$. Slightly to the left of i, the behavior of g is that of $s^{i-1}(fh)\,s^{-i+1}$ with the relevant part of fh being that in a small neighborhood of 1. But here we have $fh = s(fh)\,s^{-1}$ so the behavior of g to the left of i is also that of $s^i(fh)\,s^{-i}$ which agrees with the behavior of g to the right of i. Thus g is $\mathcal{G}$-compatible at each i. It is also $\mathcal{G}$-compatible in the interior of each $[i, i + 1]$ since $\mathcal{G}$ is s-invariant.

If h is decreasing, then $h(1) < h(0)$ and we let $a = h(1)$ and $c = h(0)$. Now $(shsh^{-1})\,(h(0)) = h(1) + 1$ and if we let $b = h(1)$ and $d = h(1) + 1$, then we have $a < c$ and $b < d$. There is an f in $H(\mathcal{G})$ whose germ at $h(1)$ is that of the identity and whose germ at $h(0)$ is that of $shsh^{-1}$. Now fh takes 1 to $h(1)$ with the germ of h at 1 and takes 0 to $h(1) + 1$ with the germ of shs at 0. The rest of the argument is similar to the paragraph above except we define g on $[-i, -(i-1)]$ to be $s^i f s^i$ and get a decreasing function that satisfies $sgs = g$.

In both cases, we get a function that defines a homeomorphism on $S^1 = \mathbf{R}/\mathbf{Z}$. This function is in the same coset as h since it is a $\mathcal{G}$-rearrangement. We must show that another function g' in the same coset as h that has $sg's^{-1} = g'$ if g' is increasing or $sg's = g'$ if g' is decreasing has the projection of g' to S^1 in the same coset as the projection of g. We know that $g^{-1}g'$ is in $H(\mathcal{G})$. Now g' is increasing if and only if g and h are increasing. If g and g' are increasing, then $g^{-1}g' = sg^{-1}s^{-1}sg's^{-1} = sg^{-1}g's^{-1}$ and $g^{-1}g'$ projects to S^1 and shows that the projection of g and g' are in the same coset of $H(\mathcal{G}')$. If h and g' are decreasing, then $g^{-1}g' = s^{-1}g^{-1}s^{-1}sg's = s^{-1}g^{-1}g's$ and the same conclusion holds.

The map clearly defines a homomorphism. We want to show that the function is one to one and onto. It is onto by lifting representatives of $N(\mathscr{G}')/H(\mathscr{G}')$ to $\mathbf{R}$. Such lifts are in $N(\mathscr{G})$ since $\mathscr{G}$ is s-invariant. It is one to one, since if h projects to a $\mathscr{G}'$-compatible homeomorphism of S^1, then h is $\mathscr{G}$-compatible.

4. Normalizers and germs of Thompson's groups

First we wish to show that all normalizers that we wish to study can be related to normalizers of T. Then we wish to show that it suffices to study conjugators of v_2. We first need a consequence of Lemma 2.1.

Lemma **4.1.** — *Given* $f \in PL_2(\mathbf{R})$ *and* $-\infty < a < b < \infty$, *there is an element* g *in* $BPL_2(\mathbf{R})$ *that agrees with* f *on* $[a, b]$.

Proof. — By decreasing a and increasing b we can assume that a and b are in $\mathbf{Z}[\frac{1}{2}]$. By the hypothesis on f we know that $f(a)$ and $f(b)$ are in $\mathbf{Z}[\frac{1}{2}]$. By Lemma 2.1, there is a g_1 in $BPL_2(\mathbf{R})$ that carries a to $f(a)$ and b to $f(b)$. Now we let g agree with f on $[a, b]$ and with g_1 elsewhere.

Lemma **4.2.** — *Let* G *be a group for which* $BPL_2(\mathbf{R}) \subseteq G \subseteq \widetilde{PL}_2(\mathbf{R})$ *or* $T \subseteq G \subseteq \tilde{T}$. *Then* $N(G) \subseteq N(PL_2(\mathbf{R}))$ *or* $N(G) \subseteq N(T)$ *whichever applies.*

Proof. — Conjugation preserves the properties of being orientation preserving and having bounded support. This gives $N(G) \subseteq N(BPL_2(\mathbf{R}))$ or $N(G) \subseteq N(T)$. Now Lemma 4.1 implies that the germs of $BPL_2(\mathbf{R})$ are exactly the germs of $PL_2(\mathbf{R})$. This shows that $N(BPL_2(\mathbf{R})) = N(PL_2(\mathbf{R}))$.

We must relate $N(PL_2(\mathbf{R}))$ to $N(T)$. Let $\mathscr{E}$ be the groupoid of germs generated by the group T. This is consistent with the terminology $\mathscr{E}$-compatible used in Section 1. Let $\mathscr{P}$ be the groupoid of germs generated by the group $PL_2(\mathbf{R})$. These groupoids were previously identified in Section 3 as $(\mathscr{A}_2(\mathbf{R}); \mathbf{Z}[\frac{1}{2}])$ and $(\mathscr{A}_2(S^1); \mathbf{Z}[\frac{1}{2}])$, but we will stick with the simpler notation.

Lemma **4.3** (i) $\mathscr{P}$ *contains the identity.* (ii) $\mathscr{P}$ *is* s*-invariant where* $s(x) = x + 1$. (iii) $\mathscr{P}$ *is interpolating.*

Proof. — We get (i) and (ii) from the fact that the identity and s are in $H(\mathscr{P}) = PL_2(\mathbf{R})$. To see (iii) we let $a < c$ and $b < d$ where $\mathscr{P}_{a,b}$ and $\mathscr{P}_{c,d}$ are non-empty. Given $\alpha \in \mathscr{P}_{a,b}$ and $\beta \in \mathscr{P}_{c,d}$, we can get $a < a' < c' < c$ where a' and c' are in $\mathbf{Z}[\frac{1}{2}]$ and $[a, a']$ is in the domain of some representative f of α and $[c', c]$ is in the domain of some representative g of β. We can move a' close to a in $\mathbf{Z}[\frac{1}{2}]$ and c' close to c in $\mathbf{Z}[\frac{1}{2}]$ to guarantee $f(a') < g(c')$. We know that f and g preserve $\mathbf{Z}[\frac{1}{2}]$. Now Lemma 2.1 allows us to fill the missing part on $[a', c']$.

We recall the following:

Lemma **4.4.** — *The groupoid* $\mathscr{E}$ *is the projection of the groupoid* $\mathscr{P}$ *to* S^1.

Corollary **4.4.1.** — (1) *The projection from* $\mathbf{R}$ *to* $S^1 = \mathbf{R}/\mathbf{Z}$ *induces an isomorphism* $N(\mathscr{P})/H(\mathscr{P}) \simeq N(\mathscr{E})/H(\mathscr{E})$. *In particular, there is a non-PL normalizer of* $PL_2(\mathbf{R})$, *if and only if there is a non-PL normalizer of* T. (2) *Let* G *be a group for which* $BPL_2(\mathbf{R}) \subseteq G \subseteq \widetilde{PL}_2(\mathbf{R})$ *or* $T \subseteq G \subseteq \widetilde{T}$. *If there is a non-PL normalizer of* G, *then there is a non-PL normalizer of* T.

Proof. — Item (1) follows from Lemmas 3.7, 4.4 and 4.3. Item (2) follows from Lemma 4.2 and item (1).

Note that $\nu_2 : S^1 \to S^1$ defined by $\nu_2(x) = 2x$ is $\mathscr{E}$-compatible.

Lemma **4.5.** — *Let* G *be a group for which* $BPL_2(\mathbf{R}) \subseteq G \subseteq \widetilde{PL}_2(\mathbf{R})$ *or* $T \subseteq G \subseteq \widetilde{T}$. *If there is a non-PL normalizer of* G, *then there is a non-PL homeomorphism* $h : S^1 \to S^1$ *for which* $h(\mathbf{Z}[\frac{1}{2}]) = \mathbf{Z}[\frac{1}{2}]$ *and for which* $h\nu_2 h^{-1}$ *and* $h^{-1}\nu_2 h$ *are* $\mathscr{E}$*-compatible.*

Proof. — By Corollary 4.4.1, there is a non-PL normalizer h of T. From Lemma 3.4 and the remarks that follow, we get $h(\mathbf{Z}[\frac{1}{2}]) = \mathbf{Z}[\frac{1}{2}]$. The function ν_2 is $\mathscr{E}$-compatible and T generates $\mathscr{E}$. Lemma 3.3 finishes the proof.

5. PL normalizers: Proof of Theorem 1 from Theorem 2

It is the goal of this section to prove Theorem 1 from Theorem 2. For the rest of this section, we assume the truth of Theorem 2. From Lemma 4.5, we know the following.

Lemma **5.1.** — *Let* G *be a group for which* $BPL_2(\mathbf{R}) \subseteq G \subseteq \widetilde{PL}_2(\mathbf{R})$ *or* $BT \subseteq G \subseteq \widetilde{T}$. *Then every element in the normalizer of* G *is PL.*

Proof of Theorem 1 from Theorem 2. — Part (i) is proven in Section 2. From Lemma 4.2 and Corollary 4.4.1, we know that if we prove that $N(PL_2(\mathbf{R})) \subseteq \widetilde{PL}_2(\mathbf{R})$, then we can conclude that $N(G) \subseteq \widetilde{PL}_2(\mathbf{R})$ or $N(G) \subseteq \widetilde{T}$ for the objects G covered in the lemmas. This will prove Part (ii) of Theorem 1.

Let h be an element of $N(PL_2(\mathbf{R}))$. As argued in the proof of Lemma 4.5, we know that $h(\mathbf{Z}[\frac{1}{2}]) = \mathbf{Z}[\frac{1}{2}]$. Given any interval on which h is affine, there are a pair of elements of $\mathbf{Z}[\frac{1}{2}]$ in the interval whose distance apart is in integral power of 2. Since the images of these two points are in $\mathbf{Z}[\frac{1}{2}]$, the distance between the images is in $\mathbf{Z}[\frac{1}{2}]$, and the slope of h on the interval must have an integral power of 2 for the denominator. The discussion also applies to h^{-1}, and it follows that the slope is an integral power of 2. If h has a break in $\mathbf{R} - \mathbf{Z}[\frac{1}{2}]$, then a simple calculation with the chain rule shows that conjugating a translation by a sufficiently small amount will produce a function with a break in $\mathbf{R} - \mathbf{Z}[\frac{1}{2}]$. This proves Part (ii) of Theorem 1.

The proof of Part (iii) of Theorem 1 is one of checking that $\widetilde{PL}_2(\mathbf{R})$ and $\widetilde{T}$ normalize the groups listed. This is quite easy and is left to the reader.

We consider (iv) of Theorem 1. We know that $N(F) \subseteq \widetilde{PL}_2(\mathbf{R})$. Let f be in $N_+(F)$, the orientation preserving elements of N(F). The germ of F at $-\infty$ is isomorphic to $\mathbf{Z}$ generated by translation to the right or left by 1. A normalizer must preserve the germ at $-\infty$ and so must take a generator to a generator. (This is identical to the concept discussed in Lemma 3.4.) An orientation preserving conjugator will take a right translation to a right translation. Thus near $-\infty$, an orientation preserving normalizer commutes with translation to the right by 1. This forces $f(x+1) = f(x) + 1$ near $-\infty$. Similarly, we get $f(x+1) = f(x) + 1$ near $+\infty$. Conversely, any f in $PL_2(\mathbf{R})$ that satisfies $f(x+1) = f(x) + 1$ near $\pm\infty$ conjugates elements of F into F. Thus we have characterized the elements of $N_+(F)$. Now by restricting an $f \in N_+(F)$ sufficiently far to the left, we get a well defined element f_- of T. The restriction of f near $+\infty$ gives a well defined element f_+ of T. The function $f \mapsto (f_-, f_+)$ is a homomorphism into $T \times T$. The kernel consists of all elements of F. Now Lemma 2.1 shows that two elements g_1 and g_2 of T will have the lift of g_1 near $-\infty$ and the lift of g_2 near $+\infty$ connectable into one function in $PL_2(\mathbf{R})$. Thus the homomorphism is onto. This completes the proof of Theorem 1.

The next rigidity statement is a rewording of parts of Theorem 1. Recall from Section 3 that if $\mathcal{G}$ is a groupoid of germs on X, then a homeomorphism $h : X \to X$ preserves $\mathcal{G}$ under conjugation if hfh^{-1} is in $F(\mathcal{G})$ for all $f \in F(\mathcal{G})$. We say that h normalizes $\mathcal{G}$ if h and h^{-1} preserve $\mathcal{G}$ under conjugation.

Theorem **5.2.** — *If an orientation preserving homeomorphism $h : \mathbf{R} \to \mathbf{R}$ normalizes $\mathcal{P}$, then h is $\mathcal{P}$-compatible. If an orientation preserving homeomorphism $h : S_1 \to S_1$ normalizes $\mathcal{E}$, then h is $\mathcal{E}$-compatible.*

PART III. PROOF OF THEOREM 2

In this part we will prove:

Theorem 2. — *Let $h : S^1 \to S^1$ be an orientation preserving homeomorphism for which $h(\mathbf{Z}[\frac{1}{2}]) = \mathbf{Z}[\frac{1}{2}]$. Assume that $h\nu_2 h^{-1}$ and $h^{-1}\nu_2 h$ are $\mathcal{E}$-compatible. Then h is PL.*

Since any rotation by an element of $\mathbf{Z}[\frac{1}{2}]$ is a normalizer of T, we can compose h by a rotation through $-h(0)$ to get an element satisfying the hypotheses that fixes 0 and that is PL if and only if h is. Thus throughout the rest of this part, we assume $h(0) = 0$. For the remainder of this part we will let $g = h\nu_2 h^{-1}$.

We need one preliminary lemma which will have several important consequences.

Lemma **5.3.** — *The homeomorphism h is a normalizer of* T.

Proof. — This follows from Lemma 3.2, from (3.1) and from Lemma 3.3.

6. Markov partitions, structures and morphisms

Let I_i^k, $0 \leqslant i < 2^k$, be the interval $[i2^{-k}, (i+1)\,2^{-k}]$ on S^1. For $0 \leqslant k < \infty$, let $P^k = \{ I_i^k \mid 0 \leqslant i < 2^k \}$. We call P^k a Markov partition for v_2 of depth k. We treat the subscripts of the intervals in P^k cyclically mod 2^k. The action of v_2 on I_i^k is to carry it onto $I_i^{k-1} = I_{2i}^k \cup I_{2i+1}^k$. We let $P = \{ P^k \mid 0 \leqslant k < \infty \}$ and we call P a Markov structure for v_2. The endpoints EP^k of P^k are the endpoints of the intervals in P^k. Note that $EP^k \subseteq EP^j$ for all $j > k$. The endpoints EP of P are the endpoints of all the P^k. We have $EP = \mathbf{Z}[\frac{1}{2}]$ in S^1.

Let $J_i^k = h(I_i^k)$, let $Q^k = \{ J_i^k \mid 0 \leqslant i < 2^k \}$ and let $Q = \{ Q^k \mid 0 \leqslant k < \infty \}$. Then each Q^k is a Markov partition for g and Q is a Markov structure for g. The action of g on J_i^k is to carry it onto $J_i^{k-1} = J_{2i}^k \cup J_{2i+1}^k$. Since $h(\mathbf{Z}[\frac{1}{2}]) = \mathbf{Z}[\frac{1}{2}]$, we have $EQ = \mathbf{Z}[\frac{1}{2}]$ in S^1. The breaks of g are a finite set of points in $\mathbf{Z}[\frac{1}{2}]$, so there is a smallest K so that the breaks of g are in EQ^K. Thus the action of g on each interval in Q^j for $j \geqslant K$ is affine. We call K the stable level of Q for reasons that will become clear later.

A morphism of P (or Q) of index $d \in \mathbf{Z}$, is a continuous function f from a connected open set in S^1 to S^1 so that for all $j \in \mathbf{Z}$ with $j \geqslant 0$ and $j + d \geqslant 0$ the function f carries the endpoints of P^j (or Q^j) in order to the endpoints of P^{j+d} (or Q^{j+d}). The function v_2 is a morphism of index -1 of P and g is a morphism of index -1 of Q. We can invent a notion of morphism from P to Q and note that this will make h an isomorphism from P to Q. However, h will be the only morphism from P to Q that we will ever look at. The next lemma is clear.

Lemma **6.1.** — (1) *A morphism is determined by its domain, its index and its value on one element in* $\mathbf{Z}[\frac{1}{2}]$. (2) *Indices of morphisms add under composition.*

A local morphism of P (or Q) is a continuous function f from S^1 to itself so that at every x in EP (or EQ), there is an $\varepsilon > 0$ so that f agrees with some morphism of P (or Q) on $[x, x+\varepsilon)$ and agrees with (perhaps another) morphism of P (or Q) on $(x-\varepsilon, x]$. A piecewise morphism of P (or Q) is a continuous function f from S^1 to itself for which there are a finite number of closed intervals with endpoints in EP (or EQ) covering S^1 so that f agrees with a morphism of P (or Q) on each interval.

Lemma **6.2.** — *The* $\mathscr{C}$*-compatible functions from* S^1 *to itself are the piecewise morphisms of* P, *and also are the piecewise morphisms of* Q.

Proof. — That the $\mathscr{C}$-compatible functions from S^1 to itself are the piecewise morphisms of P is clear. That they are also the piecewise morphisms of Q follows because h is an isomorphism from P to Q and because, by Lemma 5.3, h normalizes T and thus $\mathscr{C}$.

7. The calculus of break values

Given an element x of S^1, we associate a measure of the change of slope of g that occurs at x. It acts like a derivative, so we give it a notation that resembles g'. We define

$$g^b(x) = \log_2\left(\frac{g'_+(x)}{g'_-(x)}\right)$$

where $g'_+(x)$ represents the right derivative of g at x (the slope of g immediately to the right of x) and $g'_-(x)$ represents the left derivative of g at x. Since g is PL, this is defined for all x. We call $g^b(x)$ the break value of g at x. We use the logarithm to make the chain rule additive in that if g_1 and g_2 are two PL functions, then

$$(7.1) \qquad (g_1 \circ g_2)^b(x) = g_1^b(g_2(x)) + g_2^b(x).$$

This follows because left and right derivatives satisfy the usual chain rule. We use log base 2 to make the numbers come out nicer — a fact that we will never use except in examples. For those familiar with non-linearity, we can think of the break value as the non-linearity concentrated at a point.

Some properties of the break value are as follows.

$$(7.2) \qquad g^b(x) = 0 \text{ if and only if } g \text{ is affine in a neighborhood of } x.$$

$$(7.3) \qquad (g^n)^b(x) = \sum_{i=0}^{n-1} g^b(g^i(x)).$$

This is just the sum of the break values of g at n consecutive points in an orbit starting with x. If $f: S^1 \to S^1$ is a PL homeomorphism, then

$$(7.4) \qquad (fgf^{-1})^b(x) = -f^b(f^{-1}(x)) + g^b(f^{-1}(x)) + f^b((gf^{-1})(x)), \quad \text{and}$$

$$(7.5) \qquad (fgf^{-1})^b(f(x)) = -f^b(x) + g^b(x) + f^b(g(x))$$

where (7.4) and (7.5) have been simplified by using $(f^{-1})^b(f(x)) = -f^b(x)$ which follows from applying (7.1) to $(f^{-1}f)^b(x)$.

Since g is a PL function defined on S^1, the sum of the break values of g once around S^1 must be zero. Since EQ^k contains all the break points of g as long as $k \geqslant K$, we get

$$(7.6) \qquad \sum_{x \in EQ^k} g^b(x) = 0 \quad \text{whenever } k \geqslant K.$$

Lemma **7.1.** — *If* $b : S^1 \to \mathbf{Z}$ *is zero except on a finite set, if the sum of* $b(x)$ *is* 0 *over* S^1, *and if* d *is a positive integer, then there is a unique, strictly increasing,* PL $h : S^1 \to S^1$ *of degree* d *that fixes* 0 *and that has* $h^b(x) = b(x)$ *for all* $x \in S^1$. *If* $d = 1$, *then* h^{-1} *exists and* $(h^{-1})^b(h(x)) = -b(x)$.

This is a " converse " to (7.6) whose proof is elementary and left to the reader. The point is to build the function h on $[0, 1]$ starting with an arbitrary slope m at 0 and show that since $h(1)$ is linear in m, there is a unique m that makes $h(1) = d$.

Lemma **7.2.** — $g^b(0) = 0$ *and* $g'(0) = 2$.

Proof. — From Lemma 5.3, we know that h is a normalizer of T. From Lemma 3.4, we know that h induces an isomorphism from the group of germs $\mathcal{E}_{0,0}$ of functions of T with fixed points at 0 in S^1 to the group of germs $\mathcal{E}_{0,0}$. This group is isomorphic to $\mathbf{Z} \times \mathbf{Z}$ and the two natural generators are " slope 2 to the right of 0 " and " slope 2 to the left of 0 ". Conjugation by h keeps the " behavior to the right " separate from the " behavior to the left " in that the germ to the right of 0 of the conjugate is determined only by the germ to the right of 0 of the conjugated function, and similarly for germs to the left. Thus a germ to the right of 0 that represents " slope 2 to the right " must be carried to " slope 2 to the right " or " slope 1/2 to the right ". However, conjugation preserves the property of a fixed point being repelling. Thus " slope 2 to the right " must be carried to " slope 2 to the right ". Similarly for the slopes to the left. We now note that v_2 has slope 2 at its fixed point 0. Thus g has slope 2 at both sides of its fixed point 0.

We apply Lemma 7.2. For any $x \in \mathbf{Z}[\frac{1}{2}]$ we have $g^N(x) = 0$ for some N. Thus every $n \geqslant N$ has $g^n(x) = 0$. Since $g^b(0) = 0$, we get from (7.3) that $(g^n)^b(x)$ is independent of n as long as $n \geqslant N$. We can now define a function $\Sigma : \mathbf{Z}[\frac{1}{2}] \to \mathbf{Z}$ by

$$\Sigma(x) = \lim_{n \to \infty} (g^n)^b(x). \tag{7.7}$$

We can refer to it as the total iterated break value at x, and it is the sum of all the break values in the forward orbit of x (including x) under g. The following fact about any x and y in $\mathbf{Z}[\frac{1}{2}]$ is clear from these remarks:

$$g^i(x) = g^j(y) \Rightarrow [(g^i)^b(x) = (g^j)^b(y) \Leftrightarrow \Sigma(x) = \Sigma(y)]. \tag{7.8}$$

We know that the set of breakpoints of g is contained in EQ^K where K is the stable level for g. For $x \in \mathbf{Z}[\frac{1}{2}]$, let $\lambda(a)$ be the smallest integer i for which $x \in EQ^i$. Now all $x \in \mathbf{Z}[\frac{1}{2}]$ with $\lambda(x) > K$ have $g^b(x) = 0$. If $x \in \mathbf{Z}[\frac{1}{2}]$ has $\lambda(x) > K$, then all $(g^i)^b(x) = 0$ for $i < \lambda(x) - K$ and we have

$$\Sigma(x) = \Sigma(x') \quad \text{where } x' = g^{\lambda(x)-K}(x) \text{ has } \lambda(x') = K. \tag{7.9}$$

Since EQ^K is finite, there are only finitely many values that $\Sigma(x)$ can achieve. The property given in (7.9) is the reason for calling the level K the stable level.

Note that (7.7) and (7.3) imply that $g^b(x) = \Sigma(x) - \Sigma(g(x))$ for any x. Thus Lemma 7.2 implies that g^b is a coboundary on an appropriate complex. In smooth settings, this is often enough to reach conclusions about the conjugating homeomorphism h. We show in Example 1 of Part IV that this is not sufficient here.

We will need one consequence of (7.6).

Lemma **7.3.**

$$\sum_{\lambda(x)=K} \Sigma(x) = \sum_{\lambda(x)<K} \Sigma(x).$$

Proof. — The sum of $\Sigma(x)$ over a set of all x with $\lambda(x)$ a constant value i uses each value of $g^b(y)$ with $\lambda(y) = i - j$ exactly 2^j times. Thus each $g^b(y)$ with $\lambda(y) = K - j$ is used 2^j times by the left sum in the statement and $2^{j-1} + 2^{j-2} + \ldots + 1 = 2^j - 1$ times by the right sum in the statement. Thus the difference of the two sums in the statement is the sum in (7.6) for $k = K$ which is 0.

8. Criteria for the piecewise linearity of h

We say that Q^j has equal pairs if each pair of intervals I^j_{2i} and I^j_{2i+1} has equal lengths.

Lemma **8.1.** — *The following are equivalent:*

(1) Q^K *has equal pairs.*
(2) Q^j *has equal pairs for all* $j \geqslant K$.
(3) h *is PL.*
(4) *The values of* $\Sigma(x)$ *are equal for all* x *with* $\lambda(x) = K$.

Proof. — (1 $\Leftrightarrow$ 2) The set EQ^{j+1} equals $g^{-1}(EQ^j)$. When $j \geqslant K$, the action of g is affine on each interval of Q^j and the spacing of the even-odd pairs is preserved from Q^j to Q^{j+1}. The equivalence of 1 and 2 follows by induction.

(2 $\Leftrightarrow$ 3) If h is PL, then the equal spacing of the endpoints of P is carried over to equal spacings in sections of Q. Conversely, if (2) holds, then intervals of Q^K are evenly subdivided repeatedly to create intervals of Q^j, $j \geqslant K$. This creates equal spacings that can only be carried back to the equal spacings of P by a PL h^{-1}.

(3 $\Rightarrow$ 4) If h is PL, then as in the proof of Theorem 1 from Theorem 2, we know that h is $\mathscr{E}$-compatible and has all its breaks in $\mathbf{Z}[\frac{1}{2}]$. Now for sufficiently large n, (7.5) and the fact that v_2 has no breaks gives

$$\Sigma(x) = (g^n)^b(x) = (hv_2^n h^{-1})^b(x) = -h^b(x) + 0 + h^b(0). \tag{8.1}$$

The finite number of breaks in h now implies that all but finitely many x have the same value $h^b(0)$ for $\Sigma(x)$. Now (7.9) shows that each value of $\Sigma(\ )$ on the stable level appears infinitely often at deeper levels. This yields 4.

(4 $\Rightarrow$ 3) We reverse the process in the last argument by integration. Let Σ be the constant value of $\Sigma(x)$ for $\lambda(x) = K$. Let $b(x) = \Sigma - \Sigma(x)$ for all $x \in \mathbf{Z}[\frac{1}{2}]$. Note that our hypothesis and (7.9) imply that $b(x) = 0$ for all but finitely many $x \in \mathbf{Z}[\frac{1}{2}]$. We want to argue that h is PL with all its break points in $\mathbf{Z}[\frac{1}{2}]$ and that $h^b(x) = b(h(x))$

for all $x \in \mathbf{Z}[\frac{1}{2}]$. We will do this by building an $\bar{h}$ with these properties and then showing that $\bar{h}$ must equal h. As in (7.6), we must also have

$$\textbf{(8.2)} \qquad \sum_{x \in \mathbf{Z}[\frac{1}{2}]} h^b(x) = \sum_{x \in \mathbf{Z}[\frac{1}{2}]} b(h(x)) = 0$$

in order for this to work. Note that since h is a one to one correspondence from $\mathbf{Z}[\frac{1}{2}]$ to itself, we can replace $b(h(x))$ with $b(x)$ in (8.2).

We have $b(x) = \Sigma - \Sigma(x)$ and this is zero when $\lambda(x) \geqslant K$. Thus

$$\textbf{(8.3)} \qquad \sum_{x \in \mathbf{Z}[\frac{1}{2}]} b(x) = \sum_{\lambda < K} (\Sigma - \Sigma(x)).$$

But the 2^K endpoints of Q^K are divided into 2^{K-1} endpoints x with $\lambda(x) = K$ and 2^{K-1} endpoints x with $\lambda(x) < K$. Thus the positive terms in (8.3) add to $2^{K-1}\Sigma$. By Lemma 7.3, the negative terms in (8.3) add to

$$-\sum_{\lambda(x) = K} \Sigma(x)$$

and (8.3) adds to 0 since $\Sigma(x) = \Sigma$ for all 2^{K-1} endpoints x with $\lambda(x) = K$.

By Lemma 7.1, there is a PL, increasing, homeomorphism $\bar{h} : S^1 \to S^1$ with $\bar{h}^b(x) = b(\bar{h}(x))$. If we let $\bar{g} = \bar{h} v_2 \bar{h}^{-1}$, then we can define $\bar{\Sigma}(x)$ in analogy with $\Sigma(x)$. As in (8.1), we get $\bar{\Sigma}(\bar{h}(x)) = \bar{h}^b(0) - \bar{h}^b(x)$ or equivalently

$$\bar{\Sigma}(x) = \bar{h}^b(\bar{h}^{-1}(0)) - \bar{h}^b(\bar{h}^{-1}(x)) = b(0) - b(x)$$

$$= (\Sigma - \Sigma(0)) - (\Sigma - \Sigma(x)) = \Sigma(x)$$

since $\Sigma(0) = 0$. We recover $\bar{g}^b(x)$ as $\bar{\Sigma}(x) - \bar{\Sigma}(\bar{g}(x))$ and $g^b(x)$ as $\Sigma(x) - \Sigma(g(x))$, so $\bar{g}^b$ and g^b are identical functions. Both $\bar{g}$ and g have degree 2 and fix 0, so by the uniqueness feature of Lemma 7.1, $\bar{g} = g$ and $\bar{h}$ conjugates v_2 to g. This makes $\bar{h}$ another isomorphism fixing 0 from P to Q and $\bar{h} = h$.

9. Local morphisms of Q

From Lemma 6.2, we know that every $\mathscr{C}$-compatible function from S^1 to S^1 is a piecewise morphism of Q. We will prove Theorem 2 by showing that this is false unless h is PL. To show that a function is not a piecewise morphism, we will show that it " breaks " from one morphism to another infinitely often. We thus need to know how to detect such breaks. The reason for the next lemma will become apparent by the lemma that follows it.

Lemma **9.1.** — *Let f be a morphism of* Q *taking $a \in \mathbf{Z}[\frac{1}{2}]$ to b. Then there are only finitely many $x \in \mathbf{Z}[\frac{1}{2}]$ with $\Sigma(x) \neq \Sigma(f(x))$.*

Proof. — Take x with $\lambda(x) > \lambda(a) + K$. From (7.9) we have $\Sigma(x) = \Sigma(g^{\lambda(x)-K}(x))$ and we know that $g^{\lambda(x)-K}(a) = 0$. If f has index d, then $\lambda(f(x)) = \lambda(x) + d$ and $\lambda(b) = \lambda(a) + d$ so $\lambda(f(x)) > \lambda(b) + K$. Thus $\Sigma(f(x)) = \Sigma(g^{\lambda(f(x))-K}(f(x)))$ and $g^{\lambda(f(x))-K}(b) = 0$. Now f, $g^{\lambda(x)-K}$ and $g^{\lambda(f(x))-K}$ are morphisms of Q of indices d, $K - \lambda(x)$ and $K - \lambda(f(x)) = K - (\lambda(x) + d)$ respectively, so $g^{\lambda(f(x))-K} \circ f$ and $g^{\lambda(x)-K}$ are morphisms of Q of equal indices that agree on a. By Lemma 6.1, they are equal and $\Sigma(x) = \Sigma(f(x))$. There are now only finitely many $x \in \mathbf{Z}[\frac{1}{2}]$ with $\lambda(x) \leqslant \lambda(a) + K$.

Lemma **9.2.** — *Let f be affine on $[a, b] \subseteq S^1$ with $a \in \mathbf{Z}[\frac{1}{2}]$ and let f_1 be a morphism of index d of Q on $[a, b]$ that agrees with f on some $[a, a + \varepsilon)$. If f does not agree with f_1 on all of $[a, b]$, then the largest x in $[a, b]$ for which f agrees with f_1 on $[a, x]$ is the smallest $x \in \mathbf{Z}[\frac{1}{2}] \cap (a, b)$ for which $\Sigma(x) \neq \Sigma(f_1(x))$.*

Proof. — Note that by Lemma 9.1, if there is an $x \in \mathbf{Z}[\frac{1}{2}] \cap (a, b)$ for which $\Sigma(x) \neq \Sigma(f_1(x))$, then there is a smallest.

For an interval A, let L(A) denote its length. For 2 consecutive intervals A and B in some Q^k (with A to the left of B), let $r(A, B) = \log_2(L(B)/L(A))$. This gives a value for each consecutive pair in Q^k. We have that the affine f will agree with the morphism f_1 on $[a, y]$ if and only if for all Q^k with k sufficiently large, the sequence of r values in Q^k starting with the interval pair to the right of a and ending with the interval pair to the left of y equals the sequence of r values of the same length in Q^{k+d} starting with the interval pair to the right of $f(a)$.

As in the proof of Lemma 9.1, with k large enough, there are powers g^i and g^j of g that are morphisms respectively from Q^k and Q^{k+d} to Q^K so that a and $f(a)$ respectively are carried to 0. Let A and B be consecutive intervals in Q^k with common endpoint t. Let $A' = f_1(A)$, let $B' = f_1(B)$ and let $t' = f_1(t)$. Note that $g^i(A) = g^j(A')$, $g^i(B) = g^j(B')$ and $g^i(t) = g^j(t')$. Since all the breaks of g are in the endpoints of Q^K, the actions of g^i on A and B and of g^j on A′ and B′ are affine. A trivial calculation shows that

$$r(A, B) + (g^i)^b(t) = r(g^i(A), g^i(B)) = r(g^j(A'), g^j(B'))$$
$$= r(A', B') + (g^j)^b(t').$$

Thus $r(A, B) = r(A', B')$ if and only if $(g^i)^b(t) = (g^j)^b(t')$ which by (7.8) is equivalent to $\Sigma(t) = \Sigma(t')$.

10. Conjugates of affine functions

We will prove Theorem 2 from the next lemma which is inductive in nature and is best stated with a previously defined hypothesis. We will prove the lemma after using it to prove Theorem 2. We use (*) (f, a, b) to mean all of: (1) f is an affine function on $[a, b]$ with $a \in \mathbf{Z}[\frac{1}{2}]$, $f(a) \in \mathbf{Z}[\frac{1}{2}]$, $\lambda(a) > K$ and $\lambda(f(a)) > K$. (2) A morphism of Q of index $d \neq \lambda(f(a)) - \lambda(a)$ defined on $[a, b]$ agrees with f on some $[a, a + \varepsilon)$. (3) $[a, b]$ includes the next endpoint a' to the right of a in $Q^{\lambda(a)}$.

Lemma **10.1.** — *Assume* (4) *of Lemma* 8.1 *fails and condition* (*) (f, a, b) *holds. Then there is a smallest* $x \in \mathbf{Z}[\frac{1}{2}]$ *with* $a < x < a'$ *and with* $\Sigma(x) \neq \Sigma(f(x))$, *and further, for this* x, *the condition* (*) (f, x, b) *will hold.*

Proof of Theorem 2 from Lemma 10.1. — If h is not PL, then (4) of Lemma 8.1 fails. Take two points a and c with $\lambda(a)$ and $\lambda(c)$ larger than K. There is a morphism f_1 taking a to c with index d_1 not equal to $\lambda(c) - \lambda(a)$. This locally is $\mathcal{E}$-compatible by the arguments proving Lemma 6.2. Let f_a be affine on $[a, a']$ with slope agreeing with that of f_1 to the right of a. Regarding $[a, a']$ as a subset of [0, 1], we can use Lemma 2.1 to build an element f of T that agrees with f_a on $[a, a']$. We have now satisfied the hypotheses of Lemma 10.1. Let x_1 be the x value guaranteed by Lemma 10.1. By Lemma 6.2, f agrees with a morphism f_2 of some index d_2 to the right of x_1. By Lemma 9.2, f agrees with f_1 to the left of x_1 but not to the right of x_1. By Lemma 6.1, the morphisms f_1 and f_2 cannot have the same index. This implies that $h^{-1}fh$ has a break at $h^{-1}(x_1)$. Since the hypotheses of Lemma 10.1 are repeated with a replaced by x_1, we obtain a sequence $x_1 < x_2 < \ldots$ in $[a, a']$ so that $h^{-1}fh$ has a break at each $h^{-1}(x_i)$ in $[h^{-1}(a), h^{-1}(a')]$. This implies that $h^{-1}fh$ is not PL and contradicts Lemma 5.3.

To find an x in Lemma 10.1 with $\Sigma(x) \neq \Sigma(f(x))$, we need to compare sequences of break sum values. The next lemma is the necessary tool. If (σ_i), $0 \leqslant i < 2^k$, is a finite sequence with 2^k entries and $0 \leqslant j < k$, then a 2^j-part is a subsequence (σ_i), $n2^j \leqslant i < (n+1)\,2^j$, for some fixed n with $0 \leqslant n < 2^{k-j}$. Such a 2^j-part is odd or even depending on whether n is odd or even. Thus there are two 2^{k-1}-parts (one odd and one even) ordinarily called halves, four 2^{k-2}-parts (two odd and two even) ordinarily called quarters, and so forth. For a fixed j, two different 2^j-parts (σ_i), $n2^j \leqslant i < (n+1)\,2^j$ and (σ_i), $m2^j \leqslant i < (m+1)\,2^j$ are called equal if $\sigma_{s(i)} = \sigma_{t(i)}$ for all i with $0 \leqslant i < 2^j$ where $s(i) = n2^j + i$ and $t(i) = m2^j + i$.

Lemma **10.2.** — *If* (σ_i), $0 \leqslant i < 2^k$ *is a finite sequence so that for all* j *with* $0 \leqslant j < k$, *some even* 2^j*-part equals some odd* 2^j*-part, then the sequence is constant.*

Proof. — If all even 2^j-parts are equal and all odd 2^j-parts are equal, then the hypothesis implies that all 2^j-parts are equal and therefore that all even 2^{j-1}-parts are equal and all odd 2^{j-1}-parts are equal. The lemma follows by induction.

Proof of Lemma 10.1. — Let f agree with the morphism f_1 on $[a, a+\varepsilon)$. Either f_1 takes $Q^{\lambda(a)}$ to $Q^{\lambda(f(a))+e}$ for a positive $e = d - (\lambda(f(a)) - \lambda(a))$ or takes $Q^{\lambda(a)+e}$ to $Q^{\lambda(f(a))}$ for a positive $e = (\lambda(f(a)) - \lambda(a)) - d$. In the first case we will work with a and a' as described in $Q^{\lambda(a)}$. In the second case we will work with a and the next endpoint to its right a'' in $Q^{\lambda(a)+e}$ which will satisfy $[a, a''] \subseteq [a, a']$. If x is found with the right properties in $[a, a'']$, then we are done. We will describe the argument only in the first case and let the reader check that it applies with little change to the second.

We will consider elements $x \in \mathbf{Z}[\frac{1}{2}] \cap (a, a')$ with $\lambda(x) = \lambda(a) + k$. There are 2^{k-1}

of these and they are carried by f_1 to elements with $\lambda(f_1(x)) = \lambda(f(a)) + e + k$. Now $\Sigma(x)$ is determined by $\bar{x} = g^{\lambda(x)-K}(x) = g^{\lambda(a)+k-K}(x)$ with $\lambda(\bar{x}) = K$, and $\Sigma(f_1(x))$ is determined by $\hat{x} = g^{\lambda(f_1(x))-K}(f_1(x)) = g^{\lambda(f(a))+e+k-K}(f_1(x))$ with $\lambda(\hat{x}) = K$.

Now Q^K has 2^K endpoints of which 2^{K-1} of them have $\lambda = K$. These occur at the odd positions starting at 0 (which has position 0). Because of (7.9), the function $\Sigma(\ )$ on this sequence of endpoints forms the sequence (σ_i) of length 2^{K-1} to which we will apply Lemma 10.2. Among the endpoints of Q^K are the 2^{K-k} endpoints of Q^{K-k}. These occur at positions that are multiples of 2^k from 0, lie between endpoints with $\lambda = K$, and carve (σ_i) into 2^{K-k} different 2^{k-1}-parts. Among these 2^{K-k} endpoints of Q^{K-k}, exactly half of them have $\lambda = K - k$, and these occur at positions that are odd multiples of 2^k from 0. To the right of these endpoints we find the odd 2^{k-1}-parts of (σ_i). Now

$$\lambda(g^{\lambda(x)-K}(a)) = \lambda(g^{\lambda(a)+k-K}(a)) = \lambda(a) - [\lambda(a) + k - K] = K - k,$$

and

$$\lambda(g^{\lambda(f(a))+e+k-K}(f(a))) = \lambda(f(a)) - [\lambda(f(a)) + e + k - K] = K - (k + e).$$

Thus the 2^{k-1} elements $x \in \mathbf{Z}[\frac{1}{2}] \cap (a, a')$ with $\lambda(x) = \lambda(a) + k$ map to an odd 2^{k-1}-part of (σ_i) and their images under f_1 map to an even 2^{k-1}-part of (σ_i) since we assume $e > 0$. Since (4) of Lemma 8.1 is assumed false, the Lemma 10.2 implies that we cannot have $\Sigma(x) = \Sigma(f_1(x))$ for all $x \in \mathbf{Z}[\frac{1}{2}] \cap (a, a')$ with $\lambda(x) = \lambda(a) + k$ for all k with $1 < k < K$. Thus some $x \in \mathbf{Z}[\frac{1}{2}] \cap (a, a')$ has $\Sigma(x) \neq \Sigma(f_1(x))$.

The rest of the argument is similar to the proof of Theorem 2 from this lemma. Lemma 9.1 implies there is a minimum such x. Lemma 9.2 implies that f agrees with f_1 to the left of x and not to the right. By Lemma 6.2, f agrees with some morphism f_2 of Q to the right of x. Now $\lambda(f_1(x)) - \lambda(x) = \lambda(f(a)) + e - \lambda(a) = d$ the degree of f_1. By Lemma 6.1, this is not the degree of f_2. Since $x \in \mathbf{Z}[\frac{1}{2}] \cap (a, a')$ and a and a' are consecutive endpoints in $Q^{\lambda(a)}$, the endpoint x' to the right of x in $Q^{\lambda(x)}$ cannot exceed a'. We have verified all properties of $(*)$ (f, x, a').

PART IV. EXAMPLES

We build examples of conjugators h by building the function $g = h\nu_2 h^{-1}$. We build g by building a Markov partition for g. Let $a_0, a_1, \ldots, a_{n-1}$ be positive integers with sum S. The integers represent the lengths in units of 1/S of intervals in counterclockwise order of a Markov partition of S^1 where the first interval (with length a_0/S) has left endpoint 0. The action of g is to fix 0 and take each interval affinely to the union of two consecutive intervals. This corresponds to the structures in Section 6 if n is an integral power of 2. However for any n, a Markov structure for ν_2 based on n initial intervals can be built by starting with intervals of length $1/n$ for the first partition and then pulling back endpoints under iterates of ν_2^{-1} for the finer partitions. Once a string of positive integers is written out for the first partition for g, then the remaining partitions

are similarly obtained by pulling back endpoints under iterates of g^{-1}. The only question that remains is whether the endpoints in the Markov structure for g are dense in S^1. That this is the case is seen by noting that each interval is subdivided into two intervals that are shorter than the original by some fraction. This fraction is of the form $a_{2i}/(a_{2i} + a_{2i+1})$ or $a_{2i+1}/(a_{2i} + a_{2i+1})$ where subscripts are treated cyclically mod n. Among these ratios is one that is largest. Now we observe that all intervals shrink under repeated subdivision at least as fast as powers of this largest ratio. Once the density of the endpoints is established, then there is a unique conjugating homeomorphism $h : S^1 \to S^1$ that is an isomorphism of the Markov structure for ν_2 to that of g.

Once the values of the a_i are known, then it is a straightforward task to check if g is $\mathcal{F}$-compatible, to calculate values of g^b and $\Sigma(\ )$ at the endpoints, and so forth. Whether h is PL is detected by the equal pairs condition according to Lemma 8.1. This condition is easier to interpret if n is even, so our examples will all use an even number of initial intervals.

If g turns out to be $\mathcal{F}$-compatible and n is an integral power of 2, then conjugation by h is at least an endomorphism of T into T. This is because the endpoints of the Markov partition for ν_2 will be all of $\mathbf{Z}[\frac{1}{2}]$ and the endpoints of the Markov partition for g will be a (perhaps proper) subset of $\mathbf{Z}[\frac{1}{2}]$. This gives $h(\mathbf{Z}[\frac{1}{2}]) \subseteq \mathbf{Z}[\frac{1}{2}]$ and the rest follows from Lemma 3.2, from (3.1) and from Lemma 3.3.

Example 1. — The sequence 2, 2, 3, 1, 4, 2, 1, 1, 2, 2, 3, 1, 2, 2, 2, 2 has 16 lengths with sum 32. It is easily checked that the function g defined by this partition is $\mathcal{F}$-compatible. The equal pairs condition is violated, so h is not PL. The results in Part III were discovered by studying this example. This gives an example of a non-PL homeomorphism h that conjugates T into T and has lifts that conjugate F into F. In addition, the function g has zero break at its fixed point 0. This makes the function $\Sigma(\ )$ well defined. Since

$$g^b(x) = \Sigma(x) - \Sigma(g(x)) \tag{10.1}$$

it also makes the non-linearity of g a " coboundary ". This shows that (10.1) is not sufficient to reach the conclusion of Theorem 2. The right hypotheses to replace (10.1) seems to be that the following statement fails for only finitely many pairs x and y in S^1:

(*) If x and y have $g^p(x) = g^q(y)$ for some positive integers p and q, then $(g^p)^b(x) = (g^q)^b(y)$.

Example 2. — The sequence 2, 2, 1, 1, 1, 1 has 6 lengths with sum 8. Since it satisfies the equal pairs condition, the conjugating homeomorphism h is PL. Since the number of intervals is not a power of 2, there are endpoints of the Markov partition for ν_2 that are not in $\mathbf{Z}[\frac{1}{2}]$. Thus $h(\mathbf{Z}[\frac{1}{2}]) \subsetneq \mathbf{Z}[\frac{1}{2}]$, h conjugates T into but not onto T, and h^{-1} conjugates T onto a group of PL homeomorphisms of S_1 that properly contains T.

Remark. — We do not have an example of a non-PL homeomorphism h so that g is $\mathcal{F}$-compatible and h^{-1} conjugates ν_2 to a PL function. Thus we do not know whether

the hypothesis that $h^{-1}\nu_2 h$ be $\mathscr{E}$-compatible in Theorem 2 can or cannot be replaced by the hypothesis that $h^{-1}\nu_2 h$ be PL. The point is that under the weaker hypothesis, Lemma 5.3 and its consequences Lemma 6.2, Lemma 7.2 and (7.7) do not seem to be available.

Example 3. — The sequence 1, 1, 3, 1, 2, 4, 1, 1, 1, 1, 6, 2, 2, 2, 2, 2 has 16 lengths with sum 32. Here the conjugator h fails to have $h(\mathbf{Z}[\frac{1}{2}]) = \mathbf{Z}[\frac{1}{2}]$ in spite of the fact that there are 2^4 intervals. It is easy to check that the midpoint of the unique interval of length 4 is a point of period 2 under the action of g and cannot be the image of any element of $\mathbf{Z}[\frac{1}{2}]$.

Example 4. — The sequence 1, 3, 4, 2, 1, 3, 1, 1 has 8 lengths with sum 16. There is no well defined function $\Sigma(\ )$ since the break at the fixed point is not zero.

Example 5. — This sequence 1, 1, 3, 1, 2, 1, 3, 1, 2, 2, 1, 3, 2, 1, 1, 3, 2, 2 has 18 lengths with sum 32. Two of the endpoints (sixth and twelfth) form a cycle of period 2 and have breaks -2 and $+1$. Even though $g^b(0) = 0$, we cannot define $\Sigma(\ )$ because the sum of the breaks on a future orbit that contains these endpoints does not converge.

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