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THE $C^{1+\alpha}$ HYPOTHESIS IN PESIN THEORY

by CHARLES C. PUGH*

1. Introduction. — The stable manifold theory developed by Pesin and others [1, 4, 7, 8, 9] contains a hypothesis that the given dynamics be of differentiability class $C^{1+\alpha}$ for some $\alpha > 0$. That is, first derivatives must obey α -Hölder conditions. Here is an example showing that C^1 alone, i.e. $\alpha = 0$, is insufficient. It uses the auxiliary function

$$g(u) = \begin{cases} \frac{u}{\log(1/u)} & \text{if } 0 < u < 1 \\ 0 & \text{if } u = 0 \end{cases}$$

which is C^1 , strictly monotone increasing, has $g'(0) = 0$, and is not $C^{1+\alpha}$ for any $\alpha > 0$. Near 0, g grows faster than $u^{1+\alpha}$ for all $\alpha > 0$. Functions like g have been seen before, for instance in S. Sternberg's example of a non C^1 -linearizable C^1 contraction of $\mathbf{R}$ [10, p. 101].

Suppose $f: M \rightarrow M$ is a C^1 diffeomorphism of the compact manifold M . Let $p \in M$, $v \in T_p M$, $v \neq 0$, be given. The *Lyapunov exponents* of v are

$$\chi^-(v) = \lim_{n \rightarrow -\infty} \frac{1}{-n} \log |T_p f^{-n}(v)|$$

$$\chi^+(v) = \lim_{n \rightarrow \infty} \frac{1}{n} \log |T_p f^n(v)|$$

if the limits exist. For vectors in most tangent spaces, Oseledec [6] proves that these Lyapunov exponents do exist and that there is a kind of uniformity referred to as *regularity* of the orbit $\mathcal{O}(p) = \{f^n p\}_{n \in \mathbf{Z}}$; namely, if E_p^λ denotes $\{v : v = 0 \text{ or } \chi^-(v) = \lambda = \chi^+(v)\}$ then $T_p M = \bigoplus_\lambda E_p^\lambda$ and

$$(1) \quad \lim_{|n| \rightarrow \infty} \frac{1}{n} \log \left(\frac{\|T_p f^n| E_p^\lambda\|}{\|(T_p f^n| E_p^\lambda)^{-1}\|} \right) = 0.$$

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See also [1, 8]. (A better name for this property might be *subexponential conformality* since $Tf^n|E_p^\lambda$ grows like a sequence of dilations, up to a subexponential error.)

Definition. — The orbit $\mathcal{O}(p)$ is *asymptotically hyperbolic* if $\mathcal{O}(p)$ is regular and $\chi^\pm(v) \neq 0$ for all non-zero $v \in T_p M$.

Assume $\mathcal{O}(p)$ is asymptotically hyperbolic and write

$$E_p^u = \bigoplus_{\lambda > 0} E_p^\lambda, \quad E_p^s = \bigoplus_{\lambda < 0} E_p^\lambda.$$

Pesin's Stable Manifold Theorem [1, 5, 7, 9] asserts in part that the *asymptotically exponentially stable set* of p

$$W^s(p) = \left\{ x \in M : \lim_{n \rightarrow \infty} \frac{1}{n} \log d(f^n x, f^n p) < 0 \right\}$$

is an injectively immersed C^1 manifold tangent at p to E_p^s , provided f is $C^{1+\alpha}$ for some $\alpha > 0$.

Theorem. — *There exists a C^1 diffeomorphism of a 4-manifold having an asymptotically hyperbolic orbit $\mathcal{O}(p)$ such that $W^s(p)$ is not an injectively immersed manifold tangent to E_p^s .*

See § 3 for the example cited.

In the proof of Pesin's Theorem, f is lifted from M to TM along $\mathcal{O}(p)$ via the exponential map. The composites

$$\begin{array}{ccccccc} & & T_{f^n p} M & \xrightarrow{f_n} & T_{f^{n+1} p} M & & \\ & & \downarrow \exp & & \downarrow \exp & & \\ f_n = \exp_{f^{n+1} p}^{-1} \circ f \circ \exp_{f^n p} & & M & \xrightarrow{f} & M & & \\ n \in \mathbf{Z} & & & & & & \\ \dots & \xrightarrow{f_{-2}} & T_{f^{-1} p} M & \xrightarrow{f_{-1}} & T_p M & \xrightarrow{f_0} & T_{f p} M & \xrightarrow{f_1} & T_{f^2 p} M & \xrightarrow{f_2} & \dots \\ & & \downarrow \exp & & \downarrow \exp & & \downarrow \exp & & \downarrow \exp & & \\ \dots & \longrightarrow & \bullet & \xrightarrow{f} & \bullet & \xrightarrow{f} & \bullet & \xrightarrow{f} & \bullet & \longrightarrow & \dots \\ & & f^{-1} p & & p & & f p & & f^2 p & & \end{array}$$

locally represent f in exponential coordinates along the orbit $\mathcal{O}(p)$. The crucial consequence of $C^{1+\alpha}$ differentiability of f is that the f_n are $C^{1+\alpha}$ -equicontinuous. That is, there are uniform constants $K, \delta > 0$ such that

$$\|(Df_n)_x - (Df_n)_y\| \leq K |x - y|^\alpha$$

for all $x, y \in T_{f^n p} M$, with $|x|, |y| \leq \delta$.

In § 2, we give an example of a sequence of maps

$$\mathbf{R}^2 \xrightarrow{f_n} \mathbf{R}^2, \quad f_n(z) = A_n z + G_n(z), \quad n \in \mathbf{Z}$$

such that A_n is a diagonal 2×2 matrix, $\|A_n\|$ and $\|A_n^{-1}\|$ are uniformly bounded,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log |A_n \circ \dots \circ A_0(v)| = \lambda$$

$$v \in \mathbf{R}^2 - \{0\}$$

$$\lim_{n \rightarrow \infty} \frac{1}{-n} \log |A_n^{-1} \circ \dots \circ A_{-1}^{-1}(v)| = \lambda$$

where λ is a negative constant, G_n is C^1 , $(DG_n)_0 = 0$, and (C^1 equicontinuity of f_n)

$$\|(DG_n)_x - (DG_n)_y\| \rightarrow 0 \text{ uniformly as } |x - y| \rightarrow 0,$$

but there exist points z arbitrarily near 0 with

$$|f_n \circ \dots \circ f_0(z)| \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

Thus, 0 should be asymptotically stable under $\{f_n\}$ since $\{Tf_n\}$ contracts asymptotically, but, due to lack of smoothness of $\{f_n\}$, it is not. See Theorem 1 in § 2. (Regularity in this context is implied by the fact that A_n is diagonal.)

In § 3 the maps f_n are realized as lifts along an orbit of some C^1 diffeomorphism of a compact 4-manifold. See Theorem 2.

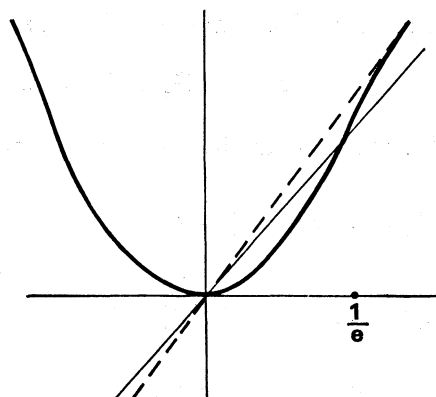
Conjecture. — If $\mathcal{O}(p)$ is an asymptotically hyperbolic orbit of the C^1 diffeomorphism $f: M \rightarrow M$, M has dimension two, and $\dim E_p^u = \dim E_p^s = 1$, then Pesin's result holds: $W^s(p)$ is C^1 and tangent at p to E_p^s . Indeed this might be true whenever E_p^s has dimension one. Regularity is automatic on one-dimensional subspaces.

Thanks. — In writing this paper, I benefitted from conversations with M. Herman and A. Fathi at the École Polytechnique in Paris. Comments by the referee were also useful.

2. Nonlinear shear. — Let $g: (0, \infty) \rightarrow (0, \infty)$ be any smooth function such that

$$(2) \quad \begin{aligned} g(u) &= \frac{u}{\log(1/u)} && \text{if } 0 < u \leq 1/e \\ g'(u) &> 1 && \text{if } u \geq 1/e \\ g'(u) &\text{ is constant} && \text{if } u \geq 1. \end{aligned}$$

Extend g to all of $\mathbf{R}$ by setting $g(-u) = g(u)$ and $g(0) = 0$. Then $g: \mathbf{R} \rightarrow (0, \infty)$ is C^1 and is C^∞ on $\mathbf{R} - \{0\}$. The graph of g is shown in Figure 1.

FIG. 1. — The graph of g

Choose constants $0 < a < ab < 1 < b$ and call

$$S = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad T = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix},$$

$$f_S = \begin{pmatrix} a & g \\ 0 & b \end{pmatrix}, \quad f_T = \begin{pmatrix} b & 0 \\ g & a \end{pmatrix}.$$

Thus, f_S and f_T are C^1 origin preserving maps $\mathbf{R}^2 \rightarrow \mathbf{R}^2$

$$f_S : z \mapsto \begin{pmatrix} ax + g(y) \\ by \end{pmatrix}, \quad f_T : z \mapsto \begin{pmatrix} bx \\ g(x) + ay \end{pmatrix}, \quad z = \begin{pmatrix} x \\ y \end{pmatrix}.$$

The linear parts of f_S, f_T at 0 are S, T and f_S, f_T are easily seen to be invertible on all of $\mathbf{R}^2$.

As $n = 1, 2, 3, \dots$ choose A_n equal to S or T according to the pattern

$$\begin{array}{cccccccccccccccc} S & T & T & S & S & S & T & T & T & T & S & S & S & S & S & \dots \\ \parallel & \parallel & \parallel & & & & & & & & & & & & \parallel & \\ A_1 & A_2 & A_3 & & & & & & & & & & & & A_{15} & \dots \end{array}$$

Thus, if L_k denotes $1 + \dots + k = k(k+1)/2$ then $0 = L_0 < L_1 < L_2 < \dots$, the sets

$$\mathcal{S} = \{n \in \mathbf{N} : L_{k-1} < n \leq L_k \text{ for some odd } k\}$$

$$\mathcal{E} = \{n \in \mathbf{N} : L_{k-1} < n \leq L_k \text{ for some even } k\}$$

decompose $\mathbf{N}$ into disjoint subsets, and

$$A_n = \begin{cases} S & n \in \mathcal{S} \\ T & n \in \mathcal{E} \end{cases}.$$

Correspondingly define

$$f_n = \begin{cases} f_S & n \in \mathcal{S} \\ f_T & n \in \mathcal{E} \end{cases}.$$

To suggest local iteration along the origin-orbit, write

$$A^n = A_{n-1} \circ \dots \circ A_0, \quad f^n = f_{n-1} \circ \dots \circ f_0, \quad n > 0,$$

where we take $A_0 = T$ and $f_0 = f_T$. To complete the picture, if $-n < 0$, set

$$\begin{aligned} A_{-n} &= A_n, & f_{-n} &= f_n, \\ A^{-n} &= A_{-n}^{-1} \circ \dots \circ A_{-1}^{-1}, & f^{-n} &= f_{-n}^{-1} \circ \dots \circ f_{-1}^{-1}, \end{aligned}$$

and call $A^0 = f^0 = \text{identity}$. Then A^n is the linear part of f^n at o , $n \in \mathbb{Z}$.

For $|n|$ large

$$A^n = \begin{pmatrix} c_n & 0 \\ 0 & d_n \end{pmatrix}$$

where c_n and d_n are products of approximately equal numbers of a 's and b 's. As $|n| \rightarrow \infty$, $c_n^{1/n}$ and $d_n^{1/n} \rightarrow (ab)^{1/2}$, so the family $\{A^n\}_{n \in \mathbb{Z}}$ has double Lyapunov exponent equal to

$$\lambda = \frac{1}{2} \log(ab) < 0.$$

Since the A_n are diagonal, regularity (see (1) in § 1) of this Lyapunov splitting is automatic. In sum, $\{A^n\}_{n \in \mathbb{Z}}$ contracts asymptotically. Note, however, that the contraction is *only* asymptotic. Along the orbit, the length of $v = (1, 0)$ expands by b , shrinks by a , grows by b at the next two points, shrinks by a at the next three points, etc.

Theorem 1. — $\{f^n\}_{n \in \mathbb{Z}}$ does not contract asymptotically although its linear part does. In fact if $z = (x, y)$ is any point in the first quadrant of $\mathbb{R}^2$, $x > 0$ and $y > 0$, then

$$f^n(z) \rightarrow \infty \quad \text{as } n \rightarrow \infty$$

although $(Df^n)_o(z) \rightarrow 0$ as $n \rightarrow \infty$.

Remarks. — a) f_S is a C^1 diffeomorphism of $\mathbb{R}^2$ onto itself leaving invariant the foliation by horizontal lines. It *shears* the y -axis onto the curve $y \mapsto g(y/b)$ and is an affine contraction of each horizontal line by the constant factor a . See Figure 2.

b) f_T is conjugate to f_S by a 90° rotation:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \circ f_S \circ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = f_T.$$

c) $\{f_n\}_{n \in \mathbb{Z}}$ is uniformly C^1 -equicontinuous.

d) The stable set of o under f^n can be shown to lie wholly in the third quadrant, $x < 0$ and $y < 0$; it seems to be a curve which converges to o in the manner of $\sin(1/x)/\log(1/x)$ as $x \rightarrow 0$. All other orbits converge to ∞ as $n \rightarrow \infty$.

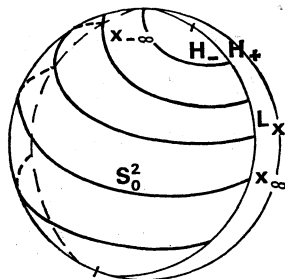


FIG. 2. — The non-linear horizontal shear f_S

Lemma 1. — If $0 < \beta_n \rightarrow 1$ as $n \rightarrow \infty$ and $b > 1$, $c > 0$ are fixed, then

$$\lim_{n \rightarrow \infty} b^{cn^2} \beta_n \beta_{n-1}^2 \dots \beta_1^n \beta_0^{n+1} = \infty.$$

Proof. — Fix s, l such that

$$\begin{aligned} n > s &\Rightarrow \beta_n \geq b^{-c} \\ \beta_0, \dots, \beta_s &\geq b^{-l}. \end{aligned}$$

Then

$$\begin{aligned} b^{cn^2} \beta_n \dots \beta_0^{n+1} &= b^{cn^2} \beta_n \dots \beta_{s+1}^{n-s} \beta_s^{n-s+1} \dots \beta_0^{n+1} \\ &\geq b^{cn^2} (b^{-c})^{1+\dots+(n-s)} (b^{-l})^{(n+1)(s+1)} \\ &> b^{(cn^2/2) - l(s+1)(n+1)} \rightarrow \infty \end{aligned}$$

as $n \rightarrow \infty$.

Q.E.D.

For any function $f(u)$ that vanishes at $u = 0$ and is defined for $u \geq 0$, let

$$\sigma_f(u) = \frac{f(u)}{u}.$$

$\sigma_f(u)$ is the *shrinking factor* of f at u : under f , the distance from u to o is shrunk by the factor $\sigma_f(u)$.

Lemma 2. — Let $g(u) = u/\log(1/u)$ as above, $0 < u < 1/e$. Call $\sigma = \sigma_g$. Then

$$\frac{\sigma(gu)}{\sigma(u)} \rightarrow 1 \quad \text{as } u \rightarrow 0.$$

Proof.

$$\frac{\sigma(gu)}{\sigma(u)} = \frac{\log(u)}{\log(gu)} = \frac{\log(u)}{\log u - \log |\log u|} = \frac{1}{1 + \frac{\log |\log u|}{|\log u|}} \rightarrow 1$$

as $u \rightarrow 0$.

Q.E.D.

Remark. — Since $g'(0) = 0$, $\sigma(gu)$ and $\sigma(u)$ both tend to 0 with u . It is somewhat remarkable that they do so at the same order because gu is *far* closer to 0 than is u . If g were $C^{1+\alpha}$, $\alpha > 0$, the lemma would fail. For example if $\tilde{g} = u^{1+\alpha}$ and $\tilde{\sigma}$ is the shrinking factor for $\tilde{g}$, then

$$\frac{\tilde{\sigma}(\tilde{g}u)}{\tilde{\sigma}(u)} = u^{\alpha} \rightarrow 0 \quad \text{as } u \rightarrow 0.$$

Likewise, in the next lemma, g must be $u/\log(1/u)$ or some similar function which grows too fast near 0 to be $C^{1+\alpha}$, $\alpha > 0$.

Lemma 3. — Let g be as in Figure 1 and (2). Fix $u_0 > 0$, $b > 1$. Set

$$u_1 = bg(u_0), \dots, u_k = b^k g(u_{k-1}).$$

Then $\lim_{k \rightarrow \infty} u_k = \infty$.

Proof. — If $u_0 > 1/e$ then $u_1 = bg(u_0) \geq b/e > 1/e$, and, by induction,

$$u_k = b^k g(u_{k-1}) \geq b^k/e \rightarrow \infty \quad \text{as } k \rightarrow \infty,$$

so we may suppose $0 < u_0 < 1/e$. On $(0, 1/e)$,

$$\sigma'(u) = \frac{1}{u(\log(1/u))^2} > 0,$$

so σ is monotone there. Clearly g and $g^k = g \circ \dots \circ g$, k times, are monotone also. Thus,

$$\begin{aligned} g^k(u_0) &= g^{k-1}(gu_0) < g^{k-1}(bg u_0) = g^{k-1}(u_1) \\ &= g^{k-2}(gu_1) < g^{k-2}(b^2 g u_1) = g^{k-2}(u_2) \\ &= \dots < u_k \end{aligned}$$

and

$$\begin{aligned} u_{k+1} &= b^{k+1} g(u_k) = b^{k+1} \sigma(u_k) u_k = b^{k+1} \sigma(u_k) b^k \sigma(u_{k-1}) u_{k-1} \\ &= \dots = b^{(k+2)(k+1)/2} \sigma(u_k) \dots \sigma(u_0) \\ &> b^{(k+2)(k+1)/2} \sigma(g^k u_0) \sigma(g^{k-1} u_0) \dots \sigma(u_0) \\ &> b^{k^2/2} \left(\frac{\sigma(g^k u_0)}{\sigma(g^{k-1} u_0)} \right) \left(\frac{\sigma(g^{k-1} u_0)}{\sigma(g^{k-2} u_0)} \right)^2 \dots \left(\frac{\sigma(g u_0)}{\sigma(u_0)} \right)^k (\sigma(u_0))^{k+1}. \end{aligned}$$

By Lemma 2, all the factors

$$\frac{\sigma(g^n u_0)}{\sigma(g^{n-1} u_0)} \rightarrow 1$$

as $n \rightarrow \infty$, so by Lemma 1, $u_{k+1} \geq 1/e$ for some first $k+1$. Beyond this $k+1$, $u_m \geq b^{m-(k+1)}/e \rightarrow \infty$ as $m \rightarrow \infty$, as observed at the outset. Q.E.D.

Proof of Theorem 1. — Recall that $0 < a < ab < 1 < b$ and

$$S = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad T = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}, \quad f_S = \begin{pmatrix} a & g \\ 0 & b \end{pmatrix}, \quad f_T = \begin{pmatrix} b & 0 \\ g & a \end{pmatrix},$$

$$A_n = \begin{cases} S & \text{if } n \in \mathcal{S} \\ T & \text{if } n \in \mathcal{E} \end{cases}, \quad A^n = A_{n-1} \circ \dots \circ A_0,$$

$$f_n = \begin{cases} f_S & \text{if } n \in \mathcal{S} \\ f_T & \text{if } n \in \mathcal{E} \end{cases}, \quad f^n = f_{n-1} \circ \dots \circ f_0,$$

where

$$\mathcal{S} = \{n \in \mathbf{N} : 1 + \dots + (k-1) < n \leq 1 + \dots + k \text{ for some even } k \in \mathbf{N}\},$$

$$\mathcal{E} = \mathbf{N} - \mathcal{S},$$

and g is the function in Figure 1 or (2). Write

$$f_n(z) = A_n z + G_n(z),$$

where $G_n: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ is given by

$$G_n(z) = \begin{cases} \begin{pmatrix} gy \\ 0 \end{pmatrix} & \text{if } n \in \mathcal{S} \\ \begin{pmatrix} 0 \\ gx \end{pmatrix} & \text{if } n \in \mathcal{E} \end{cases}, \quad z = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Let π_1 and π_2 be the projections onto the x -axis and y -axis. Identify each axis with $\mathbf{R}$. Then

$$\pi_1 \circ G_n = \begin{cases} g \circ \pi_2 & \text{if } n \in \mathcal{S}, \\ 0 & \text{if } n \in \mathcal{E}, \end{cases} \quad \pi_2 \circ G_n = \begin{cases} 0 & \text{if } n \in \mathcal{S}, \\ g \circ \pi_1 & \text{if } n \in \mathcal{E}. \end{cases}$$

Now fix some $z = (x, y)$ with $x > 0$ and $y > 0$. We must prove

$$(3) \quad f^n(z) \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

By construction, f^n carries the first quadrant into itself, so all quantities in the following estimates are positive. Set

$$s_0 = \pi_2(f^1 z) = \pi_2(f_0 z),$$

$$s_1 = \pi_1(f^3 z),$$

$$s_2 = \pi_2(f^6 z),$$

$$\vdots$$

$$s_k = \pi_i(f^m z), \quad m = 1 + \dots + (k+1), \quad i = \begin{cases} 1 & k \text{ is odd} \\ 2 & k \text{ is even} \end{cases},$$

$$\vdots$$

We claim

$$(4) \quad s_k \geq b^k g(s_{k-1}) \quad k \geq 1.$$

Suppose k is odd, for instance $k = 1$. Then

$$\left. \begin{array}{l} A_n = T \\ \pi_1 \circ G_n = 0 \\ \pi_2 \circ G_n = g \circ \pi_1 \end{array} \right\} \quad \text{for } m - (k + 1) + 1 \leq n \leq m,$$

$$\left. \begin{array}{l} A_n = S \\ \pi_1 \circ G_n = g \circ \pi_2 \\ \pi_2 \circ G_n = 0 \end{array} \right\} \quad \text{if } n = m - (k + 1).$$

Thus

$$\begin{aligned} s_k &= \pi_1(f^m z) = \pi_1(f_{m-1}(f^{m-1} z)) = \pi_1(T(f^{m-1} z) + G_{m-1}(f^{m-1} z)) \\ &= b\pi_1(f^{m-1} z) = \dots = b^{k-1}\pi_1(f^{m-k+1} z) \\ &= b^{k-1}\pi_1(f_{m-k}(f^{m-k} z)) = b^{k-1}\pi_1(T(f^{m-k} z) + G_{m-k}(f^{m-k} z)) \\ &= b^k\pi_1(f^{m-k} z) = b^k\pi_1(f_{m-k-1}(f^{m-k-1} z)) \\ &= b^k\pi_1(S(f^{m-k-1} z) + G_{m-k-1}(f^{m-k-1} z)) \\ &= b^k a\pi_1(f^{m-k-1} z) + b^k g(\pi_2(f^{m-k-1} z)) \\ &> b^k g(\pi_2(f^{m-k-1} z)) = b^k g(s_{k-1}), \end{aligned}$$

since $(1 + \dots + k) = m - k - 1$. Similarly, if k is even, $k \geq 2$, $m = 1 + \dots + (k + 1)$, then

$$\left. \begin{array}{l} A_n = S \\ \pi_1 \circ G_n = g \circ \pi_2 \\ \pi_2 \circ G_n = 0 \end{array} \right\} \quad \text{for } m - (k + 1) + 1 \leq n \leq m,$$

$$\left. \begin{array}{l} A_n = T \\ \pi_1 \circ G_n = 0 \\ \pi_2 \circ G_n = g \circ \pi_1 \end{array} \right\} \quad \text{if } n = m - (k + 1)$$

and

$$\begin{aligned} s_k &= \pi_2(f^m z) = \dots = b^k\pi_2(f_{m-k-1}(f^{m-k-1} z)) \\ &= b^k\pi_2(T(f^{m-k-1} z) + G_{m-k-1}(f^{m-k-1} z)) \\ &= b^k a\pi_2(f^{m-k-1} z) + b^k g(\pi_1(f^{m-k-1} z)) > b^k g(s_{k-1}), \end{aligned}$$

which proves (4).

Call $u_0 = s_0$ and $u_k = b^k g(u_{k-1})$, $k \geq 1$. By (4), induction, and monotonicity of g ,

$$(5) \quad s_k \geq u_k \quad \text{if } k \geq 0,$$

for $s_k \geq b^k g(s_{k-1}) \geq b^k(u_{k-1}) = u_k$. By Lemma 3, $u_k \rightarrow \infty$ as $k \rightarrow \infty$. From (5)

$$(6) \quad s_k \rightarrow \infty \quad \text{as } k \rightarrow \infty.$$

This proves (3) for a certain subsequences of $n \rightarrow \infty$, namely for $n = m$ with m of the form $1 + \dots + (k + 1)$.

To handle general n , observe that

$$(7) \quad \pi_1(f^{m+1} z) \text{ and } \pi_2(f^{m+1} z) \text{ both tend to } \infty \text{ as } m = 1 + \dots + (k + 1) \rightarrow \infty.$$

For if k is odd then

$$\begin{aligned} \pi_1(f^{m+1} z) &= \pi_1(T(f^m z) + G_m(f^m z)) \\ &= b\pi_1(f^m z) = bs_k \rightarrow \infty \quad \text{as } k \rightarrow \infty, \\ \pi_2(f^{m+1} z) &= \pi_2(T(f^m z) + G_m(f^m z)) \\ &= a\pi_2(f^m z) + g(\pi_1(f^m z)) > g(s_k) \rightarrow \infty \quad \text{as } k \rightarrow \infty. \end{aligned}$$

Similarly, if k is even. Now the $k + 1$ iterates f_n , $m < n < m + (k + 2)$, increase one of these coordinates, $\pi_1(f^{m+1} z)$ or $\pi_2(f^{m+1} z)$, by powers of b , so, for these n ,

$$\max(\pi_1(f^{n+1} z), \pi_2(f^{n+1} z)) > \min(\pi_1(f^{m+1} z), \pi_2(f^{m+1} z)).$$

But this means, by (7), that

$$|f^{n+1} z| \geq \min(\pi_1(f^{m+1} z), \pi_2(f^{m+1} z)) \rightarrow \infty \quad \text{as } k \rightarrow \infty,$$

for $1 + \dots + (k + 1) \leq n < 1 + \dots + (k + 2)$, which completes the proof of (3) and Theorem 1.

Remark. — The strategy of $\{f_n\}$ is to expand one component of z k times by b and then transfer as much as possible of this expanded component to the opposite component for the next $k + 1$ iterates. The non-smoothness of f_n permits just enough transfer.

3. Realizing the Example. — Consider the diffeomorphisms $f_n: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ as in § 2. We want to lift them to the 2-sphere by central projection, that is, by projection from the center of a unit 2-sphere whose south pole rests at the origin of $\mathbf{R}^2$. For polynomial vector fields this is a standard construction due to Poincaré [2]. Let

$$r = \frac{R}{1 + R^2}$$

$$R = \tan(\varphi)$$

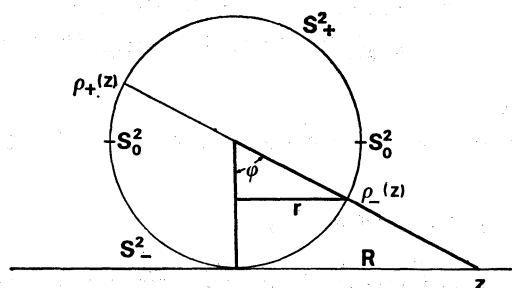


FIG. 3. — Central Projection

$\rho_- : \mathbf{R}^2 \rightarrow S_-^2$ be this projection where S_-^2 is the southern hemisphere. Let α be the antipodal map of S^2 and define $\rho_+ : \mathbf{R}^2 \rightarrow S_+^2$, the central projection to the northern hemisphere, S_+^2 , by $\rho_+ = \alpha \circ \rho_-$. See Figure 3. (Stereographic projection, by the way, is unsuitable for such lifting.)

Any map $f : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ lifts to a map $\rho_- f \cup \rho_+ f : S^2 - S_0^2 \rightarrow S^2 - S_0^2$ making

$$\begin{array}{ccc} S_{\pm}^2 & \xrightarrow{\rho_{\pm} f} & S_{\pm}^2 \\ \downarrow \rho_{\pm} & & \downarrow \rho_{\pm} \\ \mathbf{R}^2 & \xrightarrow{f} & \mathbf{R}^2 \end{array}$$

commute. (S_0^2 is the equator of S^2 .) The next lemma gives sufficient conditions that $\rho_- f \cup \rho_+ f$ extend to a map $\rho_{\#} f$ on all of S^2 . Note first, however, that any linear (or affine) isomorphism $A : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ lifts to a diffeomorphism $\rho_{\#} A : S^2 \rightarrow S^2$. See [2] or below.

Lemma 4. — Suppose $A = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$, $ab \neq 0$, and $h : \mathbf{R} \rightarrow \mathbf{R}$ is a C^1 function with compact support. Then the map

$$f = \begin{pmatrix} a & c+h \\ 0 & b \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ax + cy + h(y) \\ by \end{pmatrix}$$

lifts to a unique continuous map $\rho_{\#} f : S^2 \rightarrow S^2$ which agrees with $\rho_{\pm} f$ on S^2 . Moreover, $\rho_{\#} f$ is a C^1 diffeomorphism whose 1-jet at the equator S_0^2 is the same as that of $\rho_{\#} A$. At the points $x_{\pm\infty}$ where the x -axis-longitude L_x meets S_0^2 , this 1-jet is independent of c ; it is

$$x_{\pm\infty} \mapsto \left(x_{\pm\infty}, \begin{pmatrix} 1/a & 0 \\ 0 & b/a \end{pmatrix} \right)$$

respecting the splitting $T_{x_{\pm\infty}}(S^2) = T_{x_{\pm\infty}}(L_x) \oplus T_{x_{\pm\infty}}(S_0^2)$.

Proof. — This is basically a chain rule calculation. Let (φ, θ) be the natural angular coordinates on S_-^2 and let (r, θ) be the polar coordinates in $\mathbf{R}^2$. Then

$$\begin{aligned} r &= \tan \varphi, & \theta &= \theta, & \rho^{-1}(\varphi, \theta) &= (\tan \varphi, \theta), \\ x &= r \cos \theta, & f_1 &= ax + cy + h(y), \\ y &= r \sin \theta, & f_2 &= by. \end{aligned}$$

Express $\rho_- f : S_-^2 \rightarrow S_-^2$ in the (φ, θ) -coordinates as

$$\begin{aligned} (\varphi, \theta) &\mapsto (\Phi, \Theta) \\ \Phi &= \tan^{-1}(R), & R &= |f| = (f_1^2 + f_2^2)^{1/2}, \\ \Theta &= \tan^{-1}(f_2/f_1). \end{aligned}$$

Since $R = |f|$ we have

$$\begin{aligned} \frac{R^2}{r^2} &= a^2 \cos^2 \theta + c^2 \sin^2 \theta + \frac{(h(r \sin \theta))^2}{r^2} + 2ac \cos \theta \sin \theta \\ &\quad + 2a \cos \theta \frac{h(r \sin \theta)}{r} + 2c \sin \theta \frac{h(r \sin \theta)}{r} + b^2 \sin^2 \theta. \end{aligned}$$

Let $\varphi \rightarrow \pi/2$. Then $r = \tan \varphi \rightarrow \infty$ and

$$(8) \quad \frac{R^2}{r^2} \rightrightarrows a^2 \cos^2 \theta + c^2 \sin^2 \theta + 2ac \cos \theta \sin \theta + b^2 \sin^2 \theta > 0.$$

By $\rightrightarrows$ we denote uniform convergence respecting θ . From (8) follows

$$(9) \quad \Phi \rightrightarrows \pi/2 \quad \text{as } \varphi \rightarrow \pi/2.$$

Similarly

$$\begin{aligned} (10) \quad \Theta(\varphi, \theta) &= \tan^{-1}(f_2/f_1) = \tan^{-1} \left(\frac{b \sin \theta}{a \cos \theta + c \sin \theta + \frac{1}{r} h(r \sin \theta)} \right) \\ &\rightrightarrows \tan^{-1} \left(\frac{b \sin \theta}{a \cos \theta + c \sin \theta} \right). \end{aligned}$$

Some care is needed here since $a \cos \theta + c \sin \theta$ can equal 0. Fix some small $\theta_0 > 0$ and let $N = \{\theta : 0 \leq \theta \leq \theta_0, \text{ or } \pi - \theta_0 \leq \theta \leq \pi + \theta_0, \text{ or } 2\pi - \theta_0 \leq \theta < 2\pi\}$. If θ_0 is small and $\theta \in N$ then the argument of $\tan^{-1}$ converges uniformly and (10) is immediate. If $\theta \notin N$ and $z = (r, \theta)$, $r = \tan \varphi \rightarrow \infty$, then

$$f(z) = \begin{pmatrix} ax + cy \\ by \end{pmatrix}$$

since $h(y) = h(r \sin \theta)$, h has compact support, and $r \sin \theta \rightarrow \infty$. Since Θ refers to the angle made by $f(z)$, $\Theta(\varphi, \theta)$ converges uniformly for $\theta \notin N$ also, proving (10).

From (9) and (10) we see that $\rho_- f: S_-^2 \rightarrow S_-^2$ extends to a continuous map on $S_0^2 \cup S_-^2$, sending the equator into itself according to

$$(11) \quad \theta \mapsto \tan^{-1} \left(\frac{b \sin \theta}{a \cos \theta + c \sin \theta} \right).$$

Note that (11) changes by π if θ is replaced by $\theta + \pi$. Thus, $\rho_+ f$ extends to the same map on the equator; *i.e.* f lifts to a (necessarily unique) continuous map $\rho_* f: S^2 \rightarrow S^2$ agreeing with $\rho_{\pm} f$ on S^2 . It is easily seen to be a homeomorphism which is a diffeomorphism except perhaps at the equator.

To calculate the derivatives of $\rho_{\#}f$ we compute

$$\begin{aligned}\frac{\partial \Phi}{\partial \varphi} &= \left(\frac{1}{1+R^2} \right) \left(\frac{1}{R} \right) \left\{ f_1 \left(\frac{\partial f_1}{\partial x} \frac{\partial x}{\partial r} \frac{\partial r}{\partial \varphi} + \frac{\partial f_1}{\partial y} \frac{\partial y}{\partial r} \frac{\partial r}{\partial \varphi} \right) + f_2 \left(\frac{\partial f_2}{\partial x} \frac{\partial x}{\partial r} \frac{\partial r}{\partial \varphi} + \frac{\partial f_2}{\partial y} \frac{\partial y}{\partial r} \frac{\partial r}{\partial \varphi} \right) \right\} \\ &= \left(\frac{1+r^2}{1+R^2} \right) \left(\frac{r}{R} \right) \left\{ \left(a \cos \theta + c \sin \theta + \frac{1}{r} h(r \sin \theta) \right) \right. \\ &\quad \left. (a \cos \theta + (h'(r \sin \theta) + c) \sin \theta) + b^2 \sin^2 \theta \right\}.\end{aligned}$$

By (8), the first two factors converge uniformly. As above, the terms $h(r \sin \theta)/r$ and $h'(r \sin \theta) \sin \theta$ go to 0 when $r \rightarrow \infty$. Thus

$$\begin{aligned}(12) \quad \frac{\partial \Phi}{\partial \varphi} &\Rightarrow \lim_{\varphi \rightarrow \frac{\pi}{2}} \left(\frac{r}{R} \right)^3 \{ a^2 \cos^2 \theta + 2ac \cos \theta \sin \theta + c^2 \sin^2 \theta + b^2 \sin^2 \theta \} \\ &= \lim_{\varphi \rightarrow \frac{\pi}{2}} \left(\frac{r}{R} \right).\end{aligned}$$

Second,

$$\begin{aligned}\frac{\partial \Phi}{\partial \theta} &= \left(\frac{1}{R^2+1} \right) \left(\frac{1}{R} \right) \left\{ f_1 \left(\frac{\partial f_1}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f_1}{\partial y} \frac{\partial y}{\partial \theta} \right) + f_2 \left(\frac{\partial f_2}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f_2}{\partial y} \frac{\partial y}{\partial \theta} \right) \right\} \\ &= \left(\frac{r}{R^2+1} \right) \left(\frac{r}{R} \right) \left\{ \left(a \cos \theta + c \sin \theta + \frac{1}{r} h(r \sin \theta) \right) \right. \\ &\quad \left. (-a \sin \theta + (c + h'(r \sin \theta)) \cos \theta) + b^2 \sin \theta \cos \theta \right\}.\end{aligned}$$

As $r \rightarrow \infty$, $r/(R^2+1) \rightarrow 0$ while the other factors approach finite limits. Hence

$$(13) \quad \frac{\partial \Phi}{\partial \theta} \Rightarrow 0 \quad \text{as } \varphi \rightarrow \frac{\pi}{2}.$$

Third,

$$\begin{aligned}\frac{\partial \Theta}{\partial \varphi} &= \frac{1}{1+(f_2/f_1)^2} \left\{ \frac{\partial(f_2/f_1)}{\partial x} \frac{\partial x}{\partial r} \frac{\partial r}{\partial \varphi} + \frac{\partial(f_2/f_1)}{\partial y} \frac{\partial y}{\partial r} \frac{\partial r}{\partial \varphi} \right\} \\ &= \frac{f_1^2}{R^2} \left\{ \frac{-aby \cos \theta}{f_1^2 \cos^2 \varphi} + \frac{abx + bh(y) - byh'(y) \sin \theta}{f_1^2 \cos^2 \varphi} \right\} \\ &= \frac{1+r^2}{R^2} \{ -abr \sin \theta \cos \theta + abr \cos \theta \sin \theta + (bh(y) - byh'(y)) \sin \theta \} \\ &= \frac{1+r^2}{R^2} \{ bh(r \sin \theta) \sin \theta - br \sin \theta h'(r \sin \theta) \sin \theta \}.\end{aligned}$$

Now as $\varphi \rightarrow \frac{\pi}{2}$, either h and h' equal 0 or else $r \sin \theta$ stays bounded: $\sin \theta \rightarrow 0$ while $r \rightarrow \infty$. Thus, the bracketed terms go to 0 while $(1 + r^2)/R^2$ tends to $(\lim r/R)^2 \neq \infty$. Therefore

$$(14) \quad \frac{\partial \Theta}{\partial \varphi} \rightarrow 0 \quad \text{as } \varphi \rightarrow \frac{\pi}{2}.$$

Finally,

$$\begin{aligned} \frac{\partial \Theta}{\partial \theta} &= \frac{1}{1 + (f_2/f_1)^2} \left\{ \frac{\partial f_2/f_1}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f_2/f_1}{\partial y} \frac{\partial y}{\partial \theta} \right\} \\ &= \left(\frac{r}{R} \right)^2 \left\{ ab \sin^2 \theta + ab \cos^2 \theta \right. \\ &\quad \left. + \left(\frac{1}{r} bh(r \sin \theta) - bh'(r \sin \theta) \sin \theta \right) \cos \theta \right\}. \end{aligned}$$

As above, the terms involving h and h' tend to 0 as $\varphi \rightarrow \pi/2$. Thus

$$(15) \quad \frac{\partial \Theta}{\partial \theta} \rightarrow \left(\lim \frac{r}{R} \right)^2 ab \quad \text{as } \varphi \rightarrow \pi/2.$$

The limits (12)-(15) commute with the antipodal map, so $\rho_{\#}f$ is C^1 ; at $(\pi/2, \theta) \in S_0^2$ it has derivative

$$\begin{bmatrix} \gamma & 0 \\ 0 & ab\gamma^2 \end{bmatrix}, \quad \gamma = ((a \cos \theta + c \sin \theta)^2 + (b \sin \theta)^2)^{-1/2},$$

respecting the (φ, θ) -coordinates. This is clearly invertible and independent of h . Hence $\rho_{\#}f$ is a C^1 diffeomorphism whose 1-jet agrees with that of $\rho_{\#}A$ at the equator. The points $x_{-\infty}$, $x_{+\infty}$ correspond to $\theta = \pi$, $\theta = 0$ and give $\gamma = a^{-1}$, verifying the fact that the derivative of $\rho_{\#}f$ at $x_{\pm\infty}$ is independent of c . Q.E.D.

Now return to the proof of Theorem 1. Since f_s does not satisfy the hypotheses of Lemma 4, it is convenient to introduce the odd version of the function g in § 2,

$$g_o(u) = \begin{cases} g(u) & \text{if } u \geq 0, \\ -g(u) & \text{if } u \leq 0. \end{cases}$$

Then $g'_o(u) \equiv c$ for some constant $c > 1$, provided $|u| \geq 1$. Call

$$A_{\pm} = \begin{pmatrix} a & \pm c \\ 0 & b \end{pmatrix}, \quad f_{\pm} = \left(A_{\pm} \pm \begin{pmatrix} 0 & g_o - c \\ 0 & 0 \end{pmatrix} \right) : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ax \pm g_o(y) \\ by \end{pmatrix}.$$

Clearly

$$f_{\pm}(x, y) = f_s(x, y) \quad \text{for } \pm y \geq 0.$$

Since $g_o(y) - cy$ has compact support, $f_{\pm}$ satisfies the hypotheses of Lemma 4 and lifts to S^2 as $\rho_{\#}(f_{\pm})$. At the equator, the 1-jet of $\rho_{\#}(f_{\pm})$ agrees with that of $\rho_{\#}(A_{\pm})$.

Divide S^2 into two hemispheres $H_{\pm}$ along the x -axis longitude L_x , say

$H_{\pm}$ is the hemisphere containing the quarter sphere $\rho_{-}\{(x, y) \in \mathbf{R}^2 : \pm y > 0\}$.

See Figure 4.

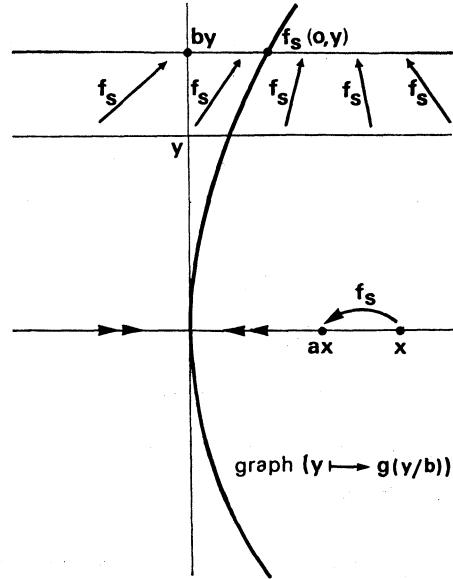


FIG. 4. — The hemispheres $H_{\pm}$ with half latitudes drawn in H_{-} .

Define $F_S : S^2 \rightarrow S^2$ by

$$F_S(z) = \begin{cases} \rho_{\#}f_+(z) & \text{if } z \in H_+, \\ \rho_{\#}f_-(z) & \text{if } z \in H_-. \end{cases}$$

Then F_S lifts f_S to S^2 , but *not* as $\rho_-f_S \cup \rho_+f_S$! In fact this canonical lift $\rho_{\#}f_S$ fails to be C^1 at the equator.

At all finite points of the x -axis, $f_+ - f_-$ vanishes to first order (since $g(0) = g'(0) = 0$) so F_S is well defined and continuous on S^2 ; in fact

$$T_z F_S = T_z(\rho_{\#}f_+) = T_z(\rho_{\#}f_-)$$

for all $z \in L_x \cap S^2_-$. Thus, $F_S|_{S^2_-}$ is a C^1 diffeomorphism of S^2_- .

Although F_S does not commute with the antipodal map α , there is enough symmetry that differentiability of F_S on S^2_- implies it on S^2_+ . If $z \in S^2_+ \cap H_+$ then

$$\begin{aligned} F_S(z) &= \rho_{\#}f_+(z) = \alpha \circ \rho_-f_+ \circ \alpha(z), \\ T_z F_S &= (T\alpha)_{\rho_-(\alpha z)} \circ T_{\alpha z}(\rho_-f_+) \circ T_z \alpha, \end{aligned}$$

while if $z' \in S_+^2 \cap H_-$ then

$$\begin{aligned} F_8(z') &= \rho_{\#} f_-(z') = \alpha \circ \rho_- f_- \circ \alpha(z'), \\ T_{z'} F_8 &= (T\alpha)_{\rho_- f_-(\alpha z')} \circ T_{\alpha z'}(\rho_- f_-) \circ T_{z'} \alpha. \end{aligned}$$

Now if $z = z' \in L_x \cap S_+^2$ then

$$\begin{aligned} \rho_- f_+(\alpha z) &= \rho_- f_-(\alpha z'), \\ T_{\alpha z}(\rho_- f_+) &= T_{\alpha z'}(\rho_- f_-), \end{aligned}$$

since $\rho_- f_+$ equals $\rho_- f_-$ to first order along $L_x \cap S_-^2$. Thus, $F_8|S_+^2$ is a well defined C^1 diffeomorphism of S_+^2 also.

Since F_8 is the C^1 diffeomorphism $\rho_{\#} f_{\pm}$ on the interior of $H_{\pm}$, it remains only to check F_8 at $S_0^2 \cap \partial H_{\pm} = x_{\pm\infty}$. But by Lemma 4, $\rho_{\#} f_{\pm}$ has at the equator a 1-jet equal to that of the diffeomorphism $\rho_{\#} A_{\pm}$ and at $x_{\pm\infty}$ the latter 1-jet does not depend on c . That is,

$$\begin{aligned} T_{x_{\pm\infty}}(\rho_{\#} f_+) &= T_{x_{\pm\infty}}(\rho_{\#} A_+), \\ T_{x_{\pm\infty}}(\rho_{\#} f_-) &= T_{x_{\pm\infty}}(\rho_{\#} A_-), \end{aligned}$$

and so $T_{x_{\pm\infty}} F_8$ exists and is invertible. Hence

$$(16) \quad \begin{aligned} f_8 \text{ lifts to a (somewhat noncanonical) } C^1 \text{ diffeomorphism } F_8 \text{ of } S^2; \\ \text{similarly } f_T \text{ lifts to } F_T. \end{aligned}$$

Remarks. — It is because the dynamics of the sequence $\{f_n\}$ is sensitive to perturbations at infinity that we took pains to lift the global map f_n to S^2 , not just its germ near 0.

We are now ready to embed the example in § 2 into a diffeomorphism of a compact manifold.

Let $h: M^2 \rightarrow M^2$ be any diffeomorphism having a hyperbolic invariant set H on which h is topologically conjugate to the full 2-shift and

$$(17) \quad T_H h \text{ dominates } TF_8.$$

By (17) we mean that if $E^{uu} \oplus E^{ss} = T_H M^2$ is the hyperbolic splitting then

$$\begin{aligned} |Th(v)| &> |TF_8(u)| \quad \text{whenever } v \in E^{uu}, \quad |v| = 1, \\ &u \in TS^2, \quad |u| = 1, \\ |Th(v)| &< |TF_8(u)| \quad \text{whenever } v \in E^{ss}, \quad |v| = 1, \\ &u \in TS^2, \quad |u| = 1. \end{aligned}$$

That is, the spectrum of $T_H h$ lies outside the annular hull of the spectrum of TF_8 . We could, for instance, take H to be a horse-shoe basic set.

Let H_0 be the set of points of H which correspond to symbol sequences with a 0 in the initial position and H_1 be those with a 1 in the initial position. Then

$$H = H_0 \sqcup H_1$$

and H_0, H_1 are compact. Choose a smooth bump function

$$\mu: M^2 \rightarrow [0, \pi/2]$$

such that $H_0 = \mu^{-1}(0) \cap H$, $H_1 = \mu^{-1}(\pi/2) \cap H$, and $\mu^{-1}(\{0, \pi/2\})$ is a neighborhood of H .

Let R_θ be the rotation of S^2 by angle θ which fixes the poles. Form the skew product of h and F_S

$$F: M^2 \times S^2 \rightarrow M^2 \times S^2$$

$$(w, z) \mapsto (h(w), R_{\mu(w)} \circ F_S \circ R_{-\mu(w)}(z)).$$

F leaves invariant the foliation $\mathcal{S}$ by 2-spheres $w \times S^2$, $w \in M^2$, and by (17), F is normally hyperbolic to $\mathcal{S}$. See [3, p. 116]. Besides

$$(18) \quad T_{(w, z)} F = \begin{bmatrix} \frac{\partial h}{\partial w} & \frac{\partial R_\mu \circ F_S \circ R_{-\mu}}{\partial w} \\ 0 & \frac{\partial R_\mu \circ F_S \circ R_{-\mu}}{\partial z} \end{bmatrix}.$$

Since $h|_H$ is the 2-shift, there is a (unique) orbit $\mathcal{O}(p)$ in H such that

$$h^n(p) \in H_0 \quad \text{iff } A_n = S, \quad h^n(p) \in H_1 \quad \text{iff } A_n = T.$$

That is, we consider the orbit $\mathcal{O}(p)$ whose symbol is

$$000110.10110001111 \dots$$

respecting the division $H = H_0 \sqcup H_1$. Let

$$P = (p, z_0)$$

where z_0 is the south pole of S^2 . The F -orbit of P is $\{(h^n p, z_0)\}$ since z_0 is fixed under F_S and R_θ . By (16), (18), and constancy of μ near H ,

$$(19) \quad T_{(h^n p, z_0)} F = \begin{bmatrix} T_{h^n p} h & 0 \\ 0 & A_n \end{bmatrix}.$$

Indeed by choice of μ and the fact that

$$f_T = R_{\pi/2} \circ f_S \circ R_{-\pi/2}$$

one sees that

$$(20) \quad \begin{array}{ccc} h^n p \times S^2 & \xrightarrow{F} & h^{n+1} p \times S^2 \\ \downarrow & & \downarrow \\ S^2 & \xrightarrow{F_n} & S^2 \end{array}$$

commutes, where $F_n = F_S$ or $F_n = F_T$ according as $A_n = S$ or $A_n = T$.

From (19) $\mathcal{O}(P)$ has one positive Lyapunov exponent corresponding to $Th|E^{uu}$ and three negative Lyapunov exponents: one corresponding to $Th|E^{ss}$ and the other two being $\lambda = \frac{1}{2} \log(ab)$ which correspond to the A_n 's as in § 2. Let E^s denote the space of vectors with negative Lyapunov exponents

$$E_p^s \supset T(p \times S^2).$$

The orbit $\mathcal{O}(P)$ is regular because $TF^n|E_p^s$ is diagonal respecting $E^{ss} \oplus (x\text{-axis}) \oplus (y\text{-axis})$.

Theorem 2. — *The stable set of P is not an immersed manifold tangent to E_p^s .*

Proof. — Since F is normally hyperbolic to $\mathcal{S}$, a point (w, z) is asymptotic with P under F if and only if (w, z) lies on the strong stable manifold of some point

$$(p, z') \in W^s(P) \cap (p \times S^2).$$

That is,

$$W^s(P) = W^{ss}(W^s(P) \cap (p \times S^2))$$

where W^{ss} denotes the strong stable manifolds. See [3, p. 71]. But, by (20), $W^s(P) \cap (p \times S^2)$ is just the stable set of o under the maps f^n as in § 2 and this set is not a neighborhood of o ; it misses the whole first quadrant. Thus, $W^s(P)$ is contained in the three dimensional set $W^{ss}(p \times S^2)$ but does not include a neighborhood of P in it. It is therefore not able to be an immersed manifold tangent to E_p^s . Q.E.D.

Remarks and Questions. — *a)* More can probably be proved about $W^s(P)$. It seems to consist of the W^{ss} fibers over a curve tending to P in $p \times S^2$ in an oscillatory fashion. In particular, it seems to have dimension two.

b) Can the dimension of M be reduced from 4 to 3 in the above example by the introduction of a solenoid?

c) Do C^1 diffeomorphisms of 2-manifolds have C^1 stable manifolds at asymptotically hyperbolic orbits?

d) Which orbits in H (*i.e.* which symbol sequences) exhibit this anti-Pesin behavior? Do they form a residual set in H ? A set of measure zero for every h -invariant probability measure on H ?

e) Is the set of points where the stable set of f is a C^1 injectively immersed manifold a set of measure one for every f -invariant probability measure on M ?

f) Does the *generic* C^1 diffeomorphism (for example, one near the F above) have C^1 stable manifolds at its *generic* asymptotically hyperbolic orbits?

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