PUBLICATIONS MATHÉMATIQUES DE L'I.H.É.S.

PIERRE VOGEL

On the obstruction group in homology surgery

Publications mathématiques de l'I.H.É.S., tome 55 (1982), p. 165-206

<http://www.numdam.org/item?id=PMIHES_1982__55__165_0>

© Publications mathématiques de l'I.H.É.S., 1982, tous droits réservés.

L'accès aux archives de la revue « Publications mathématiques de l'I.H.É.S. » (http://www.ihes.fr/IHES/Publications/Publications.html) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

ON THE OBSTRUCTION GROUP IN HOMOLOGY SURGERY

by PIERRE VOGEL

0. Introduction

The theory of homology surgery has been introduced by Cappell and Shaneson [1]. This theory plays an important role in the theory of knots and codimension 2 embeddings.

Let $(X, \partial X)$ be a pair of finite complexes and f be a normal map from the normal bundle of a (Top, PL or Diff)-manifold V to a (Top, PL or Diff)-bundle over X and let M be a $\mathbf{Z}[\pi_1 X]$-module. The problem of homology surgery is to determine the obstruction to the existence of a normal cobordism, constant over ∂X, from f to an M-homology equivalence. Clearly we must suppose that f induces an M-homology equivalence from ∂V to ∂X and that the cap-product by $f_*[V]$ is an isomorphism from $H^*(X, \partial X; M)$ to $H_{n-*}(X; M^w)$, w being the first Stiefel-Whitney class of the bundle over X.

If $M = A$ is a quotient ring with involution of $\mathbf{Z}[\pi_1 X] = \mathbf{Z}\pi$, Cappell and Shaneson have solved the problem and have constructed an obstruction group $\Gamma_n(\mathbf{Z}\pi \to A)$ defined in terms of algebraic L-theory.

In many cases, this group was known to be the L_n-group of some ring Λ. For example, if there exists a classical localization $S^{-1}\mathbf{Z}\pi$ of $\mathbf{Z}\pi$, where S is the multiplicative subset $1 + \ker(\mathbf{Z}\pi \to A)$, Smith [7] has proved that $\Gamma_n(\mathbf{Z}\pi \to A)$ is the group $L_n(S^{-1}\mathbf{Z}\pi)$. An other example is given by Hausmann [3] who proves that $\Gamma_n(\mathbf{Z}\pi \to \mathbf{Z}[\pi/N])$ is the group $L_n(\mathbf{Z}[\pi/N])$ if N is a locally perfect normal subgroup of π.

My purpose is to show that the homology surgery is possible in a more general situation and that the obstruction group is always the L_n-group of a ring with involution Λ endowed with a subgroup of $\tilde{K}_1(\Lambda)$.

For example, suppose that $\mathbf{Z}\pi \to A$ is a morphism of rings with involution (the involution of $\mathbf{Z}\pi$ is induced by w). Then we have a diagram of rings with involution

$$\begin{array}{ccc} & \Lambda & \\ \nearrow & & \searrow \\ \mathbf{Z}\pi & \longrightarrow & A \end{array}$$

well defined by the following properties:

i) For any matrix u with entries in $\mathbf{Z}\pi$, if $u \otimes A$ is invertible then $u \otimes \Lambda$ is invertible too;

ii) Λ is universal with respect to the property i).

Theorem. — *Suppose the morphism* $\Lambda \to A$ *is onto. Then any normal map* f *over a* n*-dimensional* A*-Poincaré complex* X *which is an* A*-homology equivalence over* ∂X *determines an element* $\sigma(f) \in L^h_n(\Lambda)$, *and, if* $n \geq 5$, f *is normally cobordant to an* A*-homology equivalence if and only if* $\sigma(f)$ *vanishes.*

Corollary. — *If* A *is a quotient ring with involution of* $\mathbf{Z}\pi$, *the group* $\Gamma^h_n(\mathbf{Z}\pi \to A)$ *is isomorphic to* $L^h_n(\Lambda)$.

Theorem. — *Let* D_{2n} *be the dihedral group of order* $2n$ *(n odd) and* $D_{2n} \to \mathbf{Z}/2$ *be the non zero homomorphism. Then we have the following isomorphism:*

$$\Gamma_*(\mathbf{Z}D_{2n} \to \mathbf{Z}) \xrightarrow{\sim} \Gamma_*(\mathbf{Z}[\mathbf{Z}/2] \to \mathbf{Z}) \simeq L^h_*(\Lambda),$$

where Λ *is the pull back of rings:*

$$\begin{array}{ccc} \Lambda & \longrightarrow & \mathbf{Z} \\ \downarrow & & \downarrow \\ \mathbf{Z}_{(2)}[\mathbf{Z}/2] & \longrightarrow & \mathbf{Z}_{(2)} \end{array}$$

TABLE OF CONTENTS

§ 1. Statement of the main results 166
§ 2. A first homology surgery obstruction group 170
§ 3. Algebraic surgery 173
§ 4. Geometric surgery 181
§ 5. Localization in the category of graded differential modules 186
§ 6. The ring Λ 189
§ 7. The structure of $\mathscr{W}$ 195
§ 8. The isomorphism theorem 198
§ 9. Some results about Λ and $L_n(\Lambda)$ 201

1. Statement of the main results

(1.1) Let A be a ring with involution $a \mapsto \bar{a}$. If M is a left A-module, it can be given a right A-module structure, by setting

$$ma = \bar{a}m, \quad \forall a \in A, \quad \forall m \in M.$$

Conversely any right A-module is a left A-module. From now on an A-module will mean a left or right A-module.

Denote by $\mathscr{C}(A)$ the category of $\mathbf{Z}$-graded complexes

$$\ldots \to C_{n+1} \to C_n \to C_{n-1} \to \ldots$$

such that each C_n is a finitely generated free A-module with fixed (unordered) basis and $\bigoplus_n C_n$ is finitely generated. Theses complexes will be called finite A-complexes.

We say that a sequence of finite A-complexes $0 \to C \to C' \to C'' \to 0$ is s-exact if, for any n, the complex $0 \to C_n \to C'_n \to C''_n \to 0$ is acyclic with torsion 0 in $\tilde{K}_1(A)$; see [4] and [9].

Definition (1.2). — A class $\mathscr{W} \subset \mathscr{C}(A)$ is *exact* if $\mathscr{W}$ contains any acyclic finite A-complex with torsion 0, and if, for any s-exact sequence in $\mathscr{C}(A)$

$$0 \to C \to C' \to C'' \to 0,$$

one has the following property:

If two of the complexes C, C', C'' lie in $\mathscr{W}$, then the third lies in $\mathscr{W}$ too.

Let C be a finite A-complex. Denote by $\hat{C}_n$ the dual module $\mathrm{Hom}(C_{-n}, A)$ endowed with the dual basis, and choose on $\hat{C}$ the differential so that the evaluation from $\hat{C} \otimes C$ to A is a cocycle. So we get a new finite A-complex $\hat{C}$.

Definition (1.3). — An exact class $\mathscr{W} \subset \mathscr{C}(A)$ is called *symmetric* if, for any $C \in \mathscr{W}$, $\hat{C}$ lies in $\mathscr{W}$.

Definition (1.4). — Let $\mathscr{W}$ be an exact class in $\mathscr{C}(A)$. A morphism f in $\mathscr{C}(A)$ is a *$\mathscr{W}$-equivalence* if the mapping cone of f is in $\mathscr{W}$.

Let f be a map from a finite CW-complex X to a finite connected CW-complex Y, with fundamental group π, and let $\mathscr{W}$ be an exact class in $\mathscr{C}(\mathbf{Z}\pi)$ containing any acyclic finite $\mathbf{Z}\pi$-complex with torsion in the image of $\pi \to \tilde{K}_1(\mathbf{Z}\pi)$. Then f is a *$\mathscr{W}$-equivalence* if the chain map $C_*(X, \mathbf{Z}\pi) \to C_*(Y, \mathbf{Z}\pi)$ is a $\mathscr{W}$-equivalence.

Example (1.5). — Let $A \to B$ be a ring homomorphism and β be a subgroup of $\tilde{K}_1(B)$. Let $\mathscr{W}$ be the class of finite A-complexes C such that $C \otimes_A B$ is acyclic with torsion in β. Then $\mathscr{W}$ is exact and the $\mathscr{W}$-equivalences are the B-homology equivalences with torsion in β.

If, in addition, $A \to B$ is a morphism of rings with involution and β is stable under the involution, $\mathscr{W}$ is symmetric.

Example (1.6). — Let M be an A-module. Then the class $\mathscr{W}$ of finite A-complexes C such that $H_*(C, M)$ (resp. $H^*(C, M)$) vanishes, is an exact class and the $\mathscr{W}$-equivalences are the M-homology (resp. M-cohomology) equivalences.

Notation (**1.7**). — Let $\mathscr{W}$ be an exact class in $\mathscr{C}(A)$. We denote by Σ the set of matrices u such that the direct sum of the complex $\ldots \to 0 \to A^p \xrightarrow{u} A^q \to 0 \to \ldots$ and its suspension is in $\mathscr{W}$.

In example (1.5), Σ is the set of matrices u with entries in A such that $u \otimes B$ is invertible.

Proposition (**1.8**). — *Let $\mathscr{W}$ be an exact class in $\mathscr{C}(A)$. Then there exists a ring homomorphism $A \to \Lambda$ unique up to isomorphism, which is universal with respect to the following property: For any matrix $u \in \Sigma$, $u \otimes \Lambda$ is invertible.*

If $\mathscr{W}$ is symmetric, $A \to \Lambda$ is a morphism of rings with involution.

Actually, the ring Λ is an inversive localization of A in the sense of Cohn [2].

Definition (**1.9**). — Let α be the subgroup of $\tilde{K}_1(\Lambda)$ generated by the torsion of all complexes $C \otimes \Lambda$, such that $C \in \mathscr{W}$ and $C \otimes \Lambda$ is acyclic. The pair (Λ, α) will be called the *$\mathscr{W}$-localization* of A.

Let f be a normal map from the normal bundle of a compact n-dimensional (Top, PL or Diff)-manifold V to a (Top, PL or Diff)-bundle ξ over a pair $(X, \partial X)$ of finite complexes. Suppose X is connected. The first Stiefel-Whitney class of ξ induces an involution on the ring $A = \mathbf{Z}[\pi_1 X]$.

Let $\mathscr{W}$ be an exact symmetric class in $\mathscr{C}(A)$ containing any acyclic complex with torsion in the image of $\pi_1 X \to \tilde{K}_1(A)$.

Suppose we have the following properties:

i) $(X, \partial X)$ is a $\mathscr{W}$-Poincaré complex; *i.e.* the cap-product by $f_*[V]$ is a $\mathscr{W}$-equivalence from $C^*(X; A)$ to $C_*(X, \partial X; A)$.

ii) The restricted map $f: \partial V \to \partial X$ is a $\mathscr{W}$-equivalence.

Theorem (**1.10**). — *Let (Λ, α) be the $\mathscr{W}$-localization of* A. *Suppose that any complex in $\mathscr{W}$ is Λ-acyclic. Then, the normal map f determines a well-defined element $\sigma(f) \in L_n^\alpha(\Lambda)$. And, if $n \geq 5$, f is normally cobordant, rel the boundary, to a $\mathscr{W}$-equivalence if and only if $\sigma(f)$ vanishes.*

Theorem (**1.11**). — *With the same hypothesis as above, if $n \geq 6$, and* X *is a product* $M \times I$, M *being a* (Top, PL *or* Diff)*-manifold, any element of $L_n^\alpha(\Lambda)$ is the obstruction $\sigma(f)$ of a normal map f restricting to an isomorphism over* $M \times 0 \cup \partial M \times I$.

Remark (**1.12**). — The condition of Λ-acyclicity of complexes in $\mathscr{W}$ is a very crucial point because, in the situation of (1.10), $\sigma(f)$ can be defined only if this condition is satisfied, or, more precisely, if the Poincaré duality on $(X, \partial X)$ is a Λ-homology equivalence and f restricts to a Λ-homology equivalence on the boundaries.

On the other hand, this condition is not always satisfied. For example, if $\mathscr{W}$

is the class of finite $\mathbf{Z}[t, t^{-1}]$-complex with finite homology, the ring Λ is $\mathbf{Z}[t, t^{-1}]$ and there exist many complexes in $\mathscr{W}$ which are not acyclic.

If the condition of Λ-acyclicity of complexes in $\mathscr{W}$ is not satisfied, denote by $\mathscr{W}'$ the class of Λ-acyclic complexes in $\mathscr{W}$. Then theorems (1.10) and (1.11) hold for the class $\mathscr{W}'$. Now, the last problem is to compare the surgery problems corresponding to classes $\mathscr{W}$ and $\mathscr{W}'$. But this question seems to be very difficult.

Let $A \to B$ be a ring homomorphism. Let Λ be the inversive localization of A in the sense of Cohn [2] obtained by formal inversion of the matrices u with entries in A such that $u \otimes B$ is invertible. The ring homomorphism $A \to \Lambda$ will be called the localization of $A \to B$.

Theorem **(1.13)**. — *Let* $A \to B$ *be a ring homomorphism and* β *be a subgroup of* $\widetilde{K}_1(B)$. *Denote by* $\mathscr{W}$ *the class of finite* A*-complexes which are* B*-acyclic with torsion in* β, *and by* (Λ, α) *the* $\mathscr{W}$*-localization of* A.

Then $A \to \Lambda$ *is the localization of* $A \to B$ *and* α *is the inverse image of* β *under the canonical morphism* $\varepsilon : \Lambda \to B$.

Moreover, if ε *is onto, any complex in* $\mathscr{W}$ *is* Λ*-acyclic.*

Remark **(1.14)**. — The ring Λ and the group $L_n^{\alpha}(\Lambda)$ are difficult to compute, but we have some interesting results.

Let $S \subset A$ be the set of elements in A invertible in B. Then, if there exists a classical localization $S^{-1}A$, Λ is the ring $S^{-1}A$. This holds, for example, if A is commutative or if $A \to B$ is the ring homomorphism $\mathbf{Z}\pi \to \mathbf{Z}\pi'$ induced by a group homomorphism $\pi \to \pi'$ with finitely generated nilpotent kernel onto a finite extension of a polycyclic group [7].

An other example is the following (see theorem (9.7)): Let $\pi \to G$ be a group-epimorphism with locally perfect kernel. Then the localization of $\mathbf{Z}\pi \to \mathbf{Z}G$ is $\mathbf{Z}\pi \to \mathbf{Z}G$ itself.

Anyway, the theorems (1.10), (1.11), (1.13) imply that the obstruction groups $\Gamma_n(A \to B)$ of Cappell and Shaneson [1] are always the L_n-groups of Λ (endowed with a subgroup of $\widetilde{K}_1(\Lambda)$), at least when the theory of Cappell and Shaneson holds, *i.e.* when $A \to B$ is locally epic. This was already proved in some particular cases by Cappell and Shaneson [1], Smith [7], Hausmann [3] and the author [8].

Nevertheless the condition of surjectivity of $\Lambda \to B$ holds in many other cases.

Proposition **(1.15)**. — *Let* $A \to B$ *be a ring homomorphism and* $A \to \Lambda$ *be the localization of* $A \to B$. *Let* $B_0 \subset B_1 \subset B_2 \subset \ldots$ *be subrings of* B *defined by*:

i) B_0 *is the image of* $A \to B$;

ii) *For any* $n \geq 0$, B_{n+1} *is generated by* B_n *and the inverses of the units of* B *contained in* B_n.

Then, the image of $\Lambda \to B$ *contains all the rings* B_n. *Therefore, if* B *is the union of the rings* B_n, *the morphism* $\Lambda \to B$ *is onto and the theorems* (1.10), (1.11), (1.13) *hold.*

In fact, the image of $\Lambda \to B$ can be strictly greater than the union of the rings B_n.

Example (**1.16**). — Let F be the free group with p generators, $p > 1$, and let A be the group ring $\mathbf{Z}[F]$. Let $\mathscr{W}$ be the class of finite A-complexes C such that $H_*(C)$ is finitely generated over $\mathbf{Z}$ and let (Λ, α) be the $\mathscr{W}$-localization of A. Then the localization of $A \to \Lambda$ is $A \to \Lambda$ and the morphism $\Lambda \to \Lambda$ is the identity. One can prove that any square matrix with entries in A which is invertible in Λ, is invertible in A; hence $B_n = A$ for all n, but $A \to \Lambda$ is not surjective!

Remark (**1.17**). — Let $A \to B$ be a ring homomorphism and β be a subgroup of $\tilde{K}_1(B)$. Denote by $\mathscr{W}$ the class of finite A-complexes which are B-acyclic with torsion in β and by (Λ, α) the $\mathscr{W}$-localization of A.

If the morphism $\Lambda \to B$ is not onto, the condition of Λ-acyclicity of complexes in $\mathscr{W}$ is not always satisfied.

For example, this condition holds if $A \to B$ is the ring homomorphism $\mathbf{Z} \to \mathbf{R}$, but it does not hold if A is the ring $\mathbf{Z}[t, t^{-1}]$ and B is the product of the localizations of A with respect to the non zero principal prime ideals.

2. A first homology surgery obstruction group

In a first step, we will construct a surgery obstruction group $\Gamma_n(A, \mathscr{W})$ which looks like the group $\Gamma_n(A \to B)$ constructed by Ranicki [5], but from a dual point of view.

Throughout sections 2 and 3 we assume that A is a ring with involution and that $\mathscr{W}$ is an exact symetric class in $\mathscr{C}(A)$ (see (1.2) and (1.3)).

If C and C′ are finite A-complexes, we denote by Hom(C, C′) the set of A-homomorphisms from C to C′; Hom(C, C′) can be given a graded differential $\mathbf{Z}$-module structure by setting:

$$\partial^0 f(x) = \partial^0 f + \partial^0 x, \quad \text{for any } f \in \mathrm{Hom}(C, C'),\ x \in C$$
$$d(f(x)) = (df)(x) + (-1)^{\partial^0 f} f(dx), \quad \text{for any } f \in \mathrm{Hom}(C, C'),\ x \in C.$$

Moreover, by setting

$$\hat{f}(u) = (-1)^{\partial^0 f \partial^0 u} u \circ f, \quad \text{for any } f \in \mathrm{Hom}(C, C'),\ u \in \hat{C}',$$

we get a morphism $f \to \hat{f}$ from Hom(C, C′) to Hom($\hat{C}'$, $\hat{C}$) which respects the degrees and the differentials.

Notation (**2.1**). — If C is a finite A-complex, we denote by B(C) the graded differential $\mathbf{Z}$-module Hom(C, $\hat{C}$). The composite map:

$$\mathrm{Hom}(C, \hat{C}) \to \mathrm{Hom}(\hat{\hat{C}}, \hat{C}) \xrightarrow{\sim} \mathrm{Hom}(C, \hat{C})$$

is an involution on B(C) and B(C) is a graded differential $\mathbf{Z}[\mathbf{Z}/2]$-module.

Definition (**2.2**). — Let C be a finite A-complex. We use $Q_n(C)$ to denote the group $H_n(\mathbf{Z}/2, B(C))$. By a *quadratic n-form* over C, we mean an element of $Q_n(C)$ and by a *quadratic n-complex* we mean a pair (C, q), $q \in Q_n(C)$.

Let $C \to C'$ be an epimorphism of degree o between two finite A-complexes. We use $Q_n(C \to C')$ to denote the group $H_n(\mathbf{Z}/2, B(C)/B(C'))$. By a *quadratic n-form* over $C \to C'$, we mean an element of $Q_n(C \to C')$ and by a *quadratic n-pair*, we mean a pair $(C \to C', q)$, $q \in Q_n(C \to C')$.

Definition (**2.3**). — Let (C, q) be a quadratic n-complex. We will say that q or (C, q) is *$\mathscr{W}$-non singular* if the image of q by the composite map

$$H_n(\mathbf{Z}/2, B(C)) \xrightarrow{\text{transfer}} H_n(1, B(C)) \simeq H_n(B(C))$$

is represented by a $\mathscr{W}$-equivalence from C to $\hat{C}$.

Let $(C \to C', q)$ be a quadratic n-pair. Let K be the kernel of $C \to C'$. We will say that q or $(C \to C', q)$ is *$\mathscr{W}$-non singular* if the image of q by the composite map

$$H_n(\mathbf{Z}/2, B(C)/B(C')) \xrightarrow{\text{transfer}} H_n(B(C)/B(C')) \to H_n(\mathrm{Hom}(K, \hat{C}))$$

is represented by a $\mathscr{W}$-equivalence from K to $\hat{C}$.

Remark (**2.4**). — If C is zero except in dimension $-p$, a quadratic $2p$-form over C is exactly a $(-1)^p$-quadratic from over C_{-p} in the sense of Wall [11].

Remark (**2.5**). — If $\mathscr{W}$ is the class of acyclic complexes with zero torsion, a $\mathscr{W}$-non singular quadratic n-form q over a finite A-complex C is an n-dimensional quadratic Poincaré structure on $\hat{C}$, in the sense of Ranicki [5], at least if $\hat{C}$ is (-1)-connected.

Definition (**2.6**). — We will denote by $\Gamma_n(A, \mathscr{W})$ the set of $\mathscr{W}$-non singular quadratic n-complexes subject to the following cobordism relation: (C, q) is cobordant to (C', q') if there exists a $\mathscr{W}$-non singular quadratic $(n+1)$-pair $(\Sigma \to C \oplus C', u)$ such that $\partial u = q \oplus -q'$.

Let W be the standard free resolution of the $\mathbf{Z}[\mathbf{Z}/2]$-module $\mathbf{Z}$:

$$\mathbf{Z}[\mathbf{Z}/2]e_0 \xleftarrow{1-t} \mathbf{Z}[\mathbf{Z}/2]e_1 \xleftarrow{1+t} \mathbf{Z}[\mathbf{Z}/2]e_2 \xleftarrow{1-t} \dots$$

Then $Q_n(C)$ is the n-th homology group of $W \otimes_{\mathbf{Z}/2} B(C)$.

Lemma (**2.7**). — *Two $\mathscr{W}$-non singular quadratic n-complexes* (C, q) *and* (C', q') *are cobordant if and only if there exist two s-exact sequences*

$$o \to K \to \Sigma \xrightarrow{\alpha} C \to o$$

$$o \to K' \to \Sigma \xrightarrow{\alpha'} C' \to o$$

and an element $e_0 \otimes \psi_0 + e_1 \otimes \psi_1 + \dots$ *in* $W \otimes_{Z/2} B(\Sigma)$ *such that:*

i) *If* q *and* q' *are the homology classes of* φ *and* φ', *we have*

$$d(\Sigma e_i \otimes \psi_i) = \alpha^*(\varphi) - \alpha'^*(\varphi');$$

ii) $\psi_0 + \hat{\psi}_0$ *induces a* $\mathscr{W}$*-equivalence from* K *to* $\hat{K}'$.

Proof. — Suppose that q and q' are represented by $\varphi \in W \otimes_{Z/2} B(C)$ and $\varphi' \in W \otimes_{Z/2} B(C')$. If (C, q) and (C', q') are cobordant, there exists a s-exact sequence

$$0 \to \Sigma' \to \Sigma \xrightarrow{\alpha \oplus \alpha'} C \oplus C' \to 0$$

together with an element $\Sigma e_i \otimes \psi_i \in W \otimes B(\Sigma)$ such that:

(i) $d(\Sigma e_i \otimes \psi_i) = \alpha^*(\varphi) - \alpha'^*(\varphi')$;

(ii) $\psi_0 + \hat{\psi}_0$ induces a $\mathscr{W}$-equivalence from Σ' to $\hat{\Sigma}$.

Let K (respectively K′) be the kernel of α (respectively α'). We have a homotopy commutative diagram

$$\text{(I)} \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \Sigma' & \longrightarrow & K & \longrightarrow & C' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & \hat{\Sigma} & \longrightarrow & \hat{K}' & \longrightarrow & \hat{C}' & \longrightarrow & 0 \end{array}$$

where the lines are homotopy s-exact and a and b are induced by $\psi_0 + \hat{\psi}_0$ and c is induced by the transfer of φ'.

Since a and c are $\mathscr{W}$-equivalences, b is a $\mathscr{W}$-equivalence too and the first part of the lemma is proved.

Conversely, suppose we have two s-exact sequences

$$0 \longrightarrow K \longrightarrow \Sigma \xrightarrow{\alpha} C \longrightarrow 0$$

$$0 \longrightarrow K' \longrightarrow \Sigma \xrightarrow{\alpha'} C' \longrightarrow 0$$

and an element $\Sigma e_i \otimes \psi_i$ satisfying the conditions (i) and (ii) of the lemma. Up to simple homotopy type, we may suppose that $\alpha \oplus \alpha'$ is onto with kernel $\Sigma' \in \mathscr{C}(A)$. Then we have the homotopy commutative diagram (I) where b and c are $\mathscr{W}$-equivalences and $\psi_0 + \hat{\psi}_0$ induces a $\mathscr{W}$-equivalence from Σ' to $\hat{\Sigma}$. Hence (C, q) and (C', q') are cobordant. ■

Lemma **(2.8).** — *Let* (C, q) *be a* $\mathscr{W}$*-non singular quadratic* n*-complex and* $f: C' \to C$ *be a* $\mathscr{W}$*-equivalence. Then* $(C', f^*(q))$ *is a* $\mathscr{W}$*-non singular quadratic* n*-complex cobordant to* (C, q).

Proof. — We may suppose that f is epic with kernel $K \in \mathscr{C}(A)$. Then we have the s-exact sequences

$$0 \longrightarrow K \longrightarrow C' \xrightarrow{f} C \longrightarrow 0$$

$$0 \longrightarrow 0 \longrightarrow C' \xrightarrow{1} C' \longrightarrow 0$$

and the result is an easy consequence of lemma (2.7). ■

3. Algebraic surgery

In order to kill the homology of a $\mathscr{W}$-non singular quadratic n-complex, in low dimension, we need the following:

Lemma (3.1). — *Let* $0 \to I \xrightarrow{\alpha} C \xrightarrow{\beta} J \to 0$ *be an s-exact sequence of finite* A*-complexes. Let* q *be a* $\mathscr{W}$*-non singular quadratic n-form over* C *such that* $\alpha^* q = 0$. *Then,* q *is represented by a cycle* $\Sigma e_i \otimes f_i \beta$.

Moreover if q *is represented by such a cycle,* (C, q) *is cobordant to a* $\mathscr{W}$*-non singular quadratic n-complex* (C', q') *where* C' *is the mapping cone of* $\hat{\alpha} f_0$ *(the grading of* C' *is chosen so that the map* $C' \to J$ *has degree* 0).

Proof. — Consider the following exact sequences of graded differential $\mathbf{Z}[\mathbf{Z}/2]$-modules:

$$0 \to B \to B(C) \xrightarrow{\alpha^*} B(I) \to 0$$

$$\mathrm{Hom}(C, \hat{J}) \oplus \mathrm{Hom}(J, \hat{C}) \to B \to 0.$$

If $\alpha^* q$ is zero, q is represented by a cycle in $W \otimes_{\mathbf{Z}/2} B$, and there exist morphisms f_i' and f_i'' in $\mathrm{Hom}(J, \hat{C})$ such that q is represented by

$$\Sigma e_i \otimes (f_i' \beta + \hat{\beta} \hat{f}_i'').$$

Now we have

$$d(e_{i+1} \otimes f_i'' \beta) = e_i \otimes f_i'' \beta + (-1)^{i+1} e_i \otimes \hat{\beta} \hat{f}_i'' + (-1)^{i+1} e_{i+1} \otimes d f_i'' \beta.$$

Then there exist morphisms $f_i \in \mathrm{Hom}(J, \hat{C})$ such that q is represented by $\Sigma e_i \otimes f_i \beta$. Since $\Sigma e_i \otimes f_i \beta$ is a cycle, we have

$$\forall\, i \geq 0, \quad (-1)^i d f_i \beta + f_{i+1} \beta + (-1)^{i+1} \hat{\beta} \hat{f}_{i+1} = 0,$$

whence $\quad d(\hat{\alpha} f_0) = 0, \quad \hat{\alpha} f_i = 0, \quad$ for any $i > 0$.

Let C' be the mapping cone of $\hat{\alpha} f_0$. We have a split exact sequence

$$0 \longrightarrow \hat{I} \underset{r'}{\overset{\alpha'}{\rightleftarrows}} C' \xrightarrow{\beta'} J \longrightarrow 0$$

such that

$$\partial^0 \alpha' = -n - 1, \quad \partial^0 \beta' = 0, \quad d r' = \hat{\alpha} f_0 \beta', \quad r' \alpha' = 1$$

and $\quad 0 \to S^{-n-1} \hat{I} \to C' \to J \to 0$

is s-exact.

Let Σ be the pull-back of C and C' over J:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\gamma} & C \\ {\scriptstyle \gamma'}\downarrow & & \downarrow{\scriptstyle \beta} \\ C' & \xrightarrow{\beta'} & J \end{array}$$

Let r be a retraction of α and let u be the element $e_0 \otimes \hat{\gamma}\hat{r}r'\gamma' \in W \otimes_{\mathbf{Z}/2} B(\Sigma)$. We have

$$du = e_0 \otimes \hat{\gamma} d\hat{r} r' \gamma' + e_0 \otimes \hat{\gamma} f_0 \beta\gamma + e_0 \otimes \hat{\gamma}(\hat{r}\hat{\alpha} - 1) f_0 \beta' \gamma'$$

and it is easy to see that $\gamma^*(\Sigma e_i \otimes f_i\beta) - du$ has the form $\gamma'^*(\Sigma e_i \otimes \varphi_i')$, $\varphi_i' \in B(C')$.

On the other hand, $\hat{\gamma}\hat{r}r'\gamma' + \hat{\gamma}'\hat{r}'r\gamma$ induces the identity from the kernel of γ' to the dual of the kernel of γ. Then $\Sigma e_i \otimes \varphi_i'$ represents a $\mathscr{W}$-non singular quadratic n-form q' over C' and, by (2.7), (C, q) and (C', q') are cobordant. ■

Corollary **(3.2)**. — *Any $\mathscr{W}$-non singular quadratic n-complex is cobordant to a $\mathscr{W}$-non singular quadratic n-complex (C, q) such that C is $\left(\left[\frac{-n}{2}\right] - 1\right)$-connected.*

Proof. — Just apply lemma (3.1), I being the $\left(\left[\frac{-n}{2}\right] - 1\right)$-skeleton of the complex. ■

Lemma **(3.3)**. — *Let $0 \to I \xrightarrow{\alpha} C \xrightarrow{\beta} J \to 0$ be an s-exact sequence of finite A-complexes and $\gamma : J \to K$ be an epimorphism of degree 0 which respects the differentials. Let q be a $\mathscr{W}$-non singular quadratic n-form over $C \to K$ such that $\alpha^* q = 0$. Then q is represented by $\Sigma e_i \otimes f_i\beta$.*

Moreover, if C' is the mapping cone of $\hat{\alpha}f_0$ (the grading being chosen as in lemma (3.1)), there exists a $\mathscr{W}$-non singular quadratic n-form q' over $C' \to K$ such that ∂q and $\partial q'$ coincide in $Q_{n-1}(K)$.

Proof. — We have the following exact sequences of graded differential $\mathbf{Z}[\mathbf{Z}/2]$-modules:

$$0 \to B \to B(C)/B(K) \xrightarrow{\alpha^*} B(I) \to 0$$

$$\mathrm{Hom}(C, \hat{J}) \oplus \mathrm{Hom}(J, \hat{C}) \to B \to 0.$$

Then, as in lemma (3.1), we show that q is represented by an element $\Sigma e_i \otimes f_i\beta$ and we have

$$d(\hat{\alpha}f_0) = 0, \qquad \hat{\alpha}f_i = 0, \qquad \text{for any } i > 0.$$

Consider, as above, the diagram: $0 \to \hat{I} \underset{r'}{\overset{\alpha'}{\rightleftarrows}} C' \xrightarrow{\beta'} J \to 0$ and let s be a section of β. We have

$$ds = \alpha\delta, \qquad \delta \in \mathrm{Hom}(J, I).$$

It is not difficult to see that the element

$$u = e_0 \otimes \hat{\beta}'\hat{\delta}r' + \Sigma e_i \otimes \hat{\beta}'\hat{s}f_i\beta'$$

represents a quadratic n-form q' over $C' \to K$ and that ∂q and $\partial q'$ coincide in $Q_{n-1}(K)$. Moreover, the transfer $\tilde{u}$ of u is:

$$\tilde{u} = \hat{\beta}'\hat{\delta}r' + (-1)^{n+1}\hat{r}'\delta\beta' + \hat{\beta}'\hat{s}f_0\beta' + \hat{\beta}'\hat{f}_0 s\beta'$$

and we have

$$\tilde{u}\alpha' = \hat{\beta}'\hat{\delta} \quad \text{and} \quad \hat{\alpha}'\tilde{u} = \delta\beta'.$$

Denote by $\bar{C}$, $\bar{J}$, $\bar{C}'$ the kernels of the morphisms $C \to K$, $J \to K$ and $C' \to K$. We have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \hat{I} & \xrightarrow{\alpha'} & \bar{C}' & \xrightarrow{\beta'} & \bar{J} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \hat{\delta}} & & \downarrow{\scriptstyle \tilde{u}} & & \downarrow{\scriptstyle \delta} & & \\ 0 & \longrightarrow & \hat{J} & \xrightarrow{\hat{\beta}'} & \hat{C}' & \xrightarrow{\hat{\alpha}'} & I & \longrightarrow & 0, \end{array}$$

and we obtain a s-exact sequence between the mapping cone of $\hat{\delta}$, $\tilde{u}$ and δ. Now the boundary of this s-exact sequence is homotopic to the morphism $(-1)^{n+1}(f_0\beta + \hat{\beta}\hat{f}_0)$ from $\bar{C}$ to $\hat{C}$, which is a $\mathscr{W}$-equivalence. Then the mapping cone of $\tilde{u}\colon \bar{C}' \to \hat{C}'$ is in $\mathscr{W}$ and q' is $\mathscr{W}$-non singular. ■

Corollary **(3.4)**. — *Let* (C, q) *be a* $\mathscr{W}$*-non singular quadratic* n*-complex cobordant to zero. Then there exists a* $\mathscr{W}$*-non singular quadratic* $(n+1)$*-pair* $(\Sigma \to C, u)$ *such that* q *is the boundary of* u *and the kernel of* $\Sigma \to C$ *is* $\left(\left[\frac{-n-1}{2}\right] - 1\right)$*-connected.*

Proof. — If (C, q) is cobordant to zero, there exists a $\mathscr{W}$-non singular quadratic $(n+1)$-pair $(\Sigma' \to C, u')$ such that q is the boundary of u'. Then apply lemma (3.3), I being the $\left(\left[\frac{-n-1}{2}\right] - 1\right)$-skeleton of the kernel of $\Sigma' \to C$. ■

Now, if we want to kill the homology of a $\mathscr{W}$-non singular quadratic n-form beyond the middle dimension, we must suppose that $\mathscr{W}$ satisfies some other properties. Actually, it is useful to consider the new class $\mathscr{W}'$ of all Λ-acyclic finite A-complexes.

Splitting lemma **(3.5)**. — *Let* C *be a complex in* $\mathscr{W}'$ *and let* n *be an integer. Then, there exist two finite* A*-complexes* L *and* L' *concentrated in dimension* n *and a* $\mathscr{W}'$*-equivalence from* L *to the complex*

$$L' \oplus (\ldots \to C_{n+1} \to C_n \to 0 \to \ldots).$$

This lemma will be proved in § 7.

Lemma **(3.6)**. — *Any* $\mathscr{W}'$*-non singular quadratic* n*-complex is cobordant to a* $\mathscr{W}'$*-non singular quadratic* n*-complex* (C, q) *where* C *vanishes except in dimension* $\left[\frac{-n}{2}\right]$ $\left($*and* $\left[\frac{-n}{2}\right] + 1$ *if* n *is odd*$\right)$.

Proof. — Let (C, q) be a $\mathscr{W}'$-non singular quadratic n-complex. By corollary (3.2), we may as well suppose that C_i vanishes for $i < \left[\frac{-n}{2}\right]$.

Suppose $n = -2p$. Since (C, q) is $\mathscr{W}'$-non singular, we have the following complex in $\mathscr{W}'$:

$$\ldots \to C_{p+1} \to C_p \to \hat{C}_p \to \hat{C}_{p+1} \to \ldots$$

and, by splitting lemma (3.5), there exist two complexes L and L′ concentrated in dimension p and a $\mathscr{W}'$-equivalence

$$f\colon L \to C \oplus L'.$$

Up to stabilization, we may suppose that L'_p is even dimensional. Let $q' \in Q_n(L')$ be a standard hyperbolic structure on L'_p.

Then (C, q) is cobordant to $(C \oplus L', q \oplus q')$ and by lemma (2.8), (C, q) is cobordant to $(L, f^*(q \oplus q'))$.

Suppose $n = -2p - 1$. Since (C, q) is $\mathscr{W}'$-non singular, we have the following complex in $\mathscr{W}'$:

$$\ldots \to C_{p+1} \to C_p \oplus \hat{C}_p \to \hat{C}_{p+1} \to \ldots,$$

and, by the splitting lemma (3.5), there exist two complexes L and L′ concentrated in dimension $p + 1$ and a $\mathscr{W}'$-equivalence

$$L \to L' \oplus (\ldots \to C_{p+2} \to C_{p+1} \to 0 \ldots).$$

We deduce a $\mathscr{W}'$-equivalence

$$f\colon (\ldots \to 0 \to L_{p+1} \to C_p \oplus L'_{p+1} \to 0 \ldots) \to C$$

and (C, q) is cobordant to $(\ldots \to 0 \to L_{p+1} \to C_p \oplus L'_{p+1} \to 0 \ldots, f^*q)$. ■

Lemma **(3.7)**. — *Let* (C, q) *be a* $\mathscr{W}'$*-non singular quadratic* $(-2p)$*-complex such that* C_i *vanishes for* $i \neq p$. *Then* (C, q) *is cobordant to zero if and only if there exists a* $\mathscr{W}'$*-non singular quadratic* $(-2p + 1)$*-pair* $(\Sigma \to C, u)$ *such that* q *is the boundary of* u *and* Σ_i *vanishes for* $i \neq p, p - 1$.

Proof. — Suppose (C, q) is cobordant to zero. By corollary (3.4), there exists a $\mathscr{W}'$-non singular quadratic $(-2p + 1)$-pair $(\Sigma' \to C, u')$ such that q is the boundary of u' and Σ'_i vanishes for $i < p - 1$. Let K′ be the kernel of $\Sigma \to C$.

Since u' is $\mathscr{W}'$-non singular, we have the following complex in $\mathscr{W}'$:

$$\ldots \to K'_{p+1} \to K'_p \to K'_{p-1} \oplus \hat{\Sigma}'_{p-1} \to \hat{\Sigma}'_p \to \hat{\Sigma}'_{p+1} \to \ldots$$

and, by the splitting lemma (3.5), there exist two complexes $L, L' \in \mathscr{C}(A)$ concentrated in dimension p and a $\mathscr{W}'$-equivalence

$$(\ldots \to 0 \to L_p \to K'_{p-1} \oplus L'_p \to 0 \to \ldots) \to K'.$$

Let K be the complex $\ldots 0 \to L_p \to K'_{p-1} \oplus L'_p \to 0 \to \ldots$ Since the $\mathscr{W}'$-equivalence $K \to K'$ is $(p-1)$-connected, the boundary $C \to K'$ lifts through K and we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & \Sigma & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow f & & \| & & \\ 0 & \longrightarrow & K' & \longrightarrow & \Sigma' & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

where the lines are s-exact.

Then $(\Sigma \to C, f^*u')$ is the desired quadratic pair. ■

Lemma **(3.8).** — *Let* (C, q) *be a* $\mathscr{W}'$*-non singular quadratic* $(-2p-1)$*-form such that* C_i *vanishes for* $i \neq p$, $p+1$. *Then* (C, q) *is cobordant to zero if and only if there exists a* $\mathscr{W}'$*-non singular quadratic* $(-2p)$*-pair* $(\Sigma \to C, u)$ *such that* q *is the boundary of* u *and* $\Sigma_i \to C_i$ *is a simple isomorphism for* $i \neq p$.

Proof. — Suppose (C, q) is cobordant to zero. By corollary (3.4), there exists a $\mathscr{W}'$-non singular quadratic $(-2p)$-pair $(\Sigma' \to C, u')$ such that q is the boundary of u' and Σ'_i vanishes for $i < p$.

Let K' be the kernel of $\Sigma' \to C$. We have a complex in $\mathscr{W}'$

$$\ldots \to \Sigma'_{p+1} \to \Sigma'_p \to \hat{K}'_p \to \hat{K}'_{p+1} \to \ldots$$

and, by the splitting lemma (3.5), there exist two complexes L and $L' \in \mathscr{C}(A)$ concentrated in dimension p and a $\mathscr{W}'$-equivalence

$$f\colon L \to \Sigma' \oplus L'.$$

Up to stabilization, we may suppose that L'_p is even dimensional. Let $v \in Q_{-2p}(L')$ be a standard hyperbolic structure on L'_p.

Let X be an acyclic finite A-complex with torsion zero concentrated in dimension $p-1$, p, $p+1$ and $X \to C$ be an epimorphism with kernel in $\mathscr{C}(A)$ such that $X_{p+1} \to C_{p+1}$ is an isomorphism. Let $(\Sigma'' \to C, u'')$ be the quadratic $(-2p)$-pair defined by $\Sigma'' = L \oplus X$, $u'' = f^*(u' \oplus v) \oplus 0$.

It is easy to see that u'' is $\mathscr{W}'$-non singular and that $\partial u'' = q$. Moreover the kernel K'' of $\Sigma'' \to C$ is concentrated in dimension $p-1$ and p.

Now, by lemma (3.3), we can kill the $p-1$ skeleton of K'' by surgery and we get a $\mathscr{W}'$-non singular $(-2p)$-pair $(\Sigma \to C, u)$ such that $\partial u = q$ and the kernel of $\Sigma \to C$ vanishes except in dimension p. ■

Now, with the above lemmas, it is possible to give an interpretation of $\Gamma_n(A, \mathscr{W}')$ in term of special forms in the sense of Wall [10] and Cappell and Shaneson [1].

Definition **(3.9).** — Let $\eta = \pm 1$ and $I_\eta = \{a - \eta\bar{a},\ a \in A\}$. A *$\mathscr{W}'$-special η-form* is a triple (H, λ, μ) where H is a finitely generated free A-module, λ a $\mathbf{Z}$-bilinear map from $H \otimes_{\mathbf{Z}} H$ to A and μ a map from H to A/I_η, and satisfying the following conditions:

Q_1 $\quad \lambda(ax, yb) = a\lambda(x, y)b, \quad \forall\, x, y \in H,\ \forall\, a, b \in A$

Q_2 $\quad \lambda(x, y) = \eta\overline{\lambda(y, x)}, \quad \forall\, x, y \in H$

Q_3 $\quad \mu(x) + \eta\mu(y) = \lambda(x, y), \quad \forall\, x, y \in H$

Q_4 $\quad \mu(x + y) \equiv \mu(x) + \mu(y) + \lambda(x, y) \bmod I_\eta, \quad \forall\, x, y \in H$

Q_5 $\quad \mu(xa) = \bar{a}\mu(x)a, \quad \forall\, x \in H,\ \forall\, a \in A$

Q_6 $\quad$ the morphism $\tilde{\lambda}$ induced by λ is a Λ-isomorphism (*i.e.* $\tilde{\lambda} \otimes \Lambda$ is an isomorphism).

Definition **(3.10).** — Let (H, λ, μ) be a $\mathscr{W}'$-special η-form. A *$\mathscr{W}'$-subkernel* of (H, λ, μ) is a free A-module K endowed with a morphism $f: K \to H$ satisfying the following conditions:

S_1 $\quad f^*\lambda = 0, \quad f^*\mu = 0$

S_2 $\quad$ the following complex lies in $\mathscr{W}'$: $0 \to K \xrightarrow{f} H \xrightarrow{\hat{f}\tilde{\lambda}} \hat{K} \to 0$.

(3.11) Let $\eta = (-1)^p$ and let (H, λ, μ) be a $\mathscr{W}'$-special η-form. Since H is free, there exists a map $\varphi_0 : H \to \hat{H}$ such that

$$\lambda(x, y) = \varphi_0(x)(y) + \eta\overline{\varphi_0(y)(x)}, \quad \forall\, x, y \in H$$
$$\mu(x) \equiv \varphi_0(x)(x) \bmod I_\eta, \quad \forall\, x \in H.$$

And, if φ_0 and φ_0' are such two maps, $\varphi_0 - \varphi_0'$ has the form $\psi - \eta\hat{\psi}$.

Choose a basis for H and denote by H_* the finite A-complex defined by

$$H_i = \begin{cases} H, & i = -p \\ 0, & i \neq -p \end{cases}$$

Then $e_0 \otimes \varphi_0$ represents a $\mathscr{W}'$-non singular quadratic $2p$-form q over H_* and the cobordism class of (H_*, q) is a well defined element $\omega(H, \lambda, \mu) \in \Gamma_{2p}(A, \mathscr{W}')$.

(3.12) Let $\eta = (-1)^p$ and let $f: K \to B \oplus \hat{B}$ be a $\mathscr{W}'$-subkernel of a standard η-kernel $B \oplus \hat{B}$ (B is a finitely generated free A-module). The map f is induced by maps $d: K \to B$ and $\varphi_0 : K \to \hat{B}$. Since the quadratic form is trivial over K, there exists a map $\varphi_1 : K \to \hat{K}$ such that $\hat{\varphi}_0 \circ d = \varphi_1 - (-1)^p\hat{\varphi}_1$. Choose basis for K and B. Let C be the $-p$-dimensional complex

$$\ldots \to 0 \to K \xrightarrow{d} B \to 0 \to \ldots$$

Let $\varphi_0 \mid B = 0$. We get two bilinear forms φ_0 and φ_1 on C, and we have

$$d\varphi_0 = \varphi_1 - \hat{\varphi}_1.$$

Then, $e_0 \otimes \varphi_0 - e_1 \otimes \varphi_1$ is a cycle in $W \otimes_{Z/2} B(C)$ inducing a quadratic $(2p+1)$-form q over C.

It is easy to see that q is $\mathscr{W}'$-non singular. We denote by $\omega(f) \in \Gamma_{2p+1}(A, \mathscr{W}')$ the cobordism class of (C, q). This element depends *a priori* on the choice of φ_1.

On the other hand, the tensorization by Λ induces a map from $\Gamma_n(A, \mathscr{W}')$ to $\Gamma_n(\Lambda, \mathscr{W}_1)$ where $\mathscr{W}_1$ is the class of finite acyclic Λ-complexes. But the group $\Gamma_n(\Lambda, \mathscr{W}_1)$ is isomorphic to $L^h_n(\Lambda)$. Then we get a morphism ε from $\Gamma_n(A, \mathscr{W}')$ to $L^h_n(\Lambda)$ and $\varepsilon\omega(f)$ is the class of $f \otimes \Lambda$ in $L^h_n(\Lambda)$. We deduce that $\varepsilon\omega(f)$ does not depend on the choice of φ_1. But it will be proved in § 8 that ε is an isomorphism. Therefore $\omega(f)$ is well defined.

Proposition **(3.13).** — *Any element of* $\Gamma_{2p}(A, \mathscr{W}')$ *has the form* $\omega(H, \lambda, \mu)$ *for some* $\mathscr{W}'$*-special* $(-1)^p$*-form* (H, λ, μ) *and any element of* $\Gamma_{2p+1}(A, \mathscr{W}')$ *has the form* $\omega(f)$ *for some* $\mathscr{W}'$*-subkernel* $f: K \to B \oplus \hat{B}$ *of a standard* $(-1)^p$*-kernel* $B \oplus \hat{B}$.

Proof. — In the even dimensional case, this is a trivial consequence of lemma (3.6).

In the odd dimensional case, we know by lemma (3.6) that any element of $\Gamma_{2p+1}(A, \mathscr{W}')$ is the cobordism class of a $\mathscr{W}'$-non singular $2p+1$-complexes (C, q) where C_i vanishes for $i \neq -p, -p-1$. It is not difficult to see that q is represented by $e_0 \otimes \varphi_0 + e_1 \otimes \varphi_1$, where the morphism φ_0 is trivial over C_{-p-1}. Then the cobordism class of (C, q) is $\omega(f)$ where f is the map $d \oplus \varphi_0 : C_{-p} \to C_{-p-1} \oplus \hat{C}_{-p-1}$. ■

Proposition **(3.14).** — *Let* (H, λ, μ) *be a* $\mathscr{W}'$*-special* $(-1)^p$*-form. Then* $\omega(H, \lambda, \mu)$ *is zero if and only if the direct sum of* (H, λ, μ) *and a standard kernel has a* $\mathscr{W}'$*-subkernel.*

Proof. — Suppose that (H, λ, μ) has a $\mathscr{W}'$-subkernel $f: K \to H$. Consider the quadratic $2p$-complex (H_*, q) constructed in (3.11). Choose a basis for K and denote by $K_* \in \mathscr{C}(A)$ the complex defined by

$$K_i = \begin{cases} K, & i = -p \\ 0, & i \neq -p. \end{cases}$$

Let $K_* \to H'_* \xrightarrow{g} H_*$ be a factorization of f such that g is a simple homotopy equivalence and $K_* \to H'_*$ is a monomorphism with free cokernel. After doing an algebraic surgery along $K_* \to H'_*$, we show that (H'_*, g^*q) is cobordant to (H''_*, q'') where H''_* has the simple homotopy type of

$$\ldots \to 0 \to K \to H \to \hat{K} \to 0 \to \ldots$$

The complex H''_* is thus Λ-acyclic and (H_*, q) is cobordant to zero.

Now suppose that the direct sum of (H, λ, μ) and a standard kernel H' has a $\mathscr{W}'$-subkernel. We have

$$\omega(H, \lambda, \mu) = \omega(H, \lambda, \mu) + \omega(H') = 0.$$

Conversely suppose that $\omega(H, \lambda, \mu)$ vanishes. By lemma (3.7), there exists a $\mathscr{W}'$-non singular quadratic $(2p+1)$-pair $(\Sigma \xrightarrow{\alpha} H_*, u)$ such that q is the boundary of u and Σ_i vanishes for $i \neq -p, -p-1$.

The form u can be represented by $e_0 \otimes \psi_0 + e_1 \otimes \psi_1$, ψ_0 vanishing on Σ_{-p-1}. Let K be the kernel of $\Sigma_{-p} \to H$.

Since u is $\mathscr{W}'$-non singular, the following complex is Λ-acyclic:

$$0 \longrightarrow \Sigma_{-p} \xrightarrow{d \oplus (-1)^p \psi_0} \Sigma_{-p-1} \oplus \hat{\Sigma}_{-p-1} \xrightarrow{\hat{\psi}_0 + \hat{d}} \hat{K} \longrightarrow 0,$$

and since $\tilde{\lambda} : H \to \hat{H}$ is a Λ-isomorphism, we deduce that

$$\alpha \oplus d \oplus (-1)^p \psi_0 : \Sigma_{-p} \to H \oplus \Sigma_{-p-1} \oplus \hat{\Sigma}_{-p-1}$$

is a $\mathscr{W}'$-subkernel of the direct sum of (H, λ, μ) and the standard kernel $\Sigma_{-p-1} \oplus \hat{\Sigma}_{-p-1}$.

Proposition **(3.15).** — *Let* $f : K \to B \oplus \hat{B}$ *be a* $\mathscr{W}'$*-subkernel of the standard* $(-1)^p$*-kernel* $B \oplus \hat{B}$. *Then* $\omega(f)$ *is zero if and only if there exist a kernel* $C \oplus \hat{C}$ *endowed with its standard subkernel* $g : C \to C \oplus \hat{C}$ *and an isometry* h *of* $B \oplus \hat{B} \oplus C \oplus \hat{C}$ *leaving each element of* $B \oplus \hat{C}$ *fixed, such that the composite map*

$$K \oplus C \xrightarrow{h \circ (f \oplus g)} B \oplus \hat{B} \oplus C \oplus \hat{C} \longrightarrow B \oplus \hat{C}$$

is a Λ*-isomorphism.*

Proof. — Consider the "if" part first. If g is the standard subkernel of $C \oplus \hat{C}$, the complex associated to g (see (3.12)) is acyclic and then $\omega(g)$ vanishes.

The complex associated to $f \oplus g$ is

$$0 \to K \oplus C \to B \oplus C \to 0 \to \dots$$

If we perform a surgery along B, we get a new complex

$$\dots \to K \oplus C \to \hat{B} \oplus C \to 0 \to \dots$$

and $\omega(f)$ is equal to $\omega(f')$, f' being the new $\mathscr{W}'$-subkernel

$$K \oplus C \xrightarrow{f \oplus g} \hat{B} \oplus C \oplus \widehat{(\hat{B} \oplus C)}.$$

It is easy to show that, for any isometry h of $\hat{B} \oplus C \oplus B \oplus \hat{C}$ leaving each element of $B \oplus \hat{C}$ fixed ($h \in UU_r(A)$ with the notations of [10]), the two $\mathscr{W}'$-subkernels f' and $h \circ f'$ represent the same quadratic $(2p+1)$-form over the same complex.

It suffices now to perform a surgery along $\hat{B} \oplus C$ to get a Λ-acyclic complex and $\omega(f)$ is zero.

Conversely, suppose $\omega(f)$ is zero. Let (C_*, q) be the quadratic complex associated

to f (see (3.12)). By lemma (3.8), there exists a $\mathscr{W}'$-non singular quadratic $(2p+2)$-pair $(\Sigma_* \to C_*, u)$ such that q is the boundary of u and $\Sigma_i \to C_i$ is a simple isomorphism for $i \neq -p-1$.

The map $\Sigma_* \to C_*$ has the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{1} & K & & & & \\ \downarrow & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d} & & & & \\ 0 & \longrightarrow & X & \xrightarrow{\alpha} & \Sigma & \xrightarrow{\beta} & B & \longrightarrow & 0 \end{array}$$

where $K \xrightarrow{d'} \Sigma$ is the complex Σ_*.

If u is represented by $\Sigma e_i \otimes \psi_i$, ψ_0 is a homomorphism from Σ to $\hat{\Sigma}$ satisfying

$$\tilde{\psi} \circ d' + \hat{\beta} \circ \varphi_0 = 0 \quad \text{with } \tilde{\psi} = \psi_0 - (-1)^p \hat{\psi}_0$$

and the following complex is Λ-acyclic:

$$0 \longrightarrow K \xrightarrow{d'} \Sigma \xrightarrow{\hat{\alpha} \circ \tilde{\psi}} \hat{X} \longrightarrow 0.$$

By the splitting lemma (3.5), there exist two finitely generated free A-modules C and I and a homomorphism $\gamma : C \to \Sigma \oplus I$ such that $(\gamma \oplus d') \otimes \Lambda$ is an isomorphism. After adding a kernel to Σ_*, we may suppose that I is zero and γ is a homomorphism from C to Σ.

Then the morphism $\hat{\alpha} \circ \tilde{\psi} \circ \gamma : C \to \hat{X}$ is a Λ-isomorphism, and the morphism $\tilde{\psi} \circ \gamma \oplus \hat{\beta} : C \oplus \hat{B} \to \hat{\Sigma}$ is also a Λ-isomorphism. That implies that the composite map from $C \oplus K$ to $\hat{C} \oplus B$

$$(\hat{\gamma} \circ \tilde{\hat{\psi}} \oplus \beta) \circ (\gamma \oplus d') = -(-1)^p \hat{\gamma} \circ \tilde{\psi} \circ \gamma \oplus (-1)^p \hat{\gamma} \circ \hat{\beta} \circ \varphi_0 \oplus \beta \circ \gamma \oplus d$$

is a Λ-isomorphism.

Let h be the homomorphism from $B \oplus \hat{B} \oplus C \oplus \hat{C}$ to itself defined by

$$h = 1 \oplus (-1)^p \hat{\gamma} \circ \hat{\beta} \oplus (-1)^{p+1} \beta \circ \gamma \oplus (-1)^{p+1} \hat{\gamma} \circ \tilde{\psi} \circ \gamma.$$

It is easy to check that h is an isometry leaving each element of $B \oplus \hat{C}$ fixed and that the composite map

$$K \oplus C \xrightarrow{h \circ (f \oplus g)} B \oplus \hat{B} \oplus C \oplus \hat{C} \longrightarrow B \oplus \hat{C}$$

is a Λ-isomorphism. ■

4. Geometric surgery

Throughout this section, we will suppose that A is the group ring $\mathbf{Z}\pi$ with an involution induced by a morphism $w : \pi \to \pm 1$, and that $\mathscr{W}$ is an exact symmetric class in $\mathscr{C}(A)$ containing any acyclic complex with torsion in the image of $\pi \to \tilde{K}_1(A)$.

We denote by (Λ, α) the $\mathscr{W}$-localization of A (1.9) and by $\mathscr{W}'$ the class of Λ-acyclic

complexes in $\mathscr{C}(A)$. The class $\mathscr{W}'$ is exact and symmetric and the $\mathscr{W}'$-localization of A is $(\Lambda, \tilde{K}_1(\Lambda))$. The fact that any element in $\tilde{K}_1(\Lambda)$ is the torsion of a complex $C \otimes \Lambda$, $C \in \mathscr{W}'$, will be proved in § 7.

Let f be a degree one normal map from the normal bundle of a compact n-dimensional (Top, PL or Diff)-manifold V to a (Top, PL or Diff)-bundle ξ over a connected $\mathscr{W}$-Poincaré complex with fundamental group π, such that the first Stiefel-Whitney class of ξ is w. We assume that f induces a $\mathscr{W}$-equivalence on the boundaries.

Suppose that any complex in $\mathscr{W}$ is Λ-acyclic. Then f induces a Λ-homology equivalence with torsion in α between the boundaries. Then we can use Wall's technique [10] in order to define $\sigma(f) \in L_n^{\alpha}(\Lambda)$ and $\sigma(f)$ depends only on the normal cobordism class (relative the boundary) of f, and vanishes if f is normally cobordant to a $\mathscr{W}$-equivalence.

(4.1) *Proof of theorem* (1.10) *in the case* $\mathscr{W} = \mathscr{W}'$

Suppose $n = 2p$ or $2p + 1 \geq 5$ and $\sigma(f) = 0$. After performing surgeries, we may suppose that the normal map $f: V \to X$ is p-connected.

Denote by C_* the complex $\Sigma^{-1}C_*(X, V; \mathbf{Z}\pi)$. If g is a homotopy inverse of the cap product $C^*(V; \mathbf{Z}\pi) \to C_*(V, \partial V; \mathbf{Z}\pi)$, the composite map

$$C_* \to C_*(V; \mathbf{Z}\pi) \to C_*(V, \partial V; \mathbf{Z}\pi) \xrightarrow{g} C^*(V; \mathbf{Z}\pi) \to \hat{C}_*$$

is a $\mathscr{W}'$-equivalence.

a) *The even dimensional case*

If $n = 2p$, we have a complex in $\mathscr{W}'$

$$\ldots \to C_{p+1} \to C_p \to \hat{C}_p \to \hat{C}_{p+1} \to \ldots$$

and by the splitting lemma (3.5), there exist two complexes L and L′ concentrated in dimension p and a $\mathscr{W}'$-equivalence $L \to C_* \oplus L'$.

After performing trivial surgeries, we may suppose that L′ is zero. Then the intersection and self-intersection forms on $H_{p+1}(X, V; \mathbf{Z}\pi)$ induce forms λ and μ on L_p and (L_p, λ, μ) is a $\mathscr{W}'$-special $(-1)^p$-form. Clearly, $\omega(L_p, \lambda, \mu)$ is sent to $\sigma(f)$ by the canonical map: $\varepsilon : \Gamma_n(\mathbf{Z}\pi, \mathscr{W}') \to L_n^h(\Lambda)$.

But ε is an isomorphism. This will be proved in § 8.

Then $\omega(L_p, \lambda, \mu)$ is zero and by proposition (3.14), the direct sum of (L_p, λ, μ) and a $(-1)^p$-kernel has a $\mathscr{W}'$-subkernel. We can realize the direct sum by trivial surgeries. So we may as well suppose that (L_p, λ, μ) has a $\mathscr{W}'$-subkernel $K \to L_p$. Now it suffices to perform surgeries along a basis of K, via the map $K \to L_p \to C_p \to H_{p+1}(X, V; \mathbf{Z}\pi)$, to get a $\mathscr{W}'$-equivalence.

b) *The odd dimensional case*

If $n = 2p + 1$, we have a complex in $\mathscr{W}'$

$$\ldots \to C_{p+2} \to C_{p+1} \to C_p \oplus \hat{C}_p \to \hat{C}_{p+1} \to \hat{C}_{p+2} \to \ldots$$

and by the splitting lemma (3.5), there exist two complexes L and L′ concentrated in dimension $p+1$ and a $\mathscr{W}'$-equivalence

$$L \to (\ldots \to C_{p+2} \to C_{p+1} \to 0 \to \ldots) \oplus L'.$$

So we get a $\mathscr{W}'$-equivalence $(\ldots \to 0 \to L_{p+1} \to C_p \oplus L'_{p+1} \to 0 \to \ldots) \to C$.

Denote by $K \xrightarrow{d} B$ the map $L_{p+1} \to C_p \oplus L'_{p+1}$, and consider the composite map $B \to C_p \to \pi_{p+1}(X, V)$. The basis of B induces maps from S^p to V homotopic to zero in X. These maps are covered by fibered maps and we get immersions $\alpha_i : S^p \to V$, which we can suppose to be disjoint embeddings. Let U be a regular neighborhood of the images of these embeddings, connectified with 1-handles. The group $H_{p+1}(\mathrm{pt}, \partial U; \mathbf{Z}\pi)$ endowed with intersection and self-intersection forms is the standard $(-1)^p$-kernel $B \oplus \hat{B}$.

The morphisms $K \to B$ and $K \to C_{p+1}$ induce a morphism from K to the relative homology group

$$H_{p+2}\left(\begin{array}{ccc} U & \longrightarrow & \mathrm{pt} \\ \downarrow & & \downarrow \\ V & \longrightarrow & X \end{array}; \mathbf{Z}\pi\right) = H_{p+2}\left(\begin{array}{ccc} \partial U & \longrightarrow & \mathrm{pt} \\ \downarrow & & \downarrow \\ V - \mathring{U} & \longrightarrow & X \end{array}; \mathbf{Z}\pi\right)$$

and we get, upon composing with the boundary, a morphism h from K to

$$H_{p+1}(\partial U \to \mathrm{pt}; \mathbf{Z}\pi) = H_{p+1}(\mathrm{pt}, \partial U; \mathbf{Z}\pi) = B \oplus \hat{B}.$$

It is not difficult to see that the image under h of the basis of K can be represented by spheres immersed in ∂U with zero intersections and self-intersections. To prove that h is a $\mathscr{W}'$-subkernel, it suffices to show that the complex $\ldots \to 0 \to K \to B \oplus \hat{B} \to \hat{K} \to 0 \to \ldots$ lies in $\mathscr{W}'$; and this follows from the $\mathscr{W}'$-equivalences

$$(\ldots \to 0 \to K \to B \to \ldots) \to C_* \to \hat{C}_* \to (\ldots \to 0 \to \hat{B} \to \hat{K} \to 0 \to \ldots).$$

Then we get a $\mathscr{W}'$-subkernel h and an invariant $\omega(h) \in \Gamma_n(\mathbf{Z}\pi, \mathscr{W}')$. By construction, $\omega(h)$ is sent to $\sigma(f)$ by the isomorphism $\varepsilon : \Gamma_n(\mathbf{Z}\pi, \mathscr{W}') \to L^h_n(\Lambda)$. Hence $\omega(h)$ is zero. By proposition (3.15), there exist a standard $(-1)^p$-kernel $C \oplus \hat{C}$ endowed with its standard subkernel $g : C \to C \oplus \hat{C}$ and an automorphism φ on $B \oplus \hat{B} \oplus C \oplus \hat{C}$ leaving each element of $B \oplus \hat{C}$ fixed, such that the composite map

$$K \oplus C \xrightarrow{\varphi \circ (h \oplus g)} B \oplus \hat{B} \oplus C \oplus \hat{C} \longrightarrow B \oplus \hat{C}$$

is a Λ-isomorphism.

If we add trivial disjoint embeddings β_j, from S^p to V, corresponding to the basis of C, the new $\mathscr{W}'$-subkernel is $h \oplus g$. If we perform surgeries along the spheres α_i, the $\mathscr{W}'$-subkernel $h \oplus g$ is replaced by $T \circ (h \oplus g)$, where T exchanges the factors B and $\hat{B}$. The new embedded spheres are the duals $\bar{\alpha}_i$ of α_i and β_j.

Now we can choose a regular homotopy depending on φ (see [10]) to get new disjoint embeddings α'_i and β'_j and the $\mathscr{W}'$-subkernel $T \circ \varphi \circ (h \oplus g)$.

If we perform surgeries along the spheres α_i' and β_j', we get the $\mathscr{W}'$-subkernel $T' \circ \varphi \circ (h \oplus g)$ where T' exchanges the factors C and $\hat{C}$.

So we obtain a new normal map $f' : V' \to X$ normally cobordant to f and a $\mathscr{W}'$-equivalence

$$(\ldots \to 0 \to K \oplus C \to B \oplus \hat{C} \to 0 \to \ldots) \to \Sigma^{-1} C_*(X, V'; \mathbf{Z}\pi).$$

Therefore f' is a $\mathscr{W}'$-equivalence. ■

(4.2) *Proof of theorem* (1.11) *in the case* $\mathscr{W} = \mathscr{W}'$

a) *The even dimensional case*

Suppose $n = 2p \geq 6$ and let $\sigma \in L_n^h(\Lambda)$. Since the morphism

$$\varepsilon : \Gamma_n(\mathbf{Z}\pi, \mathscr{W}') \to L_n^h(\Lambda)$$

is an isomorphism, σ is represented by a $\mathscr{W}'$-special $(-1)^p$-form (H, λ, μ) (3.13). Then we construct a normal map $f : W \to M \times I$ exactly as in ([10], p. 53). This normal map is an isomorphism over $M \times 0 \cup \partial M \times I$ and a $\mathscr{W}'$-equivalence over $M \times 1$ because λ is $\mathscr{W}'$-non singular. By construction, σ is the surgery invariant of f.

b) *The odd dimensional case*

Suppose $n = 2p + 1 \geq 7$ and let $\sigma \in L_n^h(\Lambda)$. We can represent σ by a trivial $(-1)^p$-kernel $B \oplus \hat{B}$ endowed with a $\mathscr{W}'$-subkernel $g : K \to B \oplus \hat{B}$ ((3.14)). After adding p-handles to $M \times I$ corresponding to the basis of B, we get a normal map $f_0 : W_0 \to M \times \left[0, \frac{1}{2}\right]$ which restricts to an isomorphism over $M \times 0 \cup \partial M \times \left[0, \frac{1}{2}\right]$. The inverse image M' of $M \times \frac{1}{2}$ is the connected sum of M and copies of $S^p \times S^p$ and the group $\pi_{p+1}\left(M \times \frac{1}{2}, M'\right)$ is the kernel $B \oplus \hat{B}$. Then we can perform surgeries along the image under g of the basis of K and we get a normal map

$$f_1 : W_1 \to M \times \left[\frac{1}{2}, 1\right].$$

These two normal maps induce a normal map $f : W \to M \times I$. It is easy to see that f restricts to an isomorphism over $M \times 0 \cup \partial M \times I$ and a $\mathscr{W}'$-equivalence over $M \times 1$. Moreover σ is the surgery obstruction $\sigma(f)$. ■

Actually this proof is almost identical with [10], p. 66.

Lemma **(4.3)**. — *Let* $\tau \in \tilde{K}_1(\Lambda)$. *Then there exist two matrices* u *and* v *with entries in* A *such that* $u \otimes \Lambda$ *and* $v \otimes \Lambda$ *are invertible and* $\tau = \tau(u \otimes \Lambda) - \tau(v \otimes \Lambda)$.

This lemma will be proved in § 7.

Lemma **(4.4)**. — *Let* M *be a connected compact* (Top, PL *or* Diff)*-manifold,* $\dim M \geq 5$. *Let* φ *be an epimorphism from* $\pi_1 M$ *to* π *and* τ *be an element of* $\tilde{K}_1(\Lambda)$. *Then, there exists a normal*

map $f: V \to M \times I$ *restricting to an isomorphism over* $M \times 0 \cup \partial M \times I$ *and such that* f *is a* Λ*-homology equivalence with torsion* τ.

Proof. — By lemma (4.3), there exist two matrices

$$u: \mathbf{Z}\pi^p \to \mathbf{Z}\pi^q \quad \text{and} \quad v: \mathbf{Z}\pi^r \to \mathbf{Z}\pi^s$$

such that $u \otimes \Lambda$ and $v \otimes \Lambda$ are invertible and

$$\tau = \tau(u \otimes \Lambda) - \tau(v \otimes \Lambda).$$

After adding q 1-handles to $M \times I$, we get a normal map $f_1: V_1 \to M \times I$ which is trivial on the handles. Now we add p 2-handles on V_1 along u and we get a normal map $f_2: V_2 \to M \times I$ restricting to an isomorphism over $M \times 0 \cup \partial M \times I$ and such that: $\tau(f_2) = \tau(u \otimes \Lambda) \in \widetilde{K}_1(\Lambda)$.

Let M' be the manifold $f_2^{-1}(M \times 1)$. After adding s trivial 2-handles and r 3-handles along v, we construct a normal map $f_3': V_3' \to M' \times I$ which restricts to an isomorphism over $M' \times 0 \cup \partial M' \times I$, and f_3' is a Λ-homology equivalence with torsion $-\tau(v \otimes \Lambda)$.

Then after gluing f_2 and f_3' together, we get a normal map $f: V \to M \times I$ which has the desired property. ■

(4.5) *Proof of theorem* (1.10) *in the general case*

Consider the Ranicki-Rothenberg exact sequence

$$L^h_{n+1}(\Lambda) \xrightarrow{\partial} H^n(\mathbf{Z}/2, \widetilde{K}_1(\Lambda)/\alpha) \to L^\alpha_n(\Lambda) \to L^h_n(\Lambda).$$

Suppose that $\sigma(f)$ vanishes in $L^\alpha_n(\Lambda)$. Then the surgery invariant of f is zero in $L^h_n(\Lambda)$ and f is normally cobordant (relative the boundary) to a normal map $f_1: V_1 \to X$ which is a $\mathscr{W}'$-equivalence. Moreover f_1 is $\left[\frac{n}{2}\right]$-connected.

Let $\tau \in \widetilde{K}_1(\Lambda)$ be the torsion of f_1. Since $\sigma(f)$ is zero, there exists an element $u \in L^h_{n+1}(\Lambda)$ such that ∂u is represented by τ. But f_1 is 2-connected and $\pi_1 V_1 = \pi$. Then, by theorem (1.11) (proved in the case $\mathscr{W} = \mathscr{W}'$, $M = V_1$), there exists a normal map $g_1: W_1 \to V_1 \times I$ restricting to an isomorphism over $V_1 \times 0 \cup \partial V_1 \times I$ and such that $\sigma(g) = u$. This normal map induces a normal cobordism (relative the boundary) from f_1 to a normal map $f_2: V_2 \to X$ which is a $\mathscr{W}'$-equivalence. Moreover the torsion of f_2 is zero in $H^n(\mathbf{Z}/2, \widetilde{K}_1(\Lambda)/\alpha)$.

Then, there exists $\tau' \in \widetilde{K}_1(\Lambda)$ such that: $\tau(f_2) \equiv \tau' + (-1)^n \bar{\tau}' \pmod{\alpha}$.

By lemma (4.4), there exists a normal map $g_2: W_2 \to V_2 \times I$ restricting to an isomorphism over $V_2 \times 0 \cup \partial V_2 \times I$ such that g_2 is a $\mathscr{W}'$-equivalence with torsion $-\tau'$. This normal map induces a normal cobordism from f_2 to $f_3: V_3 \to X$ and f_3 is a $\mathscr{W}'$-equivalence with torsion in $\alpha \subset \widetilde{K}_1(\Lambda)$. Thus, theorem (1.10) is a trivial consequence of the following lemma (proved in § 7):

Lemma **(4.6)**. — *Any finite* Λ*-complex which is* Λ*-acyclic with torsion in* α *lies in* $\mathscr{W}$.

(4.7) *Proof of theorem* (1.11) *in the general case*

Consider again the Ranicki-Rothenberg exact sequence

$$H^n(\mathbf{Z}/2, \widetilde{K}_1(\Lambda)/\alpha) \to L_n^\alpha(\Lambda) \to L_n^h(\Lambda) \to H^{n-1}(\mathbf{Z}/2, \widetilde{K}_1(\Lambda)/\alpha).$$

Let σ be an element of $L_n^\alpha(\Lambda)$ and σ' be the image of σ in $L_n^h(\Lambda)$. By theorem (1.11) (proved in the case $\mathscr{W} = \mathscr{W}'$) there exists a normal map $f_1 : W_1 \to M \times I$ restricting to an isomorphism over $M \times 0 \cup \partial M \times I$ and such that the surgery obstruction of f_1 is σ' in $L_n^h(\Lambda)$. Let V_1 be the inverse image of $M \times 1$. Since σ' is sent to zero in $H^{n-1}(\mathbf{Z}/2, \widetilde{K}_1(\Lambda)/\alpha)$ the torsion of $f_1 : V_1 \to M$ is congruent to $\tau - (-1)^n\bar{\tau} \pmod{\alpha}$ for some $\tau \in \widetilde{K}_1(\Lambda)$.

Then, by lemma (4.4), we can glue together f_1 and a normal map $f_1' : W_1' \to M \times I$ in order to construct a new normal map $f_2 : W_2 \to M \times I$ such that

(i) f_1 and f_2 have the same invariant in $L_n^h(\Lambda)$;
(ii) f_2 restricts over $M \times 1$ to a $\mathscr{W}'$-equivalence with torsion in α.

By construction, $\sigma(f_2) - \sigma$ is the image of an element of $H^n(\mathbf{Z}/2, \widetilde{K}_1(\Lambda)/\alpha)$ represented by $\tau' \in \widetilde{K}_1(\Lambda)$. By lemma (4.4), there exists a normal map

$$f_2' : W_2' \to f_2^{-1}(M \times 1) \times I$$

restricting to an isomorphism over $f_2^{-1}(M \times 1) \times 0 \cup \partial f_2^{-1}(M \times 1) \times I$ and such that f_2' is a $\mathscr{W}'$-equivalence with torsion $-\tau'$. Then, after gluing f_2 and f_2' together, we get a normal map $f : W \to M \times I$ with surgery obstruction σ. ■

5. Localization in the category of graded differential modules

Consider now the general case: A is a ring and $\mathscr{W}$ is an exact class in $\mathscr{C}(A)$. The $\mathscr{W}$-localization of A is (Λ, α).

Definition **(5.1)**. — A complex $C \in \mathscr{W}$ will be called *$\mathscr{W}$-splittable* if there exist, for any n, an n-dimensional complex $C' \in \mathscr{W}$ and an $(n-1)$-connected morphism from C' to C.

The class of $\mathscr{W}$-splittable complexes of $\mathscr{W}$ will be called $\mathscr{W}^s$.

Lemma **(5.2)**. — *The class $\mathscr{W}^s$ is exact.*

Proof. — The class $\mathscr{W}^s$ is clearly stable under simple homotopy equivalence and under any suspension.

Now let $0 \to C \to C' \to C'' \to 0$ be a s-exact sequence of finite A-complexes. Suppose that C and C' are $\mathscr{W}$-splittable.

Let n be an integer. There exists a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & C' & \longrightarrow & C'' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & & & \\ & & \overline{C} & & \overline{C}' & & & & \end{array}$$

such that $\overline{C}$ (respectively $\overline{C}'$) is an $(n-1)$-dimensional (respectively n-dimensional) complex in $\mathscr{W}$ and the morphism $\overline{C} \to C$ (respectively $\overline{C}' \to C'$) is $(n-2)$-connected (respectively $(n-1)$-connected). The obstructions to factoring the morphism $\overline{C} \to C'$ through $\overline{C}'$ are in the groups $H^p(\overline{C}, H_p(C', \overline{C}'))$ which are all trivial. So we get a morphism $\overline{C} \to \overline{C}'$. It is easy to see that the mapping cone $\overline{C}''$ of $\overline{C} \to \overline{C}'$ is an n-dimensional complex in $\mathscr{W}$ and the induced morphism from $\overline{C}''$ to C'' is $(n-1)$-connected.

Then C'' is $\mathscr{W}$-splittable and, since $\mathscr{W}^s$ is stable under simple homotopy equivalence and suspension, it is easy to prove that $\mathscr{W}^s$ is exact. ■

Lemma (5.3). — $\mathscr{W}^{ss} = \mathscr{W}^s$.

Proof. — The proof is by induction on the length of the complex. Clearly any complex in $\mathscr{W}^s$ of length two is $\mathscr{W}^s$-splittable. Suppose any complex in $\mathscr{W}^s$ of length $< p$ is $\mathscr{W}^s$-splittable, and let $C \in \mathscr{W}^s$ be a $\mathscr{W}$-splittable complex of length p. The complex C is n-dimensional and $(n-p)$-connected. Since C is $\mathscr{W}$-splittable, there exist an $(n-p+2)$-dimensional complex $C' \in \mathscr{W}$ and an $(n-p+1)$-connected morphism $C' \to C$.

The length of C' is 2 and C' lies in $\mathscr{W}^{ss}$. Then the mapping cone of $C' \to C$ is a complex in $\mathscr{W}^s$ of length $p-1$. By induction the mapping cone of $C' \to C$ lies in $\mathscr{W}^{ss}$ and $C \in \mathscr{W}^{ss}$. ■

We will work out a theory of localization in the category of graded differential modules. Unfortunately, the category $\mathscr{C}(A)$ is too small to do that and we must consider the category $\overline{\mathscr{C}}(A)$ of graded differential free A-modules bounded from below.

Notations (5.4). — Denote by $\mathscr{W}_0$ the exact class of finite A-complexes C such that $C \oplus \Sigma C$ lies in $\mathscr{W}$ and by $\mathscr{W}_0^s$ the class $(\mathscr{W}_0)^s$. We use $\overline{\mathscr{W}}$ to denote the class of complexes $C \in \overline{\mathscr{C}}(A)$ such that any morphism from a finite A-complex to C factorizes through a complex in $\mathscr{W}_0^s$.

A morphism f in $\overline{\mathscr{C}}(A)$ is a $\overline{\mathscr{W}}$-equivalence if the mapping cone of f lies in $\overline{\mathscr{W}}$.

Definition (5.5). — A complex $C \in \overline{\mathscr{C}}(A)$ will be called *local* if any morphism from a complex $C' \in \overline{\mathscr{W}}$ to C is null homotopic.

A morphism $f: C \to C'$ is a *localization* of C if f is a $\overline{\mathscr{W}}$-equivalence and C' is local.

Clearly, if C has a localization, this localization is unique up to homotopy.

Proposition (5.6). — *Any complex in* $\overline{\mathscr{C}}(A)$ *has a localization.*

Proof. — Let $C \in \overline{\mathscr{C}}(A)$. Suppose C is $(n-1)$-connected. Let $\mathscr{A}$ be the set of morphisms $K \to C$ such that K is a $(n-2)$-connected complex in $\mathscr{W}_0^s$. Let $\Phi(C)$ be the mapping cone of the morphism $\bigoplus_{\mathscr{A}} K \to C$.

Clearly $\Phi(C)$ is $(n-1)$-connected and we can carry on this process:

$$C \to \Phi(C) \to \Phi^2(C) \to \Phi^3(C) \to \dots$$

Denote by E(C) the limit of this system.

The complex $\Phi^{p+1}(C)/\Phi^p(C)$ is a direct sum of complexes in $\mathscr{W}_0^s$. Then, by induction, it is easy to show that $\Phi^p(C)/C$ lies in $\overline{\mathscr{W}}$. But, by construction, E(C) is $(n-1)$-connected and $E(C) \in \overline{\mathscr{C}}(A)$. Moreover $E(C)/C$ lies in $\overline{\mathscr{W}}$ and $C \to E(C)$ is a $\overline{\mathscr{W}}$-equivalence.

Now, let $\mathscr{C}$ be the class of complexes $C' \in \overline{\mathscr{C}}(A)$ such that any morphism from C' to E(C) is null homotopic. The class $\mathscr{C}$ is stable under homotopy equivalence and extension. The last problem is to prove that $\mathscr{C}$ contains $\overline{\mathscr{W}}$.

Let $K \in \mathscr{W}_0^s$. Since any complexe in $\mathscr{W}_0^s$ is $\mathscr{W}_0^s$-splittable ((5.3)), there exists a homotopy s-exact sequence $0 \to K' \to K \to K'' \to 0$ such that K' is a $n-1$-dimensional complex in $\mathscr{W}_0^s$ and K'' an $(n-2)$-connected complex in $\mathscr{W}_0^s$. Clearly $K' \in \mathscr{C}$. Let f be a morphism from K'' to E(C). Since K'' is finitely generated, the image of f is contained in some $\Phi^p(C)$ and f is homotopic to zero in $\Phi^{p+1}(C)$. Hence $K'' \in \mathscr{C}$ and $K \in \mathscr{C}$ too. Then $\mathscr{C}$ contains the class $\mathscr{W}_0^s$.

If $K \in \overline{\mathscr{C}}(A)$, denote by $\mathscr{H}^i(K)$ the group $[\Sigma^{-i}K, E(C)]$ of homotopy classes of morphisms from $\Sigma^{-i}K$ to E(C). The group $\mathscr{H}^i(K)$ vanishes for any $K \in \mathscr{W}_0^s$ and any $i \in \mathbf{Z}$, and we must prove that $\mathscr{H}^0(K)$ is zero for any $K \in \overline{\mathscr{W}}$.

If $K \in \overline{\mathscr{W}}$, K has the homotopy type of the limit of a directed system K_i, $K_i \in \mathscr{W}_0^s$, and we have a spectral sequence with the following E_2 term:

$$E_2^{pq} = \varprojlim{}^p \mathscr{H}^q(K_i).$$

The E_2 term is trivial and the spectral sequence converges to $\mathscr{H}^*(K)$. Then $\mathscr{H}^0(K)$ vanishes and $C \to E(C)$ is a localization of C. ■

The localization plays an important role in view of the following propositions:

Proposition (5.7). — *Let* C *and* C' *be two complexes in* $\mathscr{C}(A)$, *with* $\dim C = n$. *Let* $C' \xrightarrow{\varepsilon} E(C')$ *be a localization of* C'. *Then, for any morphism* $f: C \to E(C')$, *there exist an n-dimensional complex* $\overline{C} \in \mathscr{C}(A)$ *and a homotopy commutative diagram*

$$\begin{array}{ccc} \overline{C} & \longrightarrow & C' \\ \downarrow & & \downarrow \varepsilon \\ C & \xrightarrow{f} & E(C') \end{array}$$

such that $\overline{C} \to C$ *is a* $\mathscr{W}_0^s$*-equivalence.*

Proposition **(5.8).** — *Let* C *and* C′ *be two complexes in* $\mathscr{C}(A)$ *with* $\dim C = n$. *Let* $C' \xrightarrow{\varepsilon} E(C')$ *be a localization of* C′. *Let* $f: C \to C'$ *be a map such that* $\varepsilon \circ f$ *is null homotopic. Then, there exists a* $\mathscr{W}_0^s$*-equivalence* $\overline{C} \to C$ *such that* $\overline{C} \in \mathscr{C}(A)$ *is n-dimensional and the composite map* $\overline{C} \to C \xrightarrow{f} C'$ *is null homotopic.*

Proof of (5.7). — Suppose ε is monic with free cokernel. We have an exact sequence

$$0 \to C' \to E(C') \to K' \to 0, \qquad K' \in \overline{\mathscr{W}}.$$

Let us construct the homotopy commutative diagram

$$\begin{array}{ccccc} \overline{C} & \longrightarrow & C' & & \\ \downarrow & & \downarrow & & \\ C & \longrightarrow & E(C') & & \\ \downarrow & \searrow & \downarrow & & \\ K & \longrightarrow & L & \longrightarrow & K' \end{array}$$

in the following way: Since C is finitely generated, the map $C \to K'$ factorizes through a complex $L \in \mathscr{W}_0^s$ and by (5.3), there exist an $(n+1)$-dimensional complex $K \in \mathscr{W}_{0s}$ and an n-connected map $K \to L$. Then there is no obstruction to factorize the map $C \to L$ through K.

Let $\overline{C}$ be the homotopy kernel of $C \to K$. It is easy to check that $\overline{C}$ is n-dimensional and that the map $\overline{C} \to E(C')$ factorizes through C′. ■

Proof of (5.8). — Suppose ε is epic with kernel $K' \in \overline{\mathscr{W}}$. Since the composite map $C \xrightarrow{f} C' \xrightarrow{\varepsilon} E(C')$ is null homotopic, f is homotopic to a map $f': C \to K'$. Then f' factorizes through a complex $L \in \mathscr{W}_0^s$. By (5.3), there exist an $(n+1)$-dimensional complex $K \in \mathscr{W}_0^s$ and an n-connected map $K \to L$. As before the map $C \to L$ retracts in K and the homotopy kernel of $C \to K$ has the desired properties. ■

6. The ring Λ

In this section, we will compute the homology groups of the localization of a complex $C \in \overline{\mathscr{C}}(A)$ in terms of the ring Λ defined in (1.8).

Let M be a (right) A-module. This module will be said local if any $q \times p$ matrix in Σ induces an isomorphism $\mathrm{Hom}(A^q, M) \to \mathrm{Hom}(A^p, M)$.

Lemma **(6.1).** — *A module* M *is local if and only if* $H^n(C, M)$ *vanishes for any* $n \in \mathbf{Z}$ *and any* $C \in \overline{\mathscr{W}}$.

Proof. — Suppose that $H^n(C, M)$ vanishes for any $n \in \mathbf{Z}$ and any $C \in \overline{\mathscr{W}}$. If u is a matrix in Σ, denote by C the 1-dimensional complex

$$\ldots \to 0 \to A^p \xrightarrow{u} A^q \to 0 \to \ldots$$

Then $C \oplus \Sigma C$ lies in $\mathscr{W}$ (see (1.7)) and C is a complex of $\mathscr{W}_0^s \subset \overline{\mathscr{W}}$. Hence $H^*(C, M)$ vanishes and M is local.

Conversely, suppose M is local and denote by $\mathscr{C}$ the class of complexes $C \in \overline{\mathscr{C}}(A)$ such that $H^*(C, M) = 0$.

If C is a complex of length two in $\mathscr{W}_0^s$, C lies in $\mathscr{C}$ by definition.

If C is a complex in $\mathscr{W}_0^s$ of length $p > 2$, there exists a homotopy s-exact sequence

$$0 \to C' \to C \to C'' \to 0$$

such that C′ and C″ are complexes in $\mathscr{W}_0^s$ of length $< p$.

By induction, C is in $\mathscr{C}$ and $\mathscr{C}$ contains the class $\mathscr{W}_0^s$.

If $C \in \overline{\mathscr{W}}$, C is the limit of a directed system $C_i \in \mathscr{W}_0^s$ and we have a spectral sequence with E_2 term $E_2^{pq} = \varprojlim^p H^q(C_i, M)$. The E_2 term is zero and the spectral sequence converges to $H^*(C, M)$. Hence $H^*(C, M)$ vanishes and the lemma is proved.

Corollary **(6.2).** — *A complex* $C \in \overline{\mathscr{C}}(A)$ *is local if and only if* $H_n(C)$ *is local for any* $n \in \mathbf{Z}$.

Proof. — If K is a complex, denote by $\mathscr{H}^i(K)$ the group of homotopy classes of maps $\Sigma^{-i}K \to C$. We have a spectral sequence with E_2 term

$$E_2^{pq} = H^p(K, H_{-q}(C))$$

and this spectral sequence usually converges to $\mathscr{H}^*(K)$.

Suppose C is local and let $K \in \mathscr{W}_0^s$ be a complex of length 2 defined by a matrix $u \in \Sigma$. Then the above spectral sequence collapses to exact sequences

$$0 \to H^n(K, H_{-i}(C)) \to \mathscr{H}^{n+i}(K) \to H^{n-1}(K, H_{-i-1}(C)) \to 0 \quad (n = \dim K).$$

Then all the groups $H^*(K, H_i(C))$ vanish and $H_i(C)$ is local for any $i \in \mathbf{Z}$.

Conversely suppose $H_*(C)$ is local. Then for any $K \in \overline{\mathscr{W}}$, the E_2 term of the above spectral sequence vanishes and the spectral sequence converges to $\mathscr{H}^*(K)$. Hence this last group vanishes and C is local. ■

Lemma **(6.3).** — *Localization respects exact sequences.*

Proof. — Let $0 \to C \to C' \to C'' \to 0$ be a short exact sequence in $\overline{\mathscr{C}}(A)$. Take localizations $C \to E(C)$ and $C' \to E(C')$ of C and C′. We get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & C' & \longrightarrow & C'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & E(C) & \longrightarrow & E(C) & & & & \end{array}$$

Let $E(C'')$ be the mapping cone of $E(C) \to E(C')$. We have a homotopy commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & C' & \longrightarrow & C'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E(C) & \longrightarrow & E(C') & \longrightarrow & E(C'') & \longrightarrow & 0 \end{array}$$

Clearly $E(C'')$ is local and the map $C'' \to E(C'')$ is a $\overline{\mathscr{W}}$-equivalence. Then $C'' \to E(C'')$ is a localization of C'' and the result follows.

Lemma **(6.4)**. — *Localization respects direct sums.*

Proof. — Let $C_i \in \overline{\mathscr{C}}(A)$ be a class of complexes. Suppose that C_i is $(n-1)$-connected for any i, and take localizations $C_i \to E(C_i)$.

Clearly the mapping cone of $\bigoplus_i C_i \to \bigoplus_i E(C_i)$ lies in $\overline{\mathscr{W}}$ and, by (6.2), the sum $\bigoplus_i E(C_i)$ is local. Then the map $\bigoplus_i C_i \to \bigoplus_i E(C_i)$ is a localization of $\bigoplus_i C_i$. ■

Now if C is a complex in $\overline{\mathscr{C}}(A)$, denote by $\Phi_n(C)$ the group $H_n(E(C))$ where $C \to E(C)$ is a localization of C.

If M is a (right) A-module, we will also denote by $\Phi_n(M)$ the group $\Phi_n(C)$ where C is a free resolution of M. The Φ_n's are functors and we have a natural transformation $\eta : M \to \Phi_0(M)$.

Clearly, if M is local, a resolution of M is local ((6.2)). So η is bijective and $\Phi_i(M)$ vanishes for $i \neq 0$.

Lemma **(6.5)**. — *Let* M *be an* A*-module. Then, there is a natural homomorphism*

$$\varepsilon' : M \otimes_{\mathbf{Z}} \Phi_0(A) \to \Phi_0(M),$$

such that the following diagram commutes:

$$\begin{array}{ccc} M \otimes_{\mathbf{Z}} A & \longrightarrow & M \\ \downarrow{\scriptstyle 1 \otimes \eta} & & \downarrow{\scriptstyle \eta} \\ M \otimes_{\mathbf{Z}} \Phi_0(A) & \xrightarrow{\varepsilon'} & \Phi_0(M) \end{array}$$

Proof. — Let $m \in M$. Denote by $\varphi : A \to M$ the homomorphism $a \mapsto ma$. By setting $\varepsilon'(m, x) = \Phi_0(\varphi)(x)$, for any $x \in \Phi_0(A)$, we get a map $\varepsilon' : M \times \Phi_0(A) \to \Phi_0(M)$. Clearly, $\varepsilon'(m, x)$ is $\mathbf{Z}$-linear on x and, since Φ_0 respects direct sums, it is easy to see that $\varepsilon'(m, x)$ is $\mathbf{Z}$-linear on m. ■

Lemma **(6.6).** — *The module* $\Phi_0(A)$ *is a ring and* ε' *induces a homomorphism*

$$\varepsilon : M \otimes_A \Phi_0(A) \to \Phi_0(M).$$

Proof. — Let $m \in M$ and $x, y \in \Phi(A)$. Denote by $\varphi : A \to M$ the map $a \mapsto ma$ and by $\psi : A \to \Phi_0(A)$ the map $a \to xa$.

We have a commutative diagram

$$\begin{array}{ccccc}
A & \xrightarrow{\varphi} & M & & \\
\downarrow & & \downarrow & & \\
A & \xrightarrow{\psi} & \Phi_0(A) & \xrightarrow{\Phi_0(\varphi)} & \Phi_0(M) \\
\downarrow & & \sim\downarrow\eta & & \sim\downarrow\eta \\
\Phi_0(A) & \xrightarrow{\Phi_0(\psi)} & \Phi_0^2(A) & \xrightarrow{\Phi_0^2(\varphi)} & \Phi_0^2(M)
\end{array}$$

and the following formulas:

$$\Phi_0^2(\varphi) \circ \Phi_0(\psi)(y) = \Phi_0^2(\varphi)(\varepsilon'(x, y)) = \eta\varepsilon'(m, \eta^{-1}\varepsilon'(x, y))$$

$$\Phi_0[\Phi_0(\varphi) \circ \psi](y) = \varepsilon'(\varepsilon'(m, x), y)$$

whence $$\eta\varepsilon'(m, \eta^{-1}\varepsilon'(x, y)) = \varepsilon'(\varepsilon'(m, x), y).$$

Then the map $\eta^{-1}\varepsilon'$ from $\Phi_0(A) \otimes_{\mathbf{Z}} \Phi_0(A)$ to $\Phi_0(A)$ induces a ring structure on $\Phi_0(A)$ and η is a ring homomorphism from A to $\Phi_0(A)$. Moreover ε' induces a homomorphism $\varepsilon : M \otimes_A \Phi_0(A) \to \Phi_0(M)$. ■

Lemma **(6.7).** — *The ring homomorphism* $A \to \Phi_0(A)$ *is isomorphic to the homomorphism* $A \to \Lambda$.

Proof. — Let $A \to B$ be a ring homomorphism. The A-module B is local if and only if any $q \times p$ matrix $u \in \Sigma$ induces an isomorphism $u^* : \mathrm{Hom}(A^q, B) \to \mathrm{Hom}(A^p, B)$. But the matrix of u^* is the transpose of $u \otimes B$. Then, B is local if and only if, for any $u \in \Sigma$, $u \otimes B$ is invertible.

Hence, for any matrix $u \in \Sigma$, $u \otimes \Phi_0(A)$ is invertible and we will prove that $\Phi_0(A)$ is universal with respect to this property.

Let $A \to B$ be a ring homomorphism such that $u \otimes B$ is invertible for any $u \in \Sigma$. Let us choose free resolutions A_* and B_* of A and B and a localization $A_* \to E(A_*)$ of A_*. Since B is local, there exists an extension $E(A_*) \to B_*$ unique up to homotopy. Then there exists a unique extension $\Phi_0(A) \to B$ of $A \to B$.

Consider the following diagram:

$$\begin{array}{ccc} A & \longrightarrow & \Phi_0(A) \\ \downarrow & & \downarrow \\ B & \xrightarrow{\sim} & \Phi_0(B) \end{array}$$

All the morphisms of this diagram are ring homomorphisms and $B \xrightarrow{\sim} \Phi_0(B)$ is an isomorphism. Then the extension $\Phi_0(A) \to B$ is a ring homomorphism. So $A \to \Phi_0(A)$ satisfies the universal property of Λ and $A \to \Phi_0(A)$ is isomorphic to $A \to \Lambda$.

Lemma **(6.8).** — *For any module* M, *the morphism* $\varepsilon : M \otimes \Lambda \to \Phi_0(M)$ *is an isomorphism.*

Proof. — By lemma (6.4), the functor Φ_0 respects direct sums and ε is an isomorphism if M is free. Moreover, by lemma (6.5), Φ_0 is right exact and ε is an isomorphism for any M. ■

Corollary **(6.9).** — *If* M *is local, the canonical map* $M \to M \otimes \Lambda$ *is an isomorphism.*

Lemma **(6.10).** — *If* M *is local,* $\mathrm{Tor}_1(M, \Lambda)$ *is trivial.*

Proof. — Choose a free module L and an exact sequence

$$0 \to N \to L \to M \to 0.$$

By lemma (6.4), we have an exact sequence

$$\Phi_1(M) \to \Phi_0(N) \to \Phi_0(L) \to \Phi_0(M) \to 0.$$

If M is local, $\Phi_1(M)$ is zero and $\Phi_0(N) \to \Phi_0(L)$ is monic. But this map is isomorphic to the map $N \otimes \Lambda \to L \otimes \Lambda$ and its kernel is $\mathrm{Tor}_1(M, \Lambda)$. ■

Corollary **(6.11).** — *Let* $C \in \overline{\mathscr{C}}(A)$ *be an* $(n-1)$*-connected local complex. Then the canonical map* $H_i(C) \to H_i(C \otimes \Lambda)$ *is an isomorphism for* $i \leq n$ *and an epimorphism for* $i = n + 1$.

Proof. — We have a spectral sequence with E^2 term $E^2_{pq} = \mathrm{Tor}_p(H_q(C), \Lambda)$ which converges to $H_*(C \otimes \Lambda)$. Since C is local, $H_*(C)$ is local and, by (6.9) and (6.10), we have

$$E^2_{0q} = \mathrm{Tor}_0(H_q(C), \Lambda) = H_q(C),$$

$$E^2_{1q} = \mathrm{Tor}_1(H_q(C), \Lambda) = 0.$$

The result follows.

Theorem **(6.12).** — *Let* C *and* C' *be two finite* Λ*-complexes and suppose that* $C' \otimes \Lambda$ *is* $(n-1)$*-connected. Then we have the following properties*:

(i) *If* $H^i(C, \Lambda)$ *vanishes for* $i > n+1$ *and* f *is a morphism from* $C \otimes \Lambda$ *to* $C' \otimes \Lambda$, *there exist a* $\mathscr{W}_0^s$*-equivalence* $\varepsilon : \overline{C} \to C$ *with* $\dim \overline{C} = \dim C$ *and a morphism* $g : \overline{C} \to C'$ *such that* $g \otimes \Lambda$ *is homotopic to* $f \circ (\varepsilon \otimes \Lambda)$.

(ii) *If* $H^i(C, \Lambda)$ *vanishes for* $i > n$ *and* f *is a morphism from* C *to* C' *such that* $f \otimes \Lambda$ *is null homotopic, there exists a* $\mathscr{W}_0^s$*-equivalence* $\varepsilon : \overline{C} \to C$, *with* $\dim \overline{C} = \dim C$ *such that* $f \circ \varepsilon$ *is null homotopic.*

Proof. — Let $C' \to E(C')$ be a localization of C' and consider the following diagram:

$$\begin{array}{ccc} C' & \longrightarrow & C' \otimes \Lambda \\ \downarrow & & \downarrow \\ E(C') & \longrightarrow & E(C') \otimes \Lambda \end{array}$$

If f is a morphism from $C \otimes \Lambda$ to $C' \otimes \Lambda$, f is defined by an Λ-homomorphism $f' : C \to C' \otimes \Lambda$.

The obstructions to lift the composite map $f'' : C \to C' \otimes \Lambda \to E(C') \otimes \Lambda$ through $E(C')$ lie in the groups $H^p(C, H_p(E(C') \otimes \Lambda, E(C')))$. Let H_p be the module $H_p(E(C') \otimes \Lambda, E(C'))$. Since $E(C')$ is local, H_p is a Λ-module and is trivial for $p \leq n+1$, by (6.11). But $H^i(C, \Lambda)$ vanishes for $i > n+1$ and the localization $E(\hat{C})$ of $\hat{C}$ is $(-n-2)$-connected. Then we have, for $p > n+1$,

$$H^p(C, H_p) = H_{-p}(\hat{C}, H_p) = H_{-p}(E(\hat{C}), H_p) = 0.$$

Then f'' lifts through $E(C')$ and, by (5.7), there exist a complex $\overline{C} \in \mathscr{C}(\Lambda)$ with $\dim \overline{C} = \dim C$, a $\mathscr{W}_0^s$-equivalence $\varepsilon : \overline{C} \to C$ and a morphism $g : \overline{C} \to C'$ such that the following diagram is homotopy commutative:

$$\begin{array}{ccc} \overline{C} & \xrightarrow{g} & C' \\ {\scriptstyle \varepsilon}\downarrow & & \downarrow \\ C & \xrightarrow{f''} & E(C') \otimes \Lambda \end{array}$$

On the other hand, any complex in $\mathscr{W}_0^s$ of length two is Λ-acyclic and, by induction, any complex in $\mathscr{W}_0^s$ is Λ-acyclic. This implies that any complex in $\overline{\mathscr{W}}$ is Λ-acyclic and $C' \otimes \Lambda \to E(C') \otimes \Lambda$ is a homotopy equivalence.

Then the following diagram commutes up to homotopy:

$$\begin{array}{ccc} \overline{C} & \xrightarrow{g} & C' \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow \\ C & \xrightarrow{f'} & C'\otimes\Lambda \end{array}$$

and part (i) of the theorem is proved.

Suppose now f is a morphism from C to C′ with $\dim C = n$. If $f\otimes\Lambda$ is null homotopic, the composite map $C \to C' \to E(C')\otimes\Lambda$ is null homotopic and, by obstruction, the map $C \to E(C')$ is null homotopic. Then we may apply (5.8) and the theorem is proved.

7. The structure of $\mathscr{W}$

Lemma (7.1). — *The class $\mathscr{W}_0^s$ is the class $\mathscr{W}'$ of Λ-acyclic complexes in $\mathscr{C}(A)$.*

Proof. — If C is a complex in $\mathscr{W}_0$ of length two, it is Λ-acyclic by definition of Λ. Then, by induction, any complex in $\mathscr{W}_0^s$ is Λ-acyclic.

Conversely, let $C \in \mathscr{C}(A)$ be a Λ-acyclic complex and $C \to E(C)$ be a localization of C. Since C is Λ-acyclic, E(C) is Λ-acyclic too. Suppose E(C) is not acyclic and let H_n be the first non trivial homology group of E(C). The module H_n is local and

$$H_n \simeq H_n\otimes\Lambda \simeq H_n(E(C)\otimes\Lambda) = 0.$$

Hence E(C) is acyclic and $C \in \overline{\mathscr{W}}$. Since C is finite, the identity $C \to C$ factorizes through a complex $K \in \mathscr{W}_0^s$ and we get a split exact sequence

$$0 \to C' \to K \to C \to 0.$$

This implies that $C\oplus C'$ has the simple homotopy type of K and $C\oplus C'$ lies in $\mathscr{W}_0^s$.

On the other hand, ΣK has the simple homotopy type of the mapping cone of the zero map $C' \to \Sigma C$ and $C' \to \Sigma C$ is a $\mathscr{W}_0^s$-equivalence. Then $C\oplus\Sigma C$ lies in $\mathscr{W}_0^s$.

Now we will prove that C is in $\mathscr{W}_0^s$ by induction on the length of C.

If the length of C is two, $C\oplus\Sigma C$ is contained in $\mathscr{W}_0$ and $C\oplus\Sigma C\oplus\Sigma C\oplus\Sigma^2 C$ lies in $\mathscr{W}$. But $\Sigma(C\oplus\Sigma C\oplus\Sigma C\oplus\Sigma^2 C)$ is the mapping cone of the zero map $\Sigma C\oplus\Sigma C\oplus\Sigma^2 C \to \Sigma C$ which is a $\mathscr{W}$-equivalence. Then $C\oplus\Sigma C$ lies in $\mathscr{W}$ and C lies in $\mathscr{W}_0$. Since the length of C is two, C lies in $\mathscr{W}_0^s$.

If the length of C is $p > 2$, C is n-dimensional and $(n-p)$-connected. Since $C\oplus\Sigma C$ is $\mathscr{W}_0^s$-splittable, there exist an $(n-p+2)$-dimensional complex $K \in \mathscr{W}_0^s$ and an $(n-p+1)$-connected morphism $f\oplus g$ from K to $C\oplus\Sigma C$.

The morphism $f\oplus 0$ is clearly $(n-p+1)$-connected. Let M be the mapping cone of f. The complex $M\oplus\Sigma M$ is the mapping cone of $f\oplus\Sigma f$ and lies in $\mathscr{W}_0^s$. But the length of M is $p-1$. By induction, M lies in $\mathscr{W}_0^s$ and C lies in $\mathscr{W}_0^s$ too. ■

(7.2) *Proof of the splitting lemma* (3.5)

Let C be a complex in $\mathscr{W}'$ and let n be an integer. Since $\mathscr{W}' = \mathscr{W}_0^s$, C is $\mathscr{W}'$-splittable and there exist an n-dimensional complex $C' \in \mathscr{W}'$ and an $(n-1)$-connected morphism $C' \to C$.

Up to simple homotopy type, we may suppose that the map $C'_i \to C_i$ is bijective for $i < n-1$ and is epic with free kernel L'_n for $i = n-1$. Then we have the following complex in $\mathscr{W}'$:

$$\ldots \to C_{n+2} \to C_{n+1} \oplus C'_n \to C_n \oplus L'_n \to 0 \to \ldots$$

Now by setting

$$L = (\ldots \to 0 \to C'_n \to 0 \to \ldots)$$

$$L' = (\ldots \to 0 \to L'_n \to 0 \to \ldots),$$

we get a $\mathscr{W}'$-equivalence

$$L \to L' \oplus (\ldots \to C_{n+1} \to C_n \to 0 \to \ldots). \quad \blacksquare$$

Lemma **(7.3).** — *For any complex* $C \in \mathscr{W}'$, *the complex* $C \oplus \Sigma C$ *lies in* $\mathscr{W}$.

Proof. — If C is Λ-acyclic, C lies in $\mathscr{W}_0^s \subset \mathscr{W}_0$ and then $C \oplus \Sigma C \in \mathscr{W}$. ■

(7.4) We use $K(\mathscr{W})$ to denote the class of complexes $C \in \mathscr{W}'$ fulfilling the following relation:

$$C \sim C' \Leftrightarrow C \oplus \Sigma C' \in \mathscr{W}.$$

By (7.3), this relation is an equivalence relation and $K(\mathscr{W})$ is a well defined set. Moreover the direct sum of complexes induces an abelian group structure on $K(\mathscr{W})$.

If C is a Λ-acyclic complex in $\mathscr{C}(A)$, the class of C in $K(\mathscr{W})$ will be denoted by $\theta(C)$.

Lemma **(7.5).** — *Let* $0 \to C \to C' \to C'' \to 0$ *be an s-exact sequence of* Λ*-acyclic complexes in* $\mathscr{C}(A)$. *Then* $\theta(C') = \theta(C) + \theta(C'')$.

Proof. — We have an s-exact sequence

$$0 \to C \oplus \Sigma C \to C' \oplus \Sigma C \oplus \Sigma C'' \to C'' \otimes \Sigma C'' \to 0$$

and, by lemma (7.3), $C' \oplus \Sigma C \oplus \Sigma C''$ is in $\mathscr{W}$. That proves the lemma.

Now if f is a Λ-homology equivalence between two finite A-complexes, we will define $\theta(f)$ as the class of the mapping cone of f in $K(\mathscr{W})$.

Lemma **(7.6).** — *Let* $f: C \to C$ *and* $g: C' \to C''$ *be two* Λ*-homology equivalences between finite* A*-complexes. Then* $\theta(g \circ f) = \theta(f) + \theta(g)$.

Proof. — We have a short s-exact sequence between the mapping cones of f, g, $g \circ f \oplus 1_{C'}$. Then the result follows from (7.5).

(7.7) Let $f: \Lambda^p \to \Lambda^q$ be an isomorphism. Denote also by A the o-dimensional complex $\ldots \to 0 \to A \to 0 \to \ldots$ Then f is a morphism from $A^p \otimes \Lambda$ to $A^q \otimes \Lambda$, and, by (6.12), there exist a $\mathscr{W}'$-equivalence $\varepsilon : \overline{C} \to A^p$ and a map $g : \overline{C} \to A^q$ such that $f \circ (\varepsilon \otimes \Lambda)$ is homotopic to $g \otimes \Lambda$.

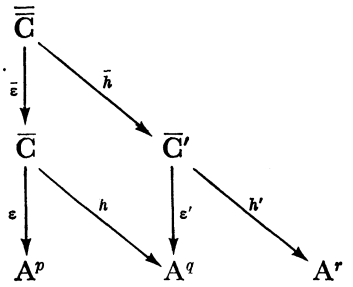
Since f is an isomorphism, g is a $\mathscr{W}'$-equivalence.

Then we define $\theta(f)$ as $\theta(g) - \theta(\varepsilon)$. By (6.12), it is easy to show that $\theta(f)$ does not depend on the choices.

Lemma **(7.8).** — *Let* $f: \Lambda^p \to \Lambda^q$ *and* $g : \Lambda^q \to \Lambda^r$ *be two isomorphisms. Then we have*

$$\theta(g \circ f) = \theta(f) + \theta(g).$$

Proof. — By theorem (6.12), there exists a homotopy commutative diagram in $\mathscr{C}(A)$



such that the morphisms are Λ-homology equivalences and $h \otimes \Lambda$ and $h' \otimes \Lambda$ are homotopic to $f \circ (\varepsilon \otimes \Lambda)$ and $g \circ (\varepsilon' \otimes \Lambda)$. Then we have

$$\theta(g \circ f) = \theta(h' \circ \bar{h}) - \theta(\varepsilon \circ \bar{\varepsilon}) = \theta(h') + \theta(\bar{h}) - \theta(\varepsilon) - \theta(\bar{\varepsilon})$$

whence $$\theta(g \circ f) = \theta(h') - \theta(\varepsilon) + \theta(h) - \theta(\varepsilon') = \theta(f) + \theta(g).$$ ■

Theorem **(7.9).** — *The torsion homomorphism* $\varepsilon : K(\mathscr{W}) \to \widetilde{K}_1(\Lambda)/\alpha$ *is an isomorphism.*

Proof. — If $x \in \widetilde{K}_1(\Lambda)/\alpha$ is represented by an isomorphism $f: \Lambda^p \to \Lambda^q$, we have

$$\varepsilon(\theta(f)) \equiv \tau(f) \bmod \alpha \Rightarrow x = \varepsilon(\theta(f))$$

and ε is surjective.

Now let θ be an element of $\operatorname{Ker} \varepsilon$, represented by a complex $C \in \mathscr{W}'$. Since $\varepsilon(\theta)$ vanishes, $\tau(C \otimes \Lambda)$ is in α and $\tau(C \otimes \Lambda)$ is the torsion of a complex $C' \otimes \Lambda$ where C' is a Λ-acyclic complex in $\mathscr{W}$. Then θ is represented by $C \oplus \Sigma C'$ and the torsion of $(C \oplus \Sigma C') \otimes \Lambda$ vanishes. Since $\mathscr{W}'$ is splittable, we can "split" $C \oplus \Sigma C'$ into complexes $C_i \in \mathscr{W}'$ of length 2. And we have

$$\theta = \Sigma\theta(C_i) \quad \text{and} \quad \Sigma\tau(C_i \otimes \Lambda) = 0.$$

On the other hand, the suspension Σ^2 does not change the invariants θ and τ. So we may as well suppose that the complexes C_i are 1 or 2-dimensional.

Then there exist two 1-dimensional complexes in $\mathscr{W}'$

$$X = (\ldots \to 0 \to A^p \xrightarrow{f} A^q \to 0 \to \ldots)$$

$$Y = (\ldots \to 0 \to A^{p'} \xrightarrow{g} A^{q'} \to 0 \to \ldots)$$

such that $\theta = \theta(X) - \theta(Y)$ and $\tau(X \otimes \Lambda) = \tau(Y \otimes \Lambda)$.

But the image of $\tau(X \otimes \Lambda) = \tau(f \otimes \Lambda)$ under the boundary $\tilde{K}_1(\Lambda) \xrightarrow{\partial} K_0(\mathbf{Z})$ is $q - p$ [9]. Then, after stabilization on X and Y, we may suppose

$$p = p' \quad \text{and} \quad q = q'.$$

Let $\varphi \in GL_q(\Lambda)$ be the map for $(f \otimes \Lambda) \circ (g \otimes \Lambda)^{-1}$. Since $\tau(f \otimes \Lambda) - \tau(g \otimes \Lambda)$ is zero, the class of φ in $K_1(\Lambda)$ is in the image of $K_1(\mathbf{Z}) \to K_1(\Lambda)$. Then, after a permutation on the basis of A^q (in X) and after stabilization on X and Y, we may suppose that φ lies in the commutator subgroup of $GL_q(\Lambda)$:

$$\varphi = \prod_i [\varphi_i, \psi_i].$$

And we have

$$\theta = \theta(X) - \theta(Y) = \theta(f) - \theta(g) = \theta(f \otimes \Lambda) - \theta(g \otimes \Lambda) = \theta(\varphi)$$

whence $\theta = \Sigma(\theta(\varphi_i) + \theta(\psi_i) - \theta(\varphi_i) - \theta(\psi_i)) = 0.$

This completes the proof.

Corollary **(7.10).** — *The class of Λ-acyclic complexes in $\mathscr{W}$ is the class of Λ-acyclic complexes C such that the torsion of $C \otimes \Lambda$ is in α.*

Now we prove lemmas (4.3) and (4.6).

Lemma (4.6) is actually the corollary (7.10).

Let $\tau \in \tilde{K}_1(\Lambda)$. By theorem (7.9), there exists a complex $C \in \mathscr{W}'$ such that τ is the torsion of $C \otimes \Lambda$. Since C is splittable ((7.1)), we can split C into Λ-acyclic complexes C_i of length two and we have $\tau = \Sigma\tau(C_i \otimes \Lambda)$. If C_i is $(n_i + 1)$-dimensional and the differential of C_i is u_i, we have:

$$\tau = \Sigma(-1)^{n_i}\tau(u_i \otimes \Lambda)$$

and lemma (4.3) follows.

8. The isomorphism theorem

Suppose now that A is a ring with involution and $\mathscr{W}$ is an exact symmetric class in $\mathscr{C}(A)$. The $\mathscr{W}$-localization of A is (Λ, α) and $A \to \Lambda$ is a morphism of rings with involution.

The class of Λ-acyclic complexes in $\mathscr{C}(A)$ is denoted by $\mathscr{W}'$ and the class of acyclic complexes in $\mathscr{C}(\Lambda)$ is denoted by $\mathscr{W}_\Lambda$.

We have a canonical map

$$\varepsilon : \Gamma_n(A, \mathscr{W}') \to \Gamma_n(\Lambda, \mathscr{W}_\Lambda) \simeq L_n^h(\Lambda).$$

In this section, we will prove that ε is an isomorphism.

Lemma **(8.1)**. — *Let* C (*respectively* Σ) *be a* p*-dimensional and* $(p-2)$*-connected complex in* $\mathscr{C}(A)$ (*respectively* $\mathscr{C}(\Lambda)$) *and* $f : \Sigma \to C \otimes \Lambda$ *be a map. Then there exist a* p*-dimensional complex* $\Sigma' \in \mathscr{C}(A)$, *a homotopy equivalence* $\varepsilon : \Sigma' \otimes \Lambda \to \Sigma$ *and a map* $g : \Sigma' \to C$ *such that* $f \circ \varepsilon$ *is homotopic to* $g \otimes \Lambda$.

Proof. — Let us consider the modules Σ_p, Σ_{p-1} as p-dimensional complexes $C'_p \otimes \Lambda$, $C'_{p-1} \otimes \Lambda$. The differential d on Σ is a map from $C'_p \otimes \Lambda$ to $C'_{p-1} \otimes \Lambda$. Then, by theorem (6.12), there exist a p-dimensional complex $\overline{C} \in \mathscr{C}(A)$, a $\mathscr{W}'$-equivalence $\bar{\varepsilon} : \overline{C} \to C'_p$ and a morphism $g : \overline{C} \to C'_{p-1}$ such that $g \otimes \Lambda$ is homotopic to $d \circ (\bar{\varepsilon} \otimes \Lambda)$.

Let M be the mapping cone of g. The $\mathscr{W}'$-equivalence $\bar{\varepsilon}$ induces a homotopy equivalence $\varepsilon' : M \otimes \Lambda \to \Sigma$. Moreover M is p-dimensional and $C \otimes \Lambda$ is $(p-2)$-connected. Then by (6.12), there exist a p-dimensional complex $\Sigma' \in \mathscr{C}(A)$, a $\mathscr{W}'$-equivalence $\varepsilon'' : \Sigma' \to M$ and a morphism $g : \Sigma' \to C$ such that $f \circ \varepsilon' \circ (\varepsilon'' \otimes \Lambda)$ is homotopic to $g \otimes \Lambda$. The result follows.

Lemma **(8.2)**. — *Let* C *be a finite* A*-complex such that* $H^i(C, \Lambda)$ *vanishes for* $i > p$ *and let* $\varphi \in B(C \otimes \Lambda)$ *be a bilinear form such that*

$$\partial^0 \varphi \leq -2p + 1, \quad d\varphi = 0.$$

Then there exist a complex $C' \in \mathscr{C}(A)$ *with* $\dim C' = \dim C$, *a* $\mathscr{W}'$*-equivalence* $\varepsilon : C' \to C$ *and a bilinear form* $\varphi' \in B(C')$ *such that* $d\varphi' = 0$ *and* $\varepsilon^*(\varphi) - \varphi' \otimes \Lambda$ *is a boundary.*

Proof. — By theorem (6.12), there exist a complex $C' \in \mathscr{C}(A)$ with $\dim C' = \dim C$, a $\mathscr{W}'$-equivalence $\varepsilon : C' \to C$ and a morphism $g : C' \to \hat{C}$ such that $\varphi \circ (\varepsilon \otimes \Lambda)$ is homotopic to $\Lambda \otimes g$. Then $\varphi' = \hat{\varepsilon} g$ is the desired form. ■

Lemma **(8.3)**. — *Let* C *be a finite* A*-complex such that* $H^i(C, \Lambda)$ *vanishes for* $i > p$ *and let* $\varphi \in B(C)$ *be a bilinear form such that*

$$\partial^0 \varphi \leq -2p, \quad d\varphi = 0.$$

Then, if $\varphi \otimes \Lambda$ *is a boundary, there exist a complex* $C' \in \mathscr{C}(A)$ *with* $\dim C' = \dim C$ *and a* $\mathscr{W}'$*-equivalence* $\varepsilon : C' \to C$ *such that* $\varepsilon^*(\varphi)$ *is a boundary.*

Proof. — If $\varphi \otimes \Lambda$ is a boundary, $\varphi \otimes \Lambda$ is null homotopic and, by (6.12), there exist a complex $C' \in \mathscr{C}(A)$ with $\dim C' = \dim C$ and a $\mathscr{W}'$-equivalence $\varepsilon : C' \to C$ such that $\varphi \circ \varepsilon$ is null homotopic. Then $\varepsilon^*(\varphi) = \hat{\varepsilon} \circ \varphi \circ \varepsilon$ is a boundary. ■

Theorem **(8.4).** — *The morphism* $\varepsilon : \Gamma_n(A, \mathscr{W}') \to L_n^h(\Lambda)$ *is an isomorphism.*

Proof. — Suppose $n = -2p$ or $n = -2p+1$, and let $\sigma \in L_n^h(\Lambda)$.

By lemma (3.6), σ is represented by a $\mathscr{W}_\Lambda$-non singular quadratic n-complex (C, q) where C is concentrated in dimension p (and $p-1$ if n is odd).

By lemma (8.1), there exist a p-dimensional complex $C' \in \mathscr{C}(A)$ and a homotopy equivalence from $C' \otimes \Lambda$ to C. Then σ is represented by $(C' \otimes \Lambda, q')$. Since C' is p-dimensional, q' is the class of $e_0 \otimes \varphi_0 + e_1 \otimes \varphi_1$ and we have

$$d\varphi_0 + \varphi_1 - \hat{\varphi}_1 = 0, \qquad d\varphi_1 = 0.$$

By lemma (8.2), we may suppose that φ_1 has the form $\psi_1 \otimes \Lambda$, $\psi_1 \in B(C')$ and $d\psi_1$ is zero. Then $(\psi_1 - \hat{\psi}_1) \otimes \Lambda$ is a boundary and, by lemma (8.3), we may suppose that $\psi_1 - \hat{\psi}_1$ is a boundary $d\xi$.

Now, $\varphi_0 + \xi \otimes \Lambda$ is a cycle and, by (8.2), we may suppose that

$$\varphi_0 + \xi \otimes \Lambda = \varphi' \otimes \Lambda + d\eta$$

where φ' is a cycle in $B(C')$ and $\eta \in B(C' \otimes \Lambda)$. Then, we have

$$e_0 \otimes \varphi_0 + e_1 \otimes \varphi_1 = (e_0 \otimes (\varphi' - \xi) + e_1 \otimes \psi_1) \otimes \Lambda + d(e_0 \otimes \eta).$$

Moreover $e_0 \otimes (\varphi' - \xi) + e_1 \otimes \psi_1$ is a cycle and represents a $\mathscr{W}'$-non singular quadratic n-form over C'. Then the morphism ε is surjective.

Now let $\sigma' \in \Gamma_n(A, \mathscr{W}')$ be an element in $\operatorname{Ker} \varepsilon$. By lemma (3.6), σ' is represented by a $\mathscr{W}'$-non singular quadratic n-complex (C, q) where C is a complex in $\mathscr{C}(A)$ concentrated in dimension p (and $p-1$ if n is odd).

Since $\varepsilon\sigma'$ is zero, $(C \otimes \Lambda, q \otimes \Lambda)$ is cobordant to zero and, by lemmas (3.7) and (3.8), there exists a $\mathscr{W}_\Lambda$-non singular quadratic $(n+1)$-pair $(\Sigma \to C \otimes \Lambda, u)$ such that q is the boundary of u and Σ_i vanishes for $i \neq p$, $p-1$.

By lemma (8.1), we may suppose that the morphism $\Sigma \to C \otimes \Lambda$ is the morphism $g \otimes \Lambda : \Sigma' \otimes \Lambda \to C \otimes \Lambda$, where Σ' is a p-dimensional complex in $\mathscr{C}(A)$. The quadratic form u is represented by

$$e_0 \otimes \psi_0 + e_1 \otimes \psi_1 + e_2 \otimes \psi_2, \qquad \psi_i \in B(\Sigma'),$$

and we have

$$\begin{aligned} d\psi_0 + \psi_1 - \hat{\psi}_1 &= \hat{g}\varphi_0 g \otimes \Lambda \\ -d\psi_1 + \psi_2 + \hat{\psi}_2 &= \hat{g}\varphi_1 g \otimes \Lambda \\ d\psi_2 &= 0 \end{aligned}$$

where $e_0 \otimes \varphi_0 + e_1 \otimes \varphi_1$ represents q.

By lemma (8.2), we may suppose that

$$\psi_2 = \psi_2' \otimes \Lambda + d\xi_1, \qquad d\psi_2' = 0$$

and, after adding to $e_0 \otimes \psi_0 + e_1 \otimes \psi_1 + e_2 \otimes \psi_2$ the boundary of $e_2 \otimes \xi_1$, we have

$$\psi_2 = \psi_2' \otimes \Lambda, \qquad d\psi_2' = 0.$$

Then $(\hat{g}\varphi_1 g - \psi_2' - \hat{\psi}_2') \otimes \Lambda$ is a boundary and, by lemma (8.3), we may suppose that

$$\hat{g}\varphi_1 g = \psi_2' + \hat{\psi}_2' + d\eta_1.$$

Since $\psi_1 + \eta_1 \otimes \Lambda$ is a cycle, we may suppose, by lemma (8.2), that

$$\psi_1 + \eta_1 \otimes \Lambda = \psi_1' \otimes \Lambda + d\xi_0, \quad d\psi_1' = 0,$$

and, after adding to $e_0 \otimes \psi_0 + e_1 \otimes \psi_1 + e_2 \otimes \psi_2$ the boundary of $-e_1 \otimes \xi_0$, we may suppose that

$$\psi_1 + \eta_1 \otimes \Lambda = \psi_1' \otimes \Lambda, \quad d\psi_1' = 0.$$

Then, we have

$$d\psi_0 + (\psi_1' - \eta_1 - \hat{\psi}_1' + \hat{\eta}_1) \otimes \Lambda = \hat{g}\varphi_0 g \otimes \Lambda.$$

Let ψ be the form $\hat{g}\varphi_0 g - \psi_1' + \eta_1 + \hat{\psi}_1' - \hat{\eta}_1$. The bilinear form ψ is a cycle of degree n and $\psi \otimes \Lambda$ is a boundary. Moreover, by Poincaré duality, $H^i(\Sigma', \Lambda)$ vanishes for $i > -n-p$. Then lemma (8.3) holds and we may suppose that

$$\hat{g}\varphi_0 g - \psi_1' + \eta_1 + \hat{\psi}_1' - \hat{\eta}_1 = d\eta_0.$$

So $\psi_0 - \eta_0 \otimes \Lambda$ is a cycle and, by (8.2), we may suppose that

$$\psi_0 - \eta_0 \otimes \Lambda = \psi_0' \otimes \Lambda + d\xi_{-1}, \quad d\psi_0' = 0,$$

and, after adding to $e_0 \otimes \psi_0 + e_1 \otimes \psi_1 + e_2 \otimes \psi_2$ the boundary of $e_0 \otimes \xi_{-1}$, we may suppose that

$$\psi_0 - \eta_0 \otimes \Lambda = \psi_0' \otimes \Lambda.$$

Now it is easy to check that

$$e_0 \otimes \psi_0 + e_1 \otimes \psi_1 + e_2 \otimes \psi_2 = [e_0 \otimes (\eta_0 + \psi_0') + e_1 \otimes (-\eta_1 + \psi_1') + e_2 \otimes \psi_2'] \otimes \Lambda$$

and

$$d[e_0 \otimes (\eta_0 + \psi_0') + e_1 \otimes (-\eta_1 + \psi_1') + e_2 \otimes \psi_2'] = g^*(e_0 \otimes \varphi_0 + e_1 \otimes \varphi_1).$$

Then $e_0 \otimes (\eta_0 + \psi_0') + e_1 \otimes (-\eta_1 + \psi_1') + e_2 \otimes \psi_2'$ represents a $\mathscr{W}'$-non singular quadratic $(n+1)$-form v over $\Sigma' \to C$ with boundary q. So σ' is zero and ε is injective. ■

9. Some results about Λ and $L_n(\Lambda)$

Throughout this section, we assume that $A \to B$ is a ring homomorphism and β is a subgroup of $\tilde{K}_1(B)$.

The class of finite A-complexes C such that $C \otimes B$ is acyclic with torsion in β is denoted by $\mathscr{W}^\beta$, and the $\mathscr{W}^\beta$-localization of A is denoted by (Λ, α).

Proposition (9.1). — *Let u be a matrix with entries in Λ. Then, if $u \otimes B$ is invertible, u is invertible too.*

Proof. — Let u be a matrix with entries in Λ. If we denote by A the 0-dimensional complex $\ldots \to 0 \to A \to 0 \to \ldots$, u is a morphism $A^p \otimes \Lambda \to A^q \otimes \Lambda$ and, by theo-

rem (6.12), there exist a 0-dimensional complex $\overline{C} \in \mathscr{C}(A)$, a $(\mathscr{W}^\beta)_0^s$-equivalence $\varepsilon : \overline{C} \to A^p$ and a morphism $g : \overline{C} \to A^q$ such that $g \otimes \Lambda$ is homotopic to $u \circ (\varepsilon \otimes \Lambda)$.

Let K be the homotopy kernel of ε. Since K is $\mathscr{W}_0^\beta$-splittable, there exist a (-1)-dimensional complex $K' \in \mathscr{W}_0^\beta$ and a (-2)-connected morphism $f : K' \to K$. The composite map $K' \to K \to \overline{C}$ is (-2)-connected. Denote by C' its mapping cone. The complex C' lies in $\mathscr{W}_0^\beta$ and has the simple homotopy type of a complex C'' such that C_i'' vanishes for $i \neq 0, -1$. Moreover ε and g factorize through C'' and we get two morphisms $\varepsilon' : C'' \to A^p$ and $g' : C'' \to A^q$ such that $g' \otimes \Lambda$ is homotopic to $u \circ (\varepsilon' \otimes \Lambda)$.

But $u \otimes B$ is invertible, then $g' \otimes B$ is a homotopy equivalence and the mapping cone of g' is B-acyclic and lies in $\mathscr{W}_0^\beta$. Since the length of this mapping cone is 2, g' is a $(\mathscr{W}^\beta)_0^s$-equivalence. Then, by (7.1), g' is a Λ-homology equivalence, and u is an isomorphism. ■

(9.2) *Proof of theorem* (1.13)

If u is a matrix with entries in A, denote by $M(u)$ the 1-dimensional complex $\ldots \to 0 \to A^p \xrightarrow{u} A^q \to 0 \to \ldots$

The set Σ is the set of matrices u such that $(M(u) \oplus \Sigma M(u)) \otimes B$ is acyclic with torsion in β. But $M(u) \oplus \Sigma M(u)$ is B-acyclic if and only if $M(u)$ is B-acyclic. Moreover if $M(u)$ is B-acyclic, we have

$$\tau[M(u) \otimes B \oplus \Sigma M(u) \otimes B] = 0.$$

Then Σ is the set of matrices u such that $u \otimes B$ is invertible and $A \to \Lambda$ is the localization of $A \to B$.

Now let τ be an element of $\tilde{K}_1(\Lambda)$. By lemma (4.3), there exists a finite A-complex C such that $C \otimes \Lambda$ is acyclic with torsion τ. Then, by lemma (7.10), τ lies in α if and only if C lies in $\mathscr{W}^\beta$. But the torsion of $C \otimes B$ is the image of τ by the morphism $\varepsilon : \Lambda \to B$. Hence α is the inverse image of β under ε.

Now suppose ε is onto, and let $C \in \mathscr{W}^\beta$. The complex $C \otimes B$ is acyclic and the identity is a homotopy: $1 = d \circ k + k \circ d$.

But $C \otimes \Lambda \to C \otimes B$ is onto and we can lift k in a map k' from $C \otimes \Lambda$ to itself. The morphism $d \circ k' + k' \circ d$ is invertible after tensorization by B. Then, by (9.1), $d \circ k' + k' \circ d$ is an isomorphism and $C \otimes \Lambda$ is acyclic. ■

(9.3) *Proof of Proposition* (1.15)

Let $B_0 \subset B_1 \subset B_2 \subset \ldots$ be subrings of B defined by:

(i) B_0 is the image of $A \to B$;

(ii) for any $n \geq 0$, B_{n+1} is generated by B_n and the inverses of the units of B contained in B_n.

Denote by B′ the image of $\Lambda \to B$. The subring B′ contains A and, by (9.1), any unit of B contained in B′ is a unit of B′. Then B′ contains all the rings B_n. ■

As a corollary of (9.1), we have:

Lemma **(9.4)**. — *If* $\Lambda \to B$ *is onto,* $\tilde{K}_1(\Lambda) \to \tilde{K}_1(B)$ *is onto.*

From now on, we will suppose that $A \to B$ is a morphism of rings with involution and that β is stable under the involution. Then $\mathscr{W}^\beta$ is symetric and Λ has an involution. We suppose also that $\Lambda \to B$ is onto.

Theorem **(9.5)**. — *If n is even, the morphism* $L_n^\alpha(\Lambda) \to L_n^\beta(B)$ *is epic. If n is odd, this morphism is monic.*

Proof. — By lemma (9.4), the relative group $L_n^{\alpha,\beta}(\Lambda \to B)$ does not depend on β. Then it suffices to prove the theorem in the case $\beta = \tilde{K}_1(B)$.

Let $n = 2p$. An element $u \in L^h_{2p}(B)$ is represented by a hermitian $(-1)^p$-form (H, λ, μ) such that the induced map $\tilde{\lambda} : H \to \hat{H}$ is an isomorphism. Since H is free over B and $\Lambda \to B$ is epic, there exists a hermitian $(-1)^p$-form (H', λ', μ') such that

$$H' \text{ is free over } \Lambda,$$
$$H' \otimes B = H, \quad \lambda' \otimes B = \lambda, \quad \mu' \otimes B = \mu.$$

Then, by lemma (9.1), λ' induces an isomorphism from H′ to $\hat{H}'$ and (H', λ', μ') represents an element $v \in L^h_{2p}(\Lambda)$ such that $\varepsilon_*(v) = u$.

Let now $n = 2p + 1$. An element $v \in L^h_{2p+1}(\Lambda)$ is represented by an isometry between two standard kernel K and K′. If v is sent to zero in $L^h_{2p+1}(B)$, $K = K'$ and $g \otimes B$ is an element of $RU^h(B)$ (with the notations of [10]).

Consider the following diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & UU(\Lambda) & \longrightarrow & TU^h(\Lambda) & \longrightarrow & GL(\Lambda) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 1 & \longrightarrow & UU(B) & \longrightarrow & TU^h(B) & \longrightarrow & GL(B) & \longrightarrow & 1 \end{array}$$

By lemma (9.1), a and c are surjective. Then b is epic and the morphism $RU^h(\Lambda) \to RU^h(B)$ is epic too. Hence v can be represented by an isometry f such that $f \otimes B$ is the identity map.

Let $H \oplus \hat{H}$ be the standard kernel K. The isometry f is defined by

$$f(x, y) = (x + a(x) + b(y), y + c(x) + d(y)), \quad \forall\, x \in H,\ y \in \hat{H}$$

and $a \otimes B$, $b \otimes B$, $c \otimes B$, $d \otimes B$ vanish. By (9.1), $1 + a$ is invertible and, after composing f with an element of $GL(\Lambda)$, we may as well suppose that a is zero.

Since f is an isometry, it is easy to see that the map g defined by

$$g(x,y) = (x, y - c(x))$$

is an isometry leaving each element of $\hat{H}$ fixed and g lies in $RU^h(\Lambda)$. We have

$$g \circ f(x,y) = (x + b(y), y + d(y) - c \circ b(y)).$$

But $1 + d - c \circ d$ is invertible and there is an isometry $h \in RU^h(\Lambda)$ such that

$$h \circ g \circ f(x,y) = (x + a'(x) + b'(y), y).$$

It is easy to see that a' is zero and $h \circ g \circ f$ lies in $RU^h(\Lambda)$. Therefore V is zero. ■

Theorem **(9.6).** — *The relative group* $L^h_{2p+1}(\Lambda \to B)$ *is the group of equivalence classes of pairs* (H, K) *where* H *is a hermitian* $(-1)^p$*-form over* Λ *and* K *a subkernel of* $H \otimes B$, *subject to the following relation:*

(H, K) *is equivalent to* (H', K') *if there exist two* Λ*-kernels* H_0 *and* H'_0 *with subkernels* S_0 *and* S'_0 *and an isometry* $\varphi : H \oplus H_0 \to H' \oplus H'_0$ *such that*

$$\varphi(K \oplus S_0 \otimes B) = K' \oplus S'_0 \otimes B.$$

Proof. — By Wall ([10], p. 72), $L^h_{2p+1}(\Lambda \to B)$ is generated by such pairs. Moreover (H, K) and (H', K') represent the same element in $L^h_{2p+1}(\Lambda \to B)$ if there exist two kernels $\overline{H}_0$ and H'_0 with subkernels $\overline{S}_0$ and $\overline{S}'_0$ and an isometry

$$\bar{\varphi} : H \oplus \overline{H}_0 \oplus -H' \to H'_0$$

such that any automorphism $\bar{\psi}$ taking $\overline{S}'_0 \otimes B$ to $\bar{\varphi}(K \oplus \overline{S}_0 \otimes B \oplus K')$ lies in $RU^h(B)$. But the map $RU^h(\Lambda) \to RU^h(B)$ is epic (see the proof of (9.5)). Hence we can lift $\bar{\psi}$ to an automorphism ψ on H'_0.

Let S'_0 be the subkernel $\psi(\overline{S}'_0)$. We have an isometry

$$\varphi : H \oplus \overline{H}_0 \oplus -H' \oplus H' \to H' \oplus H'_0$$

taking $K \oplus \overline{S}_0 \otimes B \oplus K' \oplus K'$ to $K' \oplus S'_0 \otimes B$.

On the other hand, the diagonal $\overline{K}$ is a subkernel of $-H' \oplus H'$ and there exists an automorphism in $RU^h(B)$ taking $\overline{K} \otimes B$ to $K' \oplus K'$. By lifting this automorphism in $RU^h(\Lambda)$ we get an automorphism f and $f(\overline{K})$ is a subkernel of $-H' \oplus H'$ such that $f(\overline{K}) \otimes B = K' \oplus K'$. Let H_0 be the kernel $\overline{H}_0 \oplus -H' \oplus H'$ with subkernel $S_0 = \overline{S}_0 \oplus f(\overline{K})$. Then φ is an isometry taking $K \oplus S_0 \otimes B$ to $K' \oplus S'_0 \otimes B$. ■

Now, consider the following question: Under what conditions is the map $\varepsilon : \Lambda \to B$ an isomorphism? To study this problem, it is convenient to use the following definitions:

An A-module M is called B-*perfect* if $M \otimes B$ is zero; it is called *locally* B-*perfect* if any element in M is contained in a finitely generated B-perfect submodule.

Theorem **(9.7).** — *Suppose the kernel of* $A \to B$ *is locally* B*-perfect and* B *is the localization of* $\mathrm{Im}(A \to B)$ *with respect to a multiplicative subset of the center. Then the morphism* $\varepsilon : \Lambda \to B$ *is an isomorphism.*

Proof. — Let $a \in \mathrm{Ker}(A \to B)$ and suppose that a is contained in a finitely generated B-perfect submodule I. Let us choose a free resolution of I

$$C \xrightarrow{t} A^n \to I \to 0.$$

Since I is B-perfect, $f \otimes B$ is epic and has a section s. But $\Lambda \to B$ is epic and we can lift s to a morphism $g : \Lambda^n \to C \otimes \Lambda$. By (9.1), $f \otimes \Lambda \circ g$ is an isomorphism and $f \otimes \Lambda$ is epic. Hence I is Λ-perfect and the composite map $I \to A \to \Lambda$ is zero. Then $A \to B$ and $A \to \Lambda$ have the same kernel K.

Now it is easy to see that the maps $A/K \to B$ and $A/K \to \Lambda$ have the same universal property and $\varepsilon : \Lambda \to B$ is an isomorphism. ■

This theorem is in fact a generalization of a theorem of Hausmann [3] proved also in [6] and [8], theorem (1.4).

Finally, we will give an example of computation.

Let D_{2n} be the dihedral group of order $2n$ (n odd) and let $\mathbf{Z}D_{2n} \to \mathbf{Z}$ be the evaluation map. The group D_{2n} is not perfect and not nilpotent, then we cannot use the techniques of Hausmann or Smith in order to compute the group $\Gamma_*(\mathbf{Z}D_{2n} \to \mathbf{Z})$.

Theorem **(9.8).** — *We have the isomorphisms*

$$\Gamma_*(\mathbf{Z}D_{2n} \to \mathbf{Z}) \xrightarrow{\sim} \Gamma_*(\mathbf{Z}[\mathbf{Z}/2] \to \mathbf{Z}) \xrightarrow{\sim} L^h_*(\Lambda)$$

where Λ *is the pull back of rings*

$$\begin{array}{ccc} \Lambda & \longrightarrow & \mathbf{Z} \\ \downarrow & & \downarrow \\ \mathbf{Z}_{(2)}[\mathbf{Z}/2] & \longrightarrow & \mathbf{Z}_{(2)} \end{array}$$

Proof. — The group D_{2n} is generated by t and τ with the following relations:

$$t^n = 1, \quad \tau^2 = 1, \quad \tau t = t^{-1}\tau.$$

Let $\mathbf{Z}D_{2n} \to \Lambda$ be the localization of $\mathbf{Z}D_{2n} \to \mathbf{Z}$ and let x and y be the images of t and τ in Λ. We have

$$\left[\frac{1-n}{2}(1+\tau) + 1 + t + \ldots + t^{n-1}\right](1-\tau)(1-t) = 0.$$

But $\dfrac{1-n}{2}(1+\tau) + 1 + t + \ldots + t^{n-1}$ is sent to 1 in $\mathbf{Z}$ and

$$\frac{1-n}{2}(1+y) + 1 + x + \ldots + x^{n-1}$$

is invertible. This implies that

$$(1-y)(1-x) = 0.$$

On the other hand, $\mathbf{Z}D_{2n} \to \Lambda$ is a morphism of rings with involution. So we have:

$$(1-y)(1-x) = (1-x^{-1})(1-y) = 0 \Rightarrow (1-x)(1-y) = 0.$$

And x and y commute. Then:

$$yx = x^{-1}y = xy \Rightarrow x = 1.$$

Hence t is sent to 1 in Λ and Λ is the localization of $\mathbf{Z}[\mathbf{Z}/2] \to \mathbf{Z}$. But $\mathbf{Z}[\mathbf{Z}/2]$ is commutative and Λ is the localization $S^{-1}\mathbf{Z}[\mathbf{Z}/2]$ where S is the set of elements $a + b\tau \in \mathbf{Z}[\mathbf{Z}/2]$ with $a + b = 1$. Then it is easy to see that Λ is the subring of $\mathbf{Z}_{(2)}[\mathbf{Z}/2]$ defined by

$$\Lambda = \{a + b\tau,\ a, b \in \mathbf{Z}_{(2)} \text{ and } a + b \in \mathbf{Z}\}.$$

REFERENCES

[1] S. CAPPELL and J. SHANESON, The codimension two placement problem and homology equivalent manifolds, *Ann. of Math.* (2), **99** (1974), pp. 277-348.

[2] P. M. COHN, Inversive localization in noetherian rings, *Comm. Pure Appl. Math.*, **26** (1973), pp. 679-691.

[3] J. C. HAUSMANN, Homological surgery, *Ann. of Math.* (2), **104** (1976), pp. 573-584.

[4] J. W. MILNOR, Whitehead torsion, *Bull. Amer. Math. Soc.*, **72** (1966), pp. 358-426.

[5] A. RANICKI, The algebraic theory of surgery I, Foundations, *Proc. London Math. Soc.* (3), **40** (1980), pp. 87-192.

[6] J. R. SMITH, Homology surgery and perfect groups, *Topology*, **16** (1977), pp. 461-463.

[7] J. R. SMITH, Acyclic localizations, *Journal of Pure and Applied Algebra*, **12** (1978), pp. 117-127.

[8] P. VOGEL, Un théorème de Hurewicz homologique, *Comment. Math. Helv.*, **52** (1977), pp. 393-413.

[9] P. VOGEL, Torsion de Whitehead généralisée, *C.R.A.S.*, **290** (1980), pp. 491-493.

[10] C. T. C. WALL, *Surgery on compact manifolds*, New York and London, Academic Press, 1970.

[11] C. T. C. WALL, On the axiomatic foundations of the theory of hermitian forms, *Proc. Camb. Phil. Soc.*, **67** (1970), pp. 243-250.

Université de Nantes,
Institut de Mathématiques et d'Informatique,
2, chemin de la Houssinière,
44072 Nantes Cedex (France).

Manuscrit reçu le 20 octobre 1979,
révisé le 25 mars 1981.