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RATIONAL POINTS IN HENSELIAN DISCRETE VALUATION RINGS

by MARVIN J. GREENBERG

I.

Let R be a Henselian discrete valuation ring, with t a generator of the maximal ideal, k the residue field, and K the field of fractions. Let R^* be the completion of R , K^* its field of fractions. If $F = (F_1, \dots, F_r)$ is a system of r polynomials in n variables with coefficients in R , and x is an n -tuple with coordinates in R , set $F(x) = (F_1(x), \dots, F_r(x))$. If F' is another system of r' polynomials, let FF' denote the system of rr' products. By the ideal $FR[X]$ generated by F is meant the ideal in $R[X]$ generated by $F_1, \dots, F_r$.

Theorem 1. — Assume, in case K has characteristic $p > 0$, that K^* is separable over K . Then there are integers $N \geq 1$, $c \geq 1$, $s \geq 0$ depending on $FR[X]$ such that for any $v \geq N$ and any x in R such that

$$F(x) \equiv 0 \pmod{t^v}$$

there exists y in R such that $y \equiv x \pmod{t^{[v/c]-s}}$

$$F(y) = 0$$

Corollary 1. — Let Z be a prescheme of finite type over R . Then there are integers $N \geq 1$, $c \geq 1$, $s \geq 0$ depending on Z such that for $v \geq N$ and for any point x of Z in R/t^v , the image of x in $Z(R/t^{[v/c]-s})$ lifts to a point of Z in R .

Proof. — We can take a finite covering of Z by affine opens Z_i . We have $Z(S) = \bigcup_i Z_i(S)$ for any local R -algebra S , hence the maxima of the integers for the Z_i will do for Z .

Corollary 2. — Z has a point in R if and only if Z has a point in R/t^v for all $v \geq 1$.

Let V be the algebraic set in affine n -space over K which is the locus of zeros of F . In the special case that R is complete and V is K -irreducible, non-singular, with a separably generated function field over K , Néron [4; Prop. 20, p. 38] has proved this theorem, showing that in this case one can take $c = 1$. However, in the general case we may have $c > 1$ (consider the polynomial $Y^2 - X^3$ and for any even integer $2v$ the point $x = (t^v, t^v)$). Theorem 1 implies that the hypothesis of non-singularity in [4; Prop. 22] can be dropped, so that the sets in that proposition are always constructible.

Theorem 1 is proved by induction on the dimension m of V . If $m = -1$, i.e., the ideal $FR[X]$ contains a non-zero constant, it is clear. Suppose $m > 0$.

We may assume the ideal $FR[X]$ is equal to its own radical (i.e., the scheme over R defined by F is reduced): For let E generate its radical. Then some power E^q is in $FR[X]$. From $F(x) \equiv 0 \pmod{t^v}$

we conclude t^v divides $E^q(x)$, so that

$$E(x) \equiv 0 \pmod{t^{[v/q]}}$$

If N', c', s' are integers for E , we see that $N = qN', c = qc', s = s'$ are integers for F .

We may further assume V is K -irreducible: For if $V = W \cup W'$, where W, W' are algebraic sets defined respectively by systems of polynomials G, G' with coefficients in R , let N', c', s' (resp. N'', c'', s'') be integers for G (resp. for G'). If x in R satisfies

$$F(x) \equiv 0 \pmod{t^v}$$

then either

$$G(x) \equiv 0 \quad \text{or} \quad G'(x) \equiv 0 \pmod{t^{[v/2]}}$$

since GG' is in the ideal $FR[X]$. Thus

$$N = 2\max(N', N'')$$

$$c = 2\max(c', c'')$$

$$s = \max(s', s'')$$

will work for F .

Then there are two cases:

Case 1. — V is separable over K .

Let J be the Jacobian matrix of F , and let D be the system of minors of order $n-m$ taken from $\det J$. The locus of common zeros of D and F is a proper K -closed W in V . By inductive hypothesis there are integers N', c', s' for the system (D, F) .

For each system $F_{(i)}$ of $n-m$ polynomials out of F , (i) a system of $n-m$ indices, let $V_{(i)}$ be the locus over K of zeros of $F_{(i)}$, and let $V_{(i)}^+$ be the union of the K -irreducible components of $V_{(i)}$ which have dimension m and are different from V ; let $G_{(i)}$ be a system of generators for the ideal of $V_{(i)}^+$ in $R[X]$. By inductive assumption there are integers $N_{(i)}, c_{(i)}, s_{(i)}$ for the system $(G_{(i)}, F)$.

If x is a point of $V_{(i)}$ in some extension of K such that for some (j)

$$D_{(i),(j)}(x) \neq 0$$

then the tangent hyperplanes of $F_{i_1}, \dots, F_{i_{n-m}}$ at x are transversal, and x lies on exactly one component of $V_{(i)}$, that component having dimension m .

We now invoke (see Lemma 2, n° 3)

Newton's Lemma. — If x in R is such that

$$F_{(i)}(x) \equiv 0 \pmod{t^{2\mu+1}}$$

$$D_{(i),(j)}(x) \neq 0 \pmod{t^\mu} \text{ for some } (j)$$

then there exists y in R such that

$$F_{(i)}(y) = 0$$

$$y \equiv x \pmod{t^\mu}$$

Hence

$$D_{(i),(j)}(y) \neq 0$$

If we knew also

$$G_{(i)}(y) \neq 0$$

we could deduce that y is a point of V .

Take v so large that

$$\mu = [(v-1)/2] \geq \max(N', \text{all } N_{(i)})$$

Let x in R be a zero mod t^ν of F . If

$$D(x) \equiv 0 \pmod{t^\mu}$$

our inductive hypothesis gives us y in R such that y is a singular point of V and

$$y \equiv x \pmod{t^{[\mu/c']-s'}}$$

If for some (i)

$$G_{(i)}(x) \equiv 0 \pmod{t^\mu}$$

then again by induction there is y in R which is a point of $V \cap V_{(i)}^+$ such that

$$y \equiv x \pmod{t^{[\mu/c(i)]-s(i)}}$$

Otherwise we use Newton's Lemma to find y in R which is a point of V such that

$$y \equiv x \pmod{t^\mu}$$

Thus as integers for F we can take

$$\begin{aligned} N &= 2 + 2 \max(N', \text{all } N_{(i)}) \\ c &= 2 \max(c', \text{all } c_{(i)}) \\ s &= 1 + \max(s', \text{all } s_{(i)}) \end{aligned}$$

Case 2. — V is inseparable over K .

In this case we need two facts.

Fact 1. — If K' is a finite extension of K , then the integral closure R' of R in K' is a finite R -module.

This follows from our assumption K^* separable over K (7; O_{IV}, 23.1.7 (ii)). For the convenience of the reader, we sketch the proof, valid also when R is a higher dimensional local domain: $K' \otimes_K K^*$ is a finite extension field of K^* , because of our assumption. $R' \otimes_R R^*$ is a subring of this field, integral over the complete local domain R^* , hence finite over R^* . Since R^* is faithfully flat over R , R' is a finite R -module. (The assumption that R^* is a domain, implicit in this argument, can be eliminated (*loc. cit.*)).

Fact 2. — There is a functor $\mathcal{F}$ from the category of affine schemes of finite type over R' to affine schemes of finite type over R such that $\mathcal{F}$ is right adjoint to the change of base functor from R to R' . Thus we have an isomorphism of bifunctors

$$\text{Mor}_R(Y, \mathcal{F}Z) \xrightarrow{\sim} \text{Mor}_{R'}(Y_{R'}, Z)$$

(for $Y/R, Z/R'$). Moreover, $\mathcal{F}$ preserves closed immersions.

This follows from Fact 1, and can also be established in greater generality (see [8; p. 195-193] where the notation $\mathcal{F}Z = \pi_{R'/R}Z$ is used).

Choose a basis $b_1, \dots, b_d$ for the R -module R' . Every element of R' has uniquely determined coordinates in R with respect to this basis, and the addition and multiplication in R' are given by polynomial functions in these coordinates. Hence there is a commutative ring scheme S over R , whose underlying scheme is affine d -space over R , such that for any R -algebra A ,

$$\text{Mor}_R(\text{Spec } A, S) = A \otimes_R R'$$

Now the same arguments as in [9; pp. 638-9] can be repeated word for word. The point is that by using the basis $b_1, \dots, b_d$, if P is a polynomial in n variables with coefficients in R' , the problem of finding a zero of P in $A \otimes_R R'$ is replaced by the problem of finding a common zero in A of d polynomials in nd variables with coefficients in R .

Let Y be the affine scheme over R defined by the polynomial system F ($Y = \text{Spec } R[X]/FR[X]$). Since the scheme Y_K over K obtained from Y by change of base is inseparable over K , there is a purely inseparable finite extension K' of K such that the scheme $Y_{K'}$ is not reduced, *a fortiori* $Y_{R'}$ is not reduced [5; 4.6.3].

Consider the affine scheme $\mathcal{F}Y_{R'}$ over R . There is a canonical R -morphism $\theta: Y \rightarrow \mathcal{F}Y_{R'}$ which corresponds by adjointness to the identity morphism of $Y_{R'}$. Now $\mathcal{F}((Y_{R'})_{\text{red}})$ is a closed subscheme of $\mathcal{F}Y_{R'}$; let W be its pre-image under θ . Then W is a proper closed subscheme of Y , otherwise the identity morphism of $Y_{R'}$ would factor through $(Y_{R'})_{\text{red}}$, i.e., $Y_{R'}$ would be reduced, contradicting the choice of R' . By inductive assumption, there are integers N', c', s' for W .

Suppose y is a point of Y in R/t^v . Let e be the ramification index of the discrete valuation ring R' over R , u a generator of its maximal ideal. Then y induces a point of $Y_{R'}$ in R'/u^{ev} . By a previous argument, there is an integer q (independent of y) such that the image of this point mod $u^{[ev/q]}$ is actually a point of $(Y_{R'})_{\text{red}}$. By adjointness, the image of y mod $t^{[v/q]}$ is actually a point of W . Hence $N = qN', c = qc', s = s'$ are integers for F .

Remark. — Theorem 1 is false without the separability assumption. For there exists a discrete valuation ring R whose completion R^* is a purely inseparable integral extension of R [6; o. 207]. R must therefore be its own Henselization. The minimal polynomial of an element of R^* not in R gives a counter-example to Corollary 2.

2. Applications to C_i questions.

Recall that a domain R is called C_i if any form with coefficients in R of degree d in n variables with $n > d^i$ has a non-trivial zero in R . C_0 means that the field of fractions of R is algebraically closed.

Theorem 2. — *If k is a C_i field, then the field $k((t))$ of formal power series in one variable t over k is C_{i+1} .*

This generalizes some results of Lang [3], who did the cases $i=1$, k finite, and $i=0$. Note that $[k:k^p] \leq p^i$ (take a basis).

It suffices to prove that $R = k[[t]]$ is C_{i+1} . By Lang [3], $k[t]$ is C_{i+1} . Hence the hypersurface H in projective $(n-1)$ -space defined by the given form has a point in the ring R/t^v for all v . By Corollary 2, H has a point in R .

Note 1. — The same type of argument yields a short proof of Lang's theorem that if R is a Henselian discrete valuation ring with algebraically closed residue field, such that K^* is separable over K , then R is C_1 . For by Corollary 2, we may assume R complete, and since C_1 is inherited by finite extensions, we may also assume R unramified. Then the argument given in [3; p. 384] shows H has a point in R/t^v for all v .

Note 2. — In the definition of C_i , replace the word “form” by “polynomial without constant term”; a ring with this property is called *strongly* C_i . For example, finite fields are strongly C_1 . A theorem of Lang-Nagata states that an algebraic function field in one variable over a strongly C_i field is strongly C_{i+1} . It is natural to ask whether the same statement holds for the power series field in one variable. Ax-Kochen confirm this in characteristic 0 by showing that the Henselization of $k[t]$ at the origin is elementarily equivalent to $k[[t]]$.

Note 3. — In the definition of strongly C_i , suppose we take the expression “non-trivial” to mean “some coordinate is a unit in R ”, instead of “some coordinate is non-zero”. Call this property strongly C_i^* . If R is a strongly C_i^* discrete valuation ring, then the completion of R is also strongly C_i^* , by Theorem 1. It is therefore natural to ask: If a field k is strongly C_i , is the localization of $k[t]$ at the origin strongly C_{i+1}^* ?

3. Newton's Lemma.

In this section, R will be an analytically irreducible Henselian local domain with maximal ideal $\mathfrak{m}$, F will be a system of r polynomials in n variables with coefficients in R , $1 \leq r \leq n$, J the Jacobian matrix of this system.

Lemma 1. — Assume $r=n$. Given x in R such that

$$\begin{aligned} F(x) &\equiv 0 \pmod{\mathfrak{m}} \\ \det J(x) &\not\equiv 0 \pmod{\mathfrak{m}} \end{aligned}$$

Then there is y in R such that

$$\begin{aligned} \text{(i)} \quad & y \equiv x \pmod{\mathfrak{m}} \\ \text{(ii)} \quad & F(y) = 0 \end{aligned}$$

Proof. — There is y in the completion R^* satisfying (i) and (ii), by [2; 11.13.3]. Since $r=n$ and $\det J(y) \not\equiv 0$, the domain $R[y]$ is separably algebraic over R . But R is separably algebraically closed in R^* , hence y is in R .

Lemma 2. — Let x in R be such that

$$F(x) \equiv 0 \pmod{e^2 \mathfrak{m}}$$

where $e = D(x)$, D being a subdeterminant of order r of $\det J$. Then there is y in R such that

$$\begin{aligned} y &\equiv x \pmod{e \mathfrak{m}} \\ F(y) &= 0 \end{aligned}$$

Proof. — We may assume $e \neq 0$. We may assume $x = 0$ and that D is the subdeterminant obtained from the first r variables. If $r < n$, setting

$$F_j(X) = X_j \quad j = r+1, \dots, n$$

shows we can assume $r=n$, hence $D = \det J$. Let J' be the adjoint matrix to J , so that $JJ' = DI = J'J$, with I the identity matrix. By Taylor's formula,

$$F(eX) = F(0) + eJ(0)X + e^2G(X)$$

where $G(X)$ is a vector of polynomials each beginning with terms of degree at least 2.

Using

$$e = J(o)J'(o)$$

and the hypothesis on $F(o)$, we can factor out $eJ(o)$:

$$F(eX) = eJ(o)H(X)$$

where H is a system whose Jacobian matrix at o is I , and

$$H(o) \equiv 0 \pmod{\mathfrak{m}}$$

By lemma 1, there is y' in $\mathfrak{m}$ such that $H(y') = 0$, whence $y = ey'$ does the trick.

Note. — The following argument (due to M. Artin) should eliminate the assumption that R is analytically irreducible, used in the proof of Lemma 1: Let $Y = \text{Spec } R[X]/FR[X]$, $f: Y \rightarrow \text{Spec } R$ the canonical morphism. The hypothesis of Lemma 1 gives us a point $\bar{x}$ of Y lying over the closed point of $\text{Spec } R$, such that $\bar{x}$ is isolated in its fibre and f is smooth at $\bar{x}$. Hence the local ring $\mathfrak{o}$ of x on Y is étale over R [5; 11, 1.4] with the same residue field. Since R is Henselian, $R \rightarrow \mathfrak{o}$ is an isomorphism [1], hence we have a section $\text{Spec } R \rightarrow Y$ passing through $\bar{x}$.

4. Acknowledgements.

The argument in Case 1 has been developed from ideas of P. Cohen and A. Néron. My original argument in Case 2 required the extra assumption $[k : k^p] < \infty$; the present argument is essentially due to M. Raynaud. Newton's lemma for Henselian local rings was suggested by M. Artin.

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