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# INVARIANTS OF ANALYTIC LOCAL RINGS (1)

*by* S. S. ABHYANKAR, T. T. MOH and M. VAN DER PUT

## § 1. Introduction.

This is a sequel to [2]. In Theorems (4.2), (4.3), (5.5), (5.6), (5.7), (5.8), (5.9), and (6.1) we shall prove several results concerning groups of automorphisms of analytic local rings and the rings of invariants of such groups. In the statements of all these theorems except the last one, K is any valued field and $A_m$ is the ring of convergent power series in indeterminates $X_1, \ldots, X_m$ with coefficients in K. In § 7 we shall make some remarks concerning fields of definition and their relationship with fields of invariants.

*Terminology.* — We shall use the terminology of [2, § 2]. By card we shall *denote* cardinal number. If R is any ring and S is the integral closure of R in the total quotient ring T of R, then every automorphism of R can be extended uniquely to an automorphism of S, i.e., given any $g \in G(R)$ there exists a unique $h \in G(S)$ such that $h(r) = g(r)$ for all $r \in R$; (namely, since T is the total quotient ring of R, there exists a unique $h' \in G(T)$ such that $h'(r) = g(r)$ for all $r \in R$; since S is the integral closure of R in T, we must have $h'(S) = S$, and hence we get the unique $h \in G(S)$ by taking $h(s) = h'(s)$ for all $s \in S$); the resulting map of G(R) into G(S) will be *denoted* by $I_R$, i.e., $I_R : G(R) \to G(S)$ is the unique monomorphism such that for all $g \in G(R)$ and all $r \in R$ we have $I_R(g)(r) = g(r)$; note that

$$I_R(G(R)) = \{h \in G(S) : h(R) = R\}.$$

## § 2. Integral dependance and conductor.

Recall that if R is a ring and S is an overring of R then by definition,
the *conductor* of R in S

$$= \{u \in R : us \in R \text{ for all } s \in S\}$$
$$= \text{the largest ideal in R which remains an ideal in S.}$$

*Lemma* (2.1). — *Let* R *be a ring and let* S *be an overring of* R. *Let* C *be the conductor of* R *in* S. *Then we have the following.*

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**(2.1.1)** *Let* Q *be any ideal in* S, *and let* $h \in G(S, Q)$ *and* $g \in G(R)$ *be such that* $h(r) = g(r)$ *for all* $r \in R$. *Then* $g \in G(R, Q \cap R)$.

**(2.1.2)** *For any* $h \in G(S, C)$ *we have* $h(R) = R$.

**(2.1.3)** *Let* Q *be any ideal in* R. *Then* QC *is an ideal in* S. *Moreover, for any* $h \in G(S, QC)$ *we have* $h(R) = R$, *and upon defining* $g \in G(R)$ *by taking* $g(r) = h(r)$ *for all* $r \in R$, *we have that* $g \in G(R, QC)$.

**(2.1.4)** *Let* Q *be any ideal in* S, *and let* $u \in C$. *Then* $(uS)Q$ *is an ideal in* R. *Now assume that* u *is a nonzerodivisor in* S, *and let* $h \in G(S)$ *and* $g \in G(R, (uS)Q)$ *be such that* $h(r) = g(r)$ *for all* $r \in R$. *Then* $h \in G(S, Q)$.

**(2.1.5)** *Let* Q *be any ideal in* R, *and let* $u \in C$ *be such that* u *is a nonzerodivisor in* S. *Let* $h \in G(S)$ *and* $g \in G(R, (uR)Q)$ *be such that* $h(r) = g(r)$ *for all* $r \in R$. *Then* $h \in G(S, QS)$.

*Proof of* (2.1.1). — Obvious.

*Proof of* (2.1.2). — For any $h \in G(S, C)$ and any $r \in R$ we have $h(r) - r \in C \subset R$, and hence $h(r) \in R$. Thus for any $h \in G(S, C)$ we have $h(R) \subset R$; by [2, (2.1)] we also have $h^{-1} \in G(S, C)$ and hence $h^{-1}(R) \subset R$; therefore $h(R) = R$.

*Proof of* (2.1.3). — Any element $t$ in $(QC)S$ can be expressed as a finite sum: $t = \sum_i q_i u_i s_i$ with $q_i \in Q, u_i \in C, s_i \in S$; now $u_i s_i \in C$ for all $i$, and hence $t \in QC$. This shows that QC is an ideal in S. Since $QC \subset C$, we have $G(S, QC) \subset G(S, C)$; therefore the rest now follows from (2.1.1) and (2.1.2).

*Proof of* (2.1.4). — Clearly $(uS)Q \subset R$ and hence $(uS)Q$ is an ideal in R. Now assume that $u$ is a nonzerodivisor in S, and let $h \in G(S)$ and $g \in G(R, (uS)Q)$ be such that $h(r) = g(r)$ for all $r \in R$. Given any $s \in S$ we want to show that $h(s) - s \in Q$. Now $us \in R$; since $g \in G(R, (uS)Q)$, and $u$ and $us$ are elements in R, we get

$$g(us) - us = uq \quad \text{with } q \in Q, \quad \text{and} \quad g(u) - u = uq' \quad \text{with } q' \in Q.$$

Now
$$\begin{aligned} g(us) - us &= h(us) - us \\ &= h(u)h(s) - uh(s) + uh(s) - us \\ &= h(s)(h(u) - u) + u(h(s) - s) \\ &= h(s)(g(u) - u) + u(h(s) - s) \end{aligned}$$

and hence
$$\begin{aligned} u(h(s) - s) &= (g(us) - us) - h(s)(g(u) - u) \\ &= uq - h(s)uq' \\ &= u(q - h(s)q'). \end{aligned}$$

Since $u$ is a nonzerodivisor in S, we must have $h(s) - s = q - h(s)q'$ and hence $h(s) - s \in Q$.

*Proof of* (2.1.5). — We get a proof of this by making the following changes in the proof of (2.1.4): omit the first two sentences; in the third and the last sentences change Q to QS; in the fourth sentence change $G(R, (uS)Q)$ to $G(R, (uR)Q)$. Alternatively let $Q' = QS$; then $Q'$ is an ideal in S; clearly $(uR)Q \subset (uS)Q'$ and hence $g \in G(R, (uS)Q')$; therefore by (2.1.4) we get that $h \in G(S, Q')$.

*Lemma* **(2.2)**. — *Let* R *be a ring and let* S *be the integral closure of* R *in the total quotient ring of* R. *Let* C *be the conductor of* R *in* S. *Then we have the following.*

**(2.2.1)** *If* $Q$ *is any ideal in* $S$ *then* $G(S, Q) \cap I_R(G(R)) \subset I_R(G(R, Q \cap R))$.

**(2.2.2)** $G(S, C) \subset I_R(G(R))$.

**(2.2.3)** *If* $Q$ *is any ideal in* $R$ *then* $QC$ *is an ideal in* $S$ *and* $G(S, QC) \subset I_R(G(R, QC))$.

**(2.2.4)** *Let* $Q$ *be any ideal in* $S$, *and let* $u \in C$. *Then* $(uS)Q$ *is an ideal in* $R$. *If moreover* $u$ *is a nonzerodivisor in* $R$, *then* $I_R(G(R, (uS)Q)) \subset G(S, Q)$.

**(2.2.5)** *Let* $Q$ *be any ideal in* $R$, *and let* $u \in C$ *be such that* $u$ *is a nonzerodivisor in* $R$. *Then* $I_R(G(R, (uR)Q)) \subset G(S, QS)$.

*Proof.* — (2.2.1), (2.2.2) and (2.2.3) follow respectively from (2.1.1), (2.1.2) and (2.1.3). (2.2.4) and (2.2.5) follow respectively from (2.1.4) and (2.1.5) by noting that in the present case every nonzerodivisor in R is also a nonzerodivisor in S.

*Lemma* **(2.3).** — *Let* $R$ *be a noetherian ring with* $\mathrm{rad}_R\{0\} = \{0\}$. *Let* $P_1, \ldots, P_e$ *be all the distinct prime ideals of height zero in* $R$. *Let* $T$ *be the total quotient ring of* $R$. *Let* $R^*$ *be a noetherian subring of* $T$ *such that* $T$ *is the total quotient ring of* $R^*$. (Note that then by [1, (18.9)] we have that: $P_1T, \ldots, P_eT$ are exactly all the distinct prime ideals in T, and they all have height zero; T is noetherian; $(P_1T) \cap R^*, \ldots, (P_eT) \cap R^*$ are exactly all the distinct prime ideals of height zero in $R^*$; $\mathrm{rad}_{R^*}\{0\} = \{0\}$; and for $1 \leq i \leq e$ we have $(P_iT) \cap R = P_i$ and $((P_iT) \cap R^*)T = P_iT$.) *Let* $S$ *be the integral closure of* $R$ *in* $T$. *Assume that* $R^* \subset S$ *and* $S$ *is integral over* $R^*$. *Then we have the following*:

**(2.3.1)** *For* $1 \leq i \leq e$ *we have*

$$I_R^{-1}(G[S, (P_iT) \cap S]) = G[R, P_i],$$

$$I_{R^*}^{-1}(G[S, (P_iT) \cap S]) = G[R^*, (P_iT) \cap R^*],$$

*and*

$$I_R^{-1}(I_{R^*}(G[R^*, (P_iT) \cap R^*])) \subset G[R, P_i].$$

**(2.3.2)** *Assume that the integral closure of* $R/P_i$ *in its quotient field is a finite* $(R/P_i)$*-module for* $1 \leq i \leq e$. *Let* $C$ *be the conductor of* $R$ *in* $S$. *Then* $C$ *contains a nonzerodivisor of* $R$.

**(2.3.3)** *Assume that the integral closure of* $R/P_i$ *in its quotient field is a finite* $(R/P_i)$*-module for* $1 \leq i \leq e$. *Also assume that the integral closure of* $R^*/(P_iT) \cap R^*$ *in its quotient field is a finite* $(R^*/(P_iT) \cap R^*)$*-module for* $1 \leq i \leq e$. *Let* $J$ *be any ideal in* $R$. *Then there exists an ideal* $J^*$ *in* $R^*$ *such that* $I_{R^*}(G(R^*, J^*)) \subset I_R(G(R, J))$ *and such that for* $1 \leq i \leq e$ *we have*: $J^* \subset P_iT \Leftrightarrow J \subset P_iT$.

*Proof of* (2.3.1). — The second equation follows the first equation by interchanging R and $R^*$. The last inclusion follows from the first and the second equations. To prove the first equation, given any $g \in G(R)$ let $h = I_R(g)$. What we have to show is that: $g(P_i) = P_i \Leftrightarrow h((P_iT) \cap S) = (P_iT) \cap S$. If $h((P_iT) \cap S) = (P_iT) \cap S$ then

$$g(P_i) = h(P_i) = h(R \cap ((P_iT) \cap S)) = h(R) \cap h((P_iT) \cap S) = R \cap ((P_iT) \cap S) = P_i.$$

Conversely, suppose that $g(P_i) = P_i$. Let $h'$ be the unique element in G(T) such that $h'(s) = h(s)$ for all $s \in S$. Since $P_1T, \ldots, P_eT$ are exactly all the distinct prime ideals in T and $h'$ is an automorphism of T, we see that $h'(P_iT) = P_jT$ for some $j$. Now

$$P_i = g(P_i) = h'(P_i) = h'((P_iT) \cap R) = h'(P_iT) \cap h'(R) = (P_jT) \cap R = P_j$$

and hence $j=i$. Therefore

$$h((P_iT)\cap S)=h'((P_iT)\cap S)=h'(P_iT)\cap h'(S)=(P_iT)\cap S.$$

*Proof of* (2.3.2). — The quotient ring of R with respect to $P_i$ is clearly a field and hence by [1, (19.21.2)] we get that $C\not\subset P_i$. This being so for $1\leq i\leq e$, we conclude that C contains a nonzerodivisor of R.

*Proof of* (2.3.3). — Let C be conductor of R in S; by (2.3.2) we can find $u\in C$ such that $u$ is a nonzerodivisor in T, i.e., $u\notin P_iT$ for $1\leq i\leq e$. Let $C^*$ be the conductor of $R^*$ in S; by (2.3.2) we can find $u^*\in C^*$ such that $u^*$ is a nonzerodivisor in T, i.e., $u^*\notin P_iT$ for $1\leq i\leq e$. By (2.2.3) we know that JC is an ideal in S and $G(S, JC)\subset I_R(G(R, JC))$; since $JC\subset J$, we also have $I_R(G(R, JC))\subset I_R(G(R, J))$. By (2.2.4) we get that $(u^*S)(JC)$ is an ideal in $R^*$ and $I_{R^*}(G(R^*, (u^*S)(JC)))\subset G(S, JC)$. Therefore upon letting $J^*=(u^*S)(JC)$ we get that $J^*$ is an ideal in $R^*$ and $I_{R^*}(G(R^*, J^*))\subset I_R(G(R, J))$. Now $(uu^*T)(JT)\subset J^*T\subset JT$, and $uu^*\notin P_iT$ for $1\leq i\leq e$; therefore for $1\leq i\leq e$ we have: $J^*\subset P_iT \Leftrightarrow J\subset P_iT$.

*Lemma* (**2.4**). — *Let* R *be an analytic local ring over a valued field* K. *Let* S *be an overring of* R *such that* S *is a finite* R*-module. Let* N *be any subset of* S *such that* N *is contained in every maximal ideal of* S. *Then* R[N] *is an analytic local ring over* K.

*Proof.* — We can find a finite sequence of elements $y_1, \ldots, y_m$ in N such that $R[N]=R[y_1, \ldots, y_m]$; now $R[y_1, \ldots, y_i]=(R[y_1, \ldots, y_{i-1}])[y_i]$ for $1\leq i\leq m$; consequently, by an obvious induction, the general case would follow from the case when N consists of a single element $y$. Let $X_0, X_1, X_2, \ldots$ be indeterminates. Since R is an analytic local ring over K, there exists a K-epimorphism $v: B\to R$ where $B=K[\langle X_1, \ldots, X_n\rangle]$ for some nonnegative integer $n$. Let $A=K[\langle X_0, \ldots, X_n\rangle]$ where we regard A to be an overring of B. Since $y$ is integral over R, there exists a positive integer $e$ and elements $a_0, \ldots, a_e$ in B with $a_e=1$ such that

$$\sum_{i=0}^{e} v(a_i)y^i=0. \tag{1}$$

Let $d$ be the smallest nonnegative integer $\leq e$ such that $a_d\notin M(B)$. Let $q=e-d$. Then by the Weierstrass Preparation Theorem [1, (10.3)] there exist elements $b_0, \ldots, b_d$, $t_0, \ldots, t_q$ in B such that $b_d=1=t_q$, $b_i\in M(B)$ for $0\leq i<d$, $t_0\notin M(B)$, and

$$\sum_{i=0}^{e} a_iX_0^i=\Big(\sum_{i=0}^{d} b_iX_0^i\Big)\Big(\sum_{i=0}^{q} t_iX_0^i\Big). \tag{2}$$

Now $v(t_0)\notin M(R)$. For every maximal ideal M in S we have $M\cap R=M(R)$ and hence $v(t_0)\notin M$; since by assumption $y\in M$, we get that $t_0+t_1y+\ldots+t_qy^q\notin M$. This being so for every maximal ideal M in S, we conclude that $t_0+t_1y+\ldots+t_qy^q$ is a unit in S; therefore by (1) and (2) we get that

$$\sum_{i=0}^{d} v(b_i)y^i=0.$$

Whence, in particular, $d>0$. Let

$$F=\sum_{i=0}^{d} b_i X_0^i$$

and let A′ be the set of all polynomials of degree $<d$ in $X_0$ with coefficients in B. Then by [1, (10.3)], for every $f\in A$ there exists a unique $r_f\in A'$ such that $f-r_f\in FA$. We get a map $w: A\to R[y]$ by taking

$$w(f)=\sum_{i=0}^{d-1} v(f_i)y^i \quad \text{for all } f\in A$$

where $f_0, \ldots, f_{d-1}$ are the unique elements in B with

$$r_f=\sum_{i=0}^{d-1} f_i X_0^i.$$

By [1, (10.3)] we also have that: $r_{f+f^*}=r_f+r_{f^*}$ and $r_{ff^*}-r_f r_{f^*}\in FB[X_0]$ for all $f$ and $f^*$ in A. It follows that $w(z)=v(z)$ for all $z\in R$, $w$ is a ring homomorphism of A into $R[y]$ and $w(A)=R[y]$ (note that if $d=1$ then we must have $y\in R$). Therefore $R[y]$ is an analytic local ring over K.

## § 3. Automorphisms leaving a hypersurface fixed.

Let K be a valued field, and let $A=K[\langle X\rangle]=K[\langle X_0, \ldots, X_n\rangle]$ where $X=(X_0, \ldots, X_n)$ are indeterminates and $n>0$ (the statement and proof of Lemmas (3.1) and (3.2) hold verbatim also for $n=0$).

*Lemma* **(3.1).** — *Let* $B=K[\langle X, Y_1, \ldots, Y_m\rangle]$ *where* $Y_1, \ldots, Y_m$ *are indeterminates* $(m>0)$. *Let* $V_i=V_i(X, Y_1, \ldots, Y_m)\in B$ *with*

$$V_i-Y_i\in((Y_1, \ldots, Y_m)B)^2 \quad \textit{for } 1\leq i\leq m. \tag{1}$$

*Let* $D_i\in M(A)$ *for* $1\leq i\leq m$. *Then there exist unique elements* $E_1, \ldots, E_m$ *in* $M(A)$ *such that*

$$V_i(X, E_1, \ldots, E_m)=D_i \quad \textit{for } 1\leq i\leq m. \tag{2}$$

*Moreover, we have*

$$(E_1, \ldots, E_m)A=(D_1, \ldots, D_m)A \tag{3}$$

*and*

$$E_i-D_i\in((D_1, \ldots, D_m)A)^2 \quad \textit{for } 1\leq i\leq m. \tag{4}$$

*Proof.* — In view of (1) we see that the value of the jacobian determinant

$$\frac{\partial(V_1-D_1, \ldots, V_m-D_m)}{\partial(Y_1, \ldots, Y_m)}(0, \ldots, 0)$$

equals 1, and hence by the Implicit Function Theorem [1, (10.8)] there exist unique elements $E_1, \ldots, E_m$ in $M(A)$ satisfying (2). By (1) and (2) we see that

$$(D_1, \ldots, D_m)A \subset (E_1, \ldots, E_m)A \subset (D_1, \ldots, D_m)A + ((E_1, \ldots, E_m)A)M(A)$$

and hence by Nakayama's lemma we get (3). By (1), (2), and (3) we get (4).

We shall now give an alternative proof by using the Inversion Theorem instead of the Implicit Function Theorem. In view of (1) we see that

$$\frac{\partial(V_1, \ldots, V_m)}{\partial(Y_1, \ldots, Y_m)}(0, \ldots, 0) = 1$$

and hence by the Inversion Theorem [1, (10.10)] there exists

$$W_i = W_i(X, Y_1, \ldots, Y_m) \in M(B) \quad \text{for} \quad 1 \leq i \leq m,$$

such that for $1 \leq i \leq m$ we have

$$Y_i = V_i(X, W_1(X, Y_1, \ldots, Y_m), \ldots, W_m(X, Y_1, \ldots, Y_m)) \tag{5}$$

and

$$Y_i = W_i(X, V_1(X, Y_1, \ldots, Y_m), \ldots, V_m(X, Y_1, \ldots, Y_m)). \tag{6}$$

We can write

$$W_i = W_i' + W_{i1}Y_1 + \ldots + W_{im}Y_m + W_i^*$$

with $$W_i' \in M(A), \quad W_{ij} \in A, \quad W_i^* \in ((Y_1, \ldots, Y_m)B)^2;$$

now in view of (1), by (6) we get that

$$Y_i = W_i' + W_{i1}Y_1 + \ldots + W_{im}Y_m + \text{an element in } ((Y_1, \ldots, Y_m)B)^2.$$

Considering the above as an equation between power series in $Y_1, \ldots, Y_m$ with coefficients in the quotient field of $K[[X]]$, and comparing coefficients on the two sides we see that

$$W_i' = 0, \quad W_{ii} = 1, \quad \text{and} \quad W_{ij} = 0 \quad \text{whenever} \quad j \neq i.$$

In other words,

$$W_i - Y_i \in ((Y_1, \ldots, Y_m)B)^2 \quad \text{for} \quad 1 \leq i \leq m. \tag{7}$$

Upon letting

$$E_i = W_i(X, D_1, \ldots, D_m) \quad \text{for} \quad 1 \leq i \leq m, \tag{8}$$

we get elements $E_1, \ldots, E_m$ in $M(A)$; upon substituting $D_1, \ldots, D_m$ for $Y_1, \ldots, Y_m$ in (5) we get (2); by (1), (2), (7) and (8) we get (3) and (4). Conversely, if $E_1, \ldots, E_m$ are any elements in $M(A)$ satisfying (2) then upon substituting $E_1, \ldots, E_m$ for $Y_1, \ldots, Y_m$ in (6) we get (8), which proves the uniqueness.

For the formal case the following lemma was given by Samuel [3]:

*Lemma* **(3.2).** — *Let* $F=F(X)=F(X_0, \ldots, X_n)\in A$. *Let* $F_i=\partial F/\partial X_i$. *Let* $D_{ij}\in M(A)$ *for* $0\leq i\leq n$, $0\leq j\leq n$. *Then there exist elements* $H_0, \ldots, H_n$ *in* $M(A)$ *such that*

$$F(X_0+H_0, \ldots, X_n+H_n)=F+\sum_{i=0}^{n}\sum_{j=0}^{n}D_{ij}F_iF_j \tag{1}$$

*and*

$$H_i-\sum_{j=0}^{n}D_{ij}F_j\in((F_0, \ldots, F_n)A)((D_{00}, D_{01}, \ldots, D_{nn})A)^2 \quad \textit{for} \quad 0\leq i\leq n. \tag{2}$$

*Proof.* — Let $Z_0, \ldots, Z_n$ be indeterminates. Then

$$F(X_0+Z_0, \ldots, X_n+Z_n)=F+\sum_{i=0}^{n}Z_iF_i+V' \tag{3}$$

where $V'$ is an element in $K[\langle X, Z_0, \ldots, Z_n\rangle]$ such that the order of $V'$ in $Z_0, \ldots, Z_n$ is $\geq 2$, and hence we can write

$$V'=\sum_{i=0}^{n}\sum_{j=0}^{n}V'_{ij}(X, Z_0, \ldots, Z_n)Z_iZ_j \tag{4}$$

with $V'_{ij}(X, Z_0, \ldots, Z_n)\in K[\langle X, Z_0, \ldots, Z_n\rangle]$. Let $Y_{00}, Y_{01}, \ldots, Y_{nn}$ be $(n+1)^2$ indeterminates. Upon substituting

$$\sum_{s=0}^{n}Y_{rs}F_s \quad \text{for} \quad Z_r,\ 0\leq r\leq n,$$

by (3) and (4) we get

$$F(X_0+\sum_{s=0}^{n}Y_{0s}F_s, \ldots, X_n+\sum_{s=0}^{n}Y_{ns}F_s)=F+\sum_{i=0}^{n}\sum_{j=0}^{n}V_{ij}(X, Y_{00}, Y_{01}, \ldots, Y_{nn})F_iF_j \tag{5}$$

where $V_{ij}=V_{ij}(X, Y_{00}, Y_{01}, \ldots, Y_{nn})$ is the element in $B=K[\langle X, Y_{00}, Y_{01}, \ldots, Y_{nn}\rangle]$ given by

$$V_{ij}(X, Y_{00}, Y_{01}, \ldots, Y_{nn})=Y_{ij}+\sum_{t=0}^{n}\sum_{u=0}^{n}V'_{tu}(X, \sum_{s=0}^{n}Y_{0s}F_s, \ldots, \sum_{s=0}^{n}Y_{ns}F_s)Y_{ti}Y_{uj}$$

and hence $\quad V_{ij}-Y_{ij}\in((Y_{00}, Y_{01}, \ldots, Y_{nn})B)^2 \quad$ for $0\leq i\leq n$, $0\leq j\leq n$.

By (3.1) there exist $(n+1)^2$ elements $E_{00}, E_{01}, \ldots, E_{nn}$ in $M(A)$ such that

$$V_{ij}(X, E_{00}, E_{01}, \ldots, E_{nn})=D_{ij} \quad \text{for} \quad 0\leq i\leq n,\ 0\leq j\leq n, \tag{6}$$

and

$$E_{ij}-D_{ij}\in((D_{00}, D_{01}, \ldots, D_{nn})A)^2 \quad \text{for} \quad 0\leq i\leq n,\ 0\leq j\leq n. \tag{7}$$

Let

$$H_r=\sum_{s=0}^{n}E_{rs}F_s \quad \text{for} \quad 0\leq r\leq n. \tag{8}$$

Then $H_0, \ldots, H_n$ are elements in $M(A)$ and upon substituting $E_{rs}$ for $Y_{rs}$ $(0\leq r\leq n, 0\leq s\leq n)$ in (5), by (6) and (8) we get (1). By (7) and (8) we also get (2).

*Lemma* **(3.3)**. — *Given* $F \in A$ *let* $F_i = \partial F/\partial X_i$. *Let* $D \in M(A)$ *be such that* $DF_i \in M(A)^2$ *for* $0 \leq i \leq n$. *Then there exists* $g \in G_K(A, (DF_0, \ldots, DF_n)A)$ *such that* $g(F) = F$,

$$g(X_0) - X_0 + DF_1 \in (D^2F_0, \ldots, D^2F_n)A,$$
$$g(X_1) - X_1 - DF_0 \in (D^2F_0, \ldots, D^2F_n)A,$$

*and*

$$g(X_i) - X_i \in (D^2F_0, \ldots, D^2F_n)A \quad \textit{for} \quad 2 \leq i \leq n.$$

*Proof.* — Upon taking

$$\begin{aligned} &D_{01} = -D, && D_{00} = 0 = D_{0j} && \text{for } 2 \leq j \leq n, \\ &D_{10} = D, && D_{1j} = 0 && \text{for } 1 \leq j \leq n, \\ &D_{ij} = 0 && \text{for } 2 \leq i \leq n && \text{and } 0 \leq j \leq n, \end{aligned}$$

by (3.2) we find elements $H_0, \ldots, H_n$ in $M(A)$ such that

$$F(X_0 + H_0, \ldots, X_n + H_n) = F$$

and such that the elements $H_0 + DF_1, H_1 - DF_0, H_2, \ldots, H_n$ all belong to the ideal $(D^2F_0, \ldots, D^2F_n)A$. In particular then $H_i \in (DF_0, \ldots, DF_n) \subset M(A)^2$ for $0 \leq i \leq n$, and hence by [2, (2.15)] we get a unique $g \in G_K(A)$ such that $g(X_i) = X_i + H_i$ for $0 \leq i \leq n$. Now clearly $g(F) = F$,

$$g(X_0) - X_0 + DF_1 \in (D^2F_0, \ldots, D^2F_n)A,$$
$$g(X_1) - X_1 - DF_0 \in (D^2F_0, \ldots, D^2F_n)A,$$

and

$$g(X_i) - X_i \in (D^2F_0, \ldots, D^2F_n)A \quad \text{for} \quad 2 \leq i \leq n.$$

Since $g(X_i) - X_i = H_i \in (DF_0, \ldots, DF_n)$ for $0 \leq i \leq n$, by [2, (2.9)] we see that $g \in G_K(A, (DF_0, \ldots, DF_n)A)$.

*Lemma* **(3.4)**. — *Let* $0 \neq F \in M(A)$, $L \in A$, $E_1 \in M(A), \ldots, E_d \in M(A)$ $(d > 0)$, *be such that* $L(\partial F/\partial X_0) \notin E_jA$ *for* $1 \leq j \leq d$. *Let* $P_1, \ldots, P_e$ *be all the distinct prime ideals of height one in* A *containing* F. *Let* $u$ *be a positive integer. Then there exists an infinite subset* G *of*

$$G_K(A, (LA) \cap M(A)^u) \cap G_K[A, P_1] \cap \ldots \cap G_K[A, P_e]$$

*with* $\operatorname{card}(G) \geq \operatorname{card}(K)$ *such that for all* $g \in G$ *we have* $g(F) = F$, *and for all* $g \neq h$ *in* G *we have* $g(X_1) - h(X_1) \notin E_jA$ *for* $1 \leq j \leq d$.

*Proof.* — Let J be the set of integers $1, \ldots, d$, and let $J' = \{j \in J : E_j \neq 0\}$. For every $j \in J'$ let $r_j = \operatorname{ord}_A E_j$ and let $E_j^*$ be the unique nonzero homogeneous element of degree $r_j$ in $K[X]$ with $E_j - E_j^* \in M(A)^{r_j+1}$.

Clearly there exists an infinite set N of pairwise coprime nonconstant irreducible homogeneous elements in $K[X_0, X_1]$ (namely, if K is infinite then $\{X_0 + kX_1 : k \in K\}$ is such a set; in the general case, upon letting $N_i$ to be the set of all monic irreducible polynomials of degree $i$ in $K[X_0]$ we clearly have that $\bigcup_{i=1}^{\infty} N_i$ is an infinite set, and hence $\bigcup_{i=1}^{\infty} \{X_1^i f(X_0/X_1) : f(X_0) \in N_i\}$ is an infinite set of pairwise coprime nonconstant irreducible homogeneous elements in $K[X_0, X_1]$). Moreover, for any such N we have that N

is an infinite set of pairwise coprime nonconstant irreducible homogeneous elements in K[X].

Therefore, we can find a nonconstant irreducible homogeneous element E, of some degree $q>0$, in K[X] such that for all $j\in J'$ we have $E_j^*\notin EK[X]$.

Let $w_j : A\to A/E_jA$ be the canonical epimorphism.

We claim that if $j$ is any integer in J, V is any element in A with $w_j(V)\neq 0$, $k^*$ is any nonzero element in K, and $b$ is any nonnegative integer, then

$$\operatorname{ord}_{w_j(A)}w_j(k^*VE^b)=bq+\operatorname{ord}_{w_j(A)}w_j(V). \tag{1}$$

This being obvious for $j\notin J'$, suppose that $j\in J'$ and let $v=\operatorname{ord}_{w_j(A)}w_j(V)$. Then there exists a nonzero homogeneous element $V^*$ of degree $v$ in K[X] such that

$$w_j(V)=w_j(V^*)+\text{an element in } M(w_j(A))^{v+1}. \tag{2}$$

By (2) we get

$$w_j(k^*VE^b)=w_j(k^*V^*E^b)+\text{an element in } M(w_j(A))^{bq+v+1}. \tag{3}$$

Suppose if possible that $\operatorname{ord}_{w_j(A)}w_j(k^*VE^b)>bq+v$. Then by (3) we get that $\operatorname{ord}_{w_j(A)}w_j(k^*V^*E^b)>bq+v$; and hence there exists $E_j'\in A$ such that

$$k^*V^*E^b-E_jE_j'\in M(A)^{bq+v+1}.$$

Since $k^*V^*E^b$ is a nonzero homogeneous element of degree $bq+v$ in K[X], we must now have

$$\operatorname{ord}_A E_j'=bq+v-r_j \quad\text{and}\quad k^*V^*E^b=E_j^*E_j'^*$$

where $E_j'^*$ is the unique nonzero homogeneous element of degree $bq+v-r_j$ in K[X] such that

$$E_j'-E_j'^*\in M(A)^{bq+v-r_j+1}.$$

Now E is irreducible, $E_j^*\notin EK[X]$, and $k^*V^*E^b\in E_j^*K[X]$; consequently we must have $V^*\in E_j^*K[X]$, and hence $V^*=E_j^*E_j^{**}$ for some $E_j^{**}\in K[X]$. It now follows that $V^*-E_jE_j^{**}\in M(A)^{v+1}$, and hence by (2) we get $\operatorname{ord}_{w_j(A)}w_j(V)>v$, which is a contradiction. This completes the proof of (1).

We can write

$$F=F^{(1)a_1}\dots F^{(e)a_e} \tag{4}$$

where

$$a_s>0,\quad F^{(s)}\in A,\quad\text{and}\quad F^{(s)}A=P_s\quad\text{for } 1\leq s\leq e. \tag{5}$$

For all $s\neq t$ we have

$$F^{(s)}\notin F^{(t)}A=\bigcap_{m=1}^{\infty}(F^{(t)}A+M(A)^m),$$

and hence we can find a positive integer $b^*$ such that

$$F^{(s)}\notin F^{(t)}A+M(A)^{b^*}\quad\text{whenever } s\neq t. \tag{6}$$

By assumption $w_j(\mathrm{L}(\partial \mathrm{F}/\partial \mathrm{X}_0)) \neq 0$ for $1 \leq j \leq d$, and hence we can find an integer $b_0 \geq b^* + u + 1$ such that

$$b_0 q > \mathrm{ord}_{w_j(\mathrm{A})} w_j(\mathrm{L}(\partial \mathrm{F}/\partial \mathrm{X}_0)) \quad \text{for} \quad 1 \leq j \leq d. \tag{7}$$

Given any $b \geq b_0$ and any $k \in \mathrm{K}$, by taking $\mathrm{D} = k\mathrm{LE}^b$ in (3.3) we find

$$g_{k,b} \in \mathrm{G}_{\mathrm{K}}(\mathrm{A}, (k\mathrm{LE}^b(\partial \mathrm{F}/\partial \mathrm{X}_0), \ldots, k\mathrm{LE}^b(\partial \mathrm{F}/\partial \mathrm{X}_n))\mathrm{A})$$

such that

$$g_{k,b}(\mathrm{F}) = \mathrm{F}, \tag{8}$$

and $g_{k,b}(\mathrm{X}_1) - \mathrm{X}_1 - k\mathrm{LE}^b(\partial \mathrm{F}/\partial \mathrm{X}_0) \in ((k\mathrm{LE}^b)^2(\partial \mathrm{F}/\partial \mathrm{X}_0), \ldots, (k\mathrm{LE}^b)^2(\partial \mathrm{F}/\partial \mathrm{X}_n))\mathrm{A}$; clearly then

$$g_{k,b} \in \mathrm{G}_{\mathrm{K}}(\mathrm{A}, (\mathrm{LA}) \cap \mathrm{M}(\mathrm{A})^u), \tag{9}$$

$$g_{k,b} \in \mathrm{G}_{\mathrm{K}}(\mathrm{A}, \mathrm{M}(\mathrm{A})^{b^*}), \tag{10}$$

and

$$g_{k,b}(\mathrm{X}_1) - \mathrm{X}_1 - k\mathrm{L}(\partial \mathrm{F}/\partial \mathrm{X}_0)\mathrm{E}^b \in \mathrm{M}(\mathrm{A})^{2bq}; \tag{11}$$

in view of (4), (5) and (8) we see that there exists a permutation $(\mathrm{H}(1), \ldots, \mathrm{H}(e))$ of $(1, \ldots, e)$ and units $f_1, \ldots, f_e$ in A such that

$$g_{k,b}(\mathrm{F}^{(s)}) = \mathrm{F}^{(\mathrm{H}(s))} f_s \quad \text{for} \quad 1 \leq s \leq e;$$

now in view (6) and (10) we see that $\mathrm{H}(s) = s$ for $1 \leq s \leq e$, and then by (5) we get

$$g_{k,b} \in \mathrm{G}_{\mathrm{K}}[\mathrm{A}, \mathrm{P}_1] \cap \ldots \cap \mathrm{G}_{\mathrm{K}}[\mathrm{A}, \mathrm{P}_e]. \tag{12}$$

By taking $\mathrm{V} = \mathrm{L}(\partial \mathrm{F}/\partial \mathrm{X}_0)$ in (1) we get that: if $j$ is any integer with $1 \leq j \leq d$, $k^*$ is any nonzero element in K, and $b$ is any nonnegative integer, then

$$\mathrm{ord}_{w_j(\mathrm{A})} w_j(k^*\mathrm{L}(\partial \mathrm{F}/\partial \mathrm{X}_0)\mathrm{E}^b) = bq + \mathrm{ord}_{w_j(\mathrm{A})} w_j(\mathrm{L}(\partial \mathrm{F}/\partial \mathrm{X}_0)). \tag{13}$$

It only remains to note that in view of (7), (11) and (13) we have the following: Let $b$ and $b'$ be any integers with $b \geq b_0$ and $b' \geq b_0$. Let $k$ and $k'$ be any elements in K. Assume that either: $b' = b$ and $k' \neq k$, or: $b' > b$ and $k \neq 0$. Then

$$\mathrm{ord}_{w_j(\mathrm{A})} w_j(g_{k,b}(\mathrm{X}_1) - g_{k',b'}(\mathrm{X}_1)) = bq + \mathrm{ord}_{w_j(\mathrm{A})} w_j(\mathrm{L}(\partial \mathrm{F}/\partial \mathrm{X}_0)) < \infty$$

and hence

$$g_{k,b}(\mathrm{X}_1) - g_{k',b'}(\mathrm{X}_1) \notin \mathrm{E}_j\mathrm{A}.$$

*Lemma* **(3.5).** — *Let* $v : \mathrm{A} \to \mathrm{R}$ *be a* K*-epimorphism where* R *is an overring of* K *with* $\mathrm{rad}_{\mathrm{R}}\{0\} = \{0\}$, *and* $\mathrm{Ker}\, v = \mathrm{FA}$ *with* $0 \neq \mathrm{F} \in \mathrm{M}(\mathrm{A})$. *Let* $\mathrm{P}_1, \ldots, \mathrm{P}_e$ *be all the distinct prime ideals of height zero in* R. *Let* J *be any ideal in* R *such that* J *contains a nonzerodivisor of* R. *Assume that* $v(\partial \mathrm{F}/\partial \mathrm{X}_0)$ *is a nonzerodivisor in* R. *Then there exists an infinite subset* G *of*

$$\mathrm{G}_{\mathrm{K}}(\mathrm{R}, \mathrm{J}) \cap \mathrm{G}_{\mathrm{K}}[\mathrm{R}, \mathrm{P}_1] \cap \ldots \cap \mathrm{G}_{\mathrm{K}}[\mathrm{R}, \mathrm{P}_e]$$

*with* $\mathrm{card}(\mathrm{G}) \geq \mathrm{card}(\mathrm{K})$ *such that for all* $g \neq h$ *in* G *we have* $g(v(\mathrm{X}_1)) - h(v(\mathrm{X}_1)) \notin \mathrm{P}_i$ *for* $1 \leq i \leq e$.

*Proof.* — We can take $L \in A$ such that $v(L) \in J$ and $v(L)$ is a nonzerodivisor of R. Now $v^{-1}(P_1), \ldots, v^{-1}(P_e)$ are exactly all the distinct prime ideals of height one in A containing F. We can take $E_i \in M(A)$ with $E_i A = v^{-1}(P_i)$ for $1 \leq i \leq e$. Clearly $L(\partial F/\partial X_0) \notin E_i A$ for $1 \leq i \leq e$. Therefore by (3.4) we can find an infinite subset $G^*$ of

$$G_K(A, LA) \cap G_K[A, v^{-1}(P_1)] \cap \ldots \cap G_K[A, v^{-1}(P_e)]$$

with $\mathrm{card}(G^*) \geq \mathrm{card}(K)$ such that for all $g \in G^*$ we have $g(F) = F$, and for all $g \neq h$ in $G^*$ we have $g(X_1) - h(X_1) \notin v^{-1}(P_i)$ for $1 \leq i \leq e$. Let $w : G_K[A, \mathrm{Ker}\, v] \to G_K(R)$ be the homomorphism induced by $v$. Now $G^* \subset G_K[A, \mathrm{Ker}\, v]$ and upon letting $G = w(G^*)$, in view of [2, (2.2), (2.4), (2.5)], we see that G is an infinite subset of

$$G_K(R, J) \cap G_K[R, P_1] \cap \ldots \cap G_K[R, P_e]$$

with $\mathrm{card}(G) = \mathrm{card}(G^*) \geq \mathrm{card}(K)$ and for all $g \neq h$ in G we have $g(v(X_1)) - h(v(X_1)) \in P$ for $1 \leq i \leq e$.

## § 4. Separable generation.

Let K be any valued field. Let $X_0, X_1, X_2, \ldots$ be indeterminates. For every nonnegative integer $m$ let $A_m = K[\langle X_1, \ldots, X_m \rangle]$. We shall tacitly use [2, (2.14)].

*Lemma* **(4.1).** — *Let* R *be an analytic local ring over* K *with* $\dim R = n > 0$ *and* $\mathrm{rad}_R\{0\} = \{0\}$. *Let* $P_1, \ldots, P_a$ *be all the distinct prime ideals of height zero in* R. *Assume that* $\dim R/P_i = n$ *for* $1 \leq i \leq a$. *Let* $t_i : R \to R/P_i$ *be the canonical epimorphism. Let* J *be an ideal in* R *such that* J *contains a nonzerodivisor of* R. *Assume that*

(′) *for* $1 \leq i \leq a$ : $R/P_i$ *is analytically separably generated over* K, i.e., *equivalently, there exists a local* K*-monomorphism* $v_i : A_n \to R/P_i$ *such that* $R/P_i$ *is integral over* $v_i(A_n)$ *and the quotient field of* $R/P_i$ *is separable over the quotient field of* $v_i(A_n)$.

*Now, for* $1 \leq i \leq a$, *let* $v_i$ *be any such and take any* $x_i \in R$ *with* $t_i(x_i) = v_i(X_1)$. *Then there exists an infinite subset* G *of*

$$G_K(R, J) \cap G_K[R, P_1] \cap \ldots \cap G_K[R, P_a]$$

*with* $\mathrm{card}(G) \geq \mathrm{card}(K)$ *such that for all* $g \neq h$ *in* G *we have* $g(x_i) - h(x_i) \notin P_i$ *for* $1 \leq i \leq a$.

*Proof.* — Let T be the total quotient ring of R, and let S be the integral closure of R in T.

By [1, (18.9)] we see that: $P_1T, \ldots, P_eT$ are exactly all the distinct prime ideals in T, and they all have height zero; $(P_iT) \cap R = P_i$ for $1 \leq i \leq a$; T is noetherian; and if T′ is any noetherian subring of T with total quotient ring T then $(P_1T) \cap T', \ldots, (P_aT) \cap T'$ are exactly all the distinct prime ideals of height zero in T′, $((P_1T) \cap T') \cap \ldots \cap ((P_aT) \cap T') = \mathrm{rad}_{T'}\{0\} = \{0\}$, and $((P_iT) \cap T')T = P_iT$ for $1 \leq i \leq a$.

For $1 \leq i \leq a$, we have $(TP_i) \cap K = P_i \cap K = \{0\}$ and hence we can take an overring $T_i$ of K and a K-epimorphism $w_i : T \to T_i$ with $\mathrm{Ker}\, w_i = P_iT$. By [1, (18.9)] we now see that $w_1 \oplus \ldots \oplus w_a : T \to T_1 \oplus \ldots \oplus T_a$ is an isomorphism, and $T_i$ is the quotient field of $w_i(R)$ for $1 \leq i \leq a$. Let $S_i$ be the integral closure of $w_i(R)$ in $T_i$ for $1 \leq i \leq a$.

Because of (') we have that the integral closure of $R/P_i$ in the quotient field of $R/P$ is a finite $(R/P_i)$-module for $1 \leq i \leq a$, i.e., $S_i$ is a finite $w_i(R)$-module for $1 \leq i \leq a$. Therefore by [1, (19.23)] we see that S is a finite R-module and

$$S = w_1^{-1}(S_1) \cap \ldots \cap w_a^{-1}(S_a). \tag{1}$$

Let $t_i' : R/P_i \to T_i$ be the unique monomorphism such that $t_i' t_i(y) = w_i(y)$ for all $y \in R$. Let $v_i' = t_i' v_i$. Then for $1 \leq i \leq a$ we have that: $v_i' : A_n \to T_i$ is a K-monomorphism; $w_i(x_i) = v_i'(X_1)$; $T_i$ is a finite separable algebraic extension of the quotient field of $v_i'(A_n)$; $S_i$ is the integral closure of $v_i'(A_n)$ in $T_i$; and $S_i$ is a finite $v_i'(A_n)$-module.

Let B be the quotient field of $A_n$. Then there exists a unique monomorphism $q_i : B \to T_i$ such that $q_i(y) = v_i'(y)$ for all $y \in A_n$. Since $T_i$ is separable over $q_i(B)$, there exists $0 \neq z_i' \in T_i$ such that $T_i = q_i(B)[z_i']$. Let $d_i = [T_i : q_i(B)]$, and let $f_i'(X_0)$ be the monic polynomial of degree $d_i$ in $X_0$ with coefficients in B such that upon applying $q_i$ to the coefficients of $f_i'(X_0)$ we get the minimal monic polynomial of $z_i'$ over $q_i(B)$. We can take elements $z_{ij}$ in an algebraic closure of B such that

$$f_i'(X_0) = (X_0 - z_{i1}) \ldots (X_0 - z_{id_i}) \quad \text{for } 1 \leq i \leq a.$$

Since $z_1' \neq 0, \ldots, z_a' \neq 0$, we must have $z_{ij} \neq 0$ for all $i, j$; consequently, since B is an infinite field, we can find nonzero elements $b_1, \ldots, b_a$ in B such that $b_i z_{ij} \neq b_{i'} z_{i'j'}$ for all $i, j, i', j'$ with $i \neq i'$. Now we can find $0 \neq b \in M(A_n)$ such that upon letting

$$f_i(X_0) = (bb_i)^{d_i} f_i'(X_0/(bb_i))$$

we have

$$f_i(X_0) - X_0^{d_i} \in (M(A_n))[X_0] \quad \text{for } 1 \leq i \leq a. \tag{2}$$

Let $z_i = (q_i(bb_i))z_i'$ for $1 \leq i \leq a$. Then $f_1(X_0), \ldots, f_a(X_0)$ are pairwise distinct nonconstant monic irreducible polynomials in $B[X_0]$, and for $1 \leq i \leq a$ we have that $T_i = q_i(B)[z_i]$ and upon applying $q_i$ to the coefficients of $f_i(X_0)$ we get the minimal monic polynomial of $z_i$ over $q_i(B)$; by (2) we see that $z_i \in S_i$ for $1 \leq i \leq a$. Let $u_i : B[X_0] \to T_i$ be the unique homomorphism such that $u_i(X_0) = z_i$ and $u_i(y) = q_i(y)$ for all $y \in B$; then $u_i(B[X_0]) = T_i$, $\operatorname{Ker} u_i = f_i(X_0)B[X_0]$, and $u_i(y) = v_i'(y)$ for all $y \in A_n$.

Let

$$F = f_1(X_0) \ldots f_a(X_0) \tag{3}$$

and consider the homomorphism

$$u_1 \oplus \ldots \oplus u_a : B[X_0] \to T_1 \oplus \ldots \oplus T_a.$$

Since $f_1(X_0), \ldots, f_a(X_0)$ are pairwise distinct nonconstant monic irreducible polynomials in $B[X_0]$, we see that

$$\operatorname{Ker}(u_1 \oplus \ldots \oplus u_a) = FB[X_0] \quad \text{and} \quad (u_1 \oplus \ldots \oplus u_a)(B[X_0]) = T_1 \oplus \ldots \oplus T_a.$$

Since $w_1 \oplus \ldots \oplus w_a : T \to T_1 \oplus \ldots \oplus T_a$ is an isomorphism, we get a unique homomorphism $u : B[X_0] \to T$ such that $(w_1 \oplus \ldots \oplus w_a)(u(y)) = (u_1 \oplus \ldots \oplus u_a)(y)$ for all $y \in B[X_0]$.

It follows that: $u(B[X_0])=T$; Ker $u=FB[X_0]$; $w_i(u(y))=v_i'(y)$ for all $y\in A_n$ and all $i$ with $1\leq i\leq a$; $u$ is a K-homomorphism; and $w_i(u(X_0))=z_i$ for $1\leq i\leq a$.

Since $z_i$ is separable over $u_i(B)$, we have that $\partial f_i(X_0)/\partial X_0\notin f_i(X_0)B[X_0]$ for $1\leq i\leq a$; by (3) we get

$$\partial F/\partial X_0=\sum_{i=1}^{a} f_1(X_0)\dots f_{i-1}(X_0)(\partial f_i(X_0)/\partial X_0)f_{i+1}(X_0)\dots f_a(X_0);$$

since $f_1(X_0),\dots,f_a(X_0)$ are pairwise distinct nonconstant monic irreducible polynomials in $B[X_0]$, we get that $\partial F/\partial X_0\notin f_i(X_0)B[X_0]$ for $1\leq i\leq a$; since $u(B[X_0])=T$ and Ker $u=f_1(X_0)\dots f_a(X_0)B[X_0]$, we conclude that $u(\partial F/\partial X_0)$ *is a nonzerodivisor in* T.

Let any $Z\in A_n[X_0]$ be given such that $u(Z)$ is a zerodivisor in T; since $u(B[X_0])=T$, there exists $Z'\in B[X_0]$ such that $u(Z')\neq 0=u(Z)u(Z')$; since Ker $u=FB[X_0]$, we must have $Z'\notin FB[X_0]$; we can find $0\neq Z^*\in A_n$ such that $Z'Z^*\in A_n[X_0]$; clearly $Z'Z^*\notin FB[X_0]$, and hence $u(Z'Z^*)\neq 0$; also $u(Z)u(Z'Z^*)=0$, and hence $u(Z)$ is a zerodivisor in $u(A_n[X_0])$. We conclude that every nonzerodivisor in $u(A_n[X_0])$ remains a nonzerodivisor in T. Given any $Y\in B[X_0]$, we can find $0\neq Y^*\in A_n$ such that $YY^*\in A_n[X_0]$, and then $u(Y)u(Y^*)\in u(A_n[X_0])$ and $u(Y^*)\in A_n[X_0]$; since $f_i(X_0)$ is a nonconstant polynomial in $B[X_0]$, we must have $Y^*\notin f_i(X_0)B[X_0]$ for $1\leq i\leq a$; since Ker $u=f_1(X_0)\dots f_a(X_0)B[X_0]$, and $f_1(X_0),\dots,f_a(X_0)$ are pairwise distinct nonconstant monic irreducible polynomials in $B[X_0]$, we conclude that $u(Y^*)$ is a nonzerodivisor in $u(B[X_0])$. Since $u(B[X_0])=T$, it now follows that T *is the total quotient ring of* $u(A_n[X_0])$.

For $1\leq i\leq a$ we have $w_i(u(A_n[X_0]))=v_i'(A_n)[z_i]$, $v_i'(A_n)\subset S_i$, and $z_i\in S_i$. Therefore $w_i(u(A_n[X_0]))\subset S_i$ for $1\leq i\leq a$, and hence by (1) *we get* $u(A_n[X_0])\subset S$.

Given any $s\in S$, by (1) we have $w_i(s)\in S_i$ for $1\leq i\leq a$; since $S_i$ is integral over $v_i'(A_n)$ and $v_i'(A_n)=w_i(u(A_n))$, there exists a nonconstant monic polynomial $E_i(X)$ in an indeterminate X with coefficients in $u(A_n)$ such that $w_i(E_i(s))=0$; let $E(X)=E_1(X)\dots E_a(X)$; then E(X) is a nonconstant monic polynomial in X with coefficients in $u(A_n)$, and $w_i(E(s))=0$ for $1\leq i\leq a$; consequently $E(s)=0$, and hence $s$ is integral over $u(A_n)$. This shows that S is integral over $u(A_n)$, and hence S *is integral over* $u(A_n[X_0])$.

Thus upon letting $R^*=u(A_n[X_0])$ we have that: $R^*$ is a noetherian subring of T; T is the total quotient ring of $R^*$; $K\subset R^*\subset S$; and S is integral over $R^*$. Whence, in particular, $(P_1T)\cap R^*,\dots,(P_aT)\cap R^*$ are exactly all the distinct prime ideals of height zero in $R^*$. For $1\leq i\leq a$ we have that: $w_i(R^*)=v_i'(A_n)[z_i]\subset S_i$; $T_i$ is the quotient field of $v_i'(A_n)[z_i]$; $S_i$ is the integral closure of $v_i'(A_n)$ in $T_i$; and $S_i$ is a finite $v_i'(A_n)$-module. It follows that for $1\leq i\leq a$, the integral closure of $R^*/(P_iT)\cap R^*$ in the quotient field of $R^*/(P_iT)\cap R^*$ is a finite $(R^*/(P_iT)\cap R^*)$-module. By (2.3.3) we can now find an ideal $J^*$ in $R^*$ such that $J^*\not\subset P_iT$ for $1\leq i\leq a$, and

$$I_{R^*}(G_K(R^*,J^*))\subset I_R(G_K(R,J)). \tag{4}$$

Let $d=d_1+\ldots+d_a$, let A′ be the set of all polynomials of degree $<d$ in $X_0$ with coefficients in $A_n$, and let $A=K[\langle X_0, \ldots, X_n\rangle]$ where we regard A to be an overring of $A_n$. By (2) and (3) we know that F is a monic polynomial of degree $d$ in $X_0$ with coefficients in $A_n$, and $F-X_0^d\in(M(A_n))[X_0]$. Therefore, by the Weierstrass Preparation Theorem [1, (10.3)], for every $f\in A$ there exists a unique $r_f\in A'$ such that $f-r_f\in FA$. We get a map $v: A\to T$ by taking $v(f)=u(r_f)$ for all $f\in A$. By [1, (10.3)] we also have that: $r_{f+f^*}=r_f+r_{f^*}$ and $r_{ff^*}-r_fr_{f^*}\in FA_n[X_0]$ for all $f$ and $f^*$ in A; and $f-r_f\in FA_n[X_0]$ for all $f\in A_n[X_0]$. Since $\operatorname{Ker} u=FB[X_0]$ and clearly $(FB[X_0])\cap A'=\{0\}$, we deduce that: $v(f)=u(f)$ for all $f\in A_n[X_0]$; $v$ is a ring homomorphism of A into T; $\operatorname{Ker} v=FA$; and $v(A)=u(A_n[X_0])$. Since $v(f)=u(f)$ for all $f\in A_n[X_0]$, we also get that $w_i(v(X_1))=w_i(u(X_1))=v_i'(X_1)=w_i(x_i)$ for $1\leq i\leq a$.

Thus $v: A\to T$ is a K-homomorphism such that: $\operatorname{Ker} v=FA$; $v(\partial F/\partial X_0)$ is a nonzerodivisor in T; $v(A)=R^*$; and $w_i(v(X_1))=w_i(x_i)$ for $1\leq i\leq a$. Let

$$G_0=G_K(R, J)\cap G_K[R, P_1]\cap\ldots\cap G_K[R, P_a],$$

and

$$G_0^*=G_K(R^*, J^*)\cap G_K[R^*, (P_1T)\cap R^*]\cap\ldots\cap G_K[R^*, (P_aT)\cap R^*].$$

By (3.5) we can now find an infinite subset $G^*$ of $G_0^*$ with $\operatorname{card}(G^*)\geq\operatorname{card}(K)$ such that for all $g^*\neq h^*$ in $G^*$ we have $g^*(v(X_1))-h^*(v(X_1))\notin P_iT$ for $1\leq i\leq a$. Let $G=I_R^{-1}(I_{R^*}(G^*))$. Then by (4) and (2.3.1) we see that G is an infinite subset of $G_0$ with $\operatorname{card}(G)\geq\operatorname{card}(K)$.

Finally, let any $g\neq h$ in G and any $i$ with $1\leq i\leq a$ be given. We shall show that then $g(x_i)-h(x_i)\notin P_i$ and this will complete the proof. Let $g'=I_R(g)$ and $h'=I_R(h)$. Then $g'\in I_{R^*}(G^*)$, $h'\in I_{R^*}(G^*)$, and $g'\neq h'$; consequently $g'(v(X_1))-h'(v(X_1))\notin P_iT$, i.e.,

$$w_i(g'(v(X_1)))-w_i(h'(v(X_1)))\neq 0. \tag{5}$$

Now $g'\in I_R(G_0)$ and $h'\in I_R(G_0)$, and hence by (2.3.1) we see that

$$g'\in G_K[S, (P_iT)\cap S] \quad\text{and}\quad h'\in G_K[S, (P_iT)\cap S].$$

In view of (1), we get a K-epimorphism $w_i': S\to S_i$, with $\operatorname{Ker} w_i'=(P_iT)\cap S$, by taking $w_i'(y)=w_i(y)$ for all $y\in S$. Let $w_i^*: G_K[S, \operatorname{Ker} w_i']\to G_K(S_i)$ be the homomorphism induced by $w_i'$. Now

$$\begin{aligned} w_i(g(x_i)-h(x_i)) &= w_i'(g'(x_i)-h'(x_i))\\ &= w_i'(g'(x_i))-w_i'(h'(x_i))\\ &= w_i^*(g')(w_i'(x_i))-w_i^*(h')(w_i'(x_i))\\ &= w_i^*(g')(w_i'(v(X_1)))-w_i^*(h')(w_i'(v(X_1)))\\ &= w_i'(g'(v(X_1)))-w_i'(h'(v(X_1)))\\ &\neq 0 \text{ by (5)}, \end{aligned}$$

and hence $g(x_i)-h(x_i)\notin P_i$.

*Theorem* **(4.2)**. — *Let* R *be an analytic local ring over* K *with* $\dim R>0$. *Let* $Q_1, \ldots, Q_a$ $(a>0)$, *be any distinct isolated primary components of* $\{0\}$ *in* R *such that* $\dim R/Q_1=\ldots=\dim R/Q_a$. *Let* $n=\dim R/Q_1$. *Let* $P_i=\operatorname{rad}_R Q_i$. *Let* $t_i: R\to R/P_i$

*be the canonical epimorphism. Let* $Q'_1, \ldots, Q'_b$ $(b \geq 0)$, *be any finite number of ideals in* R *such that for* $1 \leq i \leq a$ *and* $1 \leq j \leq b$ *we have* $Q'_j \not\subset P_i$. *Assume that*

(*) *there exists a* K*-epimorphism* $u : A_d \to R$, *for some d, such that* $u^{-1}(Q_i)$ *is a symbolic power of* $u^{-1}(P_i)$ *for* $1 \leq i \leq a$.

*Also assume that*

(') *for* $1 \leq i \leq a$: $R/P_i$ *is analytically separably generated over* K, i.e., *equivalently, there exists a local* K*-monomorphism* $v_i : A_n \to R/P_i$ *such that* $R/P_i$ *is integral over* $v_i(A_n)$ *and the quotient field of* $R/P_i$ *is separable over the quotient field of* $v_i(A_n)$.

*Now, for* $1 \leq i \leq a$, *let* $v_i$ *be any such and take any* $x_i \in R$ *with* $t_i(x_i) = v_i(X_1)$. *Then there exists an infinite subset* G *of*

$$\bigcap_{j=1}^{b} G_K(R, Q'_j) \cap \bigcap_{j=1}^{b} G_K[R, Q'_j] \cap \bigcap_{i=1}^{a} G_K[R, Q_i] \cap \bigcap_{i=1}^{a} G_K[R, P_i]$$

*with* $\mathrm{card}(G) \geq \mathrm{card}(K)$ *such that for all* $g \neq h$ *in* G *we have* $g(x_i) - h(x_i) \notin P_i$ *for* $1 \leq i \leq a$.

(For an intrinsic formulation of (*) see [2, (3.6)]. Note that (*) is automatically satisfied in case $Q_i = P_i$ for $1 \leq i \leq a$, because then we can take $u$ to be any K-epimorphism $A_d \to R$. Also note that (*) is automatically satisfied in case $\mathrm{endim}\, R = n+1$, because then we can take $u$ to be any K-epimorphism $A_{n+1} \to R$; (see [2, (2.16)]). Finally, note that if $(z_1, \ldots, z_m)$ is any basis of M(R) and $i$ is any integer with $1 \leq i \leq a$, then there exists an integer $q$ with $1 \leq q \leq m$ and an infinite subset G' of G with $\mathrm{card}(G') \geq \mathrm{card}(K)$ such that for all $g \neq h$ in G' we have $g(z_q) - h(z_q) \notin P_i$; namely, from the existence of G, the existence of $q$ and G' is easily deduced by using [2, (2.3) and (2.11)].)

*Proof.* — Since $Q_1, \ldots, Q_a$ are isolated primary components of $\{0\}$ in R, there exists an ideal Q in R such that $Q \cap Q_1 \cap \ldots \cap Q_a = \{0\}$ and $Q \not\subset P_i$ for $1 \leq i \leq a$. Let $J = Q \cap Q'_1 \cap \ldots \cap Q'_b$. Then $J \cap Q_1 \cap \ldots \cap Q_a = \{0\}$ and $J \not\subset P_i$ for $1 \leq i \leq a$. We can take an overring $R^*$ of K and a K-epimorphism $v : R \to R^*$ with $\mathrm{Ker}\, v = P_1 \cap \ldots \cap P_a$. Let $J^* = v(J)$. Let

$$G_0 = G_K(R, J) \cap \bigcap_{i=1}^{a} G_K[R, P_i] \cap \bigcap_{i=1}^{a} G_K[R, Q_i],$$

and

$$G_0^* = G_K(R^*, J^*) \cap \bigcap_{i=1}^{a} G_K[R^*, v(P_i)].$$

Let $w : G_K[R, \mathrm{Ker}\, v] \to G_K(S)$ be the homomorphism induced by $v$. Then by [2, (4.4)] we have $w(G_0) = G_0^*$; note that clearly

$$G_K[R, P_1] \cap \ldots \cap G_K[R, P_a] \subset G_K[R, \mathrm{Ker}\, v]$$

and hence it makes sense to talk about $w(G_0)$. Also note that, in view of [2, (2.1), (2.2)], we have $G_0 \subset G_K(R, Q'_j) \subset G_K[R, Q'_j]$ for $1 \leq j \leq b$. Now $J^*$ contains a nonzerodivisor of $R^*$, and hence by (4.1) there exists an infinite subset $G^*$ of $G_0^*$ with $\mathrm{card}(G^*) \geq \mathrm{card}(K)$ such that for all $g \neq h$ in $G^*$ we have $g(v(x_i)) - h(v(x_i)) \notin v(P_i)$ for $1 \leq i \leq a$. Since $w(G_0) = G_0^*$, for each $g \in G^*$ we can fix $g' \in G_0$ with $w(g') = g$; now it suffices to take $G = \{g' : g \in G^*\}$.

*Theorem* **(4.3).** — *Let* R *be an analytic local ring over* K *with* $\dim R>0$ *and* $\mathrm{rad}_R\{0\}=\{0\}$. *Let* $P_1, \ldots, P_e$ *be all the distinct prime ideals of height zero in* R. *Let* T *be the total quotient ring of* R, *and let* S *be the integral closure of* R *in* T.

(Note that then (see [1, (18.9)]): $P_1T, \ldots, P_eT$ are exactly all the distinct prime ideals in T, and they all have height zero; $(P_iT)\cap R=P_i$ for $1\leq i\leq e$; T is noetherian; and if T′ is any noetherian subring of T with total quotient ring T then $(P_1T)\cap T', \ldots, (P_eT)\cap T'$ are exactly all the distinct prime ideals of height zero in T′, $((P_1T)\cap T')\cap\ldots\cap((P_eT)\cap T')=\mathrm{rad}_{T'}\{0\}=\{0\}$, and $((P_iT)\cap T')T=P_iT$ for $1\leq i\leq e$.

For $1\leq i\leq e$, we have $(P_iT)\cap K=P_i\cap K=\{0\}$ and hence *we can take an overring* $T_i$ *of* K *and a* K*-epimorphism* $w_i: T\to T_i$ *with* $\mathrm{Ker}\, w_i=P_iT$. By [1, (18.9)] we now see that $w_1\oplus\ldots\oplus w_e: T\to T_1\oplus\ldots\oplus T_e$ is an isomorphism, and $T_i$ is the quotient field of $w_i(R)$ for $1\leq i\leq e$.)

*Assume that*

(′) *for* $1\leq i\leq e: R/P_i$ *is analytically separably generated over* K, i.e., *equivalently, there exists a local* K*-monomorphism* $v_i: A_{n_i}\to w_i(R)$, *where* $n_i=\dim R/P_i$, *such that* $w_i(R)$ *is integral over* $v_i(A_{n_i})$, *and* $T_i$ *is separable over the quotient field of* $v_i(A_{n_i})$.

*Now, for* $1\leq i\leq e$, *let* $v_i$ *be any such and take any* $s_i\in S$ *with* $w_i(s_i)=v_i(X_1)$.

*Let* R′ *be a subring of* T. *Assume that*: R′ *is noetherian*; $K\subset R'\subset S$; S *is integral over* R′; T *is the total quotient ring of* R′; *and the integral closure of* $R'/(P_iT)\cap R'$ *in the quotient field of* $R'/(P_iT)\cap R'$ *is a finite* $(R'/(P_iT)\cap R')$*-module for* $1\leq i\leq e$.

(Note that in the presence of (′), in view of [1, (19.23)] we see that these assumptions on R′ are automatically satisfied in case $R\subset R'\subset S$.)

*Let* J′ *be an ideal in* R′, *and let* a *be an integer with* $1\leq a\leq e$. *Assume that* $\dim R/P_1=\ldots=\dim R/P_a$ *and* $J'\not\subset P_iT$ *for* $1\leq i\leq a$. *Then there exists an infinite subset* G *of*

$$G_K(R', J')\cap G_K[R', (P_1T)\cap R']\cap\ldots\cap G_K[R', (P_eT)\cap R']$$

*with* $\mathrm{card}(G)\geq\mathrm{card}(K)$ *such that for all* $g\neq h$ *in* G *we have* $I_{R'}(g)(s_i)-I_{R'}(h)(s_i)\notin P_iT$ *for* $1\leq i\leq a$.

*Proof.* — By (2.3.3) we can find an ideal J in R such that $J\not\subset P_i$ for $1\leq i\leq a$, and

$$I_R(R, J)\subset I_{R'}(R', J'). \tag{1}$$

Let

$$G_0^*=G_K(R, J)\cap G_K[R, P_1]\cap\ldots\cap G_K[R, P_e],$$

and

$$G_0'=G_K(R', J')\cap G_K[R', (P_1T)\cap R']\cap\ldots\cap G_K[R', (P_eT)\cap R'].$$

We can take $x_i\in R$ with $w_i(x_i)=v_i(X_1)$ for $1\leq i\leq a$. By (4.2) we can now find an infinite subset $G^*$ of $G_0^*$ with $\mathrm{card}(G^*)\geq\mathrm{card}(K)$ such that for all $g^*\neq h^*$ in $G^*$ we have $g^*(x_i)-h^*(x_i)\notin P_i$ for $1\leq i\leq a$. Let $G=I_{R'}^{-1}(I_R(G^*))$. Then by (1) and (2.3.1) we see that G is an infinite subset of $G_0'$ with $\mathrm{card}(G)\geq\mathrm{card}(K)$.

Let any integer $i$ with $1\leq i\leq a$ and any elements $g$ and $h$ in G with $g\neq h$ be given;

let $g'=\mathrm{I}_{\mathrm{R}'}(g)$ and $h'=\mathrm{I}_{\mathrm{R}'}(h)$. We shall show that $g'(s_i)-h'(s_i)\notin \mathrm{P}_i\mathrm{T}$ and this will complete the proof. Clearly

$$w_i(s_i)=w_i(x_i). \tag{2}$$

Now $g'=\mathrm{I}_{\mathrm{R}}(g^*)$ and $h'=\mathrm{I}_{\mathrm{R}}(h^*)$ where $g^*$ and $h^*$ are elements in $\mathrm{G}^*$ with $g^*\neq h^*$; consequently $g^*(x_i)-h^*(x_i)\notin \mathrm{P}_i$; since $g'(x_i)=g^*(x_i)$ and $h'(x_i)=h^*(x_i)$, we conclude that $g'(x_i)-h'(x_i)\notin \mathrm{P}_i$, i.e.,

$$w_i(g'(x_i))\neq w_i(h'(x_i)). \tag{3}$$

By (2.3.1) we have $\mathrm{I}_{\mathrm{R}}(\mathrm{G}_{\mathrm{K}}[\mathrm{R}, \mathrm{P}_i])\subset \mathrm{G}_{\mathrm{K}}[\mathrm{S}, (\mathrm{P}_i\mathrm{T})\cap \mathrm{S}]$, and hence $g'$ and $h'$ are in $\mathrm{G}_{\mathrm{K}}[\mathrm{S}, (\mathrm{P}_i\mathrm{T})\cap \mathrm{S}]$. We get a K-epimorphism $w_i' : \mathrm{S}\to w_i(\mathrm{S})$, with $\operatorname{Ker} w_i'=(\mathrm{P}_i\mathrm{T})\cap \mathrm{S}$, by taking $w_i'(s)=w_i(s)$ for all $s\in \mathrm{S}$. Let $w_i^* : \mathrm{G}_{\mathrm{K}}[\mathrm{S}, \operatorname{Ker} w_i'] \to \mathrm{G}_{\mathrm{K}}(w_i(\mathrm{S}))$ be the homomorphism induced by $w_i'$. Then

$$w_i^*(g')(w_i(s))=w_i(g'(s)) \quad \text{and} \quad w_i^*(h')(w_i(s))=w_i(h'(s)) \quad \text{for all } s\in \mathrm{S};$$

consequently by (2) and (3) we get that $w_i(g'(s_i))\neq w_i(h'(s_i))$, and hence $g'(s_i)-h'(s_i)\notin \mathrm{P}_i\mathrm{T}$.

## § 5. Perfect fields.

Let K be any valued field. Let $\mathrm{X}_0, \mathrm{X}_1, \mathrm{X}_2, \ldots$ be indeterminates. For every nonnegative integer $m$ let $\mathrm{A}_m=\mathrm{K}[\langle \mathrm{X}_1, \ldots, \mathrm{X}_m\rangle]$. We shall tacitly use [2, (2.14)].

*Lemma* (5.1). — *Assume that* K *is of characteristic* $p\neq 0$. *Let*

$$\mathrm{V}(\mathrm{X}_0, \ldots, \mathrm{X}_n)\in \mathrm{K}[\langle \mathrm{X}_0, \ldots, \mathrm{X}_n\rangle], \quad (n>0),$$

*be such that* $\mathrm{V}(\mathrm{X}_0, \ldots, \mathrm{X}_n)\notin \mathrm{K}[[\mathrm{X}_0, \mathrm{X}_1^p, \ldots, \mathrm{X}_n^p]]$ *and*

$$\mathrm{V}(\mathrm{X}_0, \ldots, \mathrm{X}_n)=\mathrm{X}_0^d+\sum_{i=1}^{d}\mathrm{V}_i(\mathrm{X}_1, \ldots, \mathrm{X}_n)\mathrm{X}_0^{d-i}$$

*where* $d>0$, $\mathrm{V}_i(\mathrm{X}_1, \ldots, \mathrm{X}_n)\in \mathrm{M}(\mathrm{K}[\langle \mathrm{X}_1, \ldots, \mathrm{X}_n\rangle])$ *for* $1\leq i\leq d$, *and* $\mathrm{V}_d(\mathrm{X}_1, \ldots, \mathrm{X}_n)\neq 0$. *Then there exists an integer* $s$ *with* $1\leq s\leq m$, *and positive integers* $u_1, \ldots, u_{s-1}, u_{s+1}, \ldots, u_n$, *such that upon letting* $\mathrm{Y}_s=\mathrm{X}_s$ *and* $\mathrm{Y}_t=\mathrm{X}_t+\mathrm{X}_s^{u_t}$ *for* $t=1, \ldots, s-1, s+1, \ldots, n$, *we have that*

$$\mathrm{V}(\mathrm{X}_0, \mathrm{Y}_1, \ldots, \mathrm{Y}_n)=\mathrm{D}(\mathrm{X}_0, \ldots, \mathrm{X}_n)\mathrm{W}(\mathrm{X}_0, \ldots, \mathrm{X}_n)$$

*where* $\mathrm{D}(\mathrm{X}_0, \ldots, \mathrm{X}_n)$ *and* $\mathrm{W}(\mathrm{X}_0, \ldots, \mathrm{X}_n)$ *are elements in* $\mathrm{K}[\langle \mathrm{X}_0, \ldots, \mathrm{X}_n\rangle]$ *such that* $\mathrm{D}(0, \ldots, 0)\neq 0$, $\mathrm{W}(\mathrm{X}_0, \ldots, \mathrm{X}_n)\notin \mathrm{K}[[\mathrm{X}_0, \ldots, \mathrm{X}_{s-1}, \mathrm{X}_s^p, \mathrm{X}_{s+1}, \ldots, \mathrm{X}_n]]$, *and*

$$\mathrm{W}(\mathrm{X}_0, \ldots, \mathrm{X}_n)=\mathrm{X}_s^e+\sum_{i=1}^{e}\mathrm{W}_i(\mathrm{X}_0, \ldots, \mathrm{X}_{s-1}, \mathrm{X}_{s+1}, \ldots, \mathrm{X}_n)\mathrm{X}_s^{e-i}$$

*with* $e>0$ *and* $\mathrm{W}_i(\mathrm{X}_0, \ldots, \mathrm{X}_{s-1}, \mathrm{X}_{s+1}, \ldots, \mathrm{X}_n)\in \mathrm{M}(\mathrm{K}[\langle \mathrm{X}_0, \ldots, \mathrm{X}_{s-1}, \mathrm{X}_{s+1}, \ldots, \mathrm{X}_n\rangle])$ *for* $1\leq i\leq e$. *Moreover, if* $\mathrm{V}(\mathrm{X}_0, \ldots, \mathrm{X}_n)$ *is irreducible in* $\mathrm{K}[\langle \mathrm{X}_1, \ldots, \mathrm{X}_n\rangle][\mathrm{X}_0]$ *then* $\mathrm{W}(\mathrm{X}_0, \ldots, \mathrm{X}_n)$ *is irreducible in* $\mathrm{K}(\langle \mathrm{X}_0, \ldots, \mathrm{X}_{s-1}, \mathrm{X}_{s+1}, \ldots, \mathrm{X}_n\rangle)[\mathrm{X}_s]$.

*Proof.* — Since $V(X_0, \ldots, X_n) \notin K[[X_0, X_1^p, \ldots, X_n^p]]$, there exists an integer $j$ with $1 \leq j \leq d$ such that $V_j(X_1, \ldots, X_n) \notin K[[X_1^p, \ldots, X_n^p]]$. Now

$$V_j(X_1, \ldots, X_n) = \sum_{a=1}^{\infty} H_a(X_1, \ldots, X_n)$$

where $H_a(X_1, \ldots, X_n)$ is an element in $K[X_1, \ldots, X_n]$ which is either zero or is homogeneous of degree $a$. Since $V_j(X_1, \ldots, X_n) \notin K[[X_1^p, \ldots, X_n^p]]$, we must have $H_a(X_1, \ldots, X_n) \notin K[[X_1^p, \ldots, X_n^p]]$ for some $a$; let $b$ be the smallest such value of $a$. Now we must have $H_b(X_1, \ldots, X_n) \notin K[X_1, \ldots, X_{s-1}, X_s^p, X_{s+1}, \ldots, X_n]$ for some $s$ with $1 \leq s \leq n$. Since $V_d(X_1, \ldots, X_n) \neq 0$, by a standard argument [4, p. 147] we can find integers $u_t > 1$ for $t = 1, \ldots, s-1, s+1, \ldots, n$, such that upon letting $Y_s = X_s$ and $Y_t = X_t + X_s^{u_t}$ for $t = 1, \ldots, s-1, s+1, \ldots, n$, we have that

$$(1) \qquad V_d(Y_1, \ldots, Y_n) \notin (X_1, \ldots, X_{s-1}, X_{s+1}, \ldots, X_n)K[[X_1, \ldots, X_n]].$$

We get an element $V^*(X_0, \ldots, X_n)$ in $K[\langle X_0, \ldots, X_n \rangle]$ by setting

$$V^*(X_0, \ldots, X_n) = V(X_0, Y_1, \ldots, Y_n).$$

By (1) we see that

$$(2) \qquad V^*(X_0, \ldots, X_n) \notin (X_0, \ldots, X_{s-1}, X_{s+1}, \ldots, X_n)K[[X_0, \ldots, X_n]]$$

and hence by the Weierstrass Preparation Theorem [1, (10.3)] we have

$$(3) \qquad V^*(X_0, \ldots, X_n) = D(X_0, \ldots, X_n)W(X_0, \ldots, X_n)$$

where $D(X_0, \ldots, X_n)$ and $W(X_0, \ldots, X_n)$ are elements in $K[\langle X_0, \ldots, X_n \rangle]$ such that

$$(4) \qquad D(0, \ldots, 0) \neq 0$$

and

$$W(X_0, \ldots, X_n) = X_s^e + \sum_{i=1}^{e} W_i(X_0, \ldots, X_{s-1}, X_{s+1}, \ldots, X_n)X_s^{e-i}$$

with $e > 0$ and

$$W_i(X_0, \ldots, X_{s-1}, X_{s+1}, \ldots, X_n) \in M(K[\langle X_0, \ldots, X_{s-1}, X_{s+1}, \ldots, X_n \rangle])$$

for $1 \leq i \leq e$; by [1, (10.3) and (10.7)] we also know that if $V(X_0, \ldots, X_n)$ is irreducible in $K[\langle X_1, \ldots, X_n \rangle][X_0]$ then $W(X_0, \ldots, X_n)$ is irreducible in

$$K(\langle X_0, \ldots, X_{s-1}, X_{s+1}, \ldots, X_n \rangle)[X_s].$$

Let E be the set of all polynomials of degree $< d$ in $X_0$ with coefficients in $K[[X_1, \ldots, X_n]]$. Let $V_i^*(X_1, \ldots, X_n) \in K[[X_1, \ldots, X_n]]$ be defined by setting

$$V_i^*(X_1, \ldots, X_n) = V_i(Y_1, \ldots, Y_n).$$

Then

$$V^*(X_0, \ldots, X_n) = X_0^d + \sum_{i=1}^{d} V_i^*(X_1, \ldots, X_n)X_0^{d-i}$$

and $V_i^*(X_1, \ldots, X_n) \in M(K[[X_1, \ldots, X_n]])$ for $1 \leq i \leq d$; consequently by the uniqueness part of the Preparation Theorem [1, (10.3)] we see that

$$E \cap (V^*(X_0, \ldots, X_n)K[[X_0, \ldots, X_n]]) = \{0\},$$

and hence by (3) and (4) we get

$$\mathrm{E}\cap(\mathrm{W}(\mathrm{X}_0,\ldots,\mathrm{X}_n)\mathrm{K}[[\mathrm{X}_0,\ldots,\mathrm{X}_n]])=\{0\}. \tag{5}$$

Since $u_t>1$ for $t=1,\ldots,s-1,s+1,\ldots,n$, we see that

$$\mathrm{H}_b(\mathrm{Y}_1,\ldots,\mathrm{Y}_n)=\mathrm{H}_b(\mathrm{X}_1,\ldots,\mathrm{X}_n)+\text{terms of degree}>b \text{ in } (\mathrm{X}_1,\ldots,\mathrm{X}_n)$$

and hence

$$\begin{aligned}\mathrm{V}_j^*(\mathrm{X}_1,\ldots,\mathrm{X}_n)&=\sum_{a=1}^{\infty}\mathrm{H}_a(\mathrm{Y}_1,\ldots,\mathrm{Y}_n)\\&=\sum_{a=1}^{b-1}\mathrm{H}_a(\mathrm{Y}_1,\ldots,\mathrm{Y}_n)+\mathrm{H}_b(\mathrm{X}_1,\ldots,\mathrm{X}_n)\\&\quad+\text{terms of degree}>b \text{ in } (\mathrm{X}_1,\ldots,\mathrm{X}_n);\end{aligned}$$

since $\mathrm{H}_a(\mathrm{X}_1,\ldots,\mathrm{X}_n)\in\mathrm{K}[\mathrm{X}_1^p,\ldots,\mathrm{X}_n^p]$ for $1\leq a<b$ and

$$\mathrm{H}_b(\mathrm{X}_1,\ldots,\mathrm{X}_n)\notin\mathrm{K}[\mathrm{X}_1,\ldots,\mathrm{X}_{s-1},\mathrm{X}_s^p,\mathrm{X}_{s+1},\ldots,\mathrm{X}_n],$$

we conclude that

$$\mathrm{V}_j^*(\mathrm{X}_1,\ldots,\mathrm{X}_n)\notin\mathrm{K}[[\mathrm{X}_1,\ldots,\mathrm{X}_{s-1},\mathrm{X}_s^p,\mathrm{X}_{s+1},\ldots,\mathrm{X}_n]],$$

and hence

$$\partial\mathrm{V}_j^*(\mathrm{X}_1,\ldots,\mathrm{X}_n)/\partial\mathrm{X}_s\neq 0. \tag{6}$$

Now
$$\partial\mathrm{V}^*(\mathrm{X}_0,\ldots,\mathrm{X}_n)/\partial\mathrm{X}_s=\sum_{i=1}^{d}(\partial\mathrm{V}_i^*(\mathrm{X}_1,\ldots,\mathrm{X}_n)/\partial\mathrm{X}_s)\mathrm{X}_0^{d-i}$$

and hence by (5) and (6) we get that

$$\partial\mathrm{V}^*(\mathrm{X}_0,\ldots,\mathrm{X}_n)/\partial\mathrm{X}_s\notin\mathrm{W}(\mathrm{X}_0,\ldots,\mathrm{X}_n)\mathrm{K}[[\mathrm{X}_0,\ldots,\mathrm{X}_n]]. \tag{7}$$

By (3) we have

$$\begin{aligned}\partial\mathrm{V}^*(\mathrm{X}_0,\ldots,\mathrm{X}_n)/\partial\mathrm{X}_s&=(\partial\mathrm{D}(\mathrm{X}_0,\ldots,\mathrm{X}_n)/\partial\mathrm{X}_s)\mathrm{W}(\mathrm{X}_0,\ldots,\mathrm{X}_n)\\&\quad+\mathrm{D}(\mathrm{X}_0,\ldots,\mathrm{X}_n)(\partial\mathrm{W}(\mathrm{X}_0,\ldots,\mathrm{X}_n)/\partial\mathrm{X}_s)\end{aligned}$$

and hence by (7) we get that

$$\partial\mathrm{W}(\mathrm{X}_0,\ldots,\mathrm{X}_n)/\partial\mathrm{X}_s\neq 0,$$

i.e., $\mathrm{W}(\mathrm{X}_0,\ldots,\mathrm{X}_n)\notin\mathrm{K}[[\mathrm{X}_0,\ldots,\mathrm{X}_{s-1},\mathrm{X}_s^p,\mathrm{X}_{s+1},\ldots,\mathrm{X}_n]]$.

*Lemma* **(5.2)**. — *Assume that* K *is a perfect field of characteristic* $p\neq 0$, *and let* R *be an analytic local domain over* K *with* $\dim \mathrm{R}=n>0$. *Let* T *be the quotient field of* R. *Let* $(x_1,\ldots,x_n)$ *be a system of parameters of* R, *let* $\mathrm{R}_0=\mathrm{K}[\langle x_1,\ldots,x_n\rangle]$, *and let* $\mathrm{T}_0=\mathrm{K}(\langle x_1,\ldots,x_n\rangle)$. *Let* $z\in\mathrm{M}(\mathrm{R})$ *be such that either*: (1) $z\notin\mathrm{T}^p$, *or*: (2) $z$ *is inseparable over* $\mathrm{T}_0$. *Then there exists a basis* $(z_1,\ldots,z_n)$ *of* $\mathrm{M}(\mathrm{R}_0)$ *such that* $(z,z_2,\ldots,z_n)$ *is a system of parameters of* R *and* $z_1$ *is separable over* $\mathrm{K}(\langle z,z_2,\ldots,z_n\rangle)$.

*Proof.* — Let

$$f(\mathrm{X}_0)=\mathrm{X}_0^d+f_1\mathrm{X}_0^{d-1}+\ldots+f_d, \quad \text{with } f_i\in\mathrm{T}_0,$$

be the minimal monic polynomial of $z$ over $T_0$. Then $f_i \in M(R_0)$ for $1 \leq i \leq d$, and $f_d \neq 0$. Let $V_i(X_1, \ldots, X_n)$ be the unique element in $K[\langle X_1, \ldots, X_n \rangle]$ such that $V_i(x_1, \ldots, x_n) = f_i$, and let

$$V(X_0, \ldots, X_n) = X_0^d + \sum_{i=1}^{d} V_i(X_1, \ldots, X_n) X_0^{d-i}.$$

Then $V(X_0, \ldots, X_n) \in K[\langle X_0, \ldots, X_n \rangle]$, $V_i(X_1, \ldots, X_n) \in M(K[\langle X_1, \ldots, X_n \rangle])$ for $1 \leq i \leq d$, $V_d(X_1, \ldots, X_n) \neq 0$, and $V(X_0, \ldots, X_n)$ is irreducible in $K[\langle X_1, \ldots, X_n \rangle][X_0]$.

Suppose if possible that $V(X_0, \ldots, X_n) \in K[[X_0, X_1^p, \ldots, X_n^p]]$. Since K is perfect, by [1, (24.1)] we then have $f(X_0) \in R_0^p[X_0]$. If also $f(X_0) \in T_0[X_0^p]$ then we would get $f(X_0)^{1/p} \in T_0[X_0]$ and $f(X_0) = (f(X_0)^{1/p})^p$, which would contradict the fact that $f(X_0)$ is irreducible in $T_0[X_0]$. Consequently, $f(X_0) \notin T_0[X_0^p]$, and hence $z$ is separable over $T_0$. Therefore $z \notin T^p$. Now $z^{1/p}$ satisfies the equation

$$(z^{1/p})^d + f_1^{1/p}(z^{1/p})^{d-1} + \ldots + f_d^{1/p} = 0$$

of degree $d$ with coefficients $f_1^{1/p}, \ldots, f_d^{1/p}$ in $T_0$, and hence

$$[T_0(z^{1/p}) : T_0] \leq d = [T_0(z) : T_0].$$

Consequently $z^{1/p} \in T_0(z) \subset T$, and hence $z \in T^p$. This is a contradiction.

Thus we must have $V(X_0, \ldots, X_n) \notin K[[X_0, X_1^p, \ldots, X_n^p]]$. Therefore by (5.1) we can find $s, u_1, \ldots, u_{s-1}, u_{s+1}, \ldots, u_n, Y_1, \ldots, Y_n, D, W, e, W_1, \ldots, W_e$ as described there. Let $y_s' = x_s$, and $y_t' = x_t - x_s^{u_t}$ for $t = 1, \ldots, s-1, s+1, \ldots, n$; let $f_i' = W_i(z, y_1', \ldots, y_{s-1}', y_{s+1}', \ldots, y_n')$ for $1 \leq i \leq e$; and let

$$f'(X_s) = X_s^e + f_1' X_s^{e-1} + \ldots + f_e'.$$

Then $(y_1', \ldots, y_n')$ is a basis of $M(R_0)$, and $f'(y_s') = 0$. It follows that

$$(z, y_1', \ldots, y_{s-1}', y_{s+1}', \ldots, y_n')$$

is a system of parameters of R, and $f'(X_s)$ is the minimal monic polynomial of $y_s'$ over $K(\langle z, y_1', \ldots, y_{s-1}', y_{s+1}', \ldots, y_n' \rangle)$. Since

$$W(X_0, \ldots, X_n) \notin K[[X_0, \ldots, X_{s-1}, X_s^p, X_{s+1}, \ldots, X_n]],$$

we also have that $y_s'$ is separable over $K(\langle z, y_1', \ldots, y_{s-1}', y_{s+1}', \ldots, y_n' \rangle)$. It now suffices to take $(z_1, \ldots, z_n) = (y_s', y_1', \ldots, y_{s-1}', y_{s+1}', \ldots, y_n')$.

*Lemma* **(5.3)**. — *Assume that* K *is a perfect field of characteristic* $p \neq 0$, *and let* R *be an analytic local domain over* K *with* $\dim R = n > 0$. *Let* T *be the quotient field of* R, *let* $(x_1, \ldots, x_n)$ *be a system of parameters of* R *such that* T *is separable over* $K(\langle x_1, \ldots, x_n \rangle)$, *and let* $z \in M(R)$ *be such that* $z \notin T^p$. *Then there exists a basis* $(z_1, \ldots, z_n)$ *of* $M(K[\langle x_1, \ldots, x_n \rangle])$ *such that* $(z, z_2, \ldots, z_n)$ *is a system of parameters of* R *and* T *is separable over* $K(\langle z, z_2, \ldots, z_n \rangle)$.

*Proof.* — By (5.2) there exists a basis $(z_1, \ldots, z_n)$ of $M(K[\langle x_1, \ldots, x_n \rangle])$ such that $(z, z_2, \ldots, z_n)$ is a system of parameters of R and $z_1$ is separable over $K(\langle z, z_2, \ldots, z_n \rangle)$. It follows that T is separable over $K(\langle z, z_2, \ldots, z_n \rangle)$.

*Lemma* **(5.4).** — *Assume that* K *is perfect, and let* R *be an analytic local domain over* K. *Then* R *is analytically separably generated over* K.

*Proof.* — Let $n=\dim R$, $T=$the quotient field of R, and $p=$the characteristic of K. We have nothing to show if either $p=0$ or $n=0$. So suppose that $p\neq 0$ and $n>0$. In [1, (24.5)] we have given a proof in this case under the additional assumption of K being infinite. As an application of (5.2) we shall now give a proof which is independent of this additional assumption. Namely, it suffices to show that given any system of parameters $(x_1, \ldots, x_n)$ of R such that T is inseparable over $K(\langle x_1, \ldots, x_n\rangle)$, there exists a system of parameters $(z_1, \ldots, z_n)$ of R such that

$$(1) \qquad [T : K(\langle z_1, \ldots, z_n\rangle)]_i < [T : K(\langle x_1, \ldots, x_n\rangle)]_i$$

where $[\ ]_i$ denotes the degree of inseparability. So let $(x_1, \ldots, x_n)$ be any given system of parameters of R such that T is inseparable over $K(\langle x_1, \ldots, x_n\rangle)$. We can take $z_1\in M(R)$ such that $z_1$ is inseparable over $K(\langle x_1, \ldots, x_n\rangle)$, and then by (5.2) we can find a basis $(z, z_2, \ldots, z_n)$ of $M(K[\langle x_1, \ldots, x_n\rangle])$ such that $(z_1, z_2, \ldots, z_n)$ is a system of parameters of R and $z$ is separable over $K(\langle z_1, z_2, \ldots, z_n\rangle)$. Now we clearly have (1).

*Theorem* **(5.5).** — *Assume that* K *is perfect, and let* R *be an analytic local ring over* K *with* $\dim R>0$ *and* $\mathrm{rad}_R\{0\}=\{0\}$. *Let* $P_1, \ldots, P_e$ *be all the distinct prime ideals of height zero in* R. *Let* T *be the total quotient ring of* R, *and let* S *be the integral closure of* R *in* T.

Note that then (see [1, (18.9)]): $P_1T, \ldots, P_eT$ are exactly all the distinct prime ideals in T, and they all have height zero; $(P_iT)\cap R=P_i$ for $1\leq i\leq e$; T is noetherien; and if T′ is any noetherian subring of T with total quotient ring T then $(P_1T)\cap T', \ldots, (P_eT)\cap T'$ are exactly all the distinct prime ideals of height zero in T′, $((P_1T)\cap T')\cap\ldots\cap((P_eT)\cap T')=\mathrm{rad}_{T'}\{0\}=\{0\}$, and $((P_iT)\cap T')T=P_iT$ for $1\leq i\leq e$.

For $1\leq i\leq e$, we have $(P_iT)\cap K=P_i\cap K=\{0\}$ and hence *we can take an overring* $T_i$ *of* K *and a* K*-epimorphism* $w_i : T\to T_i$ *with* $\mathrm{Ker}\, w_i=P_iT$. By [1, (18.9)] we now see that $w_1\oplus\ldots\oplus w_e : T\to T_1\oplus\ldots\oplus T_e$ is an isomorphism, and $T_i$ is the quotient field of $w_i(R)$ for $1\leq i\leq e$.

*Let* $S_i$ *be the integral closure of* $w_i(R)$ *in* $T_i$ *for* $1\leq i\leq e$. By (5.4) we know that $R/P_i$ is analytically separably generated over K for $1\leq i\leq e$, and hence $S_i$ is a finite $w_i(R)$-module for $1\leq i\leq e$; consequently by [1, (19.23), (20.6)] we see that S is a finite R-module, $S=w_1^{-1}(S_1)\cap\ldots\cap w_e^{-1}(S_e)$, and $S_i$ is a local domain for $1\leq i\leq e$.

*Let* R′ *be a subring of* T. *Assume that*: R′ *is noetherian*; $K\subset R'\subset S$; S *is integral over* R′; T *is the total quotient ring of* R′; *and the integral closure of* $R'/(P_iT)\cap R'$ *in the quotient field of* $R'/(P_iT)\cap R'$ *is a finite* $(R'/(P_iT)\cap R')$*-module for* $1\leq i\leq e$.

(Note that, in view of what we have said above, these assumptions on R′ are automatically satisfied in case $R\subset R'\subset S$.)

*Let* K′ *be the integral closure of* K *in* T, *and let* $K_i$ *be the integral closure of* K *in* $T_i$ *for* $1\leq i\leq e$. *Then we have the following.*

**(5.5.1)** *$K_i$ is a coefficient field of $S_i$ for $1 \leq i \leq e$, and $K' = w_1^{-1}(K_1) \cap \ldots \cap w_e^{-1}(K_e)$. Given any elements $x_1, \ldots, x_e$ in S, there exists $y \in S$ such that for $1 \leq i \leq e$ we have: $w_i(y) \notin K_i$, and if $w_i(x_i) \notin K_i$ then $w_i(y) = w_i(x_i)$.*

**(5.5.2)** *Let $J'$ be an ideal in $R'$, and let $a$ be an integer with $1 \leq a \leq e$. Assume that $\dim R/P_1 = \ldots = \dim R/P_a$ and $J' \not\subset P_i T$ for $1 \leq i \leq a$. Let any elements $x_1, \ldots, x_a$ in S be given, and let W be the set of all integers $i$ with $1 \leq i \leq a$ such that $w_i(x_i) \notin K_i$. Now take any elements $y_1, \ldots, y_a$ in S such that $w_i(y_i) \notin K_i$ for $1 \leq i \leq a$, and $w_i(y_i) = w_i(x_i)$ for all $i \in W$* (note that by (5.5.1) we can actually find $y \in S$ such that $w_i(y) \notin K_i$ for $1 \leq i \leq e$, and $w_i(y) = w_i(x_i)$ for all $i \in W$). *Then there exists an infinite subset G of*

$$G_K(R', J') \cap G_K[R', (P_1 T) \cap R'] \cap \ldots \cap G_K[R', (P_e T) \cap R']$$

*with $\operatorname{card}(G) \geq \operatorname{card}(K)$ such that for all $g \neq h$ in G we have $I_{R'}(g)(y_i) - I_{R'}(h)(y_i) \notin P_i T$ for $1 \leq i \leq a$, and $I_{R'}(g)(x_i) - I_{R'}(h)(x_i) \notin P_i T$ for all $i \in W$.*

**(5.5.3)** *Let $J'$ be any ideal in $R'$ such that $J'$ contains a nonzerodivisor of $R'$. Then*

$$\operatorname{Inv} G_K(R', J') \cap G_K[R', (P_1 T) \cap R'] \cap \ldots \cap G_K[R', (P_e T) \cap R'] \subset K'.$$

*Proof of* (5.5.1). — By Hensel's lemma [1, (20.6)] we see that $K_i$ is a coefficient field of $S_i$ for $1 \leq i \leq e$.

To show that $K' = w_1^{-1}(K_1) \cap \ldots \cap w_e^{-1}(K_e)$, let any $t \in T$ be given. If $t \in K'$ then clearly $w_i(t) \in K_i$ for $1 \leq i \leq e$, i.e., $t \in w_1^{-1}(K_1) \cap \ldots \cap w_e^{-1}(K_e)$. Conversely suppose that $t \in w_1^{-1}(K_1) \cap \ldots \cap w_e^{-1}(K_e)$; then $w_i(t) \in K_i$ and hence there exists a nonconstant monic polynomial $f_i(Z)$ in an indeterminate Z with coefficients in K such that $w_i(f_i(t)) = 0$; let $f(Z) = f_1(Z) \ldots f_e(Z)$; then $f(Z)$ is a nonconstant monic polynomial in Z with coefficients in K, and $w_i(f(t)) = 0$ for $1 \leq i \leq e$; consequently $f(t) = 0$, and hence $t \in K'$.

Finally, let any elements $x_1, \ldots, x_e$ in S be given. Now $S_i$ is a local domain with $\dim S_i = \dim w_i(R) > 0$ and hence we can find $x_i' \in S_i$ with $x_i' \notin K_i$. Since $S = w_1^{-1}(S_1) \cap \ldots \cap w_e^{-1}(S_e)$, there exists a unique $y \in S$ such that for $1 \leq i \leq e$ we have: $w_i(y) = w_i(x_i)$ if $w_i(x_i) \notin K_i$, and $w_i(y) = x_i'$ if $w_i(x_i) \in K_i$.

*Proof of* (5.5.2). — Let $R^* = R[N]$ where

$$N = w_1^{-1}(M(S_1)) \cap \ldots \cap w_e^{-1}(M(S_e)).$$

Now $w_1 \oplus \ldots \oplus w_e : T \to T_1 \oplus \ldots \oplus T_e$ is an isomorphism, $S = w_1^{-1}(S_1) \cap \ldots \cap w_e^{-1}(S_e)$, and $S_i$ is a local domain for $1 \leq i \leq e$; consequently by [1, (18.8)] we see that $w_i(N) = M(S_i)$ for $1 \leq i \leq e$, and N = the intersection of all maximal ideals in S; whence, in particular, $R \subset R^* \subset S$. Since S is a finite R-module, by (2.4) we now get that $R^*$ is an analytic local ring over K. It follows that: T is the total quotient ring of $R^*$; S is the integral closure of $R^*$ in T; $(P_1 T) \cap R^*, \ldots, (P_e T) \cap R^*$ are exactly all the distinct prime ideals of height zero in $R^*$; $((P_i T) \cap R^*)T = P_i T$ for $1 \leq i \leq e$; $T_i$ = the quotient field of $w_i(R^*)$ for $1 \leq i \leq e$; and for $1 \leq i \leq e$ we have that $w_i(R^*)$ is an analytic local domain over K with $\dim w_i(R^*) = \dim R/P_i$ and $w_i(M(R^*)) = M(w_i(R^*)) = M(S_i)$.

Also note that by (5.4) we know that $R^*/(P_iT)\cap R^*$ (i.e., $w_i(R^*)$) is analytically separably generated over K for $1\leq i\leq e$.

By (5.5.1) we know that $K_i$ is a coefficient field of $S_i$ and hence there exists a unique $k_i\in K_i$ such that $w_i(y_i)-k_i\in M(S_i)$; note that now $0\neq w_i(y_i)-k_i\in M(S_i)=M(w_i(R^*))$ for $1\leq i\leq a$.

Let $n=\dim R/P_1$; note that then $n>0$ and $\dim w_i(R^*)=n$ for $1\leq i\leq a$. Let $p$ be the characteristic exponent of K, i.e., $p=1$ if K is of zero characteristic, and $p=$ the characteristic of K if K is of nonzero characteristic. *We claim that* for every $i$ with $1\leq i\leq a$, there exists a nonnegative integer $b_i$ and a local K-monomorphism $v_i: A_n\to w_i(R^*)$ such that $w_i(R^*)$ is integral over $v_i(A_n)$, $T_i$ is separable over the quotient field of $v_i(A_n)$, and

$$v_i(X_1^{q_i})=w_i(y_i)-k_i \quad \text{where} \quad q_i=p^{b_i}. \tag{1}$$

*Case of* $p=1$. — Upon letting $z_{i1}=w_i(y_i)-k_i$ we now have that $w_i(R^*)/z_{i1}w_i(R^*)$ is a local ring of dimension $n-1$, and hence we can find elements $z_{i2}, \ldots, z_{in}$ in $M(w_i(R^*))$ such that $(z_{i1}, \ldots, z_{in})$ is a system of parameters of $w_i(R^*)$; it suffices to take $b_i=1$ and $v_i: A_n\to w_i(R^*)$ to be the unique K-homomorphism with $v_i(X_j)=z_{ij}$ for $1\leq j\leq n$.

*Case of* $p\neq 1$. — Now $S_i$ is integrally closed in $T_i$, $0\neq w_i(y_i)-k_i\in M(S_i)$, and $\bigcap_{m=1}^{\infty} M(S_i)^m=\{0\}$; consequently there exists a unique nonnegative integer $b_i$ such that upon letting $q_i=p^{b_i}$ and $z_{i1}=(w_i(y_i)-k_i)^{1/q_i}$ we have that $z_{i1}\in M(S_i)$ and $z_{i1}\notin T_i^p$; since $M(S_i)=M(w_i(R^*))$, we have $z_{i1}\in M(w_i(R^*))$; now by (5.3) and (5.4) we can find elements $z_{i2}, \ldots, z_{in}$ in $M(w_i(R^*))$ such that $(z_{i1}, \ldots, z_{in})$ is a system of parameters of $w_i(R^*)$ and $T_i$ is separable over $K[\langle z_{i1}, \ldots, z_{in}\rangle]$; it suffices to take $v_i: A_n\to w_i(R^*)$ to be the unique local K-homomorphism with $v_i(X_j)=z_{ij}$ for $1\leq j\leq n$.

This completes the proof of the claim. For $1\leq i\leq a$ we can take $s_i\in S$ with

$$w_i(s_i)=v_i(X_1). \tag{2}$$

Upon taking $R^*$ for R in (4.3) we now find an infinite subset G of

$$G_K(R', J')\cap G_K[R', (P_1T)\cap R']\cap \ldots \cap G_K[R', (P_eT)\cap R']$$

with $\operatorname{card}(G)\geq\operatorname{card}(K)$ such that for all $g\neq h$ in G we have $I_{R'}(g)(s_i)-I_{R'}(h)(s_i)\notin P_iT$ for $1\leq i\leq a$. Henceforth let $i$ be any integer with $1\leq i\leq a$, let $g$ and $h$ be any elements in G with $g\neq h$, and let $g'=I_{R'}(g)$ and $h'=I_{R'}(h)$. We know that then $g'(s_i)-h'(s_i)\notin P_iT$, i.e.,

$$w_i(g'(s_i))\neq w_i(h'(s_i)). \tag{3}$$

We want to show that: $g'(y_i)-h'(y_i)\notin P_iT$; and if $i\in W$ then $g'(x_i)-h'(x_i)\notin P_iT$. Thus, what remains to be proved is that

$$w_i(g'(y_i))\neq w_i(h'(y_i)), \tag{4*}$$

and

$$\text{if } w_i(y_i)=w_i(x_i) \text{ then } w_i(g'(x_i))\neq w_i(h'(x_i)). \tag{5*}$$

Now $w_i(S)=S_i$ and hence we get a K-epimorphism $w_i' : S \to S_i$, with $\mathrm{Ker}\, w_i' = (P_i T) \cap S$, by taking $w_i'(s)=w_i(s)$ for all $s \in S$. Let $w_i^* : G_K[S, \mathrm{Ker}\, w_i'] \to G_K(S_i)$ be the homomorphism induced by $w_i'$. Now $g'=I_{R'}(g)$ and $h'=I_{R'}(h)$, and $g$ and $h$ are in $G_K[R', (P_iT) \cap R']$; hence by (2.3.1) we see that $g'$ and $h'$ are in $G_K[S, (P_iT) \cap S]$. Let $g^*=w_i^*(g')$ and $h^*=w_1^*(h')$. Then

$$g^* \in G_K(S_i) \quad \text{and} \quad w_i(g'(s))=g^*(w_i(s)) \quad \text{for all} \quad s \in S, \tag{6}$$

and

$$h^* \in G_K(S_i) \quad \text{and} \quad w_i(h'(s))=h^*(w_i(s)) \quad \text{for all} \quad s \in S. \tag{7}$$

By (2), (3), (6) and (7) we get that

$$g^*(v_i(X_1)) \neq h^*(v_i(X_1)). \tag{8}$$

In view of (6) and (7) we also see that ($4^*$) and ($5^*$) are equivalent to asserting that: $g^*(w_i(y_i)) \neq h^*(w_i(y_i))$. We now proceed to show that

$$g^*(w_i(y_i)) - h^*(w_i(y_i)) \neq 0, \tag{9*}$$

and this will complete the proof.

Now $v_i(X_1) \in M(w_i(R^*))=M(S_i)$; since $g^*$ and $h^*$ are automorphisms of $S_i$, we have $g^*(M(S_i))=M(S_i)$ and $h^*(M(S_i))=M(S_i)$; consequently

$$g^*(v_i(X_1)) \in M(S_i) \quad \text{and} \quad h^*(v_i(X_1)) \in M(S_i).$$

Therefore by (8) we get that

$$0 \neq g^*(v_i(X_1)) - h^*(v_i(X_1)) \in M(S_i). \tag{10}$$

Let

$$Z = g^*(v_i(X_1^{q_i})) - h^*(v_i(X_1^{q_i})).$$

Then by (1)

$$g^*(w_i(y_i)) - h^*(w_i(y_i)) = Z + g^*(k_i) - h^*(k_i). \tag{11}$$

Now

$$Z = (g^*(v_i(X_1)) - h^*(v_i(X_1)))^{q_i}$$

and hence by (10) we get that

$$0 \neq Z \in M(S_i). \tag{12}$$

Now $K_i$ is the integral closure of K in $S_i$, and $g^*$ and $h^*$ are K-automorphisms of $S_i$; consequently we must have $g^*(K_i)=K_i$ and $h^*(K_i)=K_i$; therefore $g^*(k_i) \in K_i$ and $h^*(k_i) \in K_i$, and hence

$$g^*(k_i) - h^*(k_i) \in K_i. \tag{13}$$

By (5.5.1) we know that $K_i$ is a coefficient field of $S_i$; therefore by (11), (12) and (13) we get ($9^*$).

*Proof of* (5.5.3). — Follows from (5.5.1) and (5.5.2).

*Theorem* (5.6). — *Assume that* K *is perfect, and let* R *be an analytic local ring over* K *with* $\dim R>0$. *Let* $Q_1, \ldots, Q_a$ $(a>0)$, *be any distinct isolated primary components of* $\{0\}$ *in* R *such that* $\dim R/Q_1=\ldots=\dim R/Q_a$. *Let* $P_i=\mathrm{rad}_R Q_i$. *Let* $Q'_1, \ldots, Q'_b$ $(b\geq 0)$, *be any finite number of ideals in* R *such that for* $1\leq i\leq a$ *and* $1\leq j\leq b$ *we have* $Q'_j \not\subset P_i$. *Assume that*:

(*) *there exists a* K*-epimorphism* $u : A_d \to R$, *for some d, such that* $u^{-1}(Q_i)$, *is a symbolic power of* $u^{-1}(P_i)$ *for* $1\leq i\leq a$.

*Let any elements* $x_1, \ldots, x_a$ *in* M(R) *be given. Clearly there then exist elements* $y_1, \ldots, y_a$ *in* M(R) *such that for* $1\leq i\leq a$ *we have*: $y_i\notin P_i$, *and if* $x_i\notin P_i$ *then* $y_i=x_i$; *now let* $y_1, \ldots, y_a$ *be any such. Let* W *be the set of all integers i with* $1\leq i\leq a$ *such that* $x_i\notin P_i$. *Then there exists an infinite subset* G *of*

$$\bigcap_{j=1}^{b} G_K(R, Q'_j) \cap \bigcap_{j=1}^{b} G_K[R, Q'_j] \cap \bigcap_{i=1}^{a} G_K[R, Q_i] \cap \bigcap_{i=1}^{a} G_K[R, P_i]$$

*with* $\mathrm{card}(G)\geq\mathrm{card}(K)$ *such that for all* $g\neq h$ *in* G *we have* $g(y_i)-h(y_i)\notin P_i$ *for* $1\leq i\leq a$, *and* $g(x_i)-h(x_i)\notin P_i$ *for all* $i\in W$.

(For an intrinsic formulation of (*) see [2, (3.6)]. Note that (*) is automatically satisfied in case $Q_i=P_i$ for $1\leq i\leq a$, because then we can take $u$ to be any K-epimorphism $A_d\to R$. Also note that (*) is automatically satisfied in case $\mathrm{emdim}\, R=n+1$ where $n=\dim R/Q_1$, because then we can take $u$ to be any K-epimorphism $A_{n+1}\to R$; see [2, (2.16)].)

*Proof.* — Since $Q_1, \ldots, Q_a$ are isolated primary components of $\{0\}$ in R, there exists an ideal Q in R such that $Q\cap Q_1\cap\ldots\cap Q_a=\{0\}$ and $Q\not\subset P_i$ for $1\leq i\leq a$. Let $J=Q\cap Q'_1\cap\ldots\cap Q'_b$. Then $J\cap Q_1\cap\ldots\cap Q_a=\{0\}$ and $J\not\subset P_i$ for $1\leq i\leq a$. We can take an overring $R^*$ of K and a K-epimorphism $v : R\to R^*$ with $\mathrm{Ker}\, v=P_1\cap\ldots\cap P_a$. Let $J^*=v(J)$. Let

$$G_0=G_K(R, J)\cap\bigcap_{i=1}^{a} G_K[R, P_i]\cap\bigcap_{i=1}^{a} G_K[R, Q_i],$$

and

$$G_0^*=G_K(R^*, J^*)\cap\bigcap_{i=1}^{a} G_K[R^*, v(P_i)].$$

Let $w : G_K[R, \mathrm{Ker}\, v]\to G_K(R^*)$ be the homomorphism induced by $v$. Then by [2, (4.4)] we have $w(G_0)=G_0^*$; note that clearly

$$G_K[R, P_1]\cap\ldots\cap G_K[R, P_a]\subset G_K[R, \mathrm{Ker}\, v]$$

and hence it makes sense to talk about $w(G_0)$. Also note that, in view of [2, (2.1), (2.2)], we have $G_0\subset G_K(R, Q'_j)\subset G_K[R, Q'_j]$ for $1\leq j\leq b$.

Let $t_i : R^*\to R^*/v(P_i)$ be the canonical epimorphism. Then for $1\leq i\leq a$, in view of [2, (2.10)], we have that $t_i(v(y_i))$ is not integral over $t_i(K)$, and: $i\in W\Leftrightarrow t_i(v(x_i))$ is not integral over $t_i(K)\Leftrightarrow t_i(v(y_i))=t_i(v(x_i))$. Also clearly $J^*\not\subset v(P_i)$ for $1\leq i\leq a$. Therefore by (5.5.2) there exists an infinite subset $G^*$ of $G_0^*$ with $\mathrm{card}(G^*)\geq\mathrm{card}(K)$ such that for all $g\neq h$ in $G^*$ we have $g(v(y_i))-h(v(y_i))\notin v(P_i)$ for $1\leq i\leq a$, and

$g(v(x_i))-h(v(x_i))\notin v(P_i)$ for all $i\in W$. Since $w(G_0)=G_0^*$, for each $g\in G^*$ we can fix $g'\in G_0$ with $w(g')=g$; now it suffices to take $G=\{g' : g\in G^*\}$.

*Theorem* **(5.7)**. — *Assume that* K *is perfect, and let* R *be an analytic local ring over* K. *Let* $Q_1, \ldots, Q_e$ *be all the distinct isolated primary components of* $\{0\}$ *in* R. *Let* $P_i=\mathrm{rad}_R Q_i$. *Let* $Q'_1, \ldots, Q'_b$ $(b\geq 0)$, *be any finite number of ideals in* R *such that for* $1\leq i\leq e$ *and* $1\leq j\leq b$ *we have* $Q'_j\not\subset P_i$. *Assume that*

(*) *there exists a* K*-epimorphism* $u : A_d\to R$, *for some* $d$, *such that* $u^{-1}(Q_i)$ *is a symbolic power of* $u^{-1}(P_i)$ *for* $1\leq i\leq e$.

*Let*

$$G=\bigcap_{j=1}^{b} G_K(R, Q'_j)\cap\bigcap_{j=1}^{b} G_K[R, Q'_j]\cap\bigcap_{i=1}^{e} G_K[R, Q_i]\cap\bigcap_{i=1}^{e} G_K[R, P_i].$$

*Then* $\mathrm{Inv}\, G\subset K+\mathrm{rad}_R\{0\}$.

(Note that by [2, (2.10)], $K+\mathrm{rad}_R\{0\}=$ the integral closure of K in R.)

(For an intrinsic formulation of (*) see [2, (3.6)]. Note that (*) is automatically satisfied in case $Q_i=P_i$ for $1\leq i\leq e$, because then we can take $u$ to be any K-epimorphism $A_d\to R$. Also note that (*) is automatically satisfied in case $\dim R/P_1=\ldots=\dim R/P_e$ and $\mathrm{emdim}\, R=n+1$ where $n=\dim R/P_1$, because then we can take $u$ to be any K-epimorphism $A_{n+1}\to R$; see [2, (2.16)].)

*Proof.* — Follows from (5.6).

*Theorem* **(5.8)**. — *Assume that* K *is perfect, and let* R *be an analytic local ring over* K *with* $\mathrm{rad}_R\{0\}=\{0\}$. *Let* $P_1, \ldots, P_e$ *be all the distinct prime ideals of height zero in* R. *Let* $Q'_1, \ldots, Q'_b$ $(b\geq 0)$ *be any finite number of ideals in* R *such that for* $1\leq i\leq b$ *we have that* $Q'_j$ *contains a nonzerodivisor of* R. *Let*

$$G=\bigcap_{j=1}^{b} G_K(R, Q'_j)\cap\bigcap_{j=1}^{b} G_K[R, Q'_j]\cap\bigcap_{i=1}^{e} G_K[R, P_i].$$

*Then* $\mathrm{Inv}\, G=K$. *Moreover, if* G′ *is any subset of* G(R) *with* $G\subset G'$, *then* $\mathrm{Inv}\, G'$ *is a subfield of* K.

(Note that by [2, (2.10)] we know that K = the integral closure of K in R.)

*Proof.* — By (5.7) we get that $\mathrm{Inv}\, G=K$. The second assertion follows from this in view of [2, (2.7)].

*Theorem* **(5.9)**. — *Assume that* K *is perfect, and let* R *be an analytic local ring over* K *with* $\mathrm{rad}_R\{0\}=\{0\}$. *Then* $\mathrm{Inv}\, G_K(R)=K$, *and* $\mathrm{Inv}\, G(R)$ *is a subfield of* K.

(Note that by [2, (2.10)] we know that K = the integral closure of K in R.)

*Proof.* — Follows from (5.8).

## § 6. Local rings in which every nonunit is a zerodivisor.

*Theorem* **(6.1)**. — *Let* R *be a local ring with coefficient field* K. *Let* $n=\mathrm{emdim}\, R$. *Assume that* $n>0$ (i.e., $R\neq K$). *Also assume that every element in* M(R) *is a zerodivisor of* R. *Let* $R_0=K[X]/X^2K[X]$ *where* X *is an indeterminate. Then we have the following*:

(1) *If* $n>1$ *then* $\mathrm{card}(G_K(R))\geq\mathrm{card}(K^{n-1})$.
(2) *If* $n=1$ *and* $M(R)^2\neq\{0\}$ *then* $\mathrm{card}(G_K(R))\geq\mathrm{card}(K)$.
(3) *If* $n=1$ *and* $M(R)^2=\{0\}$ *then* $\mathrm{card}(G_K(R))\geq\mathrm{card}(K)-1$.
(4) $G_K(R)=\{1\}$
$\Leftrightarrow n=1$, $\mathrm{card}(K)=2$, *and* $M(R)^2=\{0\}$
$\Leftrightarrow\mathrm{card}(K)=2$ *and* R *is* K-*isomorphic to* $R_0$
$\Leftrightarrow\mathrm{card}(K)=2$ *and* R *is isomorphic to* $R_0$
$\Leftrightarrow\mathrm{card}(R)=4$
$\Leftrightarrow G(R)=\{1\}$.

*Proof.* — Now M(R) is an associated prime ideal of $\{0\}$ in R, and hence there exists $0\neq y\in M(R)$ such that $(yR)M(R)=\{0\}$.

First suppose that $y\in M(R)^2$. Take a basis $(x_1, \ldots, x_n)$ of M(R). Now every $z$ in R can uniquely be expressed as

$$z=z_0+z_1x_1+\ldots+z_nx_n+z' \text{ with } z_0, \ldots, z_n \text{ in K and } z'\in M(R)^2.$$

For every $a=(a_1, \ldots, a_n)\in K^n$ we get a K-homomorphism $g_a: R\to R$ by setting:

$$g_a(z)=z+(z_1a_1+\ldots+z_na_n)y \quad \text{for all } z\in R.$$

Upon letting $-a=(-a_1, \ldots, -a_n)$, we have $g_ag_{-a}=g_{-a}g_a=$ the identity map of R, and hence $g_a\in G_K(R)$. Clearly $g_a\neq g_b$ for all $a\neq b$ in $K^n$. Thus we have shown that:

(5) If $y\in M(R)^2$ then $\mathrm{card}(G_K(R))\geq\mathrm{card}(K^n)$.

Next suppose that $y\notin M(R)^2$ and $n>1$. Let $x_n=y$. We can find elements $x_1, \ldots, x_{n-1}$ in M(R) such that $(x_1, \ldots, x_n)$ is a basis of M(R). Again, every $z$ in R can be uniquely expressed as

$$z=z_0+z_1x_1+\ldots+z_nx_n+z' \text{ with } z_0, \ldots, z_n \text{ in K and } z'\in M(R)^2.$$

For every $a=(a_1, \ldots, a_{n-1})\in K^{n-1}$ we get a K-homomorphism $g_a: R\to R$ by setting:

$$g_a(z)=z+(z_1a_1+\ldots+z_{n-1}a_{n-1})y \quad \text{for all } z\in R.$$

Upon letting $-a=(-a_1, \ldots, -a_{n-1})$, we have $g_ag_{-a}=g_{-a}g_a=$ the identity map of R, and hence $g_a\in G_K(R)$. Clearly $g_a\neq g_b$ for all $a\neq b$ in $K^{n-1}$. Thus we have shown that:

(6) If $y\notin M(R)^2$ and $n>1$ then $\mathrm{card}(G_K(R))\geq\mathrm{card}(K^{n-1})$.

Finally suppose that $y\notin M(R)^2$ and $n=1$. Now $M(R)=yR$ and

$$M(R)^2=y^2R=\{0\}.$$

Consequently, every $z$ in R can uniquely be expressed as $z=z_0+z_1y$ with $z_0$ and $z_1$ in K. For every $0\neq a\in K$ we get $g_a\in G_K(R)$ by setting: $g_a(z)=z_0+az_1y$ for all $z\in R$. Clearly $g_a\neq g_b$ for all $0\neq a\neq b\neq 0$ in K. Thus we have shown that:

(7) If $y\notin M(R)^2$ and $n=1$ then $M(R)^2=\{0\}$ and $\mathrm{card}(G_K(R))\geq\mathrm{card}(K)-1$.

(1), (2) and (3) follow from (5), (6) and (7). (4) follows from (1), (2) and (3).

## § 7. Remarks on fields of definition.

Let R be a local ring, let $S=R/M(R)$, and let $t: R\to S$ be the canonical epimorphism; now $G(R)=G[R, M(R)]$, and hence $t$ induces a homomorphism $u: G(R)\to G(S)$. Let $V'=\operatorname{Inv} G(R)$ and $V=\operatorname{Inv} u(G(R))$. Let $p$ be the characteristic of S, where $p$ may or may not be zero.

**(7.1)** *$t(V')\subset V$, and $t(V')$ and V are subfields of S. If $p\neq 0$ and $y$ is any element in S with $y^p\in V$ then $y\in V$; whence, in particular, if S is perfect then so is V.*

Obviously $t(V')\subset V$. By [2, (2.7)] we have that V is a subfield of S. If $p\neq 0$ and $y$ is any element in S with $y^p\in V$ then clearly $y\in V$. To see that $t(V')$ is a subfield of S, let any $x\in V'$ with $t(x)\neq 0$ be given; now $x\notin M(R)$ and hence $1/x\in R$; for any $g\in G(R)$ we have

$$1/x=(1/x)g(1)=(1/x)g((x)(1/x))=(1/x)g(x)g(1/x)=(1/x)(x)g(1/x)=g(1/x);$$

thus $1/x\in V'$ and clearly $t(1/x)=1/t(x)$.

**(7.2)** *If R has a coefficient field K such that $g(K)=K$ for all $g\in G(R)$, then clearly $t(K\cap V')=t(V')=V$* (note that by [2, (2.7)] we know that $K\cap V'$ is a subfield of K, and hence now $t$ induces an isomorphism of $K\cap V'$ onto V). Note that by [2, (2.12)] we see that: *if S is perfect with $p\neq 0$ and R has a coefficient field K, then $g(K)=K$ for all $g\in G(R)$.* Finally note that by (5.9) we know that: *if $\operatorname{rad}_R\{0\}=\{0\}$ and R is an analytic local ring over a perfect valued field K, then $V'$ is a subfield of K, and hence t induces an isomorphism of $V'$ onto $t(V')$.*

Henceforth assume that R is complete, R is of characteristic $p$, and S is algebraically closed. For any field H and any nonnegative integer $a$ let $H_a$ denote the ring of formal power series in indeterminates $X_1, \ldots, X_a$ with coefficients in H. By Cohen's theorem we know that R has a coefficient field, i.e., equivalently, there exists an epimorphism $b: S_a\to R$ for some $a$, such that $t(b(s))=s$ for all $s\in S$. Let $E(a)$ be the set of all epimorphisms $b: S_a\to R$ such that $t(b(s))=s$ for all $s\in S$. For every $b\in E(a)$ let $D(a, b)$ be the set of all subfields H of S such that $((\operatorname{Ker} b)\cap H_a)S_a=\operatorname{Ker} b$, and let $D'(a, b)$ be the set of all subfields H' of R such that $H'=b(H)$ for some $H\in D(a, b)$. Let

$$D=\bigcup_{a=0}^{\infty}\bigcup_{b\in E(a)} D(a, b), \qquad D'=\bigcup_{a=0}^{\infty}\bigcup_{b\in E(a)} D'(a, b),$$

$$D^*=\{H\in D: H \text{ is perfect}\}, \qquad D'^*=\{H\in D': H \text{ is perfect}\},$$

$$F=\bigcap_{H\in D} H, \quad F'=\bigcap_{H\in D'} H, \quad F^*=\bigcap_{H\in D^*} H, \quad F'^*=\bigcap_{H\in D'^*} H.$$

One might designate every member of $D^*$ (or D, or $D'^*$, or $D'$) to be a field of definition of R, and $F^*$ (or F, or $F'^*$, or $F'$) to be the field of definition of R. Note that clearly $F\subset F^*$, $F'\subset F'^*$, $t(F'^*)\subset F^*$, and $t(F')\subset F$; in view of [2, (2.12)] we also see that if $p\neq 0$ then $t(F'^*)=F^*$ and $t(F')=F$.

Thus, to R we have attached the six subfields: $t(V')$, V, $t(F')$, F, $t(F'^*)$, and $F^*$ of S. It would be interesting to investigate the properties of these fields and their relationships. For instance, one may ask: 1) $F\in D$?; 2) $F^*\in D^*$?; 3) are these various

fields, " in some sense ", finitely or countably generated over their prime subfield?; etc. In this connection we only offer the following two remarks (7.3) and (7.4):

**(7.3)** $V \subset F^*$. *Moreover, if* $H \subset V$ *for some* $H \in D^*$ *then* $H = F^* = V$.

The second assertion follows from the first. To prove the first assertion, let any perfect subfield H of S and any epimorphism $b$: $S_a \to R$ be given such that $t(b(s)) = s$ for all $s \in S$ and $((\operatorname{Ker} b) \cap H_a)S_a = \operatorname{Ker} b$; also let any $y \in S$ be given such that $y \notin H$. We want to show that then $y \notin V$. By [2, (2.8)] there exists $h' \in G_H(S)$ such that $h'(y) \neq y$. We get $h \in G_H(S_a)$ by taking

$$h(\Sigma f_{i_1 \ldots i_a} X_1^{i_1} \ldots X_a^{i_a}) = \Sigma h'(f_{i_1 \ldots i_a}) X_1^{i_1} \ldots X_a^{i_a}$$

for all

$$\Sigma f_{i_1 \ldots i_a} X_1^{i_1} \ldots X_a^{i_a} \in S_a \quad \text{with} \quad f_{i_1 \ldots i_a} \in S.$$

Since $((\operatorname{Ker} b) \cap H_a)S_a = \operatorname{Ker} b$, we see that $h \in G_H[S_a, \operatorname{Ker} b]$. Let

$$b' : G_H[S_a, \operatorname{Ker} b] \to G_{b(H)}(R)$$

be the homomorphism induced by $b$, and let $g = u(b'(h))$. Then $g \in u(G(R))$ and $g(y) = h'(y) \neq y$. Therefore $y \notin V$.

**(7.4)** Let K be any algebraically closed field of characteristic $p$, where $p$ may or may not be zero. Let $K_0$ be the prime subfield of K. Let $z_0, \ldots, z_e$ be any given finite number of elements in K with $z_0 = 1$. Let $L = K_0(z_1, \ldots, z_e)$. In other words, let L be any subfield of K such that L is finitely generated over $K_0$. Let $L^* = L$ if $p = 0$, and $L^* = L^{p^{-\infty}}$ if $p \neq 0$; note that if $p \neq 0$ then: $L^*$ is finitely generated over $K_0 \Leftrightarrow L$ is algebraic over K. We can take positive integers $m, n, q, d$ such that: $q + qe < m$; $m + q + qe < n$; $n$ is not divisible by $p$; $n + m + q + qe < d$; and $n$ and $d$ are coprime. Let X be an indeterminate, and let

$$Y = X^{n+m} + \sum_{i=0}^{e} z_i X^{n+m+q+qi} + X^d.$$

Upon taking $R = K[[X^n, Y]]$ we clearly have that R is a one-dimensional complete local domain with coefficient field K and emdim $R = 2$. Let S, $t$, V, etc., be as above. By [2, (5.3)] we have that $t(L) \subset V$ and hence by (7.1) we get that $t(L^*) \subset V$. Clearly $t(L^*) \in D^*$, and hence by (7.3) we get that $t(L^*) = F^* = V$. In view of (7.2), we now also see that, if $p \neq 0$ then $V' = L^*$.

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