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A CLASSIFICATION OF THE TOPOLOGICAL TYPES OF $\mathbf{R}^2$ -ACTIONS ON CLOSED ORIENTABLE 3-MANIFOLDS

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In this paper we shall classify the topological type of non singular actions of $\mathbf{R}^2$ on closed orientable 3-manifolds. If φ is a non singular action of $\mathbf{R}^2$ on V then we denote by $\mathcal{F}(\varphi)$ the foliation of V defined by the orbits of φ ; φ non singular means the orbits are of dimension two, therefore $\mathcal{F}(\varphi)$ is a 2-dimensional foliation of V whose leaves are planes, cylinders and tori. V is assumed orientable, therefore $\mathcal{F}(\varphi)$ is a transversally orientable foliation. We consider two non singular actions φ and ψ to be equivalent if there is a homeomorphism $h : V \rightarrow V$ which sends leaves of $\mathcal{F}(\varphi)$ to leaves of $\mathcal{F}(\psi)$. We assume throughout this paper that the actions are at least of class C^2 .

In [7], it is shown that if V admits a non singular action of $\mathbf{R}^2$ and if V is a closed orientable 3-manifold, then V is a fibre bundle over the circle $\mathbf{S}^1$ with fibre the 2-torus $\mathbf{T}^2$. Therefore V is diffeomorphic to $(\mathbf{T}^2 \times \mathbf{I})/F$ where F is a diffeomorphism $\mathbf{T}^2 \rightarrow \mathbf{T}^2$ induced by an element of $\mathbf{GL}(2, \mathbf{Z})$; $(\mathbf{T}^2 \times \mathbf{I})/F$ denotes the quotient space of $\mathbf{T}^2 \times \mathbf{I}$ where $(x, 1)$ is identified with $(F(x), 0)$ for $x \in \mathbf{T}^2$. Since V is orientable, we have $\det F = +1$. We can now announce the main results; naturally we assume φ is a non singular action on the closed orientable 3-manifold $V \approx (\mathbf{T}^2 \times \mathbf{I})/F$:

Theorem 1. — If all the orbits of φ are planes, then V is diffeomorphic to $\mathbf{T}^3$ and $\mathcal{F}(\varphi)$ is equivalent to a linear action.

Theorem 2. — If φ has no compact orbits and not all the orbits of φ are planes, then all the orbits of φ are cylinders, F has eigenvalues equal to $+1$ and φ is equivalent to the suspension of a non singular action of the circle on $\mathbf{T}^2$.

Theorem 3. — If φ has a compact orbit T , then the manifold obtained by cutting V along T is diffeomorphic to $\mathbf{T}^2 \times \mathbf{I}$. All the compact orbits of φ are isotopic in V , and if T_1 and T_2 are compact orbits of φ which bound a submanifold W of V whose interior contains no compact orbits, then $W \approx \mathbf{T}^2 \times \mathbf{I}$ and all the orbits of φ in W are either planes or cylinders (but there is no mixture of the two) which spiral in a precise manner towards T_1 and T_2 (this will be made precise in the sequel).

Theorem 1 is not new: in [4] it is shown that a closed orientable 3-manifold foliated by planes is diffeomorphic to $\mathbf{T}^3$, and in [6] it is shown that such foliations

of $\mathbf{T}^3$ are equivalent to linear foliations. Part of the interest of theorem 2 is the existence of compact orbits when F has no eigenvalue equal to $+1$.

Some notation. — Let $p: \mathbf{T}^2 \times I \rightarrow V$ be the natural projection and $T_0 = p(\mathbf{T}^2 \times \{0\})$. If $T \subset V$ is an embedded surface, we say T is incompressible if the inclusion $i: T \subset V$ induces a monomorphism $i_*: \pi_1(T) \rightarrow \pi_1(V)$. We denote by $M(T)$ the 3-manifold with boundary obtained by cutting V along T . Notice that $M(T_0)$ is diffeomorphic to $\mathbf{T}^2 \times I$; when there is no fear of confusion, we shall identify these two manifolds and call the components of the boundary of $M(T_0)$, T_0 and T_1 . We note $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ and if $p \in \mathbf{R}^2$, $[p]$ denotes the coset of p in $\mathbf{T}^2$. Let $* = p([0, 0], 0)$ be the base point in V ; we write $\pi_1(V)$ and $\pi_1(T_0)$ to mean $\pi_1(V, *)$ and $\pi_1(T_0, *)$ respectively. Let

$$\mu(t) = p([0, 0], t) \quad \text{for } t \in I,$$

and define ε to be the homotopy class of μ in $\pi_1(V)$. Let a and b be a basis of $\pi_1(T_0)$. Then $\pi_1(V)$ is the free group on a, b and c with the relations:

$$\begin{aligned} ab &= ba \\ cac^{-1} &= F_*(a) \\ cbc^{-1} &= F_*(b). \end{aligned}$$

1. In this section we shall study the manner in which the compact orbits of φ are embedded in V . We prove that $M(T) = \mathbf{T}^2 \times I$ for any compact orbit T , and if F has an eigenvalue equal to -1 , then there exist compact orbits and they are isotopic to T_0 .

(1.1) *Let T be a compact orbit of φ . Then T does not separate V and T is incompressible.*

Proof. — First we remark the foliation $\mathcal{F}(\varphi)$ contains no Reeb components, i.e. invariant submanifolds homeomorphic to $\mathbf{D}^2 \times \mathbf{S}^1$ such that $\partial(\mathbf{D}^2 \times \mathbf{S}^1)$ is a leaf; this is proved in [3]. Also, it is known that if $\mathcal{F}$ is a transversally oriented foliation of a closed 3-manifold W which contains no Reeb components, then each leaf of $\mathcal{F}$ is incompressible [5]. Therefore, if T is a compact orbit of φ , T is incompressible.

Now suppose that T does separate V ; let W be one of the connected components of $V - T$; W is a closed 3-manifold and φ acts on W so that $\partial W = T$ is an orbit. If there are no compact orbits of φ in $\text{Int } W$ then the proof of theorem (5.3) of [5] shows that all the orbits of φ in $\text{Int } W$ are $\mathbf{R}^2$. But then W is diffeomorphic to $\mathbf{D}^2 \times \mathbf{S}^1$ by theorem 1 of [5], which is impossible since an action has no Reeb components. Thus there exist compact orbits of φ in $\text{Int } W$. By lemma (5.3) of [7], there exist K compact orbits of φ in $\text{Int } W$, $T_1, \dots, T_K$, such that $A = \bigcup_{i=1}^K T_i$ does not separate W but for every other compact orbit T' of φ , $T' \cup A$ does separate W . We remark that in order to apply (5.3), one must know that not every compact orbit of φ in $\text{Int } W$ separates W . This is indeed the case (cf. remark at end of the proof of theorem 3 of [5]). Let W_1 be the manifold obtained by cutting W along $T_1, \dots, T_K$; W_1 has $2K+1$ tori in its

boundary, each an orbit of φ , and every other compact orbit of φ in W_1 separates W_1 . But it is proved in [7] (page 462) that a compact orientable 3-manifold with non empty boundary, that admits a non singular action of $\mathbf{R}^2$ such that every compact orbit in the interior separates, is necessarily $\mathbf{T}^2 \times \mathbf{I}$. Thus $W_1 \approx \mathbf{T}^2 \times \mathbf{I}$ which contradicts the fact that W_1 has an odd number of boundary components. Therefore no compact orbit of the action φ on V can separate V .

(1.2) *Let T be a torus embedded in V which is incompressible and does not separate V . Then $M(T) \approx \mathbf{T}^2 \times \mathbf{I}$.*

Before proving (1.2), we need:

Lemma (1.3). — Let T be a torus embedded in $\text{Int}(\mathbf{T}^2 \times \mathbf{I})$ such that T is incompressible and separates $\mathbf{T}^2 \times \mathbf{I}$ into two components A and B such that $\mathbf{T}^2 \times \{0\} \subset A$ and $\mathbf{T}^2 \times \{1\} \subset B$. Then $A \approx \mathbf{T}^2 \times \mathbf{I}$ and $B \approx \mathbf{T}^2 \times \mathbf{I}$ (in fact, T is necessarily incompressible if the other hypotheses are satisfied).

Proof. — Let $\mathcal{F}$ be a Reeb foliation of $\mathbf{T}^2 \times \mathbf{I}$, i.e. a C^2 -foliation such that each leaf of $\mathcal{F}$ in $\text{Int}(\mathbf{T}^2 \times \mathbf{I})$ is $\mathbf{R}^2$ and the boundary components of $\mathbf{T}^2 \times \mathbf{I}$ are leaves [cf. 5]. Since T is incompressible, T is isotopic to a torus $T' \subset \text{Int}(\mathbf{T}^2 \times \mathbf{I})$ such that T' is transverse to $\mathcal{F}$ and the foliation of T' defined by the intersection of the leaves of $\mathcal{F}$ with T' is an irrational flow (Theorem (1.1) of [6]). Therefore we can assume T is transverse to $\mathcal{F}$ and $\mathcal{F} \cap T$ is an irrational flow. Let T_0 be a torus embedded in $\text{int } A$ such that $T_0 + (\mathbf{T}^2 \times \{0\})$ bound a product cobordism in A and T_0 is transverse to $\mathcal{F}$ with $\mathcal{F} \cap T_0$ an irrational flow. Such a torus T_0 is constructed in exemple 3 of [5]. Let A_0 be the manifold with boundary $T_0 + T$; clearly $A_0 \cong A$. Now each leaf of $\mathcal{F}$ in the interior of A_0 is homeomorphic to $\mathbf{R}^2$ since every closed submanifold of $\mathbf{R}^2$ diffeomorphic to $\mathbf{R}$ separates $\mathbf{R}^2$ into two components, each homeomorphic to $\mathbf{R}^2$. Now the proof of theorem (3.5) of [5] shows that $A_0 \approx \mathbf{T}^2 \times \mathbf{I}$, hence A as well. Clearly the same reasoning applies to B .

Proof of (1.2). — Let $T \subset V$ be an incompressible torus which does not separate V . Suppose that $T \subset \text{Int } M(T_0)$. Clearly T then separates $M(T_0)$ into two connected components A and B , each of which contains one of the boundary components of $M(T_0)$. Thus A and B are both homeomorphic to $\mathbf{T}^2 \times \mathbf{I}$ by lemma (1.3). Since $M(T)$ is obtained by glueing one end of A to an end of B , it follows easily that $M(T) \approx \mathbf{T}^2 \times \mathbf{I}$.

In general we proceed by putting T into general position with respect to T_0 and mimic the argument which proves that a simple closed curve C on $\mathbf{T}^2$ which is incompressible in $\mathbf{T}^2$ has the property that $M(C) \approx \mathbf{S}^1 \times \mathbf{I}$.

To be precise, let T intersect T_0 transversally so that $T \cap T_0 = \emptyset$ or $T \cap T_0$ is a 1-manifold. We have just considered the case $T \cap T_0 = \emptyset$, therefore we may assume

$$T \cap T_0 = C_1 \cup \dots \cup C_n,$$

where each $C_i \approx S^1$ and $C_i \cap C_j = \emptyset$ if $i \neq j$. First we modify T by an isotopy, to remove those C_i which are null homotopic. Suppose C_i is null homotopic on T_0 . Then $C_i = \partial D_i$ where $D_i \subset T_0$ and $D_i \approx D^2$. By choosing C_i minimal, we can suppose $\text{Int } D_i$ contains no C_j , for $j=1, \dots, n$. Since $C_i \subset T$ and T is incompressible we know that C_i is null homotopic on T . Let $D \subset T$ satisfy $\partial D = C_i$ and $D \approx D^2$. Then $S = D \cup D_i$ is a 2-sphere embedded in V which is smooth except along the corner C_i . Since V is covered by $\mathbf{R}^3$, V is irreducible (cf. [4]), therefore S bounds a ball $B \subset V$. Now by an isotopy of D to D_i across the ball B , one removes the intersection curve C_i from $T \cap T_0$; this isotopy is described in detail in [10].

Thus we can assume $T \cap T_0 = C_1 \cup \dots \cup C_n$, where each C_i is a generator of $\pi_1(T)$ and $\pi_1(T_0)$. Two simple closed curves on a torus, which are disjoint and not null homotopic, separate the torus into two cylinders which have the curves as their common boundary. Therefore, we can label the C_i so that, for each i , C_i and C_{i+1} bound a cylinder A_i on T , whose interior contains no C_j . Choose a simple closed curve b on T which meets each C_i in exactly one point x_i . We fix an orientation of b and an orientation of the normal bundle of $T_0 \subset V$, and to each x_i we associate a $+$ or $-$ depending on whether the orientation of b at x_i coincides with the orientation of the normal bundle of T_0 at x_i .

Now suppose x_i and x_{i+1} have opposite signs. Then A_i can be considered as a cylinder embedded in $M(T_0) \approx \mathbf{T}^2 \times I$, which intersects $\partial(\mathbf{T}^2 \times I)$ in $C_i + C_{i+1}$, both of which are contained in $\mathbf{T}^2 \times \{0\}$. Let B_1, B_2 be the cylinders in $\mathbf{T}^2 \times \{0\}$, satisfying $\partial B_1 = \partial B_2 = C_i + C_{i+1}$, $B_1 \cap B_2 = C_i + C_{i+1}$. One of the B_i , B_1 say, has the property that $A_i \cup B_1$ bounds a solid torus in $\mathbf{T}^2 \times I$ and is isotopic to B_1 across this solid torus, relative to $C_i + C_{i+1}$. This is proved explicitly in [10], or one can apply theorem (5.5) of [9]. Using this isotopy one removes C_i and C_{i+1} from $T \cap T_0$. Therefore we may suppose all the x_i have the same sign, and each A_i can be considered as embedded in $\mathbf{T}^2 \times I$, having one boundary in $\mathbf{T}^2 \times \{0\}$ and the other in $\mathbf{T}^2 \times \{1\}$. Here we regard $\mathbf{T}^2 \times \{0\}$ and $\mathbf{T}^2 \times \{1\}$ as the two boundary components of a tubular neighborhood of T_0 in V .

Let $a_1, \dots, a_n$ denote the circles of intersection of T with $\mathbf{T}^2 \times \{0\}$, labelled so that $a_i \cup a_{i+1}$ bound a cylinder E_i on $\mathbf{T}^2 \times \{0\}$ whose interior is disjoint from each a_j , and $a_{n+1} = a_1$. Similarly, let $b_1, \dots, b_n$ be the circles of $T \cap (\mathbf{T}^2 \times \{1\})$, labelled so that $a_i + b_i$ bound a cylinder A_i on T such that $\text{Int } A_i \subset \mathbf{T}^2 \times (0, 1)$. Let H_i be the cylinder on $\mathbf{T}^2 \times \{1\}$ with boundary $b_i + b_{i+1}$ whose interior contains no b_j .

Now $E_i \cup H_i \cup A_i \cup A_{i+1}$ separates V into two connected components; let $M(i)$ be the component whose interior is disjoint from T_0 . It is not hard to see that $M(i)$ is homeomorphic to $S^1 \times I \times I$ by a map sending $S^1 \times I \times \{0\}$ to E_i and $S^1 \times I \times \{1\}$ to H_i . This can be proved directly (e.g. by using the theory of Reeb foliations) or one can apply [9].

Now V is the quotient space of $\mathbf{T}^2 \times I$ where $(x, 1)$ is identified with $(F(x), 0)$, for each $x \in \mathbf{T}^2$. T is embedded in V , therefore for each i there exists $\psi(i) \in I$ such that H_i is identified with $E_{\psi(i)}$ (via F).

Now suppose $n=1$, so that $\psi(1)=1$. Then $M(T)$ is the quotient space of $\mathbf{S}^1 \times I \times I$ where $(\theta, t, 1)$ is identified with $(F_1(\theta, t), 0)$, for each $(\theta, t) \in \mathbf{S}^1 \times I$;

$$F_1 : \mathbf{S}^1 \times I \rightarrow \mathbf{S}^1 \times I,$$

the diffeomorphism induced by F . Since T has a trivial normal bundle in V , $\partial M(T)$ has two connected components; therefore $F_1(\mathbf{S}^1 \times 0) = \mathbf{S}^1 \times 0$ and $F_1(\mathbf{S}^1 \times 1) = \mathbf{S}^1 \times 1$. V is orientable so F_1 is orientation preserving. Thus F_1 is homotopic to the identity map $\mathbf{S}^1 \times I \rightarrow \mathbf{S}^1 \times I$, therefore, F_1 is isotopic to the identity map. Hence

$$M(T) \approx \mathbf{S}^1 \times I \times \mathbf{S}^1 \approx \mathbf{T}^2 \times I.$$

Now suppose $n > 1$. Then $\psi(1) \neq 1$, since if $\psi(1) = 1$, $M(T)$ would have two connected components, contradicting the hypothesis that T does not separate V . Then $M(1) \bigcup_F M(\psi(1))$ is homeomorphic to $\mathbf{S}^1 \times I \times I$ since it is obtained from

$$(\mathbf{S}^1 \times I \times I) + (\mathbf{S}^1 \times I \times I)$$

where a point $(x, 1)$ in the first factor is identified with $(F(x), 0)$ in the second factor, for $x \in \mathbf{S}^1 \times I$. We observe that the numbers $1, \psi(1), \psi^2(1), \dots, \psi^{n-1}(1)$, are distinct and $\psi^n(1) = 1$, since T does not separate V . Therefore

$$M(1) \bigcup_F M(\psi(1)) \bigcup_F \dots \bigcup_F M(\psi^{n-1}(1))$$

is homeomorphic to $\mathbf{S}^1 \times I \times I$ and $M(T)$ is homeomorphic to the quotient space of $\mathbf{S}^1 \times I \times I$ where a point $(x, 1)$ is identified with $(h(x), 0)$, for $x \in \mathbf{S}^1 \times I$; $h : \mathbf{S}^1 \times I \rightarrow \mathbf{S}^1 \times I$ a diffeomorphism. Just as in the case $n=1$, we have $h(\mathbf{S}^1 \times 0) = \mathbf{S}^1 \times 0$ and $h(\mathbf{S}^1 \times 1) = \mathbf{S}^1 \times 1$ since $\partial M(T)$ has two components. Also h preserves orientation since $M(T)$ is orientable, therefore h is isotopic to the identity map and $M(T) \approx \mathbf{T}^2 \times I$.

(1.4) Let T be an incompressible torus in V which does not separate V . If F has no eigenvalue equal to $+1$ or -1 , then T is isotopic to T_0 .

Proof. — Suppose T is not isotopic to T_0 . As in the proof of (1.2), we put T into general position with respect to T_0 . Clearly T is not disjoint from T_0 , since we proved in (1.3) that this implies T is isotopic to T_0 . As before, we remove all the circles of intersection from $T \cap T_0$ which are null homotopic, and then we remove the circles C_i and C_{i+1} which have opposite sign. Thus $T \cap (\mathbf{T}^2 \times \{0\}) = a_1 \cup \dots \cup a_n$ and $T \cap (\mathbf{T}^2 \times \{1\}) = b_1 \cup \dots \cup b_n$ where a_i and b_i bound a cylinder A_i on T whose interior is contained in $\text{Int } M(T_0)$. By construction, we have $F(b_1) = a_j$ for some j , $1 \leq j \leq n$.

The cylinder A_1 in $\mathbf{T}^2 \times I$ is isotopic to $a_1 \times I$ in $\mathbf{T}^2 \times I$; one can prove this using Reeb foliation theory or [9]. Therefore, on $\mathbf{T}^2$, a_1 is isotopic to b_1 and since a_j is isotopic to a_1 we have a_1 isotopic to $F(a_1)$. Let C be a (linear) simple closed curve through the base point $(0, 0)$ of $\mathbf{T}^2$ which is isotopic to a_1 . We have $F(C)$ isotopic to C . Let $f : \mathbf{T}^2 \rightarrow \mathbf{T}^2$ be a diffeomorphism such that $f(F(C)) = C$, $f(0, 0) = (0, 0)$, and f isotopic to the identity. Then $(f \circ F)_* = F_*$ and $(f \circ F)_*[C] = \pm[C]$ where $[C]$ denotes the homotopy class of C in $\pi_1(\mathbf{T}^2)$. Therefore F_* has an eigenvalue equal to $+1$ or -1 .

(1.5) If F has an eigenvalue equal to -1 and T is an incompressible torus in V which does not separate V , then T is isotopic to T_0 .

Proof. — Suppose, on the contrary, that T is not isotopic to T_0 . As in (1.4), we put T into general position with respect to T_0 so that $T \cap T_0 = a_1 \cup \dots \cup a_n$. Let a be the homotopy class of a_1 in $\pi_1(T_0)$ and choose $b \in \pi_1(T_0)$ so that a and b form a basis of $\pi_1(T_0)$. Let c be the third generator of $\pi_1(V)$ as defined in the introduction. We know $\pi_1(V)$ is the group generated by a , b and c with the relations:

$$\begin{aligned} ab &= ba \\ cac^{-1} &= a^{-1} \\ cb c^{-1} &= a^K b^{-1}. \end{aligned}$$

This follows from the fact that $F_*(a) = a^{\pm 1}$ and since $\det F = +1$ both eigenvalues of F must be -1 ; therefore $F_*(a) = a^{-1}$. Choose a basis of $\pi_1(T)$ of the form a , $b^m c^\gamma$. We know that $M(T) \approx \mathbf{T}^2 \times \mathbf{I}$ by (1.2), so T is a fibre of a fibration of V over $\mathbf{S}^1$. Hence $\pi_1(T)$ is an invariant subgroup of $\pi_1(V)$ with quotient $\mathbf{Z}$.

First we remark that γ is even since a and $b^m c^\gamma$ commute. Next observe that $b^{2m} \in \pi_1(T)$, since $\pi_1(T)$ is invariant, for:

$$\begin{aligned} cb^m c^\gamma c^{-1} &\in \pi_1(T), \\ cb^m c^\gamma c^{-1} &= a^{mk} b^{-m} c^\gamma \quad \text{hence} \quad b^{-m} c^\gamma \in \pi_1(T), \\ b^{2m} &= b^m c^\gamma (b^{-m} c^\gamma)^{-1}. \end{aligned}$$

Also $c^{2\gamma} \in \pi_1(T)$:

$$(b^m c^\gamma)(b^{-m} c^\gamma) \in \pi_1(T)$$

$b^m c^\gamma b^{-m} c^\gamma = a^{Km} c^{2\gamma}$ since γ is even.

Now a , b^{2m} and $c^{2\gamma}$ belong to $\pi_1(T)$. We know that $\pi_1(V)/\pi_1(T)$ is isomorphic to $\mathbf{Z}$. The case $\gamma \neq 0$, $m \neq 0$ is therefore impossible. If $\gamma \neq 0$ and $m = 0$ then $\gamma = 1$ which is impossible (a and c do not commute).

The only remaining possibility is $\gamma = 0$ and $m = 1$, hence $\pi_1(T)$ is generated by (a, b) and T is isotopic to T_0 .

(1.6) Suppose F has an eigenvalue equal to -1 and φ is a non singular action of $\mathbf{R}^2$ on V . Then φ has a compact orbit, and all the compact orbits are isotopic to T_0 .

Proof. — Assume, on the contrary, that φ has no compact orbits. Then by theorem 9 of [8], all the orbits of φ are cylinders and each orbit is dense in V ; the orbits cannot all be planes since this would imply $V \approx \mathbf{T}^3$. Let X and Y be commuting, linearly independent vector fields on V which are tangent to the orbits of φ and such that all the orbits of Y are closed, of the same period [7]. Let C be a Y -orbit and L the φ -orbit which contains C . Let A be a cylinder transverse to $\mathcal{F}(\varphi)$ which is the union of Y -orbits and such that $C \subset \text{Int } A$ [cf. 7]. It is proved in [7] that $(L - C) \cap A \neq \emptyset$.

Let D be a first circle of return of $L \cap A$; i.e. $D \subset L \cap A$ and $D+C$ bound a cylinder $E \subset L$ such that $(\text{Int } E) \cap A = \emptyset$. Let B be the cylinder on A bounded by $C+D$. Then the topological torus $E \cup B$ can be smoothed in a neighborhood of A to obtain a torus T which is an orbit of a non singular $\mathbf{R}^2$ action φ_1 on V (theorem (3.1) of [7]). By (1.1) and (1.5), we know that T is isotopic to T_0 . Now T is isotopic to a torus T' such that X is transverse to T' and Y is tangent to T' . This is a slight modification of the construction of lemma (4.3) of [7]; lemma (4.3) gives a T' isotopic to T such that X is transverse to T' . To ensure that Y is tangent to T' , we define T' to be the $M(\theta_0)$ of lemma (4.3), saturated by the orbits of Y , union the annulus in $A(C)$ bounded by $(\mathbf{S}^1 \times I \times \{0\}) + (\mathbf{S}^1 \times I \times \{1\})$ (cf. (4.3) of [7]). Thus we can suppose X is transverse to T_0 and Y is tangent to T_0 .

Now consider the torus T which is a smoothing of $E \cup B$, where $C \subset T_0$ is a Y -orbit and E and B are the cylinders defined above. Each orbit of φ in $M(T_0)$ is a cylinder with one boundary in $\mathbf{T}^2 \times \{0\}$ and the other in $\mathbf{T}^2 \times \{1\}$. Therefore $\pi_1(T)$ contains an element of the form $b^m c^\gamma$ where $\gamma =$ the number of circles in $E \cap T_0$, and $\gamma > 0$. Consequently $\pi_1(T) \neq \pi_1(T_0)$. But T is an orbit of a non singular $\mathbf{R}^2$ action φ_1 on V , so by (1.1) and (1.5), T is isotopic to T_0 . This is a contradiction, therefore φ has at least one compact orbit.

Proof of Theorem 2. — Suppose φ is an action of $\mathbf{R}^2$ on V with all the orbits cylinders. In the proof of (1.6), we showed that φ can be approximated by an $\mathbf{R}^2$ action φ_1 such that φ_1 has a compact orbit T and T is not isotopic to T_0 . By (1.4), we know that F has an eigenvalue equal to $+1$ or -1 . Since the eigenvalues of F are of the same sign, we know from (1.6) that both eigenvalues of F are $+1$. Therefore, if F has no eigenvalue equal to $+1$, every $\mathbf{R}^2$ action on V has at least one compact orbit.

Now consider the action φ with all orbits cylinders. After composing φ with a diffeomorphism of V we may assume φ is transverse to T_0 and the orbits of φ in $M(T_0)$ are homeomorphic to $\mathbf{S}^1 \times I$, with one component of the boundary in T_0 and the other in T_1 (see the proof of (1.6)). Let $\mathcal{F}_0$ be the foliation of $M(T_0) \cong \mathbf{T}^2 \times I$ induced by the orbits of φ . The foliation $\mathcal{F}_0$ has no holonomy since $\mathcal{F}_0 \cap (\mathbf{T}^2 \times \{0\})$ is topologically equivalent to the foliation of $\mathbf{T}^2$ given by $\mathbf{S}^1 \times \{\theta\}$, $\theta \in \mathbf{S}^1$. Thus, by the Reeb Stability theorem, $\mathcal{F}_0$ is topologically equivalent to the foliation $\mathbf{S}^1 \times \{\theta\} \times I$, $\theta \in \mathbf{S}^1$, of $\mathbf{T}^2 \times I$. Clearly V is then homeomorphic to $(\mathbf{T}^2 \times I)/H$ where $H: \mathbf{T}^2 \rightarrow \mathbf{T}^2$ is a diffeomorphism which leaves the foliation $\mathbf{S}^1 \times \{0\}$, of $\mathbf{T}^2$ invariant. The manifold $(\mathbf{T}^2 \times I)/H$ is foliated by the cylinders $p(\mathbf{S}^1 \times \{0\} \times I)$ where $p: \mathbf{T}^2 \times I \rightarrow (\mathbf{T}^2 \times I)/H$ is the projection. Thus, the foliation of V defined by φ is topologically equivalent to this suspension.

2. The models.

In this section we shall explain theorem 3. We start with a non singular action φ of $\mathbf{R}^2$ on V which has a compact orbit T . We know that cutting V along T we obtain $\mathbf{T}^2 \times I$; therefore we shall classify the foliations of $\mathbf{T}^2 \times I$ induced by actions tangent

to the boundary. We denote by $\mathcal{F}$ the foliation of $\mathbf{T}^2 \times \mathbf{I}$ induced by φ . The classification is analogous to the classification of foliations of $\mathbf{S}^1 \times \mathbf{I}$ which are tangent to the boundary: each compact leaf is a circle isotopic to $\mathbf{S}^1 \times \{0\}$, and the complement of the set of compact leaves is the union of a countable family of open sets W_i with $\overline{W_i} \cong \mathbf{S}^1 \times \mathbf{I}$ and the foliation of $\overline{W_i}$ is of type 0 or 1 of figure 1.

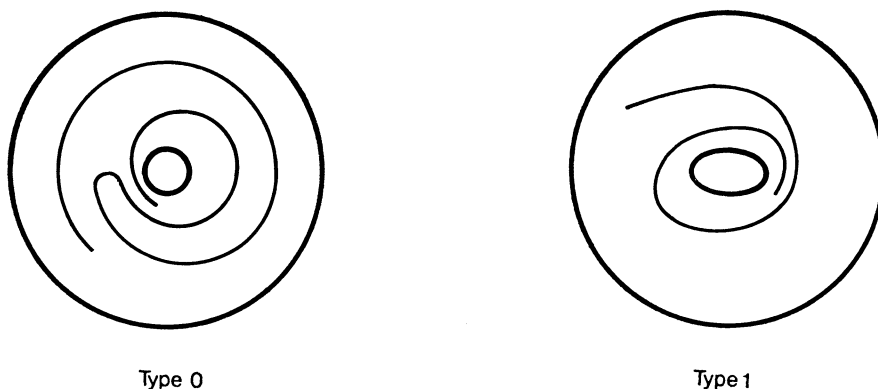


FIG. 1

(2.1) *Definition of $\mathcal{F}(\alpha, 0)$ and $\mathcal{F}(C, 0)$.*

Let X, Y and Z be the vector fields on $\mathbf{R}^2 \times \mathbf{I}$;

$$X = (\cos \pi x, 0, \sin 2\pi x(1-x))$$

$$Y = (1, \alpha, 0)$$

$$Z = (0, 1, 0),$$

(the foliation of figure 1, type 0, are the orbits of X), where $0 \leq x \leq 1$ and α is irrational. These vector fields are linearly independent and pairwise commute. Moreover the fields are invariant by the translations $(x_1, x_2) \mapsto (x_1 + 1, x_2)$ and $(x_1, x_2) \mapsto (x_1, x_2 + 1)$. Therefore (X, Y) and (X, Z) induce actions of $\mathbf{R}^2$ on $\mathbf{T}^2 \times \mathbf{I}$. It is easy to check that $\mathbf{T}^2 \times \{0\}$ and $\mathbf{T}^2 \times \{1\}$ are the compact orbits of these actions; the other orbits of the (X, Y) action are planes and the other orbits of the (X, Z) action are cylinders. We denote the corresponding foliations by $\mathcal{F}(\alpha, 0)$ and $\mathcal{F}(C, 0)$ respectively. Notice that no transversal arc joins $\mathbf{T}^2 \times \{0\}$ to $\mathbf{T}^2 \times \{1\}$ for these foliations.

(2.2) *Definition of $\mathcal{F}(\chi)$.*

Let $\mathcal{G}$ be the group of diffeomorphisms of the interval $[0, 1]$ which leave 0 and 1 fixed. Let χ be a representation of $\pi_1(\mathbf{T}^2)$ in $\mathcal{G}$. We associate an action of $\mathbf{R}^2$ to χ as follows. Let $f, g \in \mathcal{G}$ be the images of the standard basis of $\mathbf{T}^2$ by χ . Then $\mathbf{T}^2 \times \mathbf{I}$ is diffeomorphic to the quotient of $\mathbf{I} \times \mathbf{I} \times \mathbf{I}$ where $(x, 0, \lambda) \sim (x, 1, g(\lambda))$ and $(0, y, \lambda) \sim (1, y, f(\lambda))$. Since f and g commute, the vector fields $(1, 0, 0)$ and $(0, 1, 0)$ on $\mathbf{I}^3$ project to commuting vector fields X and Y on $\mathbf{T}^2 \times \mathbf{I}$. We denote the foliation

induced by this $\mathbf{R}^2$ -action on $\mathbf{T}^2 \times \mathbf{I}$ by $\mathcal{F}(\chi)$. The holonomy of this foliation on $\mathbf{T}^2 \times \{0\}$ is precisely χ . $\mathcal{F}(\chi)$ is transverse to the segments $\{\Theta\} \times \{\Theta'\} \times \mathbf{I}$ and can have compact leaves in $\text{int } \mathbf{T}^2 \times \mathbf{I}$. One can consider $\mathcal{F}(\chi)$ is the foliation canonically associated to the fibration $(\mathbf{T}^2 \times \mathbf{I}, \mathbf{I}, \mathbf{T}^2, \mathcal{G})$, $\mathbf{I}$ the fibre, $\mathbf{T}^2$ the base and $\mathcal{G}$ with the discrete topology [6]. Two such foliations $\mathcal{F}(\chi_1)$ and $\mathcal{F}(\chi_2)$ are equivalent if and only if χ_1 is conjugate to χ_2 .

(2.3) *Definition of $\mathcal{F}((1, i_1), (2, i_2), \dots, (n, i_n))$.*

This is a foliation of $\mathbf{T}^2 \times \mathbf{I}$ obtained by gluing together the preceding models (for each K , $1 \leq K \leq n$, we have $i_K = 0$ or 1). For $i_K = 1$, and $\chi_K : \pi_1(\mathbf{T}^2) \rightarrow \mathcal{G}$ a homomorphism, we define $\mathcal{F}(K, i_K) = \mathcal{F}(\chi_K)$, the foliation defined in (2.2). For $i_K = 0$, we define $\mathcal{F}(K, i_K)$ to be $\mathcal{F}(\alpha, 0)$ or $\mathcal{F}(C, 0)$, the foliations defined in (2.1). Then $\mathcal{F}((1, i_1), \dots, (n, i_n))$ is the foliation of $\mathbf{T}^2 \times \mathbf{I}$ obtained by gluing the leaf $\mathbf{T}^2 \times \{1\}$ of $\mathcal{F}(K, i_K)$ to the leaf $\mathbf{T}^2 \times \{0\}$ of $\mathcal{F}(K+1, i_{K+1})$, for each K , $1 \leq K \leq n-1$. Notice that for $i_K = 0$, no transversal of the foliation $\mathcal{F}(K, i_K)$ goes from $\mathbf{T}^2 \times \{0\}$ to $\mathbf{T}^2 \times \{1\}$; whereas, for $i_K = 1$, the segments $\{(\Theta, \Theta')\} \times \mathbf{I}$ are transversal to $\mathcal{F}(K, i_K)$.

Theorem 3. — *Let φ be a non singular action of $\mathbf{R}^2$ on $\mathbf{T}^2 \times \mathbf{I}$, with $\mathbf{T}^2 \times \{0\}$ and $\mathbf{T}^2 \times \{1\}$ orbits of φ . Then $\mathcal{F}(\varphi)$ is equivalent to $\mathcal{F}((1, i_1), \dots, (n, i_n))$, for some choice of (K, i_K) , $1 \leq K \leq n$.*

The proof will be proceeded by several lemmas.

(2.4) (Nancy Kopell [2]). *Let f and g be germs of commuting C^2 -diffeomorphisms of $\mathbf{R}^+ = \{x \geq 0\}$, such that $f(0) = g(0) = 0$. If f is a contraction (i.e. $f(x) < x$ for $x > 0$), and $g \neq \text{id}$ then 0 is the only fixed point of g .*

(2.5) *Let φ be a non singular action of $\mathbf{R}^2$ on $\mathbf{T}^2 \times \mathbf{I}$ such that $\mathbf{T}^2 \times \{0\}$ and $\mathbf{T}^2 \times \{1\}$ are the only compact orbits. There exist embedded tori T' and T'' satisfying:*

- a) T' and T'' can be chosen transverse to $\mathcal{F}(\varphi)$.
- b) T' is isotopic to $\mathbf{T}^2 \times \{0\}$ and can be chosen inside any tubular neighborhood of $\mathbf{T}^2 \times \{0\}$; in particular, one can suppose the segments $\{(\Theta, \Theta')\} \times \mathbf{I}$ are transverse to $\mathcal{F}(\varphi)$ inside the region $\mathcal{U}'$ bounded by $\mathbf{T}^2 \times \{0\}$ and T' . The same property holds for T'' , $\mathbf{T}^2 \times \{1\}$ and $\mathcal{U}''$.
- c) If L is an orbit of φ , then $L \cap T'$ (resp. $L \cap T''$) is a circle if $L \cong \mathbf{S}^1 \times \mathbf{R}$ and is the union of copies of $\mathbf{R}$ if $L \cong \mathbf{R}^2$.
- d) There exists a vector field Y on $\mathbf{T}^2 \times (0, 1)$, tangent to the (open) φ orbits, such that $Y(T', (-\infty, 0)) \subset \mathcal{U}'$, $Y(T'', (0, \infty)) \subset \mathcal{U}''$, and $Y(T', 1) = T''$ (hence the foliations of T' and T'' , induced by $\mathcal{F}(\varphi)$, are conjugate by the orbits of Y). By $Y(x, t)$ we mean the integral curve of the vector field Y at time t , which passes by x at $t = 0$.

Proof of (2.5). — If φ has a cylindrical orbit then (2.5) follows from (4.3), (4.5) and (4.6) of [7]. If all open φ orbits are planes, then (2.5) follows from the classification of Reeb foliations of $\mathbf{T}^2 \times I$ given in [1].

Corollary (2.6). — If φ is an action of $\mathbf{R}^2$ on $\mathbf{T}^2 \times I$ such that $\mathbf{T}^2 \times \{0\}$ and $\mathbf{T}^2 \times \{1\}$ are the only compact leaves, then the open leaves are planes or cylinders but there is no mixture of the two types.

Proof. — This follows from (2.4) and (2.5) where (2.4) is applied to the germs obtained by the representation $\pi_1(\mathbf{T}^2 \times \{0\}) \rightarrow g$, given by the holonomy of the foliation $\mathcal{F}(\varphi)$. Since there are no compact leaves in a neighborhood of $\mathbf{T}^2 \times \{0\}$ (other than $\mathbf{T}^2 \times \{0\}$), the generators of $\pi_1(\mathbf{T}^2 \times \{0\})$ can be chosen so that the associated germs are contractions or the identity and a contraction.

Proof of theorem 3. — Now consider the foliation $\mathcal{F} = \mathcal{F}(\varphi)$ of $\mathbf{T}^2 \times I$, tangent to the boundary. We know each compact orbit of $\mathcal{F}$ is isotopic to $\mathbf{T}^2 \times \{0\}$. Let K be the union of the set of compact orbits. We have $\overline{(\mathbf{T}^2 \times I) - K} = \bigcup_{i=1}^{\infty} W_i$ where each $W_i \cong \mathbf{T}^2 \times I$, W_i is invariant by φ and the open leaves of W_i are all planes or cylinders. We fix once and for all an orientation of $\mathcal{F}$. Let $W_1^0, \dots, W_r^0$ denote those W_i such that the orientations induced on the boundary of W_i are opposite, i.e. if on one component of ∂W_i , the normal field points to the interior of W_i (respectively the exterior) the normal field points to the interior (the exterior) on the other component. By continuity, there are at most a finite number of such W_i . Let $C_1, \dots, C_s$ be the connected components of the closure of the complement of $W_1^0 \cup \dots \cup W_r^0$ in $\mathbf{T}^2 \times I$. Let $p_K^{i_K}$ be a family of embeddings of $\mathbf{T}^2 \times I$ into $\mathbf{T}^2 \times I$, $1 \leq K \leq n$ satisfying:

- 1) if $i_K = 0$, $p_K^{i_K}(\mathbf{T}^2 \times I)$ is some W_j^0 , for $1 \leq j \leq r$;
- 2) if $i_K = 1$, $p_K^{i_K}(\mathbf{T}^2 \times I)$ is some C_j , for $1 \leq j \leq s$, and
- 3) $p_K^{i_K}(\mathbf{T}^2 \times \{0\}) = \mathbf{T}^2 \times \{0\}$,
 $p_K^{i_K}(\mathbf{T}^2 \times \{1\}) = p_{K+1}^{i_K+1}(\mathbf{T}^2 \times \{0\})$ for $1 \leq K \leq n-1$;
 $p_n^{i_n}(\mathbf{T}^2 \times \{1\}) = \mathbf{T}^2 \times \{1\}$.

We have sketched a cross section of this indexation in figure

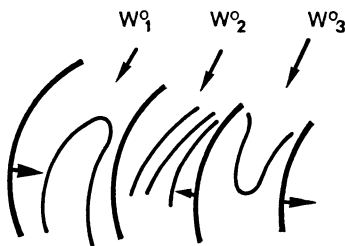


FIG. 2

We shall first construct the conjugation on the C_j and then on the W_K^0 ; the C_j are conjugate to the models of type $\mathcal{F}(\chi)$ for some representation χ ; and the W_K^0 to the models of type $\mathcal{F}(\alpha, 0)$ or $\mathcal{F}(G, 0)$.

(2.7) Let C_j be one of the manifolds defined above and denote by N the normal vector field to $\mathcal{F}$. Let K be the integer such that $p_K^1(\mathbf{T}^2 \times I) = C_j$. There exists a vector field X_j on C_j which is transverse to $\mathcal{F}$ satisfying:

- 1) $X_j = N$ on the compact orbits of C_j , and
- 2) each orbit of X_j starting at a point of $p_K^1(\mathbf{T}^2 \times \{0\})$ goes to a point of $p_K^1(\mathbf{T}^2 \times \{1\})$.

Proof of (2.7). — We may suppose N points into C_j on $p_K^1(\mathbf{T}^2 \times \{0\})$. As before, we write the complement of the compact leaves in C_j as $\bigcup_{n=1}^{\infty} W_{j,n}$ where the $W_{j,n}$ are diffeomorphic to $\mathbf{T}^2 \times I$, invariant by φ , and φ has no compact orbits in the interior of $W_{j,n}$.

We construct a vector field $X_{j,n}$ in each $W_{j,n}$ which is equal to N in a neighborhood of $\partial W_{j,n}$ as follows. Let T' and T'' be transverse tori embedded in $\text{Int } W_{j,n}$ given by (2.5), and denote by Y the vector field given by (2.5). The foliations of T' and T'' induced by $\mathcal{F}$ are conjugate by the orbits of Y and this foliation is equivalent to an irrational flow on $\mathbf{T}^2$ or the product foliation $\mathbf{S}^1 \times \{\Theta\}$ of $\mathbf{T}^2$. Now T' and T'' bound a submanifold W of $W_{j,n}$ such that the foliation of W induced by $\mathcal{F}$ is equivalent to the product of the induced foliation on T' by I ; the orbits of Y define the conjugation. Thus in W we can construct a vector field X_0 , transverse to $\mathcal{F}$ such that X_0 points into W on T' and each orbit of X_0 starting at a point of T' goes to a point of T'' . Since each orbit of N starting at $\partial W_{j,n}$ intersects T' or T'' , we can extend X_0 to $W_{j,n}$ to coincide with N in a neighborhood of $\partial W_{j,n}$ and to be transverse to $\mathcal{F}$. Denote this extension by $X_{j,n}$. Now we define X_j on C_j to equal $X_{j,n}$ on $W_{j,n}$ and N on the compact orbits of $\mathcal{F}$. Each orbit of X_j starting at a point of $p_K^1(\mathbf{T}^2 \times \{0\})$ goes to a point of $p_K^1(\mathbf{T}^2 \times \{1\})$; after reparametrizing the orbits of X_j we can assume the orbits take a time 1 to go from one boundary component of C_j to the other. This completes the proof of (2.7).

(2.8) The foliation $\mathcal{F}$ on C_j is equivalent to a foliation $\mathcal{F}(\chi)$ of $\mathbf{T}^2 \times I$, for some representation χ .

Proof. — By identifying the orbits of X_j to a point we define a fibration $C_j \rightarrow \mathbf{T}^2$ with fibre I and $\mathcal{F}$ is transverse to the fibres. Such foliations are determined by a representation $\chi : \pi_1(\mathbf{T}^2) \rightarrow \mathcal{G}$. The conjugation $H_j : C_j \rightarrow (\mathbf{T}^2 \times I, \mathcal{F}(\chi))$ can be constructed so that $H_j \circ p_K^1 = \text{identity}$ on $\partial(\mathbf{T}^2 \times I)$ (see [1]).

(2.9) The foliation $\mathcal{F}$ on W_K^0 , for K between 1 and r , is equivalent to a foliation $\mathcal{F}(\alpha, 0)$ or $\mathcal{F}(G, 0)$.

Proof. — If all the leaves of $\mathcal{F}$ in the interior of W_K^0 are planes, then we have proved in [1] that $\mathcal{F}$ is equivalent to a foliation $\mathcal{F}(\alpha, 0)$ for some irrational α . We construct

in [1] a conjugation $H_K^0 : (W_K^0, \mathcal{F}) \rightarrow (\mathbf{T}^2 \times I, \mathcal{F}(\alpha, 0))$ such that $H_K^0 p_K^0 = \text{identity}$ on $\partial(\mathbf{T}^2 \times I)$.

Now suppose the leaves of $\mathcal{F}$ in $\text{Int } W_K^0$ are cylinders. This case is much easier to deal with than the planar case because of the existence of the vector field Y given by (2.5). Let T' and T'' be the transverse tori given by (2.5). Between T' and T'' in W_K^0 we have a manifold W and the foliation $\mathcal{F}$ on W is equivalent to the foliation $\mathbf{S}^1 \times \{\emptyset\} \times I$ of $\mathbf{T}^2 \times I$; the equivalence is defined using the orbits of Y . Let A and B be the closure of the connected components of $W_K^0 - W$. The conjugation H_K^0 is defined in $A \cup B$ by the holonomy of the compact leaves, i.e., the boundary components of W_K^0 . We do this precisely in [1]; H_K^0 is defined so that $H_K^0 p_K^0 = \text{identity}$ on $\partial(\mathbf{T}^2 \times I)$. Now this gives H_K^0 on $A \cup W$ and B . The construction above might give two different values for H_K^0 on T'' (for, on $A \cup W_K^0$ its value is determined as soon as it is determined on $p_K^0(\mathbf{T}^2 \times (0))$ and, on B , it is determined by its value on $p_K^0(\mathbf{T}^2 \times (1))$).

Let H' and H'' be the restrictions of H_K^0 on T'' resulting from the two different definitions. Then $H = H'^{-1}H''$ is homotopic and hence isotopic to the identity and sends the leaves of the induced foliation $\mathcal{F} \cap T''$ onto themselves. Let then F be the diffeomorphism from T' onto T'' associated with the orbits of Y . It is clear that Y may be modified into a field Y' (tangent to the leaves) in such a way that $F' = HF$ (F' obviously means the diffeomorphism associated with the orbits of Y'). Extension of H_K^0 using the orbits of Y' gives them the same value for the definitions of H_K^0 on $A \cup W$ and B .

Now piecing together the conjugations H_j of (2.8) and H_K^0 of (2.9), theorem 3 is proved.

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