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MANIFOLDS WHICH ADMIT $\mathbf{R}^n$ ACTIONS

by G. CHATELET and H. ROSENBERG

INTRODUCTION

The purpose of this paper is to determine which n -manifolds admit smooth locally free actions of $\mathbf{R}^{n-1}$. We shall restrict ourselves to compact connected orientable manifolds V^n and locally free actions φ of $\mathbf{R}^{n-1}$ on V^n which are of class C^2 and tangent to ∂V^n , i.e. the components of ∂V^n are orbits of φ . For $n=3$, we know that V^3 admits such an $\mathbf{R}^2$ action if and only if $V^3 = \mathbf{T}^2 \times \mathbf{I}$ or V^3 is a bundle over $\mathbf{S}^1$ with fibre $\mathbf{T}^2$ [7]. Moreover, the topological type of such $\mathbf{R}^2$ actions has been completely determined [8]. We recall that the rank of V^n is the largest integer k such that V^n admits a smooth locally free action of $\mathbf{R}^k$.

Now suppose that φ is a locally free action of $\mathbf{R}^{n-1}$ on V^n . We shall prove:

Theorem 1. — If ∂V^n is not empty, then V^n is homeomorphic to $\mathbf{T}^{n-1} \times \mathbf{I}$ (here $\mathbf{T}^i$ denotes the torus of dimension i).

Theorem 2. — If ∂V^n is empty and φ has at least one compact orbit, then V^n is a bundle over $\mathbf{S}^1$ with fibre $\mathbf{T}^{n-1}$.

Theorem 3. — If ∂V^n is empty and φ has no compact orbits then V^n is a bundle over a torus $\mathbf{T}^k$ with fibre a torus $\mathbf{T}^{n-k}$.

Theorem 2 follows directly from Theorem 1 by cutting V^n along a compact orbit. Theorem 3 depends upon an observation of Novikov [4], and independently Joubert: suppose φ acts on V^n with no compact orbits. By Sacksteder [9], all the orbits of φ are $\mathbf{T}^{n-k} \times \mathbf{R}^{k-1}$ for some k . Choose linearly independent vector fields $X_1, \dots, X_{n-1}$ tangent to the orbits of φ such that all the integral curves of $X_1, \dots, X_{n-k}$ are periodic, of period one. Then $X_1, \dots, X_{n-k}$ define a locally free action of $\mathbf{T}^{n-k}$ on V^n and the orbit space M is a smooth manifold of dimension k . Also M admits an action of $\mathbf{R}^{k-1}$ with all the orbits $\mathbf{R}^{k-1}$. It follows that M is homeomorphic to $\mathbf{T}^k$, which proves Theorem 3 ([5] and [3]). Consequently, our main result is Theorem 1. Here is how we proceed to prove Theorem 1: by inductive arguments similar to those used in [7], we restrict ourselves to actions φ with no compact orbits in the interior of V^n . We then

remark that the foliation defined by the orbits of φ is almost without holonomy, i.e. the noncompact leaves have no holonomy. With this, we construct collar neighborhoods U_i of each component T_i of ∂V , such that $\partial U_i = T_i \cup T'_i$ with T'_i transverse to the foliation. We construct U_i so that some linear field Y (tangent to the orbits of φ) is transverse to each T'_i . We then prove the integral curves of Y go from T'_i to T'_j hence define a homeomorphism of V^n to $\mathbf{T}^{n-1} \times I$.

1. Some Preliminaries.

(1.1) *Let $\mathcal{F}$ be the foliation of V defined by the orbits of φ . Then each noncompact leaf of $\mathcal{F}$ has zero holonomy.*

Proof. — If T is a compact leaf of $\mathcal{F}$, then the germ of $\mathcal{F}$ in a neighborhood of T is without holonomy outside of T , provided T is an isolated compact leaf (page 13 of [8]). This is also true if T is an isolated compact leaf on one side in V and one considers the germ of $\mathcal{F}$ on this side. Now if φ has no compact orbits then $\mathcal{F}$ is without holonomy and we are done [9]. So suppose F is a noncompact leaf of $\mathcal{F}$ and $\mathcal{F}$ has compact leaves. Since $\mathcal{F}$ has no exceptional minimal sets [9], there is a compact leaf T of $\mathcal{F}$ such that T is in the closure of F . Let x be a point of F and $\alpha(x)$ a non zero element of $\pi_1(F, x)$. Let X be a vector field on V such that the integral curve of X through x is closed and homotopic to $\alpha(x)$, and all the integral curves of X on F are closed. X is easily constructed using the action φ (cf. [6]). Since T is in the closure of F , we know the integral curves of X on T are also closed. Now T is an isolated compact leaf at least on one side in V , the side where F intersects a transverse arc infinitely often. Let U be a neighborhood of T , on this side, such that all the leaves of $\mathcal{F}$ in U , except T , have zero holonomy. Then U contains closed integral curves of X which are on F , so such an integral curve C has zero holonomy. Since C is conjugate to $\alpha(x)$, it follows that $\alpha(x)$ has zero holonomy; thus F as well.

(1.2) *Suppose ∂V is not empty and φ has no compact orbits in the interior of V . Let T be a compact orbit of φ ; $T \subset \partial V$. The leaves which contain T in their closure are homeomorphic to $\mathbf{T}^k \times \mathbf{R}^{n-k-1}$ where k = the rank of the kernel of the holonomy map on T .*

Proof. — Let F be an open leaf whose closure contains T ; $F \approx \mathbf{T}^j \times \mathbf{R}^{n-j-1}$. Suppose Z^k is the kernel of the holonomy homomorphism on T . Let $\mathbf{T}^k$ be a k -torus embedded in T which lifts onto nearby leaves by the holonomy. Since $\bar{F} \supset T$, we can lift $\mathbf{T}^k$ to a k -torus T_1 in F . Also $i_\# : \pi_1(T) \rightarrow \pi_1(V)$ is injective, where $i : T \hookrightarrow V$ (cf. [4]), hence $\pi_1(T_1)$ embeds in $\pi_1(F)$ and $k \leq j$.

Next we show $j \leq k$. Let $x \in F$ and $\alpha \in \pi_1(F, x)$, $\alpha \neq 0$. Let X be a vector field tangent to the orbits of φ , such that the integral curves of X on F are closed and the integral curve of X through x is homotopic to α . Since $\bar{F} \supset T$, all the integral curves

of X on T are closed. Let C be an integral curve of X on T . We know that C lifts to a closed curve on F , so by (1.1), the holonomy of C is trivial; i.e. C is in the kernel of the holonomy homomorphism. Hence $j \leq k$.

2. The transverse torus and vector field.

Throughout this section, we suppose φ acts on V so that there are no compact orbits in the interior of V and T is a compact orbit in ∂V . Let k denote the rank of the kernel of the holonomy map associated to T ; k varies between 0 and $n-2$. Let $Y_1, \dots, Y_{n-1}$ be linearly independent commuting vector fields on V satisfying:

- (i) they are tangent to the φ -orbits;
- (ii) their integral curves are closed and of period one on T ; and
- (iii) the integral curves of $Y_1, \dots, Y_k$ represent the kernel of the holonomy map on T .

We shall construct an $(n-1)$ -torus $T' \subset \text{Int } V$ such that $T \cup T'$ bound a trivial cobordism in V , and Y_{n-1} is transverse to T' at each point.

By (1.1), we know the orbits of $Y_{k+1}, \dots, Y_{n-1}$ on T induce germs in $\text{Diffo}(\mathbf{R}^+)$ which are contractions or expansions, via the holonomy. Here $\text{Diffo}(\mathbf{R}^+)$ is the set of C^2 -germs of diffeomorphisms of $\mathbf{R}^+$ to itself, which leave 0 fixed. After reversing the sign of Y_j if necessary, we shall assume the germs are all contractions, for $k+1 \leq j \leq n-1$.

Choose a metric on V and let U_ε be a geodesic collar neighborhood of T isometric to $\mathbf{T}^{n-1} \times [0, \varepsilon]$, with the obvious product metric. Clearly, if ε is small enough, the geodesics normal to T in U_ε will be transverse to the orbits of φ . Let f_x^i be the holonomy diffeomorphism associated to the Y_i orbit through x ; f_x^i is the identity for $1 \leq i \leq k$ and a contraction for $k < i < n$.

Proposition (2.1). — *There is an $(n-1)$ -torus T' contained in U_ε such that Y_{n-1} is transverse to T' and $T \cup T'$ bound a trivial cobordism.*

In an earlier version of this paper we gave a proof of (2.1) which used calculus. Charles Pugh pointed out to us how one can use a theorem of W. Wilson on the existence of Liapounov functions for uniform stable attractors of vector fields [13]. We present this proof here and in an appendix we give our original proof.

We need some definitions before stating Wilson's theorem. Let X be a vector field on V and let A be a closed invariant subset of V (here V is a compact manifold). A is called a *uniform stable attractor* of X if the following conditions are satisfied:

- a) there exists an increasing function δ sending $\mathbf{R}^+$ into itself such that

$$d(X(p, t), A) < \varepsilon$$

whenever $d(p, A) < \delta(\varepsilon)$ and $t \geq 0$;

b) there exists a neighborhood U of A such that $\omega(p) \subset A$ whenever $p \in U$ ($\omega(p)$ is the ω -limit set of p);

c) let $D(A)$ be the set of p such that $\omega(p) \subset A$; $D(A)$ is an open set, called the *basin of attraction* of A .

Wilson has proved [13] that if A is a uniform stable attractor for X then there exists a C^∞ Liapounov function, i.e.

a) there is a C^∞ function $f: D(A) \rightarrow \mathbf{R}^+$ with $f^{-1}(0) = A$; and

b) $X(f)(p) < 0$ whenever $f(p) \neq 0$.

Hence f has no singularities outside A and all the level surfaces of f are diffeomorphic. Before proving (2.1) we need three lemmas.

Lemma (2.2). (*Action box lemma.*)

There exists a unique mapping

$$F: J^{n-1} \times [0, \varepsilon] \rightarrow U_\varepsilon \subset V$$

(where $J = [-1, 2]$) satisfying the following conditions:

- a) F is a C^2 -immersion;
- b) F sends the horizontal plaques $J^{n-1} \times \{z\}$ into the leaves of $\mathcal{F}$;
- c) F sends vertical arcs $\{\Lambda\} \times [0, \varepsilon]$ onto the geodesic arcs normal to T ;
- d) F , when restricted to $J^{n-1} \times \{0\}$, is the restriction of the natural covering map: $\mathbf{R}^{n-1} \rightarrow T$ induced by φ , which sends the i -direction line onto the Y_i circular orbit;
- e) let $x_0 \in T$; then F sends $\{0\} \times [0, \varepsilon]$ isometrically onto the geodesic arcs issued from X_0 , normal to T and pointing inside T .

Proof. — Define first F via e) and d). F obviously extends to $J^{n-1} \times [0, \varepsilon]$ using b) and c).

a) is clear, for geodesic arcs are normal to $\mathcal{F}$ in U_ε . Note that each Y_i orbit on T is covered three times by F .

Lemma (2.3). (*Commuting contraction lemma.*)

If f_1 and f_2 are commuting embeddings $[0, \varepsilon] \rightarrow [0, \infty[$ and f_2 is a contraction towards 0, then there exists a K so large that $f_1 f_2^K$ is a contraction to 0.

Proof. — f_2 commuting with $f_1 f_2^K$, $f_1 f_2^K$ is an embedding without fixed point or is the identity (N. Koppel's Thesis). For sufficiently large k , $f_1 f_2^K$ is not the identity. Hence $f_1 f_2^K$ is a contraction or an expansion. For $f_1 f_2^K[0, \varepsilon] = f_2^K f_1[0, \varepsilon]$, and K may be chosen so large that $f_2^K f_1[0, \varepsilon] \subset \left[0, \frac{\varepsilon}{2}\right]$. $f_1 f_2^K$ is therefore a contraction.

3. Attraction Lemma.

There exists ε and $\delta > 0$ such that whenever $\mathbf{X}$ is a C^1 vector field on $\mathbf{R}^{n-1}$ and $|\mathbf{X}|_0 < \delta$, then $Y = \Phi_* \left(\frac{\partial}{\partial \lambda_{n-1}} + \mathbf{X} \right)$ generates a flow having T as a uniform and stable attractor, U_ε being in the basin of attraction of T .

— Look at the application F of Lemma (2.2) (action box lemma). If

$$Y = \Phi_* \left(\frac{\partial}{\partial \lambda_{n-1}} + \mathbf{X} \right),$$

F^*Y is a C^1 vector field defined on $J^{n-1} \times [0, \varepsilon]$ (F is a C^2 -immersion); F^*Y has no vertical component and may be chosen arbitrarily close to $\frac{\partial}{\partial \lambda_{n-1}}$ for a suitable choice of δ .

Let $I = [0, 1]$, $A_0 = I^{n-2} \times \{0\} \times [0, \varepsilon]$, $A_1 = I^{n-2} \times \{1\} \times [0, \varepsilon]$ and $x \in A_0$. I being interior to J , choose δ such that the positive orbit of F^*Y through x crosses A_1 before reaching the boundary of $J^{n-1} \times [0, \varepsilon]$. Let x be the point of intersection of A_1 with the orbit. Via F , x is identified with a point $x_1 \in A_0$ and hence may be written in the form $(\lambda'_1, \dots, \lambda'_{n-1}, 0, z_1)$ where

$$z_1 = f_{K+1}^{\varepsilon_{K+1}} \circ \dots \circ f_{K+j}^{\varepsilon_{K+j}} \circ \dots \circ f_{n-1}(z) \quad \text{if} \quad x = (\lambda_1, \dots, \lambda_{n-1}, 0, z).$$

Recall that for $1 \leq j \leq n - K + 1$, f_{K+j} are the contracting holonomy diffeomorphisms associated to the circular Y_{K+j} orbits.

Using the contraction commuting lemma, we choose N such that

$$f_{K+1}^{-1} \circ \dots \circ f_{n-2}^{-1} \circ f_{n-1}^N$$

is a contraction. For ε and δ small, we may build a sequence $(x, \bar{x}, x_1, \dots, x_{N-1}, \bar{x}_{N-1}, x_N)$ where the F^*Y orbit through x_i crosses A_1 at $\bar{x}_i$ and $\bar{x}_i$ being identified via F with x_{i+1} in A_0 . So if $x = (\lambda_1, \dots, \lambda_{n-1}, 0, z)$, then $x_N = (\lambda'_1, \dots, \lambda'_{n-1}, 0, h(z))$ where

$$h(z) = \prod_j f_{K+j}^{\varepsilon_{K+j}} \circ f_{n-1}^N(z).$$

Thus we have shown that the vertical coordinate of any Y -orbit tends to 0 in a manner dominated by a fixed contraction $f_{K+1}^{-1} \circ \dots \circ f_{n-1}^N$ as we proceed along the orbit in forward times, i.e. T is a uniformly stable attractor.

Let us prove now Proposition (2.1).

— The choice of the Y_j 's on T allow us to write T as a trivial fibration $\Sigma \times \mathbf{S}_1$ where Σ is a manifold diffeomorphic to $\mathbf{T}^{n-2}$ and transversal to the circular orbits of Y_{n-1} which are the fibers of that fibration. Over these circles, consider the normal geodesic fibers of U_ε . This gives a two dimensional foliation of U_ε by cylinders. Call it $\mathcal{A}$; $\mathcal{A}$ is clearly transversal to $\mathcal{F}$.

Let $Y_{n-1} = X + Y$ where Y is tangent to $\mathcal{A} \cap \mathcal{F}$ and orthogonal to X ; clearly $Y_{n-1}(x) - Y(x) = X(x)$ tends to 0 when $d(x, T)$ tends to 0. Due to the attraction lemma, Y admits T as a uniform stable attractor. Let $V_1 = U_{\frac{\varepsilon}{3}}$, $V_2 = U_{\frac{2\varepsilon}{3}}$ and let β be a bump function such that $\beta = 1$ on V_1 and $\beta = 0$ outside V_2 . Let $Z = \beta Y + (1 - \beta)Y_{n-1}$. It is easy to check that Z admits T as a uniform stable attractor and hence there exists a Liapounov function f for Z . For $\varepsilon > \varepsilon_0 > \frac{2\varepsilon}{3}$, $Z = Y_{n-1}$ and $f^{-1}(\varepsilon_0)$ is transversal to Y_{n-1} . For $\frac{\varepsilon}{3} > \varepsilon_1 > 0$, $f^{-1}(\varepsilon_1)$ is transverse to Y ; $f^{-1}(\varepsilon_1)$ is diffeomorphic to $f^{-1}(\varepsilon_0)$. It remains to prove $f^{-1}(\varepsilon_1)$ is a $(n-1)$ -dimensional torus for $f^{-1}(\varepsilon_0)$ will be then a torus satisfying conditions of (2.1).

Y being transverse to $f^{-1}(\varepsilon_1)$, $f^{-1}(\varepsilon_1)$ is transverse to $\mathcal{A}$. Let $\mathcal{A}_x$ be the leaf of $\mathcal{A}$ through x ; $\mathcal{A}_x \cap f^{-1}(\varepsilon_1)$ is a compact one-dimensional manifold and hence diffeomorphic to a circle. Writing $T = \Sigma \times \mathbf{S}_1$ and $x = (\lambda, s)$ here $\lambda \in \Sigma$ and $s \in \mathbf{S}_1$, one produces a family of embeddings of $\mathbf{S}_1$, $(\pi_\lambda)_{\lambda \in \Sigma}$ such that $\pi_\lambda(\mathbf{S}_1) = \mathcal{A}_x \cap f^{-1}(\varepsilon_1)$. We define now an application $\pi : \Sigma \times \mathbf{S}_1 \rightarrow f^{-1}(\varepsilon_1)$ by $\pi(\lambda, s) = \pi_\lambda(s)$ which is clearly an embedding. Proposition (2.1) is thereby proved for Σ is diffeomorphic to $\mathbf{T}^{n-2}$.

Proof of Theorem 1. — We now assume ∂V is not empty and φ has no compact orbits in the interior of V . Let T, T' , and $Y_1, \dots, Y_{n-1}$ be as in section 2; so that Y_{n-1} is transverse to T' and pointing into V along T' , i.e. Y_{n-1} points out of the tubular neighborhood of T . Let F be an orbit of φ which intersects T' and let L be a connected component of $F \cap T'$.

Lemma (3.1). — $\bigcup_{t \in \mathbf{R}} Y_{n-1}(t, L) = F$.

Proof. — We know F is diffeomorphic to $\mathbf{T}^k \times \mathbf{R}^{n-k-1}$ (in the leaf topology) and we have a covering map $\pi : \mathbf{R}^{n-1} \rightarrow F$ induced by φ . Since $Y_1, \dots, Y_{n-1}$ define the action φ , we can take $\pi^*(Y_{n-1}) = \frac{\partial}{\partial x_{n-1}}$ where $(x_1, \dots, x_{n-1})$ denote the usual coordinates in $\mathbf{R}^{n-1}$. Let X denote $\frac{\partial}{\partial x_{n-1}}$, and let W be a connected component of $\pi^{-1}(L)$. It suffices to prove that each orbit of X starting at a point of the hyperplane $x_{n-1} = 0$, intersects W , since this implies $\bigcup_t X(t, W) = \mathbf{R}^{n-1}$.

Now W is a closed submanifold of $\mathbf{R}^{n-1}$, of codimension one, and X is transverse to W , and makes an angle with W that is strictly bounded away from zero, since Y_{n-1} is transverse to T' . Clearly, the set of points of the hyperplane $x_{n-1} = 0$, whose X orbits intersect W , is an open non empty set Ω . It suffices to show Ω is closed. Let $z \in \overline{\Omega}$, and $z_n \in \Omega$, satisfying: $\lim_{n \rightarrow \infty} z_n = z$ and for each n , there exists $t_n \in \mathbf{R}$, such that $X(t_n, z_n) \in W$. If some subsequence of (t_n) converges to a number t then we have $X(t, z) \in W$; hence we can suppose no subsequence converges. Let (s_n) be a subsequence of (t_n) such

that $|s_n - s_{n+1}| \geq 1$ and $|z_n - z_{n+1}| < \frac{1}{n}$. Let $E(n)$ denote the line segment joining z_n to z_{n+1} and consider $(E(n) \times \mathbf{R}) \cap W$. This is a curve in W with endpoints $X(S_n, z_n)$ and $X(S_{n+1}, z_{n+1})$. There exists a point U_n on this curve where the tangent to the curve is parallel to the cord joining the endpoints. The angle this cord makes with X tends to zero as $n \rightarrow \infty$, which contradicts the fact that the angle between X and W is strictly positive.

Lemma (3.2). — *Let F , W , L , T and T' be as in (3.1). Then there exists a compact orbit T_1 of φ such that $\bar{F} \supset T_1$ and $T_1 \neq T$.*

Proof. — Let $W_0 = W$ and $W_n = X(n, W_0)$ for each positive integer n . By an argument analogous to that of (3.1), one sees that the distances $d(W_k, W_{k+s})$ tend to infinity as $s \rightarrow \infty$. Let $L_0 = L$ and $L_n = Y_{n-1}(n, L_0)$, so that $\lim_{s \rightarrow \infty} d(L_k, L_{k+s}) = \infty$, where the metric is that induced by π . We define $\Omega = \bigcap_n \bar{E}_n$, where E_n is the connected component of $F - L_n$ towards which Y_{n-1} points on L_n . Ω is an intersection of a nested family of compact sets, hence Ω is not empty and compact. We claim Ω is invariant under the φ action: clearly $\Omega = \{y \in V \mid \text{there exists } x_n \in E_n \text{ and } x_n \rightarrow y\}$. Let $F(y)$ be the orbit of φ by $y \in \Omega$ and let $y' \in F(y)$. Let $[y, y']$ denote a path in $F(y)$ joining y to y' and let $[x_n, x'_n]$ be the holonomy lifting of this path to the leaf of x_n . By construction we have $d(x_n, x'_n)$ bounded above by some number ℓ , independent of n . Since

$$d(L_n, L_{n+s}) \rightarrow \infty$$

as $s \rightarrow \infty$, we can choose a subsequence of (x'_n) , call it (y_n) , such that $y_n \in E_n$. Thus $y' \in \Omega$ and Ω is invariant. Thus Ω contains a φ -minimal set, which must be a compact orbit by Sacksteder's theorem. Since Y_{n-1} points away from T , this compact leaf $T_1 \subset \Omega$, is different from T .

(3.3) *Let V^n be of rank $n-1$ and let φ be an action of $\mathbf{R}^{n-1}$ on V such that the only compact orbits of φ are in ∂V and ∂V is not empty. Then V is homeomorphic to $\mathbf{T}^{n-1} \times \mathbf{I}$.*

Proof. — We use the notation of (3.1) and (3.2). From these lemmas, it follows that the open leaves having T in their closure are homeomorphic to the open leaves having T_1 in their closure, i.e. to $\mathbf{T}^k \times \mathbf{R}^{n-k-1}$, where k is the rank of the kernel of the holonomy map of T . Now since all the integral curves of $Y_1, \dots, Y_k$ are closed in F , and $\bar{F} \supset T_1$, we know they are also closed in T_1 ; hence the k -tori in T_1 spanned by the orbits of $Y_1, \dots, Y_k$ represents the kernel of the holonomy map of T_1 . Now the orbits of $Y_{k+1}, \dots, Y_{n-1}$ are not necessarily closed but we can choose vector fields (from lines through the origin in $\mathbf{R}^{n-1}$) $\tilde{Y}_{k+1}, \dots, \tilde{Y}_{n-1}$, such that $Y_1, \dots, Y_k, \tilde{Y}_{k+1}, \dots, \tilde{Y}_{n-1}$ are linearly independent, commute, are tangent to the φ orbits, and all the integral curves of $\tilde{Y}_{k+1}, \dots, \tilde{Y}_{n-1}$ in T_1 are closed. Clearly this can be done so that $\tilde{Y}_{n-1}$ is

C^0 -close to Y_{n-1} . We choose $\tilde{Y}_{n-1}$ so close that $\tilde{Y}_{n-1}$ is also transverse to T' . Now we go through the construction of a torus T'_1 , bounding a collar neighborhood with T_1 , such that $\tilde{Y}_{n-1}$ is transverse to T'_1 ; this is (2.1). Letting Y denote $\tilde{Y}_{n-1}$, we now have a linear vector field Y transverse to both tori T' and T'_1 . We know the set of points A in T' whose Y -integral curve intersects T'_1 is an open non empty set. By the same reasoning, the complement of A in T' is open; hence $A=T'$. Now using the integral curves of Y , it is easy to construct a homeomorphism between V and $\mathbf{T}^{n-1} \times I$.

Proof of Theorem 1. — The proof follows from (3.3), and a reasoning identical to that on page 462 of [7].

Remarks 1. — A basic question remains unanswered: suppose φ is a locally free action of $\mathbf{R}^{n-1}$ on a closed manifold V^n , with no compact orbits. Then we know V^n fibres over a torus with fibre a torus, hence V^n fibres over $\mathbf{S}^1$ with fibre F (this also follows from [10]). Is F homeomorphic to $\mathbf{T}^{n-1}$?

2. Suppose V^n is a closed, orientable, bundle over $\mathbf{S}^1$ with fibre M . Then there exists a diffeomorphism $f: M \rightarrow M$ such that V is obtained from $M \times I$ by identifying points $(f(x), 1)$ with $(x, 0)$ for $x \in M$. We claim that if $f^*: H^1(M, \mathbf{R}) \rightarrow H^1(M, \mathbf{R})$ does not have one as an eigenvalue, then every locally free action of $\mathbf{R}^{n-1}$ on V has a compact orbit. To see this, first observe that f^* does not have 1 as an eigenvalue if and only if $\text{rank } H^1(V, \mathbf{R}) = 1$ [11]. Now suppose $\mathcal{F}$ is any foliation of V of codimension one, class C^2 and with no compact leaves. By [12], we can suppose L is a covering space of M for L a leaf of $\mathcal{F}$. We have an exact sequence of free abelian groups:

$$0 \rightarrow \pi_1(F)/\pi_1(L) \rightarrow \pi_1(V)/\pi_1(L) \rightarrow \frac{\pi_1(V)}{\pi_1(F)} \rightarrow 0.$$

Since $H^1(V, \mathbf{R}) \approx \mathbf{R}$, the last two groups are of rank one. Hence $\pi_1(L) = \pi_1(F)$ and L must be compact.

APPENDIX

Proof of (2.1)

Notation. — If X is a vector field on V , $t \mapsto X(t, x)$ will denote the integral curve of X passing through x at $t=0$. For $A \subset V$, $X(t, A) = \{X(t, x) | x \in A\}$, and

$$X([a, b], A) = \bigcup_{a \leq t \leq b} X(t, A).$$

If $x \in T$, we define $\alpha_i(x) = Y_i([0, 1], x)$ for $i=1, \dots, n-1$, and $T_i(x)$ is the i -torus in T which is the orbit through x of the $\mathbf{R}^i$ -action determined by $Y_1, \dots, Y_i$. If $\bar{x}$ is on the normal arc through $x \in T$ and if the holonomy germs are defined on $\bar{x}$,

then we denote by $\bar{T}_k(\bar{x})$ the lifting of $T_k(x)$ into the leaf of $\bar{x}$, given by the holonomy.

Let N be a unit vector field on V , normal to the orbits of φ and pointing into V along T (with respect to some metric on V). Let $U = N(I, T)$ where $I = [0, 1]$. We may suppose U is a tubular neighborhood of T in which the holonomy liftings of $\alpha_1(x), \dots, \alpha_{n-1}(x)$ are defined, for $x \in T$. Let f_x^i be the holonomy diffeomorphism of $\alpha_{k+i}(x)$; $1 \leq i \leq n-k-1$.

Let $\pi: U \rightarrow T$ be the projection along N orbits. If $x \in T$ and $\bar{x} \in \pi^{-1}(x)$, let $\bar{\alpha}_i(\bar{x})$ denote the holonomy lifting of $\alpha_i(x)$ starting at $\bar{x}$; for $1 \leq i \leq k$, $\bar{\alpha}_i(\bar{x})$ is an embedded circle, and for $i > k$, $\bar{\alpha}_i(\bar{x})$ is diffeomorphic to I . For $x \in T$ and for all $\bar{x} \in \pi^{-1}(x)$, the $\bar{\alpha}_i(\bar{x})$ form a one dimensional foliation of U . Let C_i be a vector field in U , tangent to this foliation, and coinciding with Y_i on T .

We fix a base point $x_0 \in T$ and we let $\alpha_i = \alpha_i(x_0)$, $T_i = T_i(x_0)$, etc., and define $A_i = N(I, T_i)$.

Let $E_\ell(A_j)$ be the vector bundle of exterior products of order ℓ of vectors tangent to A_j . We identify $E_\ell(A_j)$ with $A_j \times \wedge^\ell \mathbf{R}^j$; so sections of $E_\ell(A_j)$ are functions from A_j to $\wedge^\ell \mathbf{R}^j$. We give these sections the canonical norm.

Let f be a function defined in a neighborhood of 0 such that $\lim_{x \rightarrow 0} f(x) = 0$. We write $f = \sigma_1(x)$ if

$$f(x) = ax + x\mathcal{O}(x),$$

with $a \neq 0$ and $\mathcal{O}(x) \rightarrow 0$ when $x \rightarrow 0$. Finally, we let $\beta_{k+j} = Y_{k+j} \wedge \dots \wedge Y_{n-1}$.

Proposition (2.1). — For each j , $1 \leq j \leq n-k-1$, there is a family of tori $G(k+j)$, satisfying:

c_1) there is a neighborhood U_j of T_{k+j} and the $G(k+j)$'s are a foliation of U_j by tori of dimension $k+j$;

c_2) there is a section g_{k+j} of $E_{k+j}(A_{k+j})$ such that $g_{k+j}(x)$ represents the tangent space at x to $G(k+j)(x)$ and

$$(g_{k+j} \wedge \beta_{k+j})_p \neq 0$$

for all $p \in U_j - T_{k+j}$;

c_3) on T , $G_{k+s}(x) = T_{k+s}$.

Remark. — In particular c_2) implies

$$g_{n-1} \wedge Y_{n-1} \neq 0 \quad \text{in} \quad U_{n-1} - T.$$

Hence there exist $(n-1)$ -tori, transverse to Y_{n-1} , as close to T as we wish.

Proof of (2.1). — We proceed by induction on j ; first “cylinders” are constructed in $\mathbf{T}^j \times I$, $k+1 \leq j \leq n-2$, and then these cylinders are closed, to give tori, by the map F_j defined by the holonomy of α_j .

We start by constructing the foliation $G(k+1)$. Let $U_1 = \pi^{-1}(T_{k+1})$, and

let (θ, z, λ) be coordinates for $\mathbf{T}^k \times \mathbf{I} \times \mathbf{J}$ where $\theta = (\theta_1, \dots, \theta_k) \in \mathbf{T}^k$, and $\mathbf{I} = \mathbf{J} = [0, 1]$. Let $F_1 : \mathbf{T}^k \times \mathbf{I} \times \mathbf{J} \rightarrow U_1$ be defined by:

$$\begin{aligned} F_1(o, o, o) &= x_0 \\ F_1(o, z, o) &= N(z, x_0) \end{aligned}$$

$F_1(\theta, z, \lambda)$ is the endpoint of the holonomy lifting of the arc in T given by:

$$(t, Y(\lambda t, F_1(\theta, o, o))),$$

$0 \leq t \leq 1$, starting at $F_1(\theta, z, o) = N(z, F_1(\theta, o, o))$.

Here we have identified $\mathbf{T}^k = \mathbf{R}^k / \mathbf{Z}^k$ with T_k by the linear diffeomorphism $(o, \dots, \theta_i, \dots, o) \mapsto Y_i(\theta_i)(x_0)$.

By definition of F_1 we have:

- $DF_1\left(\frac{\partial}{\partial \theta_j}\right)$ is colinear with C_j for $1 \leq j \leq k$,
- F_1 sends the tori $\mathbf{T}^k \times \{z\} \times \{\lambda\}$ to the holonomy liftings of the tori $T_k(F_1(o, o, \lambda))$ to the point $F_1(o, z, \lambda)$,
- $DF_1\left(\frac{\partial}{\partial \lambda}\right) = C_{k+1} = Y_{k+1}$ on T ,
- F_1 send the segments $\{\theta\} \times \mathbf{I} \times \{\lambda\}$ to the orbits of N starting at $F_1(\theta, o, \lambda)$,
- the segments $\{\theta\} \times \{z\} \times \mathbf{J}$ are sent to $\overline{\alpha_{k+1}(F_1(\theta, z, o))}$,
- F_1 is a local diffeomorphism to U_1 .

From these remarks, it is easy to see that the map $z \mapsto F_1(\theta, z, \lambda)$ (respectively $\lambda \mapsto F_1(\theta, z, \lambda)$) is a reparametrization of the N -orbits (orbits of C_{k+1}). Hence there exist functions φ_1 and ψ_1 , invertible in z and λ such that

$$\begin{aligned} DF_1\left(\frac{\partial}{\partial z}\right) &= \frac{\partial \varphi_1}{\partial z} N \\ DF_1\left(\frac{\partial}{\partial \lambda}\right) &= \frac{\partial \varphi_1}{\partial \lambda} C_{k+1} \end{aligned}$$

(both φ_1 and ψ_1 have strictly positive derivatives on $\mathbf{T}^k \times \mathbf{I} \times \mathbf{J}$).

Now we construct a family of curves, $\gamma_1(\theta, z)$, in $\mathbf{T}^k \times \mathbf{I} \times \mathbf{J}$

$$\gamma_1(\theta, z) : \lambda \mapsto (Z_1(\theta, Z, \lambda), \lambda)$$

satisfying conditions $A)$ and $B)$:

$A)$ For fixed θ, z , $F_1(\gamma_1(\theta, z))$ is a closed curve in U_1 , of class C^1 . For θ and z in a neighborhood of $\mathbf{T}^k \times \{o\}$, the $F_1(\gamma_1(\theta, Z))$ form a one dimensional foliation of a neighborhood of T_{k+1} in A_{k+1} ,

B) Let $\Delta_1(\theta, z) = z - f_x^1(z)$, where $x = F_1(\theta, o, o)$, and let $\Delta_1(z) = \Delta_1(o, z)$. Then we require that:

$$\frac{\partial Z_1}{\partial \lambda} = \sigma_1(\Delta_1)$$

(here Δ_1 is the function $z \mapsto \Delta_1(z)$). The condition B) is not necessary to construct $G(k+1)$; however, it is necessary to insure the transversality relation c_2) when we construct $G(k+j)$, $j > 1$.

Lemma (2.2). — *There exists in $\mathbf{T}^k \times I \times J$, a family of curves $\gamma_1(\theta, z)$, satisfying conditions A) and B).*

Proof of (2.2). — Let γ' be the tangent vector field to the γ_1 curves, with the λ -parametrization. Then condition A) can be written:

$$(1) \quad (DF_1)_a \gamma'_a \wedge (DF_1)_b \gamma'_b = 0$$

where $b = (\theta, f_x^1(z), 1)$, $x = F_1(\theta, o, o)$, and $a = (\theta, z, o)$ (cf. figure 1).

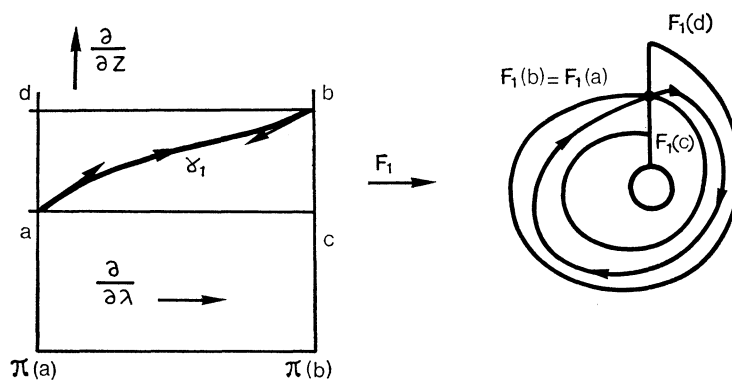


FIG. 1

An easy calculation shows that (1) can be written:

$$\left. \frac{\partial Z_1}{\partial \lambda} \right|_b = K(\theta, z) \left. \frac{\partial Z_1}{\partial \lambda} \right|_a$$

where K is a strictly positive function. Therefore, we can rewrite A) and B) as:

$$\left. \frac{\partial Z_1}{\partial \lambda} \right|_b = K \left. \frac{\partial Z_1}{\partial \lambda} \right|_a$$

$$\frac{\partial Z_1}{\partial \lambda} = \sigma_1(\Delta_1).$$

A tedious, but simple calculation, shows that the cubics (see fig. 1):

$$\lambda \rightarrow \left(\lambda, Z_1 = \Delta_1(\theta, z) \left[\left(\frac{1+K}{1+K_0} - 2 \right) \lambda^3 + \left(3 - \frac{K+2}{K_0+1} \right) \lambda^2 + \frac{\lambda}{1+K_0} \right] + f_x^1(z) \right),$$

satisfy these equations, where

$$K_0 = \sup K(\theta, z), \quad (\theta, z) \in \mathbf{T}^k \times \mathbf{I} \times \{1\}.$$

Now, one can write:

$$\frac{\partial Z_1}{\partial \lambda} = \Delta_1(\theta, z) g(\theta, z, \lambda),$$

where $g > 0$ on $\mathbf{T}^k \times \mathbf{I} \times \mathbf{J}$. Also $\frac{\partial Z_1}{\partial z} > 0$ on $\mathbf{T}^k \times [0, h] \times \mathbf{J}$ for a suitable h , $0 < h \leq 1$.

Hence the curves γ_1 form a foliation of $\mathbf{T}^k \times [0, h] \times \mathbf{J}$. Their image by F_1 is a foliation (of class C^1) of a neighborhood $V_1 \subset U_1$, of T_{k+1} in A_{k+1} . This completes the proof of (2.2) (see fig. 2).

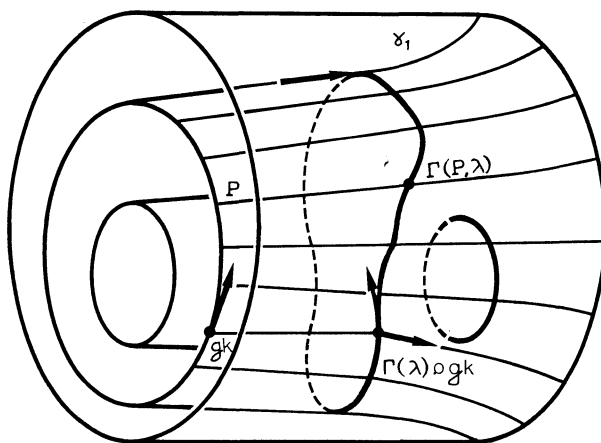


FIG. 2

We can now define $G(k+1)$. The submanifolds:

$$H(z) = \bigcup_{\theta \in \mathbf{T}^k} \gamma_1(\theta, z) \times \{\theta\}$$

are diffeomorphic to $\mathbf{T}^k \times [0, 1]$ and form a foliation of $\mathbf{T}^k \times [0, h] \times \mathbf{J}$. Hence their image by F_1 is a foliation of V_1 by tori $G(k+1)$ (figure 2). We now check conditions c_2) and c_3).

We have a C^2 vector field γ' in $\mathbf{T}^k \times [0, h] \times \mathbf{J}$; γ' is the tangent field to the γ_1 curves with the λ -parametrization. This field induces a natural action of $\mathbf{R}$ on the exterior products of vector fields, which we note by $\Gamma(\lambda)$: $\Gamma(\lambda)$ is the differential of the map induced by γ' of $\mathbf{T}^k \times [0, h] \times \{0\}$ to $\mathbf{T}^k \times [0, h] \times \{\lambda\}$. Then the tangent space to $H(z)$ at the point (θ, z, λ) is given by:

$$\Gamma(\lambda) \left(\frac{\partial}{\partial \theta_1} \wedge \dots \wedge \frac{\partial}{\partial \theta_k} \right)_{(\theta, z, 0)} \wedge \gamma'_{(\theta, z, \lambda)}.$$

Now F_1 sends the tori $\mathbf{T}^k \times \{z\} \times \{0\}$ to the trivial holonomy liftings of the T_k ; hence the tangent space to $G(k+1)$ at $p = F_1(\theta, z, \lambda)$ is given by:

$$(g_{k+1})_p = (DF_1 \circ \Gamma(\lambda) \left(\frac{\partial}{\partial \theta_1} \wedge \dots \wedge \frac{\partial}{\partial \theta_k} \right) \wedge DF_1 \circ \gamma')_{(\theta, z, \lambda)}$$

$$\gamma' = \frac{\partial}{\partial \lambda} + \sigma_1(\Delta_1) \frac{\partial}{\partial Z}.$$

We recall that:

$$DF_1 \left(\frac{\partial}{\partial Z} \right) = \frac{\partial \theta_1}{\partial Z} N, \quad DF_1 \left(\frac{\partial}{\partial \lambda} \right) = \frac{\partial \psi_1}{\partial \lambda} C_{k+1}.$$

Let $\tilde{\sigma}(\Delta_1)$ denote a function such that $\frac{\tilde{\sigma}(\Delta_1)}{\Delta_1}$ tends towards a limit a ; then

$$\tilde{\sigma}(\Delta_1) = a\Delta_1 + \sigma(\Delta_1),$$

(with a not necessarily different from 0 and $\sigma(\Delta_1) = \Delta_1 \varepsilon(\Delta_1)$, $\varepsilon(\Delta_1) \rightarrow 0$ whenever $\Delta_1 \rightarrow 0$). We take the tangent spaces to the trivial holonomy liftings of the T_k , to be given by a section of $E_k(A_k)$, equal to $Y_1 \wedge \dots \wedge Y_k$ on T_k .

Let $C_{k+1}^*(\lambda)$ denote the action induced by C_{k+1} on the vectors tangent to A_k (if Y is tangent to the φ -orbits, then so is $C_{k+1}^*(\lambda)(Y)$). Then we obtain for g_{k+1} :

$$(g_{k+1})_p = (C_{k+1}^*(\lambda) \circ (g_k)_{F_1(\theta, z, 0)} + \tilde{\sigma}(\Delta_1) \Omega_p) \wedge (C_{k+1} + \sigma_1(\Delta_1) N)_p,$$

where $p = F_1(\theta, z, \lambda)$ and Ω is a section of $E_{k+1}(A_{k+1})$ defined on V_1 . We can rewrite this as:

$$C_{k+1} \wedge C_{k+1}^*(\lambda) \circ g_k + \sigma_1(\Delta_1) C_{k+1}^*(\lambda) \circ g_k \wedge N + \sigma(\Delta_1) C_{k+1} \wedge \Omega + \sigma(\Delta_1) N \wedge \Omega.$$

Now $\beta_{k+1} \wedge C_{k+1} \wedge C_{k+1}^*(\lambda) \circ g_k$ is zero, since it is a linear combination of exterior products of n vectors tangent to the φ -orbits. Hence:

$$g_{k+1} \wedge \beta_{k+1} = \sigma_1(\Delta_1) C_{k+1}^*(\lambda) \circ g_k \wedge N \wedge \beta_{k+1} + \sigma(\Delta_1) C_{k+1} \wedge \Omega \wedge \beta_{k+1} + \sigma(\Delta_1) N \wedge \Omega \wedge \beta_{k+1}.$$

We have:

$$C_{k+1}^*(\lambda) \circ g_k \wedge N \wedge \beta_{k+1} = Y_1 \wedge \dots \wedge Y_k \wedge N \wedge Y_{k+1} \wedge \dots \wedge Y_{n-1}$$

on T_{k+1} . Hence for all points p in a neighborhood V_2 of T_{k+1} , we have:

$$|C_{k+1}^*(\lambda) g_k \wedge N \wedge \beta_{k+1}|_p > \alpha > 0.$$

We want to show $(g_{k+1} \wedge \beta_{k+1})_p \neq 0$, for p in a suitable neighborhood of T_{k+1} in A_{k+1} . Dividing by $\sigma_1(\Delta_1)$ ($\neq 0$ if $z \neq 0$):

$$\frac{g_{k+1} \wedge \beta_{k+1}}{\sigma_1(\Delta_1)} = \rho_k + C_{k+1} \wedge \Omega' \wedge \beta_{k+1} + \mathcal{E}(\Delta_1) N \wedge \Omega \wedge \beta_{k+1},$$

where $|\rho_k|_p > \alpha > 0$ in V_2 , and Ω' is a bounded section on $V_2 - T_{k+1}$, $\mathcal{E}(\Delta_1) \rightarrow 0$ as $\Delta_1 \rightarrow 0$. Since $C_{k+1} = Y_{k+1}$ on T_{k+1} , the second term is less than $\alpha/3$ if p is in some

neighborhood V_3 of T_{k+1} . Also $|\mathcal{E}(\Delta_1)| |N \wedge \Omega \wedge \beta_{k+1}| < \alpha/3$ if p is in some neighborhood V_4 of T_{k+1} . Hence for $p \in V_2 \cap V_3 \cap V_4$,

$$\frac{|g_{k+1} \wedge \beta_{k+1}|}{\sigma_1(\Delta_1)} > \alpha/3 > 0,$$

which proves the theorem for $j=1$.

It is useful for the induction to write g_{k+1} in the form:

$$g_{k+1} = \alpha_{k+1} + \sigma(\Delta_1)\Omega''$$

where α_{k+1} is a section of $E_{k+1}(A_{k+1})$, defined in a neighborhood of T_{k+1} and equal to $Y_1 \wedge \dots \wedge Y_k \wedge Y_{k+1}$ on T_{k+1} .

Construction of $G(k+j+1)$. — Let $\Delta_{k+j}(s) = s - f^{k+j}(s)$, where $s \geq 0$ denotes the normal N-coordinate.

Fundamental little lemma:

$$\lim_{s \rightarrow 0} \frac{\Delta_{k+j}}{\Delta_{k+j+1}} \text{ exists.}$$

Following (1), one may find an homeomorphism $H : [0, \varepsilon] \rightarrow [0, \varepsilon']$ such that $H^{-1}f_{K+j}H = \lambda_{K+j}$ and $H^{-1}f_{K+j+1}H = \lambda_{K+j+1}$ where λ_{K+j} and λ_{K+j+1} are the homotheties the ratio of which are λ_{K+j} and λ_{K+j+1} (recall f_{K+j} and f_{K+j+1} are contractions). Define on $[0, \varepsilon]$ a metric δ such that $\delta(x, x') = |H^{-1}(x) - H^{-1}(x')|$ — this metric is topologically equivalent to the classical one — and $\delta(x, 0) \rightarrow 0$ whenever $x \rightarrow 0$. We prove $\frac{\delta(x, f_{K+j}(x))}{\delta(x, f_{K+j+1}(x))}$ has a limit when $x \rightarrow 0$ (with respect to δ); we shall then be over. Then let $f = f_{K+j}$, $g = f_{K+j+1}$.

$$\begin{aligned} \frac{\delta(x, f(x))}{\delta(x, g(x))} &= \frac{|H^{-1}(x) - H^{-1}f(x)|}{|H^{-1}(x) - H^{-1}g(x)|} = \frac{|H^{-1}(x) - \lambda_{K+j} \circ H^{-1}(x)|}{|H^{-1}(x) - \lambda_{K+j+1} \circ H^{-1}(x)|} \\ \rho &= \frac{\delta(x, f(x))}{\delta(x, g(x))} = \frac{1 - \lambda_{K+j}}{1 - \lambda_{K+j+1}} \times H^{-1}(x) \end{aligned}$$

when $x \rightarrow 0$, $\delta(x, 0) \rightarrow 0$ and $\rho \rightarrow \frac{1 - \lambda_{K+j}}{1 - \lambda_{K+j+1}}$.

Our inductive hypothesis asserts the existence of the foliation $G(k+j)$ and a section g_{k+j} satisfying:

$$g_{k+j} = \alpha_{k+j} + \sigma(\Delta_{k+j})\Omega,$$

where α_{k+j} is a section of $E_{k+j}(A_{k+j})$, defined in a neighborhood U of T_{k+j} in A_{k+j} , which is a linear combination of vectors tangent to the φ -orbits, and equal to $Y_1 \wedge \dots \wedge Y_{k+j}$ on T_{k+j} . Ω is a section of $E_{k+j}(A_{k+j})$ defined in U . Henceforth, we work in U .

Since the $G(k+j)$ form a foliation of U transverse to the normals, we can construct, by the holonomy, a map F_{j+1} satisfying:

F_{j+1} sends $\mathbf{T}^{k+j} \times I \times J$ to U and is of maximal rank;

$F_{j+1}(o, z, o) = N(z, x_0)$;

F_{j+1} sends the tori $\mathbf{T}^{k+j} \times \{z\} \times \{o\}$ to the tori $G(k+j)$ passing by $F_{j+1}(o, z, o)$;

restricted to $\mathbf{T}^{k+j} \times \{o\} \times J$, we have:

$$DF_{j+1}\left(\frac{\partial}{\partial \theta_\ell}\right) = Y_\ell, \quad 1 \leq \ell \leq k+j,$$

$$DF_{j+1}\left(\frac{\partial}{\partial \lambda}\right) = Y_{k+j+1};$$

$$DF_{j+1}\left(\frac{\partial}{\partial \theta_1} \wedge \dots \wedge \frac{\partial}{\partial \theta_{k+j}}\right) = g_{k+j} \text{ in } A_{k+j};$$

F_{j+1} is the holonomy lifting, restricted to the plaques $\{\theta\} \times I \times J$, i.e. $(F = F_{j+1})$ $F(\theta, z, \lambda)$ is the endpoint of the holonomy lifting of the path $Y_{k+j+1}([o, \lambda], F(\theta, o, o))$ to the point $F(\theta, z, o)$.

Exactly as the case $j=1$, we have a family of curves $\gamma_{j+1}(\theta, z)$ satisfying the conditions *A*) and *B*), with F_1 replaced by $F = F_{j+1}$. This gives us a foliation of $\mathbf{T}^{k+j} \times I \times J$ by submanifolds diffeomorphic to $\mathbf{T}^{k+j} \times I$, and closing the cylinders by F we obtain a foliation by tori $G(k+j+1)$. We must verify c_2).

Let $g = g_{k+j+1}$, $C = C_{k+j+1}$, $\Delta = \Delta_{k+j+1}$ and $C^* = C_{k+j+1}^*$. Then we have:

$$\begin{aligned} g &= (C + \sigma_1(\Delta)N) \wedge (C^*(\lambda) \circ g_{k+j} + \sigma(\Delta)\Omega') \\ &= C \wedge C^*(\lambda) \circ g_{k+j} + \sigma(\Delta)N \wedge \Omega' + \sigma_1(\Delta)N \wedge C^*(\lambda) \circ g_{k+j} + \sigma(\Delta)C \wedge \Omega'. \end{aligned}$$

Now we write $g_{k+j} = \alpha_{k+j} + \sigma(\Delta_{k+j})\Omega$ (a section defined in A_{k+j}), to obtain (defined in A_{k+j+1}):

$$\begin{aligned} g &= C \wedge C^*(\lambda) \circ \alpha_{k+j} + (C \wedge C^*(\lambda)\Omega) \sigma(\Delta_{k+j}) + \sigma_1(\Delta)N \wedge C^*(\lambda) \circ \alpha_{k+j} \\ &\quad + \sigma(\Delta)\sigma(\Delta_{k+j})\Omega + \sigma(\Delta)C \wedge \Omega' + \sigma(\Delta)N \wedge \Omega''. \end{aligned}$$

Notice that $C \wedge C^*(\lambda) \circ \alpha_{k+j}$ is a linear combination of products of order $k+j+1$ of vectors tangent to the leaves, and on T_{k+j+1} , it equals $Y_{k+j+1} \wedge Y_1 \wedge \dots \wedge Y_{k+j}$. Composing g with β_{k+j+1} :

— the term $C \wedge C^*(\lambda) \alpha_{k+j} \wedge \beta_{k+j+1} = 0$ since it is a multiple of n vectors tangent to the φ orbits;

— $N \wedge C^*(\lambda) \circ \alpha_{k+j} = N \wedge Y_1 \wedge \dots \wedge Y_{k+j}$ on T_{k+j+1} .

Dividing by $\sigma_1(\Delta)$ we obtain:

$$\frac{g \wedge \beta_{k+j+1}}{\sigma_1(\Delta)} = N \wedge C^*(\lambda) \alpha_{k+j} + \frac{\sigma(\Delta_{k+j})}{\sigma_1(\Delta)} C \wedge C^*(\lambda)\Omega + \mathcal{E}(\Delta)\Omega''.$$

Now, as $s \rightarrow 0$, Δ_{k+j}/Δ is bounded hence $\sigma(\Delta_{k+j})/\sigma_1(\Delta)$ is bounded. Ω'' is a bounded section of $E_{k+j+1}(A_{k+j+1})$ and $\mathcal{E}(\Delta) \rightarrow 0$ as $\Delta \rightarrow 0$. Hence, in a small enough neighborhood of T_{k+j+1} , we have $g \wedge \beta_{k+j+1} \neq 0$, which completes the proof of (2.1).

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