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JURII I. MANIN

# Rational surfaces over perfect fields (résumé anglais)

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# RATIONAL SURFACES OVER PERFECT FIELDS

*by* Ju. I. MANIN

## RÉSUMÉ

Let $k$ be a perfect field of arbitrary characteristic. The main object of this paper is to establish some new objects associated with algebraic surfaces F defined over $k$ which are invariants for birational transformations defined over $k$. There are two main applications. The first is that if K is any extension of $k$ of degree 2, then there are infinitely many birationally inequivalent rational surfaces (¹) defined over $k$ which all become birationally equivalent to the plane over K. The second application is to a partial classification of the del Pezzo surfaces for birational equivalence over $k$. For our purposes a del Pezzo surface defined over $k$ is a nonsingular rational surface with a very ample anticanonical system, so the nonsingular cubic surfaces are a special case. As we use the language of schemes (²), we have to prove some classical results in the new framework, notably some results of Enriques [7] on the classification of rational surfaces. In the last section we produce evidence for the conjecture that if the field $k$ is quasi-algebraically closed (in the sense of Lang [11]), then a rational surface defined over $k$ always has a point on it defined over $k$.

We shall now describe the contents of our paper in more detail.

**Section 0. — Preliminaries.**

Subsections 0.1 and 0.2 recall and reformulate theorems about the resolution of the singularities of a surface and about the removal of the points of indeterminacy of a rational map (cf. Abhyankar [1]). In subsection 0.3 a curve on a surface F is defined to be an effective Cartier divisor and also the subscheme belonging to that divisor (cf. M. Artin [3]). If F is defined over $k$, we denote the corresponding surface defined over $\bar{k}$ by $F\otimes\bar{k}$. Lemma 0.3 states that a necessary and sufficient condition for a divisor $\bar{X}$ on $F\otimes\bar{k}$ to arise from a divisor X on F (defined over $k$) is that it should be invariant under

(¹) A surface defined over $k$ is rational if when considered over the algebraic closure $\bar{k}$ of $k$ it becomes birationally equivalent to the plane.

(²) In particular, we use the terms " proper " (собственный) and " regular " (регулярный) instead of the classical " complete " and " nonsingular ", respectively.

the action of every element $s$ of the galois group $G=G(\bar{k}/k)$ (cf. Cartier [5]). Section 0.4 gives the following necessary and sufficient condition for a proper regular $k$-surface (¹) to be $k$-minimal, i.e. such that every birational $k$-morphism $f: F \to F'$, where $F'$ is a regular surface, is an isomorphism:

*Lemma* **(0.4)**. — *A proper regular k-surface* F *is k-minimal if and only if for every irreducible exceptional curve* (²) X *of the first kind on the* $\bar{k}$*-surface* $F\otimes\bar{k}$ *there is an element s of the Galois group* G *such that* $s(X) \neq X$ *and the curve* $X+s(X)$ *is connected.*

The necessity of the condition follows from the Lemma of Mumford [13] about the negative definiteness of the intersection matrix of the irreducible components of the kernel of the corresponding $\bar{k}$-morphism $\bar{f}: F\otimes\bar{k} \to F'\otimes\bar{k}$. Conversely, if there is an exceptional curve X of the first kind which does not meet any of its conjugates $s(X) \neq X$, then a standard argument provides a $k$-surface $F'$ and a $k$-morphism $f: F \to F'$ whose kernel is precisely the union of X with its conjugates (cf. M. Artin [3]). Finally the Lemma of Subsection 0.5 states that the monoidal transformations $F' \to F$ whose centre is a closed point of F correspond precisely to the monoidal transformations $\bar{f}: F'\otimes\bar{k} \to F\otimes\bar{k}$ whose centre is of the type $\bigcup_{1\leqslant i\leqslant n} x_i$, where $x_1, \ldots, x_n \in F\otimes\bar{k}$ is a complete set of closed points conjugate over $k$.

## Section 1. — Enriques' Theorem.

In this section we generalize results of Enriques to surfaces defined over a general perfect field $k$.

*Theorem* **(1.2)**. — *Let* F *be a rational k-surface. Then there exists a proper regular k-surface* $F'$ *quasirationally equivalent* (³) *to* F, *a k-curve* C *and a k-morphism* $f: F' \to C$ *with the following properties:*

a) *The curve* C *is proper, regular, geometrically irreducible and reduced* (⁴), *with arithmetic genus* $p_a(C)=0$.

b) *Let x be the generic point of* C. *Then the generic fibre* $F'_x$ *of f is a proper, geometrically regular, geometrically irreducible* $\mathbf{k}(x)$*-curve, with arithmetic genus* $p_a(F'_x)$ *either* 0 *or* 1.

For, following Serre [19], we show that there is an integer $n \geqslant 0$ such that (1.2.1) holds, where $\omega_F$ is the canonical sheaf. Then a pair of linearly independent sections $s_0, s_1 \in H^0(\omega_F^{\otimes n}\otimes\mathcal{O}(1))$ determine a $k$-map $g: F \to \mathbf{P}^1_k$, which may be supposed to be a $k$-morphism on applying appropriate monoidal transformations to F. The curve C is then taken to be the integral closure of the scheme $\mathbf{P}^1_k$ in the field of functions R(F) of F. Then $p_a(C)=0$ because F is a rational surface. The arguments of Serre [19] show

(¹) I.e. a surface defined over $k$.

(²) I.e. with arithmetic genus $p_a(X)=0$ and self-intersection $(X, X)=-1$.

(³) In characteristic zero this is the same concept as birationally equivalent over $k$. In the general case, two surfaces F and $F'$ are said to be quasirationally equivalent if there is a $k$-surface $F''$ and two radicial dominating $k$-morphisms

$$f: F'' \to F;\ g: F'' \to F'.$$

(⁴) I.e. no nilpotent elements in the structure sheaf.

that (1.2.2) holds. If $F_x$ is reduced, this completes the proof. In the general case one has to invoke a result of Grothendieck (Prop. 4.6.6 on p. 69 of [8], Part IV).

In all that follows we shall consider only birational classes of surfaces for which there is a map onto a curve with the properties enunciated in Theorem 1.2. In characteristic $\neq 0$ this is equivalent to neglecting radicial morphisms.

A $k$-morphism $f$ of the type described gives F the structure of a C-scheme. We shall say that F is C-minimal if every diagram

$$\begin{array}{ccc} F & \xrightarrow{f} & C, \\ {\scriptstyle g}\downarrow & \nearrow_{h} & \\ F' & & \end{array}$$

where $g$, $h$ are $k$-morphisms and $g$ is birational, implies that $g$ is an isomorphism. Over the algebraic closure $\bar{k}$ we define the $\bar{k}$-morphism $\bar{f}=f\otimes\bar{k} : F\otimes\bar{k} \to C\otimes\bar{k}$ in the obvious way. Then the $(C\otimes\bar{k})$-minimality of $F\otimes\bar{k}$ is equivalent to the absence of exceptional curves in the fibres, whereas, by Lemma 0.4, the C-minimality of F is equivalent to the statement that every exceptional curve in a geometrical fibre intersects one of its conjugates.

*Theorem* (**1.5**). — *If the* C-*surface* F *is* C-*minimal and the generic fibre has genus* 1, *then* $F\otimes\bar{k}$ *is* $(C\otimes\bar{k})$-*minimal.*

The proof depends on the

*Lemma* (**1.5**). — *Let* $F\otimes\bar{k}\to F'$ *be some* $(C\otimes\bar{k})$-*birational morphism and for a closed point* $x\in C\otimes\bar{k}$ *suppose that the fibre* $F'_x$ *has the two following properties:*

a) *Each irreducible component* X *has either* $(X, X)<0$ *or* $p_a(X)\neq 0$.

b) *If* $X_i$, $X_j$ *are distinct components of the fibre and also exceptional curves of the first kind, then* $X_i+X_j$ *is not connected.*

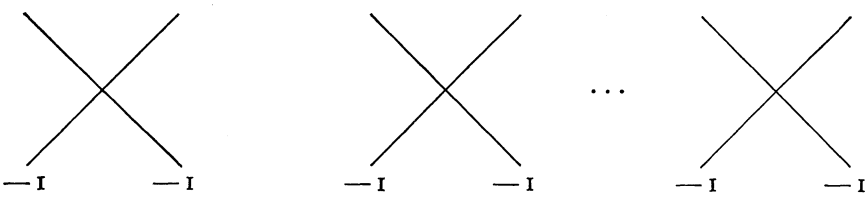
*Then the fibre* $(F\otimes\bar{k})_x$ *enjoys the same properties.*

The proof of Lemma 1.5 follows by induction from the easy case of a monoidal morphism. To prove the Theorem 1.5 we note that if the $\bar{k}$-morphism $F'\to C\otimes\bar{k}$ with generic fibre of genus 1 gives a $(C\otimes\bar{k})$-minimal surface, then the geometric fibres satisfy the conditions of Lemma 1.5 from the classification of the possible fibres given by Kodaira and Néron [15]. By the Lemma, the geometric fibres of $\bar{f}$ must satisfy the same conditions: and on comparing condition *b)* with Lemma 0.4 we deduce that the geometric fibres of $\bar{f}$ cannot contain exceptional curves of the first kind, so $F\otimes\bar{k}$ is $(C\otimes\bar{k})$-minimal, as required.

*Theorem* (**1.6**). — *Suppose that the surface* F *is* C-*minimal and that the genus of the generic fibre is zero. Then there is a* $(C\otimes\bar{k})$-*surface* F' *and a birational* $(C\otimes\bar{k})$-*morphism* $g : F\otimes\bar{k}\to F'$ *such that:*

a) *The structural morphism* $F'\to C\otimes\bar{k}$ *makes* F' *a ruled surface with fibre* $\mathbf{P}^1_{\bar{k}}$.

b) *The morphism* $g$ *is either an isomorphism or a monoidal transformation with its centre at a finite number of closed points of* F' *lying on distinct fibres. The sum of the g-inverse images* *of those fibres on* $F\otimes\bar{k}$ *is* $G(\bar{k}/k)$*-invariant and is a collection of a finite number of orbits each of the form*




(i.e. *a set of pairs of intersecting irreducible curves of genus* 0, *each with self-intersection* $-1$).

For $F'$ is obtained by blowing down the exceptional curves contained in the fibres of $F\otimes\bar{k}\to C\otimes\bar{k}$.

*Theorem* **(1.7)**. — *Let* $F\otimes\bar{k}\to C\otimes\bar{k}$ *be a rational* $(C\otimes\bar{k})$*-surface, where* $C\otimes\bar{k}=\mathbf{P}^1_{\bar{k}}$ *and the generic fibre has genus* 1. *Then the rank* $n(F\otimes\bar{k})$ *of the group* $\mathrm{Num}(F\otimes\bar{k})$ *of classes for numerical equivalence is* 10 *and there is a birational* $\bar{k}$*-morphism* $F\otimes\bar{k}\to\mathbf{P}^2_{\bar{k}}$. *For every irreducible exceptional curve of the first kind* $X\subset F\otimes\bar{k}$ *we have* $(X, F_x)=a$, *where the number* $a$ *is defined by the condition*

$$\omega_{F\otimes\bar{k}}^{\otimes a}=\mathcal{O}_{F\otimes\bar{k}}(-F_x).$$

For any irreducible reduced curve $X\subset F\otimes\bar{k}$ with $p_a(X)=0$ satisfies

$$a((X, X)+2)=(X, F_x),$$

where $x\in C\otimes\bar{k}$ is arbitrary and $a$ is defined above. Hence there are no such curves with $(X, X)<-2$, all the curves with $(X, X)=2$ are components of fibres, and all exceptional curves of the first kind satisfy $(X, F_x)=-a$. Let $g: F\otimes\bar{k}\to F'$ be a birational morphism onto a $\bar{k}$-minimal surface. Then it follows that $F'$ is isomorphic to $\mathbf{P}^2$, $\mathbf{P}^1\otimes\mathbf{P}^1$ or to the ruled surface $F_2$, the standard section of which has self-intersection $-2$. The existence of the $\bar{k}$-morphism $F\otimes\bar{k}\to\mathbf{P}^2_{\bar{k}}$ in the latter two cases follows from a detailed discussion. Finally, Noether's formula implies $(\omega_F, \omega_F)+n(F\otimes\bar{k})=10$, and so $n(F\otimes\bar{k})=10$.

Finally we discuss ruled rational $k$-surfaces. Each of these is the $k$-form of one of the surfaces $F_n$, where $F_n$ in Grothendieck's notation ([8], Chap. II) is $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(n)\oplus\mathcal{O}_{\mathbf{P}^1}(-n))$. For $n\geqslant 1$ there is a canonical section $s_n:\mathbf{P}^1\to F_n$, the image of which is the unique curve on $F_n$ with index of self-intersection $-n$. In the remaining case $n=0$ we have $F_0=\mathbf{P}^1\times\mathbf{P}^1$, and so the forms of $F_0$ are just the 2-dimensional quadrics. Otherwise we have

*Theorem* **(1.10)**. — a) (1) *For* $n\equiv 1 \pmod 2$ *the only* $k$*-form of* $F_n$ *is* $F_n$ *itself.*

b) *For* $n\equiv 0 \pmod 2$, $n\geqslant 2$ *the* $k$*-forms of* $F_n$ *are in* $1-1$ *correspondence with the* $k$*-forms of the projective line* $\mathbf{P}^1$, *the correspondence being between the surface* $F$ *and the curve with self-intersection* $-n$ *lying in it.*

(1) In [12] it was erroneously asserted that the statement in *b)* is true for all $n$.

## Section 2. — Some birational invariants.

For a regular $k$-surface F we shall denote by $N(F)=\mathrm{Num}(F\otimes\bar{k})$ the group of divisor classes on $F\otimes\bar{k}$ for numerical equivalence. It can be regarded as a G-module (where G is the Galois group of $\bar{k}/k$) and has a non-degenerate G-invariant pairing into $\mathbf{Z}$ given by the index of intersection. A $k$-morphism $f: F\to F'$ determines a G-homomorphism

$$f^*: N(F)\to N(F').$$

If F and F′ are proper and $f$ is birational, then $f^*$ is a monomorphism and preserves the index of intersection. In this case we can define the canonical homomorphism

$$f_*: N(F')\to N(F)$$

by $(f_*(\xi),\eta)=(\xi,f^*(\eta))$ for all $\eta\in N(F)$.

We denote by $\mathfrak{B}(k)$ the category whose objects are the proper regular $k$-surfaces and whose morphisms are the birational $k$-morphisms.

We recall the well-known

*Lemma* (**2.1**). — *Let $f: F'\to F$ be a morphism in the category $\mathfrak{B}(k)$. Then there is a direct decomposition*

$$N(F')=\mathrm{Im}\,f^*+\mathrm{Ker}\,f_*$$

*of* G-*modules* (cf., e.g. Nagata [14]). *If, further, $f$ is a monoidal transformation with centre at a point $x\in F$, then* $\mathrm{Ker}\,f_*$ *is generated by the components of the geometric fibre* $f^{-1}(x)\otimes\bar{k}$; *and with respect to this basis the intersection matrix on* $\mathrm{Ker}\,f_*$ *is* $-E$. *Further* $\mathrm{Im}\,f^*$ *and* $\mathrm{Ker}\,f_*$ *are orthogonal with respect to the intersection index.*

We denote by $\mathfrak{C}(k)$ the category of continuous $\mathbf{Z}$-free modules of finite rank. We shall call an element of $\mathfrak{C}(k)$ trivial if it is isomorphic to a direct sum of a finite number of modules of the shape $\mathbf{Z}[G]\otimes_{\mathbf{Z}[H]}\mathbf{Z}$ where H runs through the open subgroups of G and H acts trivially on $\mathbf{Z}$.

We can now enunciate our key

*Theorem* (**2.2**). — *A necessary condition that the $k$-surfaces* F, F′ *in the category* $\mathfrak{B}(k)$ *be birationally equivalent over $k$ is that there exist trivial* G-*modules* M *and* M′ *such that*

$$N(F)+M'\approx N(F')+M.$$

For since a birational equivalence can be decomposed into monoidal transformations it is enough to consider these: and for these Theorem 2.2 is an almost immediate consequence of Lemma 2.1. In what follows we shall consider only the G-module structure of N(F). It would be possible to give finer invariants by considering the index of intersection and the behaviour of the canonical class, but we have not seen how to exploit this. In fact we use only the following

*Corollary* (**2.3**). *A necessary condition that* F, F′ *be birationally equivalent over $k$ is that*

$$H^1(K, N(F))=H^1(K, N(F'))$$

*for all finite extensions* $K\supset k$.

We note in passing that the category $\mathfrak{C}(k)$ is dual to the category of $k$-tori, and so the work of Ono [16] provides further invariants, for example the Tamagawa number of the torus $N(F)^0$ dual to $N(F)$. In fact the zeta-function of the torus $N(F)^0$ and the surface F itself are connected as intimately as the zeta-function of a curve and its Jacobi variety.

As a first application of Theorem 2.2 we prove

*Theorem* (**2.5**). — *Suppose that $k$ possesses a quadratic extension* K. *Then there exist infinitely many rational $k$-surfaces* F *which are birationally inequivalent over $k$, but which all become birationally equivalent to the plane over* K. *In other words the kernel of the map*

$$H^1(k, \mathrm{Cr}) \xrightarrow{\mathrm{res}} H^1(K, \mathrm{Cr})$$

*is infinite, where* $\mathrm{Cr} = \mathrm{Aut}\, \overline{k}(x, y)/\overline{k}$ *is the Cremona group.*

For the proof, we consider the surface

$$x_0 y_0 f_m(z_0, z_1) = x_1 y_1 g_m(z_0, z_1)$$

on the direct product $\mathbf{P}^1_k \times \mathbf{P}^1_k \times \mathbf{P}^1_k$ of 3 projective lines with respective homogeneous coordinates, where $f_m$, $g_m$ are forms of degree $m$ with coefficients in $k$ such that $f_m g_m$ has no multiple factors. Then F is birationally equivalent to the plane (for consider the map onto $\mathbf{P}^1_k \times \mathbf{P}^1_k$ got by throwing away the $y$-coordinates). The surface F has a biregular automorphism of order 2, namely intechangeing $(x_0, x_1)$ and $(y_0, y_1)$. Let $F^h$ be the $k$-surface obtained by twisting F with respect to the cocycle which maps the nonidentical automorphism of $K/k$ into this automorphism of F. It turns out that

$$H^1(k, N(F^h)) = \begin{cases} (\mathbf{Z}_2)^{\varkappa-2} & \text{if } f_m g_m \text{ has a divisor of odd degree} \\ (\mathbf{Z}_2)^{\varkappa-1} & \text{otherwise;} \end{cases}$$

where $\varkappa$ is the number of divisors of $f_m g_m$. In particular, if $k$ has infinitely many elements we may chose $f_m$, $g_m$ which factorize into linear factors over $k$, and then

$$H^1(k, N(F)) \approx (\mathbf{Z}_2)^{2m-2}$$

By Theorem 2.3 the $F^h$ obtained with different values of $m$ are birationally inequivalent. This proves Theorem 2.5 except when $k$ is a finite field, when a slightly more subtle argument is needed.

In evaluating $N(F^h)$ we use the following Lemma, which is perfectly general and also required in Section 3.

*Lemma* (**2.9**). — *Let* $F \in \mathfrak{B}(k)$ *and let* $\{X_i\} \subset F \otimes \overline{k}$ *be a* G*-invariant set of irreducible curves the classes of which generate* N(F). *Let* S *be the group of divisors generated by the* $X_i$ *and let* $S_0 \subset S$ *be the subgroup of divisors numerically equivalent to* 0. *Let* $H \subset G$ *be a normal subgroup of finite index which acts trivially on the* $X_i$ *and put* $N = \sum_{g \in G/H} g \in \mathbf{Z}[G/H]$. *Then*

$$H^1(k, N(F)) \approx H^{-1}(G/H, N(F)) \approx (NS \cap S_0)/NS_0.$$

Here the second isomorphism is a consequence of the exact sequence

$$0 \to S_0 \to S \to N(F) \to 0.$$

The first isomorphism is not a natural one, but follows from the case $p=1$ of the natural isomorphism

$$\operatorname{Hom}(H^{-p}(G/H, N(F)), \mathbf{Q}/\mathbf{Z}) \to H^p(G/H, N(F));$$

which itself follows from the fact that N(F) is **Z**-free and has a nondegenerate pairing into **Z** (cf. [4], Chapter XII, Corollary 6.5).

In the application of Lemma 2.9 to Theorem 2.5 one takes for S the $4m$ lines on which $f_m(z_0, z_1)g_m(z_0, z_1)=0$ together with the two lines $X_0 : x_0=y_1=0$; $X_1 : x_1=y_0=0$.

## Section 3. — Del Pezzo surfaces.

A regular rational $k$-surface F is called a del Pezzo surface if the sheaf $\omega_F^{-1}$ is very ample and the anticanonical system has no fixed components. We first discuss the geometry over an algebraically closed field and require

*Lemma* **(3.2)**. — *Let* F *be a regular rational* $\bar{k}$*-surface for which* $(\omega_F, \omega_F) \geqslant 1$ *and let* X *be an irreducible curve on* F *for which* $(X, X) < 0$. *Then only the three following cases are possible:*

a) X *is an exceptional curve of the first kind and* $(X, \omega_F^{-1})=1$.

b) X *is a component of a fixed curve of the anticanonical system.*

c) $(X, X)=-2$ *and* $p_a(X)=0$, $(X, \omega_F^{-1})=0$.

The proof is a fairly straightforward computation using the Riemann-Roch theorem on F and the formula for $p_a(X)$ in terms of $(X, X)$ and $(X, \omega_F)$. An immediate consequence is

*Corollary* **(3.3)**. — *An irreducible curve with negative self-intersection on a* $\bar{k}$*-del Pezzo surface is an exceptional curve of the first kind. The injection* $F \to \mathbf{P}^n$ *with* $n=(\omega_F, \omega_F)$ *defined by the anticanonical sheaf takes the exceptional curves of the first kind on* F *precisely into the straight lines in the image.*

This gives us at once the geometric form of the del Pezzo surfaces:

*Theorem* **(3.4)**. — *Let* F *be a* $\bar{k}$*-del Pezzo surface and put* $n=(\omega_F, \omega_F)$. *Then* $3 \leqslant n \leqslant 9$ *and we have*

a) *When* $n=9$, *then* F *is isomorphic to* $\mathbf{P}^2$.

b) *When* $n=8$, *then* F *is isomorphic either to* $\mathbf{P}^1 \times \mathbf{P}^1$ *or the image of* $\mathbf{P}^2$ *under a monoidal transformation with centre at a single closed point.*

c) *When* $3 \leqslant n \leqslant 7$, *then* F *is isomorphic to the image of* $\mathbf{P}^2$ *under a monoidal transformation with centre at* $9-n$ *closed points, no three of which lie on a straight line and no six on a conic.*

d) *Every exceptional curve on* F *is the image of either a point of the centre of the monoidal transformation, or of the straight line through two such points, or of the conic through five such points.*

For one considers the birational morphism $p : F \to F'$ onto a minimal model.

The only $F'$ possible are $\mathbf{P}^2$ and $\mathbf{P}^1 \times \mathbf{P}^1$. For $n \leqslant 7$ there is in any case a morphism $F \to \mathbf{P}^2$ by the arguments of Theorem 1.7. The rest follows from a computation of indices, using Noether's formula.

We later need the

*Corollary* (**3.4**). — *The exceptional curves of the first kind on a del Pezzo surface* F *generate* N(F).

We now consider del Pezzo surfaces over a general perfect field $k$. Before considering the different values of $n$ separately we make some general remarks.

Let F be a del Pezzo surface canonically embedded in $\mathbf{P}^n$. On intersecting F with a sufficiently general hyperplane $\mathbf{P}^{n-2}$ and performing a monoidal transformation with centre $F \cap \mathbf{P}^{n-2}$ on F, we obtain a surface $F'$ with a pencil of elliptic curves over the projective line B, the basis of the pencil of hyperplanes in $\mathbf{P}^n$ through the $\mathbf{P}^{n-2}$: and $F'$ is B-minimal. Hence the birational $k$-forms of a del Pezzo surface are particular cases of the $k$-forms of surfaces with an elliptic pencil: they are obtained by blowing down a $k$-curve which splits over $\bar{k}$ into the union of $n$ nonintersecting irreducible curves of the first kind which are also sections of the elliptic pencil.

By Theorem 3.4 all del Pezzo surfaces with fixed $n \geqslant 5$, $n \neq 8$ are $\bar{k}$-isomorphic and for $n=8$ there are only two isomorphism classes. Hence we can classify their biregular $k$-forms. After that a special investigation, different for each case, shows that the existence of a $k$-rational point is equivalent to the birational triviality of the surface. Further, at least for $n \geqslant 6$, the question of the existence of a rational point in the most interesting case, namely when $k$ is an algebraic number field, turns out to be a purely local one: the Hasse principle is valid.

The cases $n=3, 4$ are much more difficult. Now the criterion of Corollary 2.3 permits us to establish the existence of birationally nontrivial $k$-forms, even over a finite field $k$ or for $k=\mathbf{R}$, which possess $k$-rational points. Here we limit ourselves to a detailed exposition of the case $n=4$. The detailed computation of the cohomology in the general case becomes here very cumbersome. It may be that one should use the connection between this class of surface and the Weyl groups of the exceptional simple Lie groups. I hope to return to this question later.

To compute the group $H^1(k, N(F))$ for the application of Corollary 2.3 we must know something about the intersection properties of the exceptional curves. For $3 \leqslant n \leqslant 6$ let $\mathscr{E}_n$ denote the graph whose vertices correspond to the lines on a del Pezzo surface of degree $n$, and where a pair of vertices are joined if the corresponding lines have an intersection. It is easily seen from Theorem 3.4 that $\mathscr{E}_n$ is unique up to isomorphism (1). Let $\Gamma_n$ be the group of automorphisms of $\mathscr{E}_n$.

*Proposition* (**3.6**). — a) *There is a natural embedding* $\mathscr{E}_n \subset \mathscr{E}_{n-1}$ *in which* $\mathscr{E}_n$ *is identified with the set of elements not meeting a fixed element of* $\mathscr{E}_{n-1}$.

(1) Of course $\mathscr{E}_n$ does not exhaust the possible information about the intersections of the lines. For example on a cubic surface it may or may not happen that three lines are concurrent.

b) $\Gamma_n$ *acts transitively on* $\mathcal{E}_n$ *and the stability group of any element is isomorphic to* $\Gamma_{n-1}$.

c) *The orders* $[\mathcal{E}_n]$, $[\Gamma_n]$ *of* $\mathcal{E}_n$, $\Gamma_n$ *respectively are given in the following table:*

| $n$ | 6 | 5 | 4 | 3 |
|---|---|---|---|---|
| $[\mathcal{E}_n]$ | 6 | 10 | 16 | 27 |
| $[\Gamma_n]$ | $2^2\cdot 3$ | $2^3\cdot 3\cdot 5$ | $2^7\cdot 3\cdot 5$ | $2^7\cdot 3^4\cdot 5$ |

All this is classical and straightforward. We give a proof in the Russian version only for lack of a suitable reference.

One may ask whether the full group $\Gamma_n$ can be realized as a Galois group acting on the lines (1). For $n=6, 5, 4, 3$ this follows from the case $n=3$, the non-singular cubic surface in $\mathbf{P}^3$. Let U be the complement on the surface of the discriminantal hypersurface in the space of the coefficients of cubic forms in 4 variables. The group $\pi_1(\mathrm{U}, u)$ clearly acts on the set of lines of the cubic surface corresponding to a point $u \in \mathrm{U}$. Segre [18] asserts that the image of $\pi_1(\mathrm{U}, u)$ is the complete group $\Gamma_3$ of automorphisms of $\mathcal{E}_n$, at least over the complex field.

We now discuss the separate *n*.

*Theorem* **(3.7)**. — *Let* F *be a k-del Pezzo surface of degree n.*

a) *For* $n=9$, F *is k-isomorphic to a Severi-Brauer surface.*

b) *For* $n=8$, F *is isomorphic either to a quadric or to the image of* $\mathbf{P}^2$ *under a monoidal transformation with centre at a k-point.*

c) *For* $n=7$, F *is k-isomorphic to the image of* $\mathbf{P}^2$ *under a monoidal transformation with centre at two k-points or with centre at a closed point* $x \in \mathbf{P}^2$ *for which* $[\boldsymbol{k}(x) : k]=2$.

*Corollary 1.* — *A del Pezzo surface of degree* 7 *always has a rational point.*

*Corollary 2.* — *Let k be a field of algebraic numbers (of finite degree) or a field of functions of transcendence degree* 1 *over a finite field. If a del Pezzo surface of degree* 7 *or* 8 *has a rational point over all local completions* $k_{\mathfrak{p}}$ *of k, then it has a rational point over k.*

*Corollary 3.* — *A necessary and sufficient condition for a k-del Pezzo surface of degree* 7, 8 *or* 9 *to be k-birationally trivial is that it possess a k-point.*

All this follows immediately from Theorem 3.4, Lemmas 0.4 and 0.5 and well known results.

The treatment of the case $n=6$ is more complicated and depends on the fact that $\mathrm{F}\backslash\mathrm{X}$ has the structure of $k$-homogeneous space over a certain 2-dimensional torus, where X is the union of the exceptional curves. Before enunciating this result in detail we must describe a general construction which generalizes one of Serre [22].

(1) The possibility discussed in the last footnote of the concurrence of three lines certainly limits the possible galois groups.

Let V be a proper irreducible and reduced algebraic $k$-scheme and let $\{X_i\} \subset V \otimes \bar{k}$ be a finite set of divisors each of which is an irreducible and reduced subscheme of $V \otimes \bar{k}$. Let $S_0$ be the group of principal divisors generated by $\{X_i\}$ and let $R \subset R(V \otimes \bar{k})^*$ be the group of rational functions on $V \otimes \bar{k}$ whose divisors are in $S_0$, so $R \subset \Gamma(V \otimes \bar{k} \setminus \bigcup_i X_i, \mathcal{O}_{V \otimes \bar{k}})$. A section $\Phi : S_0 \to R$ of the exact sequence

$$0 \to \bar{k}^* \to R \to S_0 \to 0 \tag{3.8.1}$$

determines a ring homomorphism

$$\bar{k}[S_0] \to \bar{k}[R] \to \Gamma(V \otimes \bar{k} \setminus \bigcup_i X_i, \mathcal{O}_{V \otimes \bar{k}}),$$

and so the $\bar{k}$-morphism

$$f : V \otimes \bar{k} \setminus \bigcup_i X_i \to \operatorname{Spec} \bar{k}[S_0] = T \otimes \bar{k},$$

where $T = \operatorname{Spec} k[S_0]$.

Suppose now that the divisor $\sum_i X_i$ is G-invariant (G as always the Galois group of $\bar{k}/k$). Then (3.8.1) gives the exact sequence

$$0 \to \operatorname{Hom}_{\mathbf{Z}}(S_0, \bar{k}^*) \to \operatorname{Hom}_{\mathbf{Z}}(S_0, R) \to \operatorname{Hom}_{\mathbf{Z}}(S_0, S_0) \to 0,$$

in which $\operatorname{Hom}_{\mathbf{Z}}(S_0, \bar{k}^*) = T(\bar{k})$ is the group of geometric points of T. We denote by $h \in H^1(G_1, T(\bar{k}))$ the image of $\mathrm{id}_{S_0}$ under the connecting homomorphism

$$\delta : \operatorname{Hom}_G(S_0, S_0) \to H^1(G, T(\bar{k})).$$

This is the characteristic class of the G-extension (3.8.1) and at the same time it defines a principal homogeneous space $T^h$ of T over $k$. Finally let $X \subset V$ be a divisor such that $X \otimes \bar{k} = \bigcup_i X_i$ on $V \otimes \bar{k}$.

*Proposition* (**3.9**). — *With the above notation define a k-morphism*

$$g : V \setminus X \to T^h$$

*by means of the ring homomorphism*

$$(\bar{k}[S_0])^G \to (\bar{k}[R])^G \to \Gamma^G(V \otimes \bar{k} \setminus \bigcup_i X_i, \mathcal{O}_{V \otimes \bar{k}}) = \Gamma(V \setminus X, \mathcal{O}_V),$$

*where the action of* G *on* $\bar{k}[S_0]$ *is determined by means of the cocycle h in such a way that* $\operatorname{Spec}(\bar{k}[S_0])^G = T^h$. *Then* $g \otimes \bar{k} = f$ *with the appropriate identification of* $T \otimes \bar{k}$, $T^h \otimes \bar{k}$. *Further, g does not depend on the choice of section* $\Phi$.

The proof is essentially a formal calculation with the explicit forms of cocycles. In our special case we have

*Theorem* (**3.10**). — *Let* F *be a k-del Pezzo surface of degree* 6 *and let* $X_1, \dots, X_6 \subset F \otimes \bar{k}$ *be the exceptional curves of the first kind. Let* $X \subset F$ *be a divisor such that* $X \otimes \bar{k} = \sum_i X_i$ *and put* $U = F \setminus X$. *Then the torus* $T = \operatorname{Spec} k[S_0]$ *is* 2*-dimensional and the map of the scheme* U *into a principal homogeneous space over* T *described in Proposition* 3.9 *is an isomorphism.*

Since all del Pezzo surfaces of degree 6 are $k$-forms of each other, it is enough to verify Theorem 3.10 for one of them, say the $F \subset \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ given by the equation

$$x_0 y_0 z_0 = x_1 y_1 z_1$$

where $(x_0, x_1)$, $(y_0, y_1)$, $(z_0, z_1)$ are the three sets of homogenous coordinates. In this case we take for $\varphi(S_0) \subset R$ the group of functions generated by $\frac{x_0}{x_1}, \frac{y_0}{y_1}$ and $\frac{z_0}{z_1}$. It is clear that $F \setminus X \to \operatorname{Spec} k\left[\frac{x_0}{x_1}, \frac{x_1}{x_0}, \frac{y_0}{y_1}, \frac{y_1}{y_0}\right]$ is an isomorphism.

*Corollary 1. — A necessary and sufficient condition for a del Pezzo surface of degree* 6 *to be birationally equivalent to the plane over k is that it possess a rational k-point.*

We may suppose that F is $k$-minimal, since otherwise we are reduced to a del Pezzo surface of higher degree for which the result is already proved (Cor. 3 of Theorem 3.7). Lemma 0.4 then limits the possible orbits of G on the $X_i$: in particular they cannot contain any G-invariant points. If now $x$ is a rational point on F one can obtain a birational equivalence with a quadric by first blowing up $x$, so getting a del Pezzo surface $F'$ of degree 5, and then blowing down a set of three appropriate lines on $F'$. Since the quadric has a $k$-point, it is birationally trivial, so F is trivial.

*Corollary 2* (¹). — *A del Pezzo surface of degree* 6 *defined over an algebraic number field or over a function field k has a k-point if and only if there is a $k_\mathfrak{p}$-point for every local completion $k_\mathfrak{p}$.*

This follows from Voskresenskij's result [25] that the Hasse Principle holds for 2-dimensional tori.

To deal with $n=5$ we need another general theorem about rational points on del Pezzo surfaces.

*Theorem* (**3.12**). — *Let* F *be a k-del Pezzo surface of degree n.*

a) *If the field k is infinite, suppose further that there is a* $z \in N(F)^G$ *for which* $(Z, \omega_F^{-1})$ *is relatively prime to n. Then* F *has a k-point. If, further, there are no* G*-invariant exceptional curves of the first kind on* $F \otimes \bar{k}$ (e.g. *when* F *is k-minimal*) *then there is a k-point not on an exceptional curve.*

b) *If k is finite, then every rational k-surface has a k-point. On a del Pezzo surface of degree n one can find a k-point not on an exceptional curve provided that the number q of elements of k satisfies*

$$q > 10 + e_n - n,$$

*where $e_n$ is the number of irreducible exceptional curves of the first kind on* $F \otimes \bar{k}$.

The finite field case *b)* follows from Weil's results [24] about the number of $k$-points on rational $k$-surfaces. It would doubtless be possible to improve the bound for $q$

(¹) This implies in particular Selmer's result [23] that over an algebraic number field a surface of the type

$$x^3 + ay^3 = b(z^3 + at^3)$$

satisfies the Hasse principle. For as Segre [17] remarks there is a triplet of lines which can be blown down over the ground field. The same result is true for any cubic surface with such a triplet.

somewhat, but that some bound is needed is shown by the del Pezzo surface of degree 5 over the field $k$ of 2 elements obtained by blowing up four $k$-points in the plane.

In the general case *a)* we need the following Lemma due essentially to Igusa [9].

*Lemma* **(3.13)**. — *Suppose that $k$ is infinite and let* F *be a regular projective $k$-surface of degree $n$. Then there is a $k$-morphism $f: \mathrm{F}' \to \mathrm{F}$ such that $f \otimes \bar{k}$ is a monoidal transformation with centre at closed points of $\mathrm{F} \otimes \bar{k}$ and a $k$-morphism $p: \mathrm{F}' \to \mathbf{P}^1$ such that*

$$p^*(\mathcal{O}_{\mathbf{P}^1}(1)) = f^*(\mathcal{O}_{\mathrm{F}}(1)) \otimes \mathcal{O}_{\mathrm{F}'}(-\mathrm{X}),$$

*where* X *is a divisor on* F′ *blown down by $f$. Further, we may suppose that all the fibres of $p$ are geometrically irreducible.*

For one considers the pencil of intersections with a sufficiently general pencil of hyperplanes and blows up the base points of the pencil.

To prove Theorem 3.12 we show that there is a G-invariant section $\mathbf{P}^1_{\bar{k}} \to \mathrm{F}^1 \otimes \bar{k}$. We omit the rather elaborate details. The image of the section in F is a $k$-curve of genus 0 (with a $k$-rational point) which is distinct from the exceptional curves. The proof may be seen to generalize to surfaces with an anticanonical system without fixed components of degree 2.

The application to del Pezzo surfaces is

*Theorem* **(3.15)** [1]. — *Let* F *be a $k$-del Pezzo surface of degree* 5, *and* K *a normal extension of $k$ of degree $d$ prime to* 5. *If* $\mathrm{F} \otimes \mathrm{K}$ *has a* K*-point, then* F *is birationally equivalent to* $\mathbf{P}^2_k$.

Suppose first that there is a $k$-point $x$ on F. If $x$ is not on an exceptional curve, let F′ be obtained by blowing up F. The image of $x$ is met by a set of 5 nonintersecting irreducible exceptional curves on $\mathrm{F}' \otimes \bar{k}$. The union of these 5 curves is thus G-invariant and by blowing them down we obtain a Severi-Brauer surface, which is trivial since it contains rational points. The cases when $x$ is on precisely one exceptional curve and on the intersection of two are dealt with by variants of this technique.

By Theorem 3.12 *b)* this concludes the proof for finite $k$. Otherwise we have to show that the hypotheses of Theorem 3.15 imply the existence of a $k$-point. It is enough to show that the hypotheses of Theorem 3.12 *a)* follow from those of Theorem 3.15. Indeed arguments similar to those above show that the existence of a K-rational point implies the existence of a curve X with $(\mathrm{X}, \omega_{\mathrm{F}}) \not\equiv 0\ (s)$ on $\mathrm{F} \otimes \mathrm{K}$ (which we now know is birationally equivalent to $\mathbf{P}^2_{\mathrm{K}}$). Let $\mathrm{Z} \subset \mathrm{F}$ be such that $\mathrm{Z} \otimes \mathrm{K}$ is the sum of all the $k$-conjugates of X. Then $(\mathrm{Z}, \omega_{\mathrm{F}}) \not\equiv 0\ (5)$, which is all that was needed.

Finally we come to the case $n = 4$. This is distinguished from the cases already discussed not merely by its greater complexity but because (i) a surface of degree 4 with a rational point need not be birationally trivial [2] (ii) the surfaces are not necessarily

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(1) I have been unable to verify the assertion of Enriques [7] that every del Pezzo surface of degree 5 has a rational point.

(2) Similarly Segre [17] showed that a minimal cubic surface is birationally nontrivial, although it is easy to construct examples with rational points.

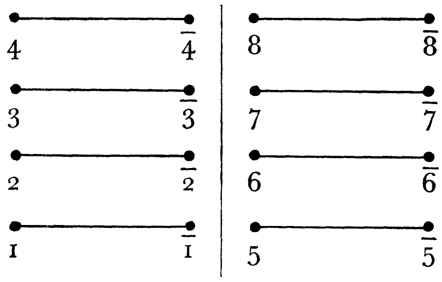
$k$-forms of each other and (iii) the criterion of Corollary 2.3 can be used to demonstrate the nontriviality of certain surfaces (while it cannot for $n \geqslant 5$).

The del Pezzo surfaces of degree 4 are of special interest because they are connected with one of the simplest Diophantine systems of equations: two quadratics in 5 variables (cf. Section 4 below). It would be interesting to know whether the Hasse Principle holds for it. Again the question of the birational triviality of a cubic surface with a single rational line, left open by Segre [17], reduces to this case. Our criterion gives a partial answer (twelve cases out of 19, see below).

In order to apply Corollary 2.3 we have to compute $H^1(k, N(F))$ for del Pezzo minimal surfaces of degree 4. It is first necessary to compute the number of possible dissections into orbits [1] of the set of lines on $F \otimes \bar{k}$ under the action of the Galois group G. It turns out that there are 19 types. For each it is automatic to compute $H^1(k, N(F))$ by means of Lemma 2.9. For 10 of the 19 types it is nontrivial, so F is nontrivial and for the remaining 9 it is trivial. In two of these cases F is nontrivial because $H^1(K, N(F))$ is nontrivial for an appropriate extension K of $k$. I do not know whether F can be trivial in the remaining 7 cases, though we shall show in one of them that $H^1(K, N(F)) = 0$ for all extensions K of $k$.

We shall use the following representation of the graph $\mathscr{E}_4$:

(3.17.1)




Here, in addition to the edges shown, each point is connected with precisely one of the vertices of each of the pairs on the other side of the vertical line, and, in particular, a left (right) vertex is connected with the left (right) vertex of the pair on the same line and with the right (left) vertex of the pairs on different lines. Thus the vertex 1 is joined to $\bar{1}$, 5, $\bar{6}$, $\bar{7}$, $\bar{8}$.

*Lemma* (**3.18**). — *The graph just described is isomorphic to* $\mathscr{E}_4$.

*Corollary.* — *The group* $\Gamma_4$ *of automorphisms of* $\mathscr{E}_4$ *acts transitively on the subgraphs of the type* •———•

The proofs are routine. We shall describe (3.17.1) as the canonical graph. We shall call a subgroup $\Gamma \subset \Gamma_4$ admissible if no subgraph corresponding to an orbit consists of isolated vertices. By Lemma 0.4 and the remark preceding Theorem 3.7

(1) Fortunately this suffices to compute $H^1(k, N(F))$, and we do not need to find the number of types of group action.

in this résumé, the admissible groups are just the possible images of Galois groups acting on minimal surfaces.

*Lemma* (**3.19**). — *Let* $Z_1, \ldots, Z_n$ *be the subgraphs corresponding to the orbits of some group acting on* $\mathscr{E}_4$. *Let* $a_r$ *be the number of vertices of* $Z_r$ *and let* $l_{rs}$ *be the number of edges going from a fixed vertex of* $Z_r$ *to a vertex of* $Z_s$. *Then* $l_{rs}$ *is independent of the choice of vertex and we have the equations*

$$\sum_s l_{rs} = 5$$
$$a_r l_{rs} = a_s l_{sr}$$
$$\sum_r a_r = 16$$

*The group is admissible if and only if* $l_{rs} \geqslant 1$ *for all r: and then* $a_r \geqslant 2$.

*Lemma* (**3.20**). — *Let* $\Gamma \subset \Gamma_4$ *be an admissible group. Then each orbit consists of* 6, 12 *or* $2^\alpha$ $(1 \leqslant \alpha \leqslant 4)$ *vertices.*

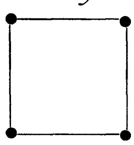
*Lemma* (**3.21**). — *Suppose that* $\Gamma$ *is admissible and that there is an orbit of* 6 *or* 12 *vertices. Then after a suitable identification of* $\mathscr{E}_4$ *with the canonical graph the orbits* $Z_i$ *are given by one of the three diagrams in* (3.21.1) (see Russian text) (1).

*Lemma* (**3.22**). — *Suppose that* $\Gamma$ *is admissible and that all the orbits have* $2^\alpha$ *vertices for some* $\alpha$. *Then after a suitable identification of* $\mathscr{E}_4$ *with the canonical graph, the orbits* $Z_i$ *are given by the* 16 *diagrams in* (3.22.1) (see Russian text).

*Lemma* (**3.23**). — *Let* $\Gamma$ *be a group of automorphisms of the canonical graph corresponding to the case* XVIII (diagram (3.22.1)). *Then* $\Gamma$ *has a subgroup of type* IV.

*Lemma* (**3.24**). — *Let* $\Gamma$ *be a group of automorphisms of the canonical graph of type* XII. *Then it has a subgroup either of type* IV *or of type* VI.

These five lemmas are all finite combinatorial statements which can therefore be verified in a finite amount of time.

*Lemma* (**3.25**). — *Let each subgraph of the canonical graph correspond to the divisors on* $F \otimes \bar{k}$ *given by the sum of the lines corresponding to the vertices. Then the group of principal divisors spanned by the lines is generated by the differences of pairs of cycles corresponding to squares*  .

The proof is straightforward.

We can now compute $H^1(k, N(F))$ in terms of the dissection $D : \bigcup_{1 \leqslant i \leqslant n} Z_i$ of the canonical graph into orbits. Let $a_i$ be the number of vertices of $Z_i$ and let $d = \operatorname{lcm}(a_i)$. Let $C^0$ be the group of 0-dimensional chains of $\mathscr{E}_4$ and let $N_D : C^0 \to C^0$ be the homomorphism defined by $N_D x = \frac{d}{a_i} \sum_{y \in Z_i} y$ for $x \in Z_i$. Denote by P the kernel of the pairing on $C^0$ induced by the index of intersection.

(1) In this case one can show that 5 does not divide the order of $\Gamma$, which must therefore be of the type $3 \cdot 2^\alpha$.

*Lemma* (3.27). — *Let* D *be the dissection of the canonical graph into orbits corresponding to a k-del Pezzo surface of degree* 4. *Then in the above notation*

$$H^1(k, N(F)) = N_D C^0 \cap P / N_D P.$$

This follows easily from Lemma 2.9.

We are now in a position to construct table 3.28 of the Russian text. The first column gives the type, the second a set of generators of $N_D C^0 \cap P$, where $Z_i$ denotes $\sum_{x \in Z_i} x$. The third column gives $H^1(k, N(F))$. When this is nonzero, the surface is birationally nontrivial by Corollary 2.3. In types XVIII and XII the surface is nontrivial in virtue of Lemmas 3.23 and 3.24 and because types IV and VI are nontrivial.

Finally we note that it can be shown that $\Gamma$ has order 6 in case II, and a further exmination shows that $H^1(K, N(F)) = 0$ for all extensions $K \supset k$.

## Section 4. — Remark on rational points.

The following statement appears to me to be probable.

*Conjecture* (4.1). — *Every rational k-surface* F *has a k-point if k is quasi algebraically closed* [1].

After Theorem 3.12 it is enough to consider infinite fields. For these we have

*Theorem* (4.2). — *Conjecture* (4.1) *is true for rational surfaces with a pencil of curves of genus zero, for forms of the absolute minimal models and for del Pezzo surfaces of degree* $n \neq 5$.

For surfaces with a pencil of curves of genus 0 this follows from a repeated application of the fact that the Brauer group of $k$ is trivial.

The absolutely minimal models are forms of $\mathbf{P}^2$, $\mathbf{P}^1 \times \mathbf{P}^1$ or ruled surfaces. The forms of $\mathbf{P}^2$ are the Severi-Brauer varieties, which are trivial because the Brauer group is. The forms of $\mathbf{P}^1 \times \mathbf{P}^1$ are embeddable as quadrics in $\mathbf{P}^3$, so have a rational point by the definition of $k$. The third case has already been dealt with.

By Theorem 3.7 the del Pezzo surfaces with $n = 9,8$ have already been dealt with and $n = 7$ is trivial. By Theorem 3.10 the case $n = 6$ reduces to that of a homogeneous space over a 2-dimensional torus, which has been dealt with by Serre [20]. The del Pezzo surfaces with $n = 3$ are the cubic surfaces in $\mathbf{P}^3$: and for these the theorem follows from the definition of $k$.

Finally, a del Pezzo surface with $n = 4$ is the complete intersection of two quadrics in the anticanonical embedding [2]. Hence the theorem follows from a theorem of Lang [11] (and a remark of Nagata freeing it from auxiliary restrictions on $k$).

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(1) I.e. there is a $k$-point on every $k$-hypersurface in $\mathbf{P}^n_k$ of degree $\leqslant n$ for all $n$ (cf. Lang [11]).

(2) For it is easy to deduce that $\dim H^0(F, \omega_F^{-2}) = 13$ from the representation of $F \otimes \bar{k}$ as the blowing up of five points of the plane.