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ON CONTRAVARIANT FUNCTORS FROM THE CATEGORY OF PRESCHEMES OVER A FIELD INTO THE CATEGORY OF ABELIAN GROUPS (WITH AN APPLICATION TO THE PICARD FUNCTOR)

by J.-P. MURRE

INTRODUCTION

In this paper we study contravariant functors from the category of preschemes over an arbitrary groundfield into the category of abelian groups. The main result is the characterization of those functors of the above type which are representable by a scheme which is locally of finite type over the groundfield (theorem 1, see I.2.1). The first six conditions are very natural ones; the main tools to verify these axioms in the case of the Picard functor of a proper algebraic scheme, are the finiteness, the comparison and the existence theorem from the *Élements de géométrie algébrique* (*E.G.A.*, III) and the descent theorems from the *Séminaire de géométrie algébrique* (*S.G.A.*, VIII) of Grothendieck. The last condition (axiom P_7) is of a more delicate nature, it corresponds with the theorem of Rosenlicht dealing with a morphism of a curve into a commutative group variety. We note that the condition P_7 is automatically satisfied in case the functor is a subfunctor of a representable functor of the same type. Following Chevalley and Seshadri we have first considered the functor on the schemes of groups (see [4]); the possibility to pass from the schemes of groups to arbitrary schemes is precisely given by the above mentioned axiom P_7 and the theory of generalized jacobians.

As an application we show in part II that the Picard functor of a proper algebraic scheme is representable. This problem was raised by Grothendieck in his Bourbaki séminaire ([5], V, 6.6). Grothendieck has pointed out to us that it is possible to embed the Picard functor of such a scheme into a representable functor (by using Chow's lemma and his existence theorem for the projective case, see part II). This means that the problem of representation of this Picard functor is reduced to the problem of proving the corollary in I.2.5.

We remark that in the meantime Grothendieck has obtained (by a different method) a more general result for the Picard functor of a proper scheme, namely if the groundscheme is integral and noetherian then the Picard functor is representable " above an open set " of the groundscheme.

Finally we note that the construction of the Picard scheme in part II still depends on projective methods (Hilbert schemes) but it is possible to avoid these. In order to verify axiom P_7 one has then to use (instead of the above mentioned embedding) the result of Seshadri ([4], 8).

We use the notations from *E.G.A.* (in particular if $\varphi : A \to B$ is a ring homomorphism then ${}^a\varphi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is the corresponding morphism of the spectra).

I am indebted to A. Grothendieck for valuable advice during the preparation of this paper; not only that I have used several unpublished results of him (such as, for instance, the above mentioned embedding of the Picard functor) but also many of the ideas of this paper have arisen from discussions with him on this subject.

I am also indebted to the referee; his comments have clarified many points in the exposition and I have followed his suggestion to rearrange the proof around a number of " key lemmas " (the propositions) which has improved the intelligibility considerably.

PART I

ON CONTRAVARIANT FUNCTORS FROM THE CATEGORY OF PRESCHEMES OVER A FIELD INTO THE CATEGORY OF ABELIAN GROUPS

I.1. Notations and preliminary remarks.

(I.1.1) Let S be a prescheme; (Sch/S) denotes the category of preschemes over the prescheme S. We always assume that a fixed *groundprescheme* S is given. If A is a ring and if $S = \mathrm{Spec}(A)$ then we write (Sch/A) instead of (Sch/Spec(A)). Furthermore **Ens** (resp. **Ab**) denotes the category of sets (resp. of abelian groups). If X is an object of a category **C** then we write $X \in \mathbf{C}$ instead of $X \in \mathrm{Ob}(\mathbf{C})$.

(I.1.2) Let **C** be a category and let P be a contravariant functor from **C** to **Ens**; i.e., $P \in \mathbf{Hom}(\mathbf{C}^0, \mathbf{Ens})$ (cf. *E.G.A.*, $\mathbf{0}_{\mathrm{III}}$, 8.1.1). Let $X \in \mathbf{C}$ and let $\xi \in P(X)$; in the following we always write $\xi : X \to P$ instead of $\xi \in P(X)$ and, if $T \in \mathbf{C}$ and $f : T \to X$ is a morphism, we write $\xi \circ f$ instead of $P(f)(\xi)$. Consider the functor $h_X = \mathrm{Hom}_{\mathbf{C}}(\ , X)$ and let $\xi : X \to P$, then ξ determines a functor morphism $\xi(T) : h_X(T) \to P(T)$ defined by $\xi(T)(f) = \xi \circ f$ for $f \in \mathrm{Hom}_{\mathbf{C}}(T, X)$. The functor P is called *representable* (*E.G.A.*, $\mathbf{0}_{\mathrm{III}}$, 8.1.8) if there exists an object $X \in \mathbf{C}$ and a *functor isomorphism* $\Phi : h_X \to P$ (i.e., $\Phi(T)$ is bijective for all $T \in \mathbf{C}$). Let $\rho : X \to P$ be defined by $\rho = \Phi(X)(1_X)$; then we see that $\Phi(T) = \rho(T)$. The couple (X, ρ) is uniquely determined up to an isomorphism; ρ is called the *canonical element* of P(X). In the following we often write X(T) instead of $h_X(T) = \mathrm{Hom}_{\mathbf{C}}(T, X)$.

(I.1.3) Again let **C** be a category and $G \in \mathbf{C}$. We recall that G is called a **C**-group (*E.G.A.*, $\mathbf{0}_{\mathrm{III}}$, 8.2.3) if for every $T \in \mathbf{C}$ the set G(T) has a group structure and if for every **C**-morphism $j : T \to S$ the corresponding map $h_G(j) : G(S) \to G(T)$ is a homomorphism of groups. If G and H are **C**-groups and if $f : G \to H$ is a morphism in **C**, then f is called a **C**-group morphism (or shortly a homomorphism) if the corresponding map $f(T) : G(T) \to H(T)$ is a group homomorphism for every $T \in \mathbf{C}$. If **C** is the category (Sch/S), where S is a prescheme, then we call G a *prescheme of groups over* S and f is a homomorphism of the preschemes of groups G and H. We remark that if P is a contravariant functor from the category **C** to the category of (for instance) abelian groups **Ab** and if P is representable by a couple (X, ρ) (or more precisely if the functor $P' : T \rightsquigarrow$ (underlying set of P(T)) from **C** to **Ens** is representable by the couple (X, ρ) then X is, according to the above definitions, a (commutative) **C**-group [1].

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[1] Also we note that ρ is a homomorphism in the sense of I.5.1.

(I.1.4) Next we come to the concept of *prorepresentability of a functor* ([5], II, A 2). Let Λ be a noetherian ring. Consider the subcategory $(\mathrm{Sch}/\Lambda)'$ of (Sch/Λ) consisting of the schemes of type $\mathrm{Spec}(A)$ where A is a Λ-algebra of finite Λ-length, i.e., A is an Λ-algebra which is a Λ-module of finite length (an artinian Λ-algebra in the terminology of [5], II, p. 195-07). Let F be a contravariant functor from $(\mathrm{Sch}/\Lambda)'$ to **Ens**; F is called *strictly prorepresentable* if the following is true. There exists a projective system $(\mathcal{O}_i, \psi_{ji})$ of Λ-algebras of finite Λ-length and an element $(\sigma_i) \in \varprojlim \mathrm{F}(\mathrm{Spec}(\mathcal{O}_i))$ such that the ψ_{ji} are surjective and such that, for every Λ-algebra A of finite Λ-length, the maps

$$\sigma_i(\mathrm{Spec}(A)) : \mathrm{Hom}_\Lambda(\mathrm{Spec}(A), \mathrm{Spec}(\mathcal{O}_i)) \to \mathrm{F}(\mathrm{Spec}(A))$$

induce a bijection

$$\varinjlim \mathrm{Hom}_\Lambda(\mathrm{Spec}(A), \mathrm{Spec}(\mathcal{O}_i)) \to \mathrm{F}(\mathrm{Spec}(A)).$$

We say that the system $(\mathcal{O}_i, \psi_{ji}, \sigma_i)$ prorepresents F. Since the set

$$\varinjlim \mathrm{Hom}_\Lambda(\mathrm{Spec}(A), \mathrm{Spec}(\mathcal{O}_i))$$

can be identified with the set $\varinjlim \mathrm{Hom}_\Lambda(\mathcal{O}_i, A)$, and therefore with the set $\mathrm{Hom}_c(\mathcal{O}, A)$ of continuous homomorphisms of the topological Λ-algebra $\mathcal{O} = \varprojlim \mathcal{O}_i$ [1] into A, it follows that the system $(\mathcal{O}_i, \psi^{ji}, \sigma_i)$ determines a bijection

$$\mathrm{Hom}_c(\mathcal{O}, A) \to \mathrm{F}(\mathrm{Spec}(A))$$

For this reason we also say that $\mathcal{O}$ *prorepresents* F.

(I.1.5) In our case we are interested in a contravariant functor P from the category (Sch/Λ) itself into **Ens**. Assume now that the restriction P′ of the functor P to the subcategory $(\mathrm{Sch}/\Lambda)'$ is prorepresentable and let, as above, $\mathcal{O}$ be the topological k-algebra which prorepresents P′ (sometimes we say — par abus de langage — that P itself is prorepresentable by $\mathcal{O}$). Consider a maximal ideal $\mathfrak{n}$ of Λ and an element $\xi : \mathrm{Spec}(\Lambda/\mathfrak{n}) \to \mathrm{P}$ (ξ is called a " rational point " of P). Let P_ξ be the subfunctor of P′ which associates with every Λ-algebra A of finite Λ-length the subset of $\mathrm{P}(\mathrm{Spec}(A))$ consisting of those elements η which have the following property: for every Λ-morphism $\alpha : \mathrm{Spec}(L) \to \mathrm{Spec}(A)$, where L is a field, there exists a commutative diagram:

$$\begin{array}{ccc} \mathrm{Spec}(A) & \xrightarrow{\eta} & \mathrm{P} \\ \uparrow{\scriptstyle a_\alpha} & & \uparrow{\scriptstyle \xi} \\ \mathrm{Spec}(L) & \xrightarrow{\text{can.}} & \mathrm{Spec}(\Lambda/\mathfrak{n}). \end{array}$$

[1] The topology of $\mathcal{O}$ is determined by the condition that it is the least fine topology which makes the maps $\mathcal{O} \to \mathcal{O}_i$ continuous; the topology in the $\mathcal{O}_i$ is discrete.

If P′ *is strictly prorepresentable then* P_ξ *is also strictly prorepresentable* in the sense described above, i.e., $P_\xi(\mathrm{Spec}(A))$ is in bijection with the set of continuous Λ-homomorphisms of a certain topological Λ-algebra into A. To be precise, we have the following: let $\xi' : \mathcal{O} \to \Lambda/\pi$ be the continuous homomorphism which corresponds, in the way described in I.1.4, with the element ξ and let $\mathfrak{m} = \mathrm{Ker}(\xi')$, $R_\xi = \mathcal{O}_\mathfrak{m}$ and $\mathfrak{m}_\xi = \mathfrak{m}.\mathcal{O}_\mathfrak{m}$. It follows from the structure of $\mathcal{O}$ that R_ξ is a direct factor of $\mathcal{O}$, called *the local component of* $\mathcal{O}$ *at " the point "* ξ. Moreover, if again A is an Λ-algebra of finite Λ-length and if we identify, in the way described in I.1.4, the set $\mathrm{Hom}_c(\mathcal{O}, A)$ with $P(\mathrm{Spec}(A))$, then it follows from the above definition of the functor P_ξ that the projection $\mathcal{O} \to R_\xi$ induces a bijection of $\mathrm{Hom}_c(R_\xi, A)$ on the subset $P_\xi(\mathrm{Spec}(A))$ of $P(\mathrm{Spec}(A))$; i.e., P_ξ is prorepresentable by R_ξ.

(I.1.6) Let the situation be as in I.1.5 and assume, moreover, that *the local component* R_ξ *is noetherian.* Then we have that, as a topological Λ-algebra, $R_\xi = \varprojlim (R_\xi/\mathfrak{m}_\xi^n)$. In order to see this we first note that from the construction of R_ξ follows that $R_\xi = \varprojlim R_i$, where (R_i, λ_{ji}) is a projective system of Λ-algebras R_i of finite Λ-length which are local rings with maximal ideals $\mathfrak{m}_i$ and the λ_{ji} are local, surjective Λ-homomorphisms (I is a pre-ordered, filtered set). We have $R_i/\mathfrak{m}_i = R_\xi/\mathfrak{m}_\xi = \mathcal{O}/\mathfrak{m} = \Lambda/\pi$ and we put for abbreviation $\Lambda/\pi = K$. Since R_ξ is noetherian we have that $\dim_K(\mathfrak{m}_\xi/\mathfrak{m}_\xi^2)$ is finite; since the λ_{ji} are surjective it follows that the $\dim_K(\mathfrak{m}_i/\mathfrak{m}_i^2)$ are bounded. Let i_1 be such that $\dim_K(\mathfrak{m}_{i_1}/\mathfrak{m}_{i_1}^2)$ is maximal; then, if we put $R_1' = R_{i_1}/\mathfrak{m}_{i_1}^2$, we have that for $i \geqslant i_1$ the homomorphisms $R_i \to R_1'$ (obtained from λ_{ii_1}) are isomorphisms. Clearly also the $\dim_K(\mathfrak{m}_i^2/\mathfrak{m}_i^3)$ are bounded; let $i_2 \geqslant i_1$ be such that this dimension is maximal for $i = i_2$ and put $R_2' = R_{i_2}/\mathfrak{m}_{i_2}^2$ then again for $i \geqslant i_2$ the homomorphisms $R_i \to R_2'$ are isomorphisms. Proceeding in this way we obtain a projective system $(R_n'; n \in \mathbf{Z}_+)$, with obvious homomorphisms, and clearly there is a continuous Λ-homomorphism $R_\xi \to \varprojlim R_n'$. Next we note that for every $i \in I$ there exists an integer q_i such that $\mathfrak{m}_i^{q_i} = (0)$ and from this follows easily that the above continuous Λ-homomorphism $R_\xi \to \varprojlim R_n'$ is in fact an isomorphism of the topological Λ-algebras. The fact that, finally, $\varprojlim R_n'$ can be identified with $\varprojlim (R_\xi/\mathfrak{m}_\xi^n)$ follows from (*E.G.A.*, $\mathbf{0}_\mathrm{I}$, Prop. 7.2.7). Instead of saying that P_ξ is prorepresented by R_ξ we sometimes say that P *(itself) is prorepresented " at the point* ξ *" by* R_ξ, or more explicitly by the system $(R_\xi/\mathfrak{m}_\xi^n, \varphi_{mn}, \tau_n)$ where the $\varphi_{mn} : R_\xi/\mathfrak{m}_\xi^m \to R_\xi/\mathfrak{m}_\xi^n$ are the natural homomorphisms and the $\tau_n : \mathrm{Spec}(R_\xi/\mathfrak{m}_\xi^n) \to P$ are the elements determined by the natural continuous Λ-homomorphisms $\mathcal{O} \to R_\xi/\mathfrak{m}_\xi^n$.

(I.1.7) Let the situation be as in I.1.5, but assume for simplicity that Λ is a field k. Consider the functor $P \times P$ determined by the conditions that $(P \times P)(T) = P(T) \times P(T)$ for $T \in (\mathrm{Sch}/k)$ and $(P \times P)(j) = P(j) \times P(j)$ for a morphism $j : T' \to T$. It follows immediately from the definitions that the restriction of the functor $P \times P$ to $(\mathrm{Sch}/k)'$ is prorepresentable by the topological k-algebra $\mathcal{O} \hat{\otimes}_k \mathcal{O}$. Let ξ

and R_ξ be as in I.1.5; since $R_\xi/\mathfrak{m}_\xi = k$ we have that $R_\xi \hat{\otimes}_k R_\xi$ is a local ring and it follows that $R_\xi \hat{\otimes}_k R_\xi$ prorepresents $P \times P$, in the sense of I.1.6, at the point $(\xi, \xi) : \mathrm{Spec}(k) \to P \times P$.

(I.1.8) Let the situation be as in I.1.7 but assume moreover that $P \in \mathbf{Hom}((\mathrm{Sch}/k)^0, \mathbf{Ab})$ and let ξ be the unit element of $P(\mathrm{Spec}(k))$. Let B be a local k-algebra of finite k-length and let $P_\xi(\mathrm{Spec}(B))$ be the subset introduced in I.1.5. Then it follows that $P_\xi(\mathrm{Spec}(B))$ is a subgroup of $P(\mathrm{Spec}(B))$ and hence $\mathrm{Hom}_c(R_\xi, B)$ has a commutative group structure for every local k-algebra of finite k-length (and if $B' \to B$ is a local homomorphism then the corresponding map $\mathrm{Hom}_c(R_\xi, B') \to \mathrm{Hom}_c(R_\xi, B)$ is a homomorphism). The formal spectrum $\mathrm{Spf}(R_\xi)$ is called a commutative " formal group " (1). We note that there exists a continuous k-homomorphism $\mu : R_\xi \to R_\xi \hat{\otimes}_k R_\xi$ (resp. $\nu : R_\xi \to R_\xi$) corresponding with the group multiplication (resp. with the group inverse).

(I.1.9) Let $S = \mathrm{Spec}(k)$ where k is an algebraically closed field and let $C \in (\mathrm{Sch}/k)$ be a *complete, non singular curve.* Let $T = \{t_1, \ldots, t_q\}$ be a *finite set of closed points on* C; write $C_1 = C - T$ and let $\Phi : \mathrm{Hom}_k(\mathrm{Spec}(k), C_1) \to G$ be a map, where G is an abelian group. Let $D \in \mathrm{Div}(C_1/k)$, i.e., D is a divisor on C_1 rational over k; then D can be considered as a formal sum $\Sigma_i n_i \alpha_i$ where $n_i \in \mathbf{Z}$ and $\alpha_i \in \mathrm{Hom}_k(\mathrm{Spec}(k), C_1)$. The map Φ can be extended to a homomorphism $\Psi : \mathrm{Div}(C_1/k) \to G$ by putting $\Psi(D) = \Sigma_i n_i \Phi(\alpha_i)$. We say ([9], p. 38) that Φ has a module $\mathfrak{m}$ with support on T if there exist positive integers m_i such that for every divisor D on C_1 which is of the form $D = (f)$, where (f) is the divisor of a function on C with $f \equiv 1 \pmod{m_i t_i}$ $(i = 1, \ldots, q)$ (see [9], p. 11) we have $\Psi(D) = 0$. More generally, let S be an arbitrary ground scheme, $\mathrm{Spec}(k)$ a prescheme over S, with k an algebraically closed field, and let $C \in (\mathrm{Sch}/k)$, T and G be as above. We say that a map $\Phi : \mathrm{Hom}_S(\mathrm{Spec}(k), C_1) \to G$ has a module $\mathfrak{m}$ with support on T if the restriction Φ' of Φ to $\mathrm{Hom}_k(\mathrm{Spec}(k), C_1)$ has such a module $\mathfrak{m}$.

(I.1.10) Let $T \in (\mathrm{Sch}/S)$ and $\xi : T \to P$ (where $P \in \mathbf{Hom}((\mathrm{Sch}/S)^0, \mathbf{Ens})$). If $j : T' \to T$ is an immersion then we often write ξ/T' instead of $\xi \circ j$. Also if t is a point of T and $j : \mathrm{Spec}(k) \to T$ (resp. $j_1 : \mathrm{Spec}(\mathcal{O}_{t,T}) \to T$, resp. $j_2 : \mathrm{Spec}(\hat{\mathcal{O}}_{t,T}) \to T$) are the natural morphisms then we write $\xi(t)$ instead of $\xi \circ j$ (resp. ξ_t instead of $\xi \circ j_1$, resp. $\hat{\xi}_t$ instead of $\xi \circ j_2$).

(I.1.11) Let $T \in (\mathrm{Sch}/S)$, $P \in \mathbf{Hom}((\mathrm{Sch}/S)^0, \mathbf{Ens})$ and $\xi : T \to P$. The element ξ is called *injective* if the corresponding map $\xi(T') : \mathrm{Hom}_S(T', T) \to P(T')$ is injective for all $T' \in (\mathrm{Sch}/S)$.

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(1) Let **C** be the category of finite k-schemes with a topological space consisting out of one point. Define **Ind**(**C**) similar as **Pro**(**C**) is defined in ([5], II, A 2). A formal group (in Cartier's sense) is an **Ind**(**C**)-group.

I.2. The main result.

(I.2.1) *Theorem 1.* — *Let k be a field; consider a functor* $P \in \mathbf{Hom}((\mathrm{Sch}/k)^0, \mathbf{Ab})$. *This functor is representable by a scheme of commutative groups, locally of finite type over k, if and only if* P *satisfies the following conditions:*

P_1: P *is strictly prorepresentable on the category* $(\mathrm{Sch}/k)'$ *consisting of schemes of the type* $\mathrm{Spec}(A)$ *where* A *is a k-algebra of finite k-length; moreover the local components* R_ξ *which prorepresent* P *at the " rational points ξ of* P " *are noetherian (see I.1.4 and I.1.5).*

P_2: *Let* A *be a noetherian, local k-algebra which is complete and separated with respect to the $\mathfrak{m}$-adique topology ($\mathfrak{m}$ is the maximal ideal of* A). *Write* $A_n = A/\mathfrak{m}^{n+1}$ $(n = 0, 1, 2, \ldots)$. *Then the natural homomorphism* $P(\mathrm{Spec}(A)) \to \varprojlim P(\mathrm{Spec}(A_n))$ *is an isomorphism.*

P_3: *Let* $\{A_\alpha\}$ *be an inductive system of k-algebras; let* $A = \varinjlim A_\alpha$. *Then the natural homomorphism* $\varinjlim P(\mathrm{Spec}(A_\alpha)) \to P(\mathrm{Spec}(A))$ *is an isomorphism.*

P_4: *Let* T *and* T' *be objects in* (Sch/k); *let* $f : T' \to T$ *be a faithfully flat and quasi-compact morphism. Let* p_1 (resp. p_2) $: T' \times_T T' \to T'$ *be the projection on the first* (resp. *second*) *factor. Then the following diagram is exact* ([5], I, p. 190-03):

$$P(T) \xrightarrow[P(f)]{} P(T') \underset{P(p_2)}{\overset{P(p_1)}{\rightrightarrows}} P(T' \times_T T').$$

P_5: P *is of local nature* ([6], IV, Def. 5.4); i.e., *if* $T \in (\mathrm{Sch}/k)$ *and if* $\{U_\alpha\}$ *is an open covering of* T *then the following diagram is exact (where the maps are the natural maps):*

$$P(T) \to \prod_\alpha P(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} P(U_\alpha \cap U_\beta).$$

P_6 [1]: *Let* $T \in (\mathrm{Sch}/k)$ *and* $\xi : T \to P$. *There exists a closed subscheme* $N(\xi)$ *of* T *with the following property: for every* $T' \in (\mathrm{Sch}/k)$ *and every morphism* $\alpha : T' \to T$ *we have* $\xi \circ \alpha = 0$ *if and only if α factors through* $N(\xi)$:

$$\begin{array}{ccc} T' & \xrightarrow{\alpha} & T \\ & \searrow \quad \nearrow & \\ & N(\xi) & \end{array}$$

P_7: *Let* $\xi : C - T \to P$, *where* C *is a complete, non-singular curve in* $(\mathrm{Sch}/\bar{k})$, *where $\bar{k}$ is the algebraic closure of k, and* T *is a finite set of closed points on* C. *Then the map* $\xi(\mathrm{Spec}(\bar{k})) : \mathrm{Hom}_k(\mathrm{Spec}(\bar{k}), C - T) \to P(\mathrm{Spec}(\bar{k}))$ *has a module with support on* T.

Remark **(I.2.2)**. — In view of the isomorphism mentioned in P_2 we identify (if the functor satisfies P_2) $P(\mathrm{Spec}(A))$ with $\varprojlim P(\mathrm{Spec}(A_n))$; a similar remark holds for the situation mentioned in P_3.

Remark **(I.2.3)**. — Assume that the functor P satisfies P_1 and P_2; let $\varepsilon : \mathrm{Spec}(k) \to P$ be a rational point of P. Let P_ε (with the notations of I.1.6) be prorepresented by

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[1] I owe this formulation to D. Mumford, my original formulation was somewhat weaker but less natural.

the local ring R_ε, or more explicitly by the system $(R_\varepsilon/\mathfrak{m}_\varepsilon^n, \varphi_{mn}, \tau_n)$ (see I.1.6), where $\mathfrak{m}_\varepsilon$ denotes the maximal ideal of R_ε. The element $(\tau_n) \in \varprojlim P(\mathrm{Spec}(R_\varepsilon/\mathfrak{m}_\varepsilon^n))$ determines uniquely an element τ in $P(\mathrm{Spec}(R_\varepsilon))$ called the *canonical* element of $P(\mathrm{Spec}(R_\varepsilon))$. The bijection $\mathrm{Hom}_c(R_\varepsilon, A) \to P_\varepsilon(\mathrm{Spec}(A))$ is such that $\eta \in P_\varepsilon(\mathrm{Spec}(A))$ corresponds with $\varphi \in \mathrm{Hom}(R_\varepsilon, A)$ if and only if $\eta = \tau \circ^a \varphi$.

Remark **(I.2.4)**. — The axiom P_6 is equivalent with the following one:

P_6'. Let X and Y be preschemes over k and $\xi : X \to P$ and $\eta : Y \to P$; then the fiber product $h_X \times_P h_Y$ [1] is representable by a closed subscheme of $X \times_k Y$.

$P_6 \Rightarrow P_6'$: for, consider on $X \times_k Y$ the element $\xi \circ p_1 - \eta \circ p_2 = \zeta$, where $p_1 : X \times_k Y \to X$ and $p_2 : X \times_k Y \to Y$ are the projections; then $N(\zeta)$ represents $h_X \times_P h_Y$.

$P_6' \Rightarrow P_6$: for consider the element $\xi : T \to P$ and the zero element $\varepsilon : \mathrm{Spec}(k) \to P$, then the closed subscheme on $T \times_k k = T$ which represents $h_X \times_P h_{\mathrm{Spec}(k)}$ has the required properties for $N(\xi)$.

(I.2.5) *Corollary of theorem 1. — Let* $P \in \mathbf{Hom}((\mathrm{Sch}/k)^0, \mathbf{Ab})$ *be a subfunctor of a functor* $h_G = \mathrm{Hom}(\ , G)$ *where* G *is a scheme of commutative groups, locally of finite type over k. Then* P *is representable by a scheme of commutative groups, locally of finite type over k, if and only if* P *satisfies* P_1, P_2, P_3, P_4 *and* P_5.

Proof. — According to theorem 1 we have only to show that P satisfies P_6 and P_7. P_6 follows (using *E.G.A.*, **I**, 4.4.1) from the fact that the unit element e of G is a closed point (e is rational over k) and P_7 follows from Rosenlicht's theorem ([9], p. 11) applied to the morphism $\xi \times 1_{\mathrm{Spec}(\bar{k})} : C - T \to G \times_k \bar{k}$.

Remark **(I.2.6)**. — The proof of the corollary is substantially easier than the proof of theorem 1, namely it does not make use of the theory of generalized jacobians. We shall indicate a simple proof of the corollary in section I.9.

I.3. The conditions are necessary.

(I.3.1) In fact we shall prove a slightly more general result, namely: *if* P *is a contravariant functor from* (Sch/S) *to* **Ab**, *where* S *is a prescheme which is locally noetherian, and if* P *is represeutable by an* S*-scheme of commutative groups* G *which is locally of finite type over* S, *and the unit section of which is closed, then* P *satisfies the conditions* $P_1, \ldots, P_7$, where now in P_7 we assume that C is a non-singular, complete curve in (Sch/k), where k is an algebraically closed field and $\mathrm{Spec}(k) \in (\mathrm{Sch}/S)$.

For the fact that P_1 is true see ([5], II, C, § 1); P_2 is immediate. The property P_3 depends essentially on the fact that G is locally of finite presentation over S (cf. [6], II, p. 9.08) (which follows in our case from the fact that S is locally noetherian and G locally of finite type over S) and follows from *E.G.A.*, chap. IV, where statements of the

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[1] The functor $h_X \times_P h_Y$ is the functor $Q : (\mathrm{Sch}/k)^0 \to \mathbf{Ab}$ determined by the condition that for $T \in (\mathrm{Sch}/k)$ the set $Q(T)$ consists out of all pairs of morphisms $\alpha : T \to X, \beta : T \to Y$ such that $\xi \circ \alpha = \eta \circ \beta$.

type as P_3 are proved in great generality. For the sake of completeness we give here a proof of P_3. Let $\lambda_\alpha : A_\alpha \to A$ and $\lambda_{\beta\alpha} : A_\alpha \to A_\beta$ $(\beta \geqslant \alpha)$ be the natural homomorphisms and let $\Phi : \varinjlim P(\mathrm{Spec}(A_\alpha)) \to P(\mathrm{Spec}(A))$ be the natural homomorphism obtained from the homomorphisms $P({}^a\lambda_\alpha) : P(\mathrm{Spec}(A_\alpha)) \to P(\mathrm{Spec}(A))$. Then we have:

Lemma (**I.3.2**). — *If* G *is locally of finite type over* S, *then* Φ *is injective, and if* G *is locally of finite presentation over* S, *then* Φ *is surjective.*

Proof. — Write $\tau : \mathrm{Spec}(A) \to S$, $\tau_\alpha : \mathrm{Spec}(A_\alpha) \to S$ and $\pi : G \to S$ for the structure morphisms. We identify $P(T)$ and $\mathrm{Hom}_S(T, G)$ in the following.

a) Φ *is injective.* — Let $\{\xi'_\alpha\}$ and $\{\xi''_\alpha\}$ be in $\varinjlim P(\mathrm{Spec}(A_\alpha))$ such that $\Phi(\{\xi'_\alpha\}) = \Phi(\{\xi''_\alpha\})$. We can take α so large that both ξ'_α and ξ''_α are defined; we want to prove that there exists an index β such that $\xi'_\beta = \xi''_\beta$.

First of all we want to show that there is no loss in generality if we assume that $S = \mathrm{Spec}(R)$ and $G = \mathrm{Spec}(B)$ (where B is an R-algebra of finite type). In order to see this write $\xi = \Phi(\{\xi'_\alpha\}) = \Phi(\{\xi''_\alpha\}) : \mathrm{Spec}(A) \to G$ and let $y \in \mathrm{Spec}(A)$. There exists an affine neighbourhood $\mathrm{Spec}(R_y)$ of $\tau(y)$ in S and an affine neighbourhood $\mathrm{Spec}(B_y)$ of $\xi(y)$ in G such that:

i) $\pi^{-1}(\mathrm{Spec}(R_y)) \supset \mathrm{Spec}(B_y)$;
ii) B_y is an R_y-algebra of finite type.

There exists $f_{\alpha y} \in A_\alpha$ such that, writing $A'_{\alpha y} = (A_\alpha)_{f_{\alpha y}}$, the point ${}^a\lambda_\alpha(y)$ is in the open set $\mathrm{Spec}(A'_{\alpha y})$ and such that both $\xi'_\alpha(\mathrm{Spec}(A'_{\alpha y}))$ and $\xi''_\alpha(\mathrm{Spec}(A'_{\alpha y}))$ are in $\mathrm{Spec}(B_y)$. We write (for $\beta \geqslant \alpha$) $f_{\beta y} = \lambda_{\beta\alpha}(f_{\alpha y})$, $A'_{\beta y} = (A_\beta)_{f_{\beta y}}$, $f_y = \lambda_\alpha(f_{\alpha y})$ and $A'_y = A_{f_y}$; clearly $A'_y = \varinjlim A'_{\beta y}$. There exist a finite number of points $y_1, y_2, \ldots, y_n$ in $\mathrm{Spec}(A)$ such that $\{\mathrm{Spec}(A_{y_i})\}$ $(i = 1, \ldots, n)$ is a covering of $\mathrm{Spec}(A)$; this implies that there exist elements $a_i \in A$ such that $\sum_i a_i f_{y_i} = 1$. Clearly we can assume that α is so large that there exist elements $a_{\alpha i}$ in A_α such that $\lambda_\alpha(a_{\alpha i}) = a_i$ and such that $\sum_i a_{\alpha i} f_{\alpha y_i} = 1$. Then $\{\mathrm{Spec}(A'_{\beta y_i})\}$ $(i = 1, \ldots, n)$ is, for $\beta \geqslant \alpha$, a covering of $\mathrm{Spec}(A_\beta)$ and it suffices to prove the lemma " for every index " separately; therefore we can assume $S = \mathrm{Spec}(R)$ and $G = \mathrm{Spec}(B)$ with an R-algebra B which is of finite type over R.

Let φ'_α and $\varphi''_\alpha : B \to A_\alpha$ be the R-homomorphisms corresponding with ξ'_α and ξ''_α; we have $\lambda_\alpha \circ \varphi'_\alpha = \lambda_\alpha \circ \varphi''_\alpha$ by assumption. Let $b_1, \ldots, b_s$ be a set of generators of B over R. Since $\lambda_\alpha \circ \varphi'_\alpha(b_i) = \lambda_\alpha \circ \varphi''_\alpha(b_i)$ $(i = 1, \ldots, s)$, there exists an index β such that $\varphi'_\beta(b_i) = \lambda_{\beta\alpha} \circ \varphi'_\alpha(b_i) = \lambda_{\beta\alpha} \circ \varphi''_\alpha(b_i) = \varphi''_\beta(b_i)$ $(i = 1, \ldots, s)$ (where φ'_β and φ''_β correspond with ξ'_β and ξ''_β). Hence $\varphi'_\beta = \varphi''_\beta$, i.e., $\xi'_\beta = \xi''_\beta$.

b) Φ *is surjective.* — (G is now assumed to be locally of finite presentation; see proof below.) There is given $\xi : \mathrm{Spec}(A) \to G$; we want to show that there exists an index α and a morphism $\xi_\alpha : \mathrm{Spec}(A_\alpha) \to G$ such that $\xi = \xi_\alpha \circ {}^a\lambda_\alpha$.

First of all we want to show that there is no loss in generality if we assume that $S = \mathrm{Spec}(T)$ and $G = \mathrm{Spec}(B)$ with a T-algebra B which is of finite presentation over T,

i.e., which is the quotient of a finite polynomial algebra over T by a finitely generated ideal. In order to see this, let $x \in \mathrm{Spec}(A)$; there exist an affine neighbourhood $\mathrm{Spec}(T_x)$ of $\tau(x)$ in S and an affine neighbourhood $\mathrm{Spec}(B_x)$ of $\xi(x)$ in G such that

i) $\pi^{-1}(\mathrm{Spec}(T_x)) \supset \mathrm{Spec}(B_x)$;

ii) B_x is a T_x-algebra of finite presentation.

There exist an $f_x \in A$ such that, if we write $A'_x = A_{f_x}$, then $\xi(\mathrm{Spec}(A'_x)) \subset \mathrm{Spec}(B_x)$. This can be done for every x in $\mathrm{Spec}(A)$; let $x_1, \ldots, x_n$ be such that:

$$\{\mathrm{Spec}(A'_{x_i})\} \qquad (i = 1, \ldots, n)$$

is a covering of $\mathrm{Spec}(A)$. We write f_i (resp. A'_i) instead of f_{x_i} (resp. A'_{x_i}). There are elements $a_i \in A$ $(i = 1, \ldots, n)$ such that $\sum_i a_i f_i = 1$. Let α be an index so large that there are elements $f_{\alpha i} \in A_\alpha$ and $a_{\alpha i} \in A_\alpha$ such that $\lambda_\alpha(f_{\alpha i}) = f_i$, $\lambda_\alpha(a_{\alpha i}) = a_i$ and $\sum_i a_{\alpha i} f_{\alpha i} = 1$; we use also the notations $A'_{\alpha i} = (A_\alpha)_{f_{\alpha i}}$, $f_{\beta i} = \lambda_{\beta\alpha}(f_{\alpha i})$ and $A'_{\beta i} = (A_\beta)_{f_{\beta i}}$ (with $\beta \geqslant \alpha$). Then $\{\mathrm{Spec}(A'_{\beta i})\}$ $(i = 1, \ldots, n)$ is a covering of $\mathrm{Spec}(A_\beta)$ and $A'_i = \varinjlim A'_{\beta i}$. It suffices to prove the existence of $\xi_{\alpha i} : \mathrm{Spec}(A'_{\alpha i}) \to G$ such that, for the restriction ξ_i of ξ to $\mathrm{Spec}(A'_i)$, we have $\xi_i = \xi_{\alpha i} \circ {}^a\lambda_{\alpha i}$, where $\lambda_{\alpha i} : A'_{\alpha i} \to A'_i$ is obtained from λ_α. For, if we have such morphisms $\xi_{\alpha i}$ $(i = 1, \ldots, n)$, then we consider for $\beta \geqslant \alpha$ the intersection $\mathrm{Spec}(A'_{\beta i}) \cap \mathrm{Spec}(A'_{\beta j}) = \mathrm{Spec}(A'_{\beta ij})$ where $A'_{\beta ij} = (A_\beta)_{f_{\beta i} \circ f_{\beta j}}$ and since $\varinjlim A'_{\beta ij} = A'_{ij}$ $(= A_{f_i f_j})$ we can apply the part of the lemma proved under a); i.e., there exists an index $\beta(i, j) = \gamma$ such that the restriction of $\xi_{\gamma i}$ to $\mathrm{Spec}(A'_{\gamma ij})$ equals the restriction of $\xi_{\gamma j}$ to $\mathrm{Spec}(A'_{\gamma ij})$, and this clearly proves the lemma.

Hence we can make the assumptions stated in the beginning; let $\varphi : B \to A$ be such that ${}^a\varphi = \xi$. Since B is the quotient of a polynomial ring $T[u_1, \ldots, u_s]$ by an ideal which is finitely generated, it follows that there exist an index α and a homomorphism $\varphi_\alpha : B \to A_\alpha$ such that $\varphi = \lambda_\alpha \circ \varphi_\alpha$ and ${}^a\varphi_\alpha = \xi_\alpha$ fulfills the requirements. This completes the proof of lemma I.3.2.

(I.3.3) Returning to the verification of the axioms $P_1, \ldots, P_7$ we note that P_4 follows from (*S.G.A.*, VIII, prop. 5.1 *b*, cor. 4.3 and cor. 1.7). P_5 follows from the definitions. P_6 follows (using *E.G.A.*, I, prop. 4.4.1) from the fact that the unit section is closed and finally P_7 follows from Rosenlicht's theorem ([9], p. 11) applied to the map $(\xi, 1_{\mathrm{Spec}(k)}) : C - T \to G \times_S \mathrm{Spec}(k)$.

I.4. Some remarks on prorepresentable functors.

(I.4.1) Consider the category (Sch/k), where k is a field and let, as in I.1.4, $(\mathrm{Sch}/k)'$ denote the subcategory consisting of schemes of type $\mathrm{Spec}(A)$ where A is a k-algebra which is a finite dimensional vectorspace over k. Let **C** be the category of contravariant functors from $(\mathrm{Sch}/k)'$ to **Ens** which are prorepresentable and let, on the other hand, $\mathbf{C}_1$ be the category of topological k-algebras of type $\mathcal{O} = \varprojlim \mathcal{O}_i$ with

$\mathrm{Spec}(\mathcal{O}_i) \in (\mathrm{Sch}/k)'$ (the topology in $\mathcal{O}$ is determined by the condition that it is the least fine one, making the homomorphism $\mathcal{O} \to \mathcal{O}_i$ continuous and the topology in the $\mathcal{O}_i$ is discrete). Let P and $\mathrm{P}^* \in \mathbf{C}$, prorepresented by $\mathcal{O}$ and $\mathcal{O}^*$ in $\mathbf{C}_1$; we want to show that there is a one-to-one correspondence between the functor morphisms $\Phi : \mathrm{P} \to \mathrm{P}^*$ and the continuous k-homomorphisms $\varphi : \mathcal{O}^* \to \mathcal{O}$ (this establishes in fact an equivalence between $\mathbf{C}^0$ and $\mathbf{C}_1$). In order to make this correspondence explicit let more precisely P (resp. P^*) be prorepresented by the projective system $(\mathcal{O}_i, \psi_{ji}, \sigma_i; i \in \mathrm{I})$ and $\mathcal{O} = \varprojlim \mathcal{O}_i$ (resp. $(\mathcal{O}_i^*, \psi_{ji}^*, \sigma_i^*; i \in \mathrm{I}^*)$ and $\mathcal{O}^* = \varprojlim \mathcal{O}_i^*$). Let $\Phi : \mathrm{P} \to \mathrm{P}^*$ be given, then we have elements $\Phi(\mathrm{Spec}(\mathcal{O}_i))(\sigma_i)$ in $\mathrm{P}^*(\mathrm{Spec}(\mathcal{O}_i))$, hence $\varphi_i \in \mathrm{Hom}_c(\mathcal{O}^*, \mathcal{O}_i)$; since $(\sigma_i) \in \varprojlim \mathrm{P}(\mathrm{Spec}(\mathcal{O}_i))$ we have $\varphi_j = \psi_{ji} \cdot \varphi_i$ for $j \leqq i$ and therefore we get a $\varphi \in \mathrm{Hom}_c(\mathcal{O}^*, \mathcal{O})$. Conversely, when $\varphi \in \mathrm{Hom}_c(\mathcal{O}^*, \mathcal{O})$ is given, then, for every $\mathrm{Spec}(\mathrm{A}) \in (\mathrm{Sch}/k)'$ and every $\xi : \mathrm{Spec}(\mathrm{A}) \to \mathrm{P}$, we have a $\gamma \in \mathrm{Hom}_c(\mathcal{O}, \mathrm{A})$, hence a map $\gamma \circ \varphi \in \mathrm{Hom}_c(\mathcal{O}^*, \mathrm{A})$, hence a $\xi^* : \mathrm{Spec}(\mathrm{A}) \to \mathrm{P}^*$ and this determines, as is easily checked, a functor morphism $\Phi : \mathrm{P} \to \mathrm{P}^*$. It is easily seen that the correspondence between Φ and φ is one to one.

(I.4.2) Let $\mathrm{T} \in (\mathrm{Sch}/k)$ be locally of finite type over k and consider the functor $h_\mathrm{T} = \mathrm{Hom}_k(\ , \mathrm{T})$. We denote (similarly as in I.1.5) the restriction of h_T to $(\mathrm{Sch}/k)'$ by h'_T; let t be a rational point of T and consider, with the notations of I.1.5, the subfunctor $(h_\mathrm{T})_t$ of h'_T (i.e. for $\mathrm{Spec}(\mathrm{A}) \in (\mathrm{Sch}/k)'$ the $(h_\mathrm{T})_t(\mathrm{Spec}(\mathrm{A}))$ consists of those morphism $\mathrm{Spec}(\mathrm{A}) \to \mathrm{T}$ which map Spec(A) set theoretically onto the point $t \in \mathrm{T}$). It is clear ([5], II, C 1, 195-11) that this subfunctor is prorepresentable and is prorepresented by $\hat{\mathcal{O}}_{t,\mathrm{T}}$.

Lemma **(I.4.3)**. — *Let* $\mathrm{P} \in \mathbf{Hom}((\mathrm{Sch}/k)^0, \mathbf{Ens})$ *be prorepresentable on* $(\mathrm{Sch}/k)'$ *and let* $\mathrm{R}_\varepsilon = \mathrm{R}$ *be the ring prorepresenting* P *at the rational point* $\varepsilon : \mathrm{Spec}(k) \to \mathrm{P}$. *Let* $\mathrm{T} \in (\mathrm{Sch}/k)$ *be locally of finite type over k, let t be a rational point on* T *and let* $\xi : \mathrm{T} \to \mathrm{P}$ *be such that* $\xi(t) = \varepsilon$. *Then there exists a continuous homomorphism* $\beta(\xi) : \mathrm{R} \to \hat{\mathcal{O}}_{t,\mathrm{T}}$ *uniquely determined by the functor morphism* $(h_\mathrm{T})_t \to \mathrm{P}_{\xi(t)}$ *(which is itself the "restriction" of the functor morphism* $\xi(\mathrm{X}) : h_\mathrm{T}(\mathrm{X}) \to \mathrm{P}(\mathrm{X})$ *for* $\mathrm{X} \in (\mathrm{Sch}/k)$*). If* ξ *is injective, then* $\beta(\xi)$ *is surjective.*

Proof. — The existence and uniqueness of $\beta(\xi)$ follows immediately from the remarks made in I.4.1 and from the fact that $\hat{\mathcal{O}}_{t,\mathrm{T}}$ is noetherian and prorepresents $(h_\mathrm{T})_t$. It remains to be shown that $\beta(\xi)$ is surjective if ξ is injective. First we note that $\beta(\xi)$ is surjective in the category $\mathbf{C}_1$ introduced in I.4.1 (i.e. the topological k-algebras of type $\mathcal{O} = \varprojlim \mathcal{O}_i$) because this follows from the observation made in I.4.1 (the duality between $\mathbf{C}$ and $\mathbf{C}_1$) and the fact that ξ is injective. It remains to be shown that surjectivity in the sense of this category $\mathbf{C}_1$ coincides with surjectivity in the set theoretic sense. We write $\mathcal{O}$ (resp. $\mathfrak{m}$) instead of $\mathcal{O}_{t,\mathrm{T}}$ (resp. $\mathfrak{m}_{t,\mathrm{T}}$). Put $\mathcal{O}_n = \mathcal{O}/\mathfrak{m}^{n+1}$ $(n = 0, 1, 2, \ldots)$ and let $\psi_n : \hat{\mathcal{O}} \to \mathcal{O}_n$ be the natural homomorphisms. Consider $\beta_1 = \psi_1 \circ \beta(\xi)$; β_1 is clearly still surjective in the sense of the category $\mathbf{C}_1$, but now, since $\mathcal{O}_1 = k \oplus \mathfrak{m}/\mathfrak{m}^2$ it is immediately seen that this implies that β_1 is surjective (in the usual set theoretic sense). Then we have, putting $\mathfrak{n}$ for the maximal ideal of $\mathrm{R} = \mathrm{R}_\varepsilon$, that $\mathfrak{m} \subset \mathfrak{n} . \hat{\mathcal{O}} + \mathfrak{m}^2$ and

hence $\mathfrak{m}=\mathfrak{n}.\hat{\mathcal{O}}$ (as is well known). We have in particular that $\hat{\mathcal{O}}$ is a quasi-finite R-module and hence an R-module of finite type (*E.G.A.*, $\mathbf{0}_{\mathrm{I}}$, Cor. 7.4.3). Since $\hat{\mathcal{O}} \subset \beta(\xi)(\mathrm{R})+\mathfrak{n}.\hat{\mathcal{O}}$ it then follows that $\hat{\mathcal{O}}=\beta(\xi)(\mathrm{R})$ by Nakayama's lemma.

Remark **(I.4.4)**. — Let the situation be as in lemma I.4.3 and assume that P satisfies also axiom P_2. If τ is the canonical element in $\mathrm{P}(\mathrm{Spec}(\mathrm{R}))$ (see remark I.2.3) then the continuous homomorphism $\beta(\xi) : \mathrm{R} \to \hat{\mathcal{O}}_{t,\mathrm{T}}$ is uniquely determined by the condition that $\tau \circ {}^a\beta(\xi) = \hat{\xi}_t$ (see I.1.10 for notations); this follows immediately from the construction of $\beta(\xi)$ (see I.4.3 and I.4.1), from the last lines of I.2.3 and from P_2.

I.5. On functors on the category of schemes of groups.

(I.5.1) *Again assume that the groundscheme* $\mathrm{S}=\mathrm{Spec}(k)$ *where* k *is a field and that there is given a functor* $\mathrm{P} \in \mathbf{Hom}((\mathrm{Sch}/k)^0, \mathbf{Ab})$. Furthermore we assume in this and the following two sections that k *is algebraically closed.* Let (Ab/k) be the category of preschemes of abelian groups which are locally of finite type over k and $(\mathrm{Ab}/k)_{\mathrm{red}}$ is the subcategory of reduced schemes of abelian groups, locally of finite type over k; also we introduce the subcategories $(\mathrm{Ab}/k)^*$ (resp. $(\mathrm{Ab}/k)^*_{\mathrm{red}}$) consisting of connected (resp. reduced and connected) objects of (Ab/k). Let $\mathrm{G} \in (\mathrm{Ab}/k)$ and $\alpha : \mathrm{G} \to \mathrm{P}$; we say that α *is a homomorphism* if the map $\alpha(\mathrm{T}) : \mathrm{G}(\mathrm{T}) \to \mathrm{P}(\mathrm{T})$, defined by $\alpha(\mathrm{T})(j)=\alpha \circ j$ for $j \in \mathrm{Hom}_k(\mathrm{T}, \mathrm{G})$, is a homomorphism, of groups for every $\mathrm{T} \in (\mathrm{Sch}/k)$ (1). We note that if $\alpha : \mathrm{G} \to \mathrm{P}$ is a homomorphism, then the closed subscheme $\mathrm{N}(\alpha)$ of G, the properties of which are described in axiom P_6 (i.e. $\mathrm{N}(\alpha)$ represents the kernel of the functor morphism $h_{\mathrm{G}} \to \mathrm{P}$), is a *subscheme of groups of* G; this follows immediately from the definitions. We call $\mathrm{N}(\alpha)$ *the kernel* of $\alpha : \mathrm{G} \to \mathrm{P}$.

Our main object in section I.5 is to prove the following:

Proposition 1. — *Let* $\mathrm{P} \in \mathbf{Hom}((\mathrm{Sch}/k)^0, \mathbf{Ab})$ *fulfil the conditions* P_1, P_4, P_5 *and* P_6 *and let* $\mathrm{P}'(\mathrm{G})$ *for* $\mathrm{G} \in (\mathrm{Ab}/k)_{\mathrm{red}}$ *be the set of all homomorphisms* $\alpha : \mathrm{G} \to \mathrm{P}$. *Then the functor* $\mathrm{G} \rightsquigarrow \mathrm{P}'(\mathrm{G})$ *is representable on* $(\mathrm{Ab}/k)_{\mathrm{red}}$. *If* (G, α) *is a couple which represents this functor (and which is clearly unique up to an isomorphism), then* α *is injective.*

(I.5.2) The proof of the above proposition is split up in several lemmas.

Lemma **(I.5.2)**. — *Let* $\mathrm{G} \in (\mathrm{Ab}/k)^*$ *and* $\alpha : \mathrm{G} \to \mathrm{P}$ *a homomorphism. There exists (if* P *satisfies* P_4 *and* P_6*) a* $\mathrm{G}_1 \in (\mathrm{Ab}/k)^*$, *an injective homomorphism* $\alpha_1 : \mathrm{G}_1 \to \mathrm{P}$ *and a homomorphism of the schemes of groups* $\varphi : \mathrm{G} \to \mathrm{G}_1$ *such that* $\alpha=\alpha_1 \circ \varphi$. *Moreover if* $\mathrm{G} \in (\mathrm{Ab}/k)^*_{\mathrm{red}}$ *then we can take* $\mathrm{G}_1 \in (\mathrm{Ab}/k)^*_{\mathrm{red}}$.

Proof. — Let $\mathrm{N}=\mathrm{N}(\alpha)$ be the kernel of α (axiom P_6). Consider the quotient $\mathrm{G}_1=\mathrm{G}/\mathrm{N}$ (see [5], III, cor. 7.4) (2). The natural homomorphism $\varphi : \mathrm{G} \to \mathrm{G}_1$ is

(1) The $\alpha : \mathrm{G} \to \mathrm{P}$ is a homomorphism if and only if $\alpha.\mu - \alpha.p_1 - \alpha.p_2$ is the zero-element in $\mathrm{P}(\mathrm{G} \times_k \mathrm{G})$, where $\mu : \mathrm{G} \times_k \mathrm{G} \to \mathrm{G}$ is the group multiplication and p_i $(i=1, 2)$ is the projection.

(2) See also the exposition of P. Gabriel in the Séminaire " Schémas en Groupes ", S.G.A., Exp. V, I.H.E.S., 1963.

faithfully flat and quasi-compact (*loc. cit.*, théorème 7.2); if G is connected the same holds for G_1 and if G is reduced then G_1 is reduced. Also G_1 if of finite type over k (cf. *loc. cit.*, théorème 6.2 (i)). Consider the diagram

$$G\times_{G_1}G \underset{p_2}{\overset{p_1}{\rightrightarrows}} G \xrightarrow{\varphi} G_1$$

Let $\alpha'=\alpha\circ p_1$ and $\alpha''=\alpha\circ p_2$; then (since α is a homomorphism) $\alpha'-\alpha''=\alpha\circ(p_1-p_2)$; since $G\times_{G_1}G\tilde{\to}G\times_k N$ it follows that $p_1-p_2 : G\times_{G_1}G\to G$ factors through N; therefore, by the definition of N, we have $\alpha'=\alpha''$. Hence by axiom P_4 there exists $\alpha_1 : G_1\to P$ such that $\alpha=\alpha_1\circ\varphi$. It follows easily that α_1 is a homomorphism and that α_1 is injective.

(I.5.3) Consider the set E of couples (G, α) with $G\in(Ab/k)^*$ and $\alpha : G\to P$ a homomorphism. We preorder this set as follows: $(G, \alpha)>(G', \alpha')$ if there exists a homomorphism (of schemes of groups) $\varphi : G'\to G$ such that $\alpha'=\alpha\circ\varphi$. *This set is filtered under this preordering*; for let (G_1, α_1) and (G_2, α_2) be two couples, consider the product $G_1\times_k G_2$, the projections $p_i : G_1\times_k G_2\to G_i$ $(i=1, 2)$ and the homomorphism $\alpha=\alpha_1.p_1+\alpha_2.p_2 : G_1\times_k G_2\to P$. Clearly $(G_1\times_k G_2, \alpha)>(G_i, \alpha_i)$ $(i=1, 2)$. Furthermore the subset $\{(G, \alpha)\}$ with α *injective* is *cofinal* in this preordering. This follows immediately from lemma I.5.2 (we assume that P_4 and P_6 hold). Finally we note that the above facts are also true if we restrict to couples (G, α) with $G\in(Ab/k)^*_{red}$ (here we use the fact that k is algebraically closed, or at least that k is perfect, in order to conclude that $G_1\times_k G_2$ is again in $(Ab/k)^*_{red}$).

Lemma **(I.5.4)**. — *Consider couples* (G_i, α_i) $(i=1, 2, \ldots)$ *with* $G_i\in(Ab/k)^*_{red}$ *and* $\alpha_i : G_i\to P$ *injective homomorphisms. Suppose that there exist homomorphisms of the schemes of groups* $\varphi_i : G_i\to G_{i+1}$ $(i=1, 2, \ldots)$ *such that* $\alpha_i=\alpha_{i+1}\circ\varphi_i$. *If the functor* P *satisfies* P_1, *then there exists an integer* i_0 *such that* φ_i *is an isomorphism for* $i>i_0$.

Proof. — Since the α_i are injective, the φ_i are injective and hence closed immersions ([5], III, Cor. 7.4). The corresponding ring homomorphisms $\psi_i : \mathcal{O}_{e,G_{i+1}}\to\mathcal{O}_{e,G_i}$ are therefore surjective and it clearly suffices to prove that the ψ_i are injective for i larger than a certain integer i_0. Consider the continuous prolongations $\hat{\psi}_i : \hat{\mathcal{O}}_{e,G_{i+1}}\to\hat{\mathcal{O}}_{e,G_i}$; it suffices to show that these are injective for $i>i_0$. Consider also the noetherian ring $R=R_\varepsilon$ which prorepresents P at the unit element ε of $P(\mathrm{Spec}(k))$; by lemma I.4.3 there exist continuous surjective homomorphisms $\beta_i : R\to\hat{\mathcal{O}}_{e,G_i}$ $(i=1, 2, \ldots)$ corresponding with the functor morphism $(h_{G_i})_e\to P_\varepsilon$ (with the notations of I.4.2 and I.1.5). From the fact that, clearly, the continuous homomorphisms $\hat{\psi}_i : \hat{\mathcal{O}}_{e,G_{i+1}}\to\hat{\mathcal{O}}_{e,G_i}$ correspond with the functor morphisms $(h_{G_i})_e\to(h_{G_{i+1}})_e$ and from $\alpha_i=\alpha_{i+1}\circ\varphi_i$ it follows that $\hat{\psi}_i.\beta_{i+1}=\beta_i$ $(i=1, 2, \ldots)$. Let $\mathfrak{R}_i=\mathrm{Kernel}(\beta_i)$; then $\mathfrak{R}_i\supseteq\mathfrak{R}_{i+1}$ and we must show that $\mathfrak{R}_i=\mathfrak{R}_{i+1}=\mathfrak{R}_{i+2}=\ldots$ for i larger than a certain i_0. However $\mathcal{O}_{e,G_i}$ and hence $\hat{\mathcal{O}}_{e,G_i}$ is a regular local ring (G_i is reduced!); hence $\mathfrak{R}_i$ is a prime ideal and our assertion

follows immediately since a sequence of decreasing prime ideals $\mathfrak{R}_i$ in a noetherian ring R is finite.

(I.5.5) *Proof of proposition 1.* — Consider first the restriction P'' of the functor P' (from proposition 1) to $(\mathrm{Ab}/k)^*_{\mathrm{red}}$. Let E_{red} be the set of couples (H, β) with $H \in (\mathrm{Ab}/k)^*_{\mathrm{red}}$ and $\beta : H \to P$ a homomorphism. By I.5.3 the set E_{red} is filtered with respect to the preordering introduced there. It suffices to show that there is a couple (G_0, α_0) in E_{red} such that $(G_0, \alpha_0) > (H, \beta)$ for every (H, β) in E_{red} and with an injective α_0. By lemma I.5.2 the subset E' of E_{red} consisting of couples (H, β) with injective β is cofinal; therefore we can restrict our attention to this subset E'. E' itself is by I.5.3 and lemma I.5.2 also filtered and the existence of (G_0, α_0) follows now immediately from lemma I.5.4.

Next we turn to the category $(\mathrm{Ab}/k)_{\mathrm{red}}$ itself. Using the injective homomorphism $\alpha_0(\mathrm{Spec}(k)) : \mathrm{Hom}_k(\mathrm{Spec}(k), G_0) = G_0(k) \to P(\mathrm{Spec}(k))$ we identify $G_0(k)$ with a subgroup of $P(\mathrm{Spec}(k))$. Let $\mathfrak{g} = P(\mathrm{Spec}(k))/G_0(k)$ and let a_σ ($\sigma \in \mathfrak{g}$) be a set of representatives of $\mathfrak{g}$ in $P(\mathrm{Spec}(k))$ (we take for the representative a_0 of the unit element of $\mathfrak{g}$ the unit element ε of $P(\mathrm{Spec}(k))$). Let G_σ be the same scheme as G_0 and $G = \coprod_{\sigma \in \mathfrak{J}} G_\sigma$; we want to introduce a group structure on G. Let $\tilde{\mu} : G_0 \times_k G_0 \to G_0$ be the group multiplication on G_0. We define $\mu : G \times_k G \to G$ as follows: $G \times_k G = \coprod_{\sigma,\tau \in \mathfrak{J}} G_\sigma \times_k G_\tau$; let $\mu_{\sigma\tau} = \mu / G_\sigma \times_k G_\tau \to G_{\sigma+\tau}$ be defined by $\mu_{\sigma\tau} = t_{a_\sigma + a_\tau - a_{\sigma+\tau}} \circ \tilde{\mu}$ where t_b for $b \in G_0$ denotes the translation by b (we note that $a_\sigma + a_\tau - a_{\sigma+\tau}$ is an element of the subgroup $G_0(k)$ of $P(\mathrm{Spec}(k))$ and determines therefore a unique rational point on G_0). An inverse $\iota : G \to G$ is introduced in a similar fashion and in this way we obtain a group structure on G which on the connected component G_0 of the unit element coincides with the original group structure on G_0. Next we define a homomorphism $\alpha : G \to P$. By axiom P_5 it suffices to define $\alpha_\sigma = \alpha / G_\sigma$; let $\alpha_\sigma = \alpha_0 + a_\sigma \circ \pi$ where $\pi : G_0 \to \mathrm{Spec}(k)$ is the structure morphism. It is easily checked that α is indeed a homomorphism, that α is injective (one uses P_6) and that $\alpha(\mathrm{Spec}(k)) : G(k) = \mathrm{Hom}_k(\mathrm{Spec}(k), G) \to P(\mathrm{Spec}(k))$ is an isomorphism (we identify both groups in the following by this isomorphism).

Finally we want to show that (G, α) represents the functor P' on $(\mathrm{Ab}/k)_{\mathrm{red}}$. Let (H, β) be a couple such that $H \in (\mathrm{Ab}/k)_{\mathrm{red}}$ and $\beta : H \to P$ a homomorphism. We have to show that there exists a unique homomorphism of groups $\varphi : H \to G$ such that $\beta = \alpha \circ \varphi$. Now $H = \coprod_{j \in J} H_j$, where H_0 is the connected component of the identity and H_j are the translations of H_0. Since (G_0, α_0) represents the restriction of P' to $(\mathrm{Ab}/k)^*_{\mathrm{red}}$, there exists a homomorphism $\varphi_0 : H_0 \to G$ such that $\beta_0 = \beta / H_0 = \alpha_0 \circ \varphi_0$. Let b_j be a rational point of H_j and let $\sigma(j) \equiv \beta(b_j) \bmod G_0(k)$; consider the morphism $\varphi_j : H_j \to G_{\sigma(j)}$ defined by $\varphi_j = t_{\beta(b_j)} \circ \varphi_0 \circ t_{b_j}$ (t denotes, as above, the translation). Since the H_j are open in H we obtain a morphism $\varphi : H \to G$ such that $\varphi / H_j = \varphi_j$. It is easily checked that φ is a homomorphism of the schemes of groups and that $\alpha \circ \varphi = \beta$. The uniqueness of φ follows from the fact that α is injective.

I.6. On functors on the category of reduced schemes.

(I.6.1) Let again k be an algebraically closed field. By $(\mathrm{Sch}/k)_{\mathrm{red}}$ we denote the subcategory of (Sch/k) consisting of reduced preschemes. In this section our main object is to prove the following:

Proposition 2. — Let $\mathrm{P} : (\mathrm{Sch}/k)^0 \to \mathbf{Ab}$ *be a functor which satisfies* P_4, P_5, P_6 *and* P_7. *Let* G *be a prescheme of abelian groups which is reduced and locally of finite type over k and let* $\alpha : \mathrm{G} \to \mathrm{P}$ *be an injective homomorphism with the following properties:*

(i) *For every* $\mathrm{H} \in (\mathrm{Ab}/k)_{\mathrm{red}}$ *and homomorphism* $\beta : \mathrm{H} \to \mathrm{P}$ *there exists a homomorphism* $\varphi : \mathrm{H} \to \mathrm{G}$ *such that* $\beta = \alpha \circ \varphi$.

(ii) $\alpha(\mathrm{Spec}(k)) : \mathrm{G}(\mathrm{Spec}(k)) \to \mathrm{P}(\mathrm{Spec}(k))$ *is bijective.*

Then (G, α) *represents the restriction of the functor* P *to the category of the preschemes which are direct sums of preschemes which are reduced and of finite type over k.*

Again the proof will be split up in several lemmas. *We assume tacitly in section I.6 that we have a functor* $\mathrm{P} : (\mathrm{Sch}/k)^0 \to \mathbf{Ab}$ *which satisfies* P_4, P_5 *and* P_6.

(I.6.2) Let T be an affine scheme, of finite type over k and let $\xi : \mathrm{T} \to \mathrm{P}$. The symmetric group $\mathfrak{S}_n$ operates on the n-fold product $\mathrm{T}_n = \mathrm{T} \times_k \mathrm{T} \times_k \ldots \times_k \mathrm{T}$. If $\sigma \in \mathfrak{S}_n$ then we denote by the same letter σ the morphism $\mathrm{T}_n \to \mathrm{T}_n$ obtained from the permutation σ of the factors; we denote the multiplication morphism corresponding with the operation of $\mathfrak{S}_n$ on T_n by $\pi_n : \mathrm{T}_n \times_k \mathfrak{S}_n \to \mathrm{T}_n$ (with the notations of *S.G.A.*, V, p. 6); the projection on the first factor of $\mathrm{T}_n \times_k \mathfrak{S}_n$ we denote by $q_1 : \mathrm{T}_n \times_k \mathfrak{S}_n \to \mathrm{T}_n$. We remark that $\mathrm{T}_n \times_k \mathfrak{S}_n$ is the product of T_n with the direct sum of $n!$ copies of $\mathrm{Spec}(k)$; it can therefore also be considered as $\coprod_{\sigma \in \mathfrak{S}_n} \mathrm{T}_n$; then the restriction of π_n to the copy of T_n corresponding with $\sigma \in \mathfrak{S}_n$ is nothing else but the morphism $\sigma : \mathrm{T}_n \to \mathrm{T}_n$. If $\eta : \mathrm{T}_n \to \mathrm{P}$, then $\mathfrak{S}_n$ operates on η by $\eta^\sigma = \eta \circ \sigma$; it follows from the above remarks and from the axiom P_5 that the statement " η is *invariant* under $\mathfrak{S}_n$ " (i.e. $\eta^\sigma = \eta$ for all $\sigma \in \mathfrak{S}_n$) is equivalent with the statement $\eta \circ \pi_n = \eta \circ q_1$ (because both statements are equivalent on each copy T_n of $\mathrm{T}_n \times_k \mathfrak{S}_n$). Consider the element $\xi_n = \Sigma_{i=1}^n \xi \circ p_i$ of $\mathrm{P}(\mathrm{T}_n)$ (where $p_i : \mathrm{T}_n \to \mathrm{T}$ is the projection on the i-th factor); since $\xi_n \circ \sigma = \Sigma_{i=1}^n \xi \circ p_i \circ \sigma = \Sigma_{i=1}^n \xi \circ p_{\sigma(i)} = \xi_n$ for $\sigma \in \mathfrak{S}_n$ we have that ξ_n is invariant under $\mathfrak{S}_n$.

Lemma **(I.6.3)**. — *Let* T *be an affine scheme,* T_n *the n-fold product,* $\mathrm{T}_{(n)}$ *the symmetric n-fold product (both over k) and* $\varphi : \mathrm{T}_n \to \mathrm{T}_{(n)}$ *the natural morphism. Let* $\xi :$ T $\to$ P *and* $\xi_n = \Sigma_{i=1}^n \xi \circ p_i$. *There exists an affine open set* U *in* $\mathrm{T}_{(n)}$ *and a* $\zeta : \mathrm{U} \to \mathrm{P}$ *such that* $\zeta \circ (\varphi/\mathrm{V}) = \xi_n/\mathrm{V}$ *where* $\mathrm{V} = \varphi^{-1}(\mathrm{U})$.

Proof. — There exists an affine open set U on $\mathrm{T}_{(n)}$ such that for every closed point $s \in \mathrm{U}$ the set $\varphi^{-1}(s)$ consists of $n!$ different points. The groups of inertia (*S.G.A.*, V, p. 7) for the points of $\mathrm{V} = \varphi^{-1}(\mathrm{U})$ are trivial and therefore (*S.G.A.*, V, prop. 2.6) the restriction $\varphi/\mathrm{V} : \mathrm{V} \to \mathrm{U}$ is faithfully flat (and quasi-compact) and V is formally a principal homo-

geneous space under $(\mathfrak{S}_n)_U$ (with the notations of *S.G.A.*, V, p. 10). Therefore $V \times_U V \tilde{\to} V \times_U (\mathfrak{S}_n)_U$. Consider the two projections $p_1, p_2 : V \times_U V \to V$; under the above isomorphism p_1 corresponds with $q_1/V \times_U (\mathfrak{S}_n)_U$ and p_2 with $\pi_n/V \times_U (\mathfrak{S}_n)_U$ (q_1 and π_n are as in section I.6.2). Since we have seen that ξ_n is invariant we have $\xi_n \circ q_1 = \xi_n \circ \pi_n$; hence also $(\xi_n/V) \circ p_1 = (\xi_n/V) \circ p_2$. The existence of $\zeta : U \to P$ with the required properties follows then immediately from the fact that $\varphi/V : V \to U$ is faithfully flat and quasi-compact and from P_4.

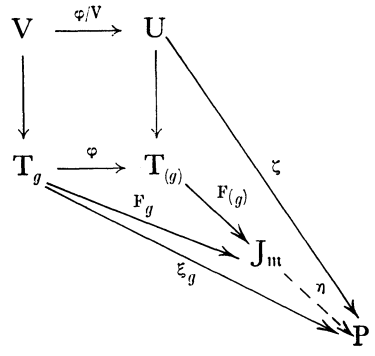
Lemma **(I.6.4)** [1]. — *Let* $P \in \mathbf{Hom}((\mathrm{Sch}/k)^0, \mathbf{Ab})$ *satisfy (as agreed upon)* P_4, P_5 *and* P_6. *Let* C *be a non-singular, complete curve (in* (Sch/k)*) and* S *a finite set of closed points on* C; *let* $\mathfrak{m}$ *be a module with support on* S. *Let* $f : C - S \to J_{\mathfrak{m}}$ *be the canonical morphism of* $C - S$ *into the generalized jacobian defined by the module* $\mathfrak{m}$ ([9], p. 89), *normalized by the condition* $f(t_0) = e$ *(the unit of* $J_{\mathfrak{m}}$*) where* t_0 *is a closed point on* $C - S$. *If* $\xi : C - S \to P$ *is such that* $\xi(t_0) = 0$ *(the unit of* $P(\mathrm{Spec}(k))$*), then the following two conditions are equivalent:*

(i) *There exists a homomorphism* $\eta : J_{\mathfrak{m}} \to P$ *such that* $\xi = \eta \circ f$.

(ii) *The map* $\xi(\mathrm{Spec}(k)) : \mathrm{Hom}_k(\mathrm{Spec}(k), C - S) \to P(\mathrm{Spec}(k))$ *has the module* $\mathfrak{m}$ *(in the sense of I.1.9).*

Proof. — (i) $\Rightarrow$ (ii) Follows immediately from the definitions and ([9], Th. 1 *b* on p. 95).

(ii) $\Rightarrow$ (i) Let $g = \dim J_{\mathfrak{m}}$. Put $T = C - S$; consider, with the notations of I.6.2, the element $\xi_g = \Sigma_{i=1}^{g} \xi \circ p_i$ on the g-fold product T_g (over k). Let $T_{(g)}$ be the symmetric g-fold product and $\varphi : T_g \to T_{(g)}$ the canonical morphism. Also we have to consider the morphism $F_g : T_g \to J_{\mathfrak{m}}$ defined by $F_g = \Sigma_{i=1}^{g} f \circ p_i$ ($p_i : T_g \to T$ the projection on the $i-th$ factor); by ([9], Chap. V, § 9, Th. 1 *c*) there exists a birational transformation $F_{(g)} : T_{(g)} \to J_{\mathfrak{m}}$ such that (as rational maps) $F_g = F_{(g)} \circ \varphi$ (in fact it is well known that $F_{(g)}$ is a morphism). By lemma I.6.3 there is an affine open set U on $T_{(g)}$ and a $\zeta : U \to P$ such that $\zeta \circ (\varphi/V) = \xi_g/V$ where $V = \varphi^{-1}(U)$; clearly we can restrict U in such a way that U is isomorphic under $F_{(g)}$ with an open set U′ of $J_{\mathfrak{m}}$ and we identify U and U′ in the following. We have now the following commutative diagram:



[1] Compare with [4], Th. 2 on p. 8-19.

We want to fill in the dotted arrow with a homomorphism $\eta : J_m \to P$. Consider the morphism $\lambda : U \times_k U \to J_m$ obtained by composition of the open immersion

$$U \times_k U \to J_m \times_k J_m$$

and the group multiplication $\mu : J_m \times_k J_m \to J_m$. Both there morphisms are simple; this is clear for the open immersion; in order to see that μ is simple, we remark that the structure morphism $J_m \to \mathrm{Spec}(k)$ is simple, hence also the morphism $\mu' : J_m \times_k J_m \to J_m$ obtained from this structure morphism by base extension is simple, and finally μ itself is simple because μ is obtained from μ' by applying on $J_m \times_k J_m$ the isomorphism $(a, b) \to (a+b, b)$. Hence λ is simple (*S.G.A.*, II, prop. 1.3). Write $U_2 = U \times_k U$, let $q_i : U_2 \to U$ $(i = 1, 2)$ be the projections and let $\zeta_2 = \zeta \circ q_1 + \zeta \circ q_2$. Consider the diagram

$$U_2 \times_{J_m} U_2 \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} U_2 \xrightarrow{\lambda} J_m$$

where π_1 and π_2 are the projections; π_1 and π_2 are obtained from λ by base extension and are therefore simple. We want to show that $\zeta_2 \circ \pi_1 = \zeta_2 \circ \pi_2$. First of all we shall prove that if a is a closed point on $U_2 \times_{J_m} U_2$, then $\zeta_2 \circ \pi_1(a) = \zeta_2 \circ \pi_2(a)$. In order to see this we note that such a point a is of the form (b, d), with $b = (b_1, b_2)$ and $d = (d_1, d_2)$ and b_1, b_2, d_1 and d_2 are closed points on U. Let B_i (resp. D_j) be the divisors on T corresponding with b_i (resp. d_j). We have $\lambda(b) = \lambda(d)$, hence by the properties of generalized jacobians ([9], Chap. V, § 9, Th. 1 *b*) this implies $B_1 + B_2$ is $\mathfrak{m}$-linearly equivalent with $D_1 + D_2$. We have $\zeta_2 \circ \pi_1(b, d) = \zeta_2(b) = \zeta(b_1) + \zeta(b_2)$; since the map $\xi(\mathrm{Spec}(k))$ has the module $\mathfrak{m}$ and since $\xi_g/V = \zeta \circ (\varphi/V) = \zeta \circ (F_g/V)$, this is equal to $\zeta(d_1) + \zeta(d_2) = \zeta_2(d) = \zeta_2 \circ \pi_2(b, d)$. Therefore $\zeta_2 \circ \pi_1(b, d) = \zeta_2 \circ \pi_2(b, d)$. Since U_2 is reduced (k is algebraically closed) and since π_1 is simple we have that $U_2 \times_{J_m} U_2$ is reduced (*S.G.A.*, II, prop. 3.1). The assertion $\zeta_2 \circ \pi_1 = \zeta_2 \circ \pi_2$ follows now from the following trivial lemma I.6.5:

Lemma **(I.6.5)**. — *Let* X *be a prescheme, of finite type over k and reduced and let* $\xi : X \to P$ (P *satisfies* P_6). *If* $\xi(x) = 0$ *for all closed points x on* X *then* $\xi = 0$.

Proof. — Consider the closed subscheme $N(\xi)$ of X such that for $\alpha : X' \to X$ we have $\xi \circ \alpha = 0$ if and only if α factors through $N(\xi)$. Since the closed points are everywhere dense on X we have that set theoretically $N(\xi) = X$. Since X is reduced we have $N(\xi) = X$.

(I.6.6) *Continuation of the proof of lemma I.6.4.* — The morphism $\lambda : U_2 \to J_m$ is, as is well-known, surjective, and since λ is simple (and hence flat), it follows from $\zeta_2 \circ \pi_1 = \zeta_2 \circ \pi_2$ by axiom P_4 that there exists a $\eta : J_m \to P$ such that $\eta \circ \lambda = \zeta_2$. Next we want to show that η is a homomorphism; for this we must prove that $\eta_0 = \eta \circ \mu - \eta \circ p_1 - \eta \circ p_2$ (where μ is, as above, the group multiplication $J_m \times_k J_m \to J_m$ and $p_i : J_m \times_k J_m \to J_m$

the projection on the i-th factor) is the zero element of $P(J_m \times_k J_m)$. Since $J_m \times_k J_m$ is reduced it suffices, again by lemma I.6.5, to check $\eta_0(x) = 0$ for every closed point x on $J_m \times_k J_m$. Let $x = (y, z)$ with y and z closed on J_m. Let furthermore $a = (a_1, a_2)$, $b = (b_1, b_2)$ and $d = (d_1, d_2)$ be closed points on U_2 such that $\lambda(a) = y$, $\lambda(b) = z$ and $\lambda(d) = y + z$. The points a_i, b_j and d_l are closed on U; let A_i, B_j and D_l be the corresponding divisors on T (i, j and $l = 1, 2$) and denote finally by E the divisor on T consisting of $2g$-times the point t_0. Then $\eta_0(x) = \eta_0(y, z) = \zeta_2(d) - \zeta_2(a) - \zeta_2(b) = \zeta(d_1) + \zeta(d_2) - \zeta(a_1) - \zeta(a_2) - \zeta(b_1) - \zeta(b_2)$ and the fact that this is zero follows, again using that the map $\xi(\mathrm{Spec}(k))$ has the module $\mathfrak{m}$, from the fact that $D_1 + D_2 + E$ is $\mathfrak{m}$-linearly equivalent with $A_1 + A_2 + B_1 + B_2$. Finally we want to show that $\eta/U = \zeta$. Again it is only necessary to check this at closed points and this is done in a similar way as above, always using the module $\mathfrak{m}$. Then we have $(\eta/U) \circ (F_g/\varphi^{-1}(U)) = \xi_g/\varphi^{-1}(U)$; the equality $\eta \circ F_g = \xi_g$ follows then from P_6. The fact that $\eta \circ f = \xi$ follows then immediately if we remark that f is the composition of the morphism $t \to (t, t_0, \ldots, t_0)$ from T to T_g and the morphism $F_g : T_g \to J_m$. This completes the proof of lemma I.6.4.

(I.6.7) For the proof of proposition 2 we also need the following two lemmas:

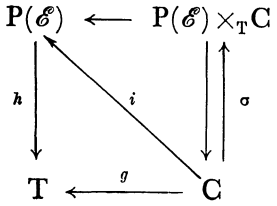
Lemma **(I.6.7)**. — *Let* $f : Z \to T$ *be a separated morphism of algebraic k-preschemes. Suppose that for every normal integral curve* $g : C \to T$ *we have that* $f \times_T 1_C = f_{(C)} : Z \times_T C \to C$ *is proper. Then f itself is proper.*

Proof. — There is no loss in generality if we drop the assumption that C is normal, for let C be an arbitrary integral curve over k and let $\lambda : C_1 \to C$ be the normalization. Consider the diagram:

$$\begin{array}{ccccc} Z & \longleftarrow & Z \times_T C & \xleftarrow{\mu} & Z \times_T C \times_C C_1 = Z \times_T C_1 \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_{(C)}} & & \downarrow{\scriptstyle f_{(C_1)}} \\ T & \xleftarrow{g} & C & \xleftarrow{\lambda} & C_1 \end{array}$$

We have that $\lambda \circ f_{(C_1)}$ is proper (*E.G.A.*, II, prop. 5.4, (ii)), $\mu = \lambda_{(Z)}$ is surjective (*E.G.A.*, I, prop. 3.5.2, (ii)), $f_{(C)}$ is separated (*E.G.A.*, I, prop. 5.5.1, (iv)) and hence $f_{(C)}$ is proper (*E.G.A.*, II, cor. 5.4.3, (ii)). Next we note that we can assume that Z is quasi-projective over T; this follows from the lemma of Chow (*E.G.A.*, II, Th. 5.6.1) and (*E.G.A.*, II, prop. 5.4.2, (ii) and cor. 5.4.3, (ii)). Let $j : Z \to P(\mathscr{E})$ (*E.G.A.*, II, prop. 5.3.2) be an immersion in a projective fiber space (where $\mathscr{E}$ is a quasi-coherent $\mathcal{O}_T$-Module of finite type); our problem is to prove that Z is projective over T (*E.G.A.*, II, Th. 5.5.3, (i)), i.e., we must show that Z is closed in $P(\mathscr{E})$ (we identify Z with a subscheme of $P(\mathscr{E})$). Write $h : P(\mathscr{E}) \to T$ for the structure morphism and $\overline{Z}$ for the closure of Z in $P(\mathscr{E})$. Suppose $\overline{Z} \neq Z$ then there is a closed point $Q \in \overline{Z}$ and $Q \notin Z$; there exists an integral curve ([3], III, § III, prop. 1) C in $\overline{Z}$ through Q such that

$C \cap Z \neq \emptyset$. Let $i : C \to P(\mathscr{E})$ be the immersion and $g = h \circ i$. Consider the following diagram:



where σ is the section $C \to P(\mathscr{E}) \times_T C$ corresponding with $i : C \to P(\mathscr{E})$. Clearly $\sigma(Q) \notin \sigma(C \cap Z)$ and in fact $\notin Z \times_T C$, hence $Z \times_T C$ is not closed in $P(\mathscr{E}) \times_T C$. Therefore (*E.G.A.*, II, cor. 5.4.4) the quasi-projective morphism $f_{(C)} : Z \times_T C \to C$ is not projective, hence not proper, which contradicts our assumption (cf. beginning of the proof) since C is an integral curve. Hence $\overline{Z} = Z$, i.e. $f : Z \to T$ is proper.

Lemma (**I.6.8**). — *Let* $q : Z \to T$ *be a morphism of algebraic k-preschemes. If* q *is a proper monomorphism then* q *is a closed immersion.*

Proof. — From the fact that q is a monomorphism follows that set theoretically q is injective; from the fact that q is proper follows that q is closed. In order to prove the lemma it suffices to prove (*E.G.A.*, I, prop. 4.2.2, *b)*) that $\mathscr{O}_T \to q_*(\mathscr{O}_Z)$ is surjective; from q proper follows that $q_*(\mathscr{O}_Z)$ is a $\mathscr{O}_T$-Module of finite type and it therefore suffices (by Nakayama's lemma) to show that $k(t) \to q_*(\mathscr{O}_Z) \otimes_{\mathscr{O}_T} k(t) = A$ is surjective for every $t \in T$. From the fact that q is a monomorphism follows by base extension that $Z \times_T \mathrm{Spec}(k(t)) \to \mathrm{Spec}(k(t))$ is a monomorphism; but this means that the natural map $A \otimes_{k(t)} A \to A$ is injective and hence $A = k(t)$.

(**I.6.9**) *The proof of proposition 2.* — Let T be a reduced prescheme which is a direct sum of preschemes of finite type over k. We have to show that the map $\alpha(T) : G(T) \to P(T)$, defined by $\alpha(T)(\varphi) = \alpha \circ \varphi$ for $\varphi : T \to G$, is bijective. The injectivity follows from the fact that α is injective. There remains to show that $\alpha(T)$ is surjective. Let $\xi : T \to P$ be given.

First we assume that T *is of the type* C—S, where C is a non-singular, complete curve defined over k and S a finite set of closed points on C. Let t_0 be a closed point on T and $\xi_1 = \xi - \xi(t_0)$. From the fact that the map $\xi_1(\mathrm{Spec}(k)) : T(\mathrm{Spec}(k)) \to P(\mathrm{Spec}(k))$ has, by P_7, a module $\mathfrak{m}$ with support on S, it follows by lemma I.6.4 the existence of a homomorphism $\eta : J_{\mathfrak{m}} \to P$ ($J_{\mathfrak{m}}$ is the generalized jacobian of C with respect to $\mathfrak{m}$) such that $\xi_1 = \eta \circ f$ ($f : T \to J_{\mathfrak{m}}$ is the canonical map normalized by $f(t_0) = e$). The existence of a morphism $\varphi_1 : T \to G$ such that $\alpha \circ \varphi_1 = \xi_1$ follows now from property (i) of the couple (G, α) applied to $J_{\mathfrak{m}}$ and the homomorphism $\eta : J_{\mathfrak{m}} \to P$. It is now immediately checked that the morphism $\varphi = t_{\xi(t_0)} \circ \varphi_1$ (where $t_{\xi(t_0)}$ is the translation on G by the point on G corresponding with $\xi(t_0)$; the existence of such a point follows from property (ii) of the couple (G, α)) is such that $\alpha \circ \varphi = \xi$.

Next let T *be reduced and of finite type over k.* Consider the closed subscheme Z on $T \times_k G$ which represents the functor $h_T \times_P h_G$ (see remark I.2.4; the structure morphisms over P are $\xi : T \to P$ and $\alpha : G \to P$), let $q : Z \to T$ (resp. $p : Z \to G$) be the projection (or more precisely the composition of the embedding of Z into $T \times_k G$ and the projection). If we show that q is an isomorphism, then clearly $\varphi = p \circ q^{-1}$ is such that $\alpha \circ \varphi = \xi$. From the fact that α is injective follows easily q *is a monomorphism.* Next we want to show that q is proper. In order to see this we first note that q is certainly proper if T is a normal curve, for we have just seen that then there exists a morphism $\varphi : T \to G$ such that $\alpha \circ \varphi = \xi$, and from the injectivity of α it follows that $\varphi \circ q = p$ and hence Z is nothing else but the graph of φ. Let $g : C \to T$, where C is a normal integral curve; then it is a straightforward matter to check that $Z \times_T C$ represents the functor $h_C \times_P h_G$, where now the structure morphisms are $\xi \circ g : C \to P$ and $\alpha : G \to P$. By the remark just made we have that $q \times_T I_C : Z \times_T C \to C$ is proper. Since clearly q is separated it follows by lemma I.6.7 that q *is proper.* Hence by lemma I.6.8 q *is a closed immersion.* However $q(Z) = T$, for otherwise we could take a closed point $t_0 \in T$ and $t_0 \notin q(Z)$ (this is possible, T is of finite type over k); but this contradicts the fact that there corresponds, by property (ii) of the couple (G, α), a point on G with the element $\xi(t_0)$ of $P(\mathrm{Spec}(k))$. Hence q *is an isomorphism.*

Let finally $T = \coprod T_i$, with all T_i reduced and of finite type over k. Then there exists by what we have just proved $\varphi_i : T_i \to G$ such that $\alpha \circ \varphi_i = \xi / T_i$. Let $\varphi : T \to G$ be such that $\varphi / T_i = \varphi_i$; then $\alpha \circ \varphi = \xi$ by axiom P_5.

I.7. The step from reduced to arbitrary preschemes.

(I.7.1) In this section k is an algebraically closed field and our main objet is to prove the following proposition:

Proposition 3. — Let $P : (\mathrm{Sch}/k)^0 \to \mathbf{Ab}$ *be a functor which satisfies* P_1, P_2, P_3, P_4, P_5 *and* P_6 *and suppose that the restriction of* P *to the category of the preschemes which are a direct sum of reduced preschemes of finite type over k is representable. Then* P *itself is representable by a scheme of abelian groups which is locally of finite type over k.*

(I.7.2) The proof of proposition 3 is spread over several lemmas and is finished in lemma I.7.12. *We assume tacitly in section I.7 that* P *has the properties mentioned in prop. 3. Also we denote throughout I.7 by* (G, α) *the couple which represents the restriction of* P *to the subcategory* **C** *of* (Sch/k) *consisting of the preschemes which are a direct sum of reduced preschemes of finite type over k.* We claim that G *is a scheme of abelian groups* and $\alpha : G \to P$ is *an injective homomorphism.* It is clear that G has a group structure in the subcategory **C** mentioned above; however then there exist a group multiplication $\mu : G \times_k G \to G$, an inverse $G \to G$ and a unit section with the right properties and therefore G is a scheme of groups. Also $\alpha : G \to P$ is a homomorphism in the sense of I.5.1 with respect to the subcategory **C**, but then $\alpha \circ \mu - \alpha \circ p_1 - \alpha \circ p_2$ is the zero element of $P(G \times_k G)$ (p_1 and p_2

are, as usual, the projections); however this implies immediately that α is a homomorphism in (Sch/k) itself. Finally the kernel $N(\alpha)$ is clearly set theoretically the point e on G; however also as a subscheme $N(\alpha)$ coincides with the " reduced subscheme consisting of the point e ". We have to show this only for $N(\alpha_0)$ where $\alpha_0 = \alpha/G_0$ and G_0 is the connected component of the unit. The fact that α_0 is injective follows from lemma I.5.2 and from the fact that (G, α) represents the functor on **C**.

Lemma (**I.7.3**). — *Let* $T = \mathrm{Spec}(A)$ *with* A *reduced and* $\xi : \mathrm{Spec}(A) \to P$. *Then there exists a unique morphism* $\varphi : \mathrm{Spec}(A) \to G$ *such that* $\alpha \circ \varphi = \xi$.

Proof. — Let $A = \varinjlim A_i$, with A_i a k-algebra of finite type over k. By P_3 there exixts an index i and a $\xi_i : \mathrm{Spec}(A_i) \to P$ such that $\xi = \xi_i \circ^a \lambda_i$ where $\lambda_i : A_i \to A$ is the canonical ring homomorphism. Since A_i is reduced there exists a $\varphi_i : \mathrm{Spec}(A_i) \to G$ such that $\xi_i = \alpha \circ \varphi_i$ and clearly $\varphi = \varphi_i \circ^a \lambda_i$ fulfils our requirements; the uniqueness of φ follows from the fact that α is injective.

Lemma (**I.7.4**). — *Let the situation be as in I.7.2; let* R *be the topological k-algebra prorepresenting* P *at the unit element of* $P(\mathrm{Spec}(k))$ *and let* $\beta(\alpha) : R \to \hat{\mathcal{O}}_{e,G}$ *be as in lemma I.4.3. Consider the map* $\mu : R \to R_{red}$ *and the factorization*

$$\begin{array}{ccc} R & \xrightarrow[\beta(\alpha)]{} & \hat{\mathcal{O}}_{e,G} \\ & {\scriptstyle\mu}\searrow \quad \nearrow{\scriptstyle\beta_{red}(\alpha)} & \\ & R_{red} & \end{array}$$

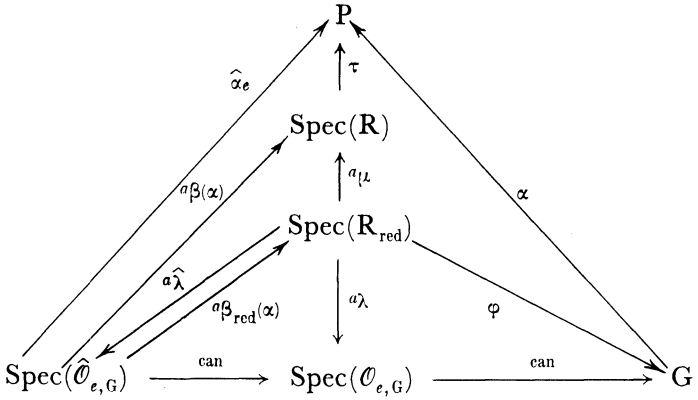
Then $\beta_{red}(\alpha)$ *is an isomorphism.*

Proof. — We note that the existence of the above factorization follows from the fact that $\hat{\mathcal{O}}_{e,G}$ is reduced (G is a reduced groupscheme; $\hat{\mathcal{O}}_{e,G}$ is even a regular local ring). $\mathrm{Spec}(R_{red})$ is reduced; consider the canonical element τ of Spec(R) (remark I.2.3) and $\tau \circ^a \mu = \tau_{red} : \mathrm{Spec}(R_{red}) \to P$. From lemma I.7.3 follows that there exists a morphism $\varphi : \mathrm{Spec}(R_{red}) \to G$ such that $\alpha \circ \varphi = \tau_{red}$. Since R_{red} is a local ring we have a factorization

$$\begin{array}{ccc} \mathrm{Spec}(R_{red}) & \xrightarrow{\varphi} & G \\ & {\scriptstyle a\lambda}\searrow \quad \nearrow{\scriptstyle \mathrm{can.}} & \\ & \mathrm{Spec}(\mathcal{O}_{e,G}) & \end{array}$$

where $\lambda : \mathcal{O}_{e,G} \to R_{red}$ is a local homomorphism; let $\hat{\lambda} : \hat{\mathcal{O}}_{e,G} \to R_{red}$ be the continuous prolongation to the completion. We have $\alpha \circ \varphi = \tau_{red}$, hence $\alpha_e \circ^a \lambda = \tau_{red}$ (see I.1.10 for notations) and hence also $\hat{\alpha}_e \circ^a \hat{\lambda} = \tau_{red}$. On the other hand we have $\hat{\alpha}_e = \tau \circ^a \beta(\alpha)$ (see I.4.4), hence $\hat{\alpha}_e = \tau \circ^a \mu \circ^a \beta_{red}(\alpha)$. Therefore $\tau_{red} = \tau \circ^a \mu$ and $\tau_{red} = \tau \circ^a \mu \circ^a \beta_{red}(\alpha) \circ^a \hat{\lambda}$. Since $\tau : \mathrm{Spec}(R) \to P$ is the canonical element it follows from axiom P_2 that there is only one continuous homomorphism $\psi : R \to R_{red}$ such that $\tau \circ^a \psi = \tau_{red}$, hence $\mu = \hat{\lambda} \circ \beta_{red}(\alpha) \circ \mu$.

Since μ is surjective, we have that $\hat{\lambda} \circ \beta_{red}(\alpha)$ is the identity. Since $\beta(\alpha)$ is surjective (lemma I.4.3; α is injective) we have that both $\hat{\lambda}$ and $\beta_{red}(\alpha)$ are isomorphisms. (See diagram.)



Lemma **(I.7.5).** — *Let the functor* P *be as in proposition 3. Consider the category* $(Ab/k)^*$ *of schemes of abelian groups* H *which are connected and of finite type over k. Let* P'(H) *be the set of homomorphisms* $\gamma : H \to P$. *The functor* $P' : H \rightsquigarrow P'(H)$ *is representable on* $(Ab/k)^*$ *by a couple* (H_0, γ_0) *with an injective* $\gamma_0 : H \to P$.

Proof. — First of all we note that if we consider the restriction P'' of P' to the category $(Ab/k)^*_{red}$ of schemes of abelian groups, connected and of finite type over k and which are moreover reduced, and if (G, α) is as in I.7.2, then P'' is representable by the couple (G_0, α_0) where G_0 is the connected component of G and $\alpha_0 = \alpha/G_0$.

Let the set E of couples (H, γ), with $H \in (Ab/k)^*$ and $\gamma \in P'(H)$, be preordered as in I.5.3. Precisely as in I.5.5 it suffices to show the existence of a couple (H_0, γ_0) in E such that $(H_0, \gamma_0) > (H, \gamma)$ for every (H, γ) in E and such that γ_0 is injective. By lemma I.5.2 it suffices to restrict our attention to the subset E' of E with couples (H, γ) with injective γ. The existence of such a (H_0, γ_0) follows then from the fact that E' is filtered and from the following statement:

(I.7.6) Let (G_0, α_0) be as above and (G_i, α_i) $(i = 1, 2, \ldots)$ be couples in the subset E' introduced above. Suppose that there exist homomorphisms of schemes of groups $\varphi_i : G_i \to G_{i+1}$ such that $\alpha_i = \alpha_{i+1} \circ \varphi_i$ $(i = 0, 1, \ldots)$. Then there exists an integer i_0 such that φ_i is an isomorphism for $i > i_0$.

Proof. — First of all we note that the φ_i are monomorphisms (the α_i are injective), hence closed immersions ([5], III, cor. 7.4) and the G_i have all the same underlying space (namely G_0). Write $\mathcal{O}_i = \mathcal{O}_{e,G_i}$ and consider the local homomorphism $\lambda_i : \mathcal{O}_{i+1} \to \mathcal{O}_i$ corresponding with $\varphi_i : G_i \to G_{i+1}$; let $\mu_i = \lambda_0 \circ \lambda_1 \circ \ldots \circ \lambda_{i-1} : \mathcal{O}_i \to \mathcal{O}_0$. The φ_i are closed immersions, the λ_i and μ_i are therefore surjective and clearly *it suffices to prove that the* λ_i *are injective for* $i >$ *some* i_0 (because then the local homomorphisms of the corresponding local rings are isomorphisms in all closed points and then the φ_i are themselves isomorphisms by ([1], Chap. II, § 3, Th. 1)). Write $\mathfrak{p}_i = \mathrm{Ker}(\mu_i)$; $\mathfrak{p}_i$ is prime since G_0 is

an integral scheme. Denote the zero ideal of $\mathcal{O}_i$ by $\mathfrak{q}_i$; we have $\mathfrak{p}_i = \mathfrak{r}(\mathfrak{q}_i)$ (the " root " of $\mathfrak{q}_i$; [1], Chap. II, § 2.6, def. 4) because $(G_i)_{red} = G_0$ (cf. *E.G.A.*, I, prop. 5.1.1). Moreover we claim that $\mathfrak{q}_i$ *is* $\mathfrak{p}_i$*-primary*. This follows from the fact that the G_i is a scheme of groups (for, if in the decomposition of $\mathfrak{q}_i$ into primary ideals there enter primary ideals belonging to embedded primes, then the same is true for the zero ideal in A, where Spec(A) is some affine neighbourhood of the unit element e in G_i; however, if we take a closed point $a \in G_i$ outside the subspaces corresponding with these embedded primes we have that in the decomposition of the zero ideal of $\mathcal{O}_{a,G_i}$ there do not enter embedded primes, which clearly is impossible because $\mathcal{O}_{a,G_i} \tilde{\to} \mathcal{O}_{e,G_i} = \mathcal{O}_i$). Next we turn to the completions $\hat{\mathcal{O}}_i$, $\hat{\mathfrak{p}}_i$, $\hat{\mathfrak{q}}_i$, $\hat{\lambda}_i$ and $\hat{\mu}_i$ of $\mathcal{O}_i$, $\mathfrak{p}_i$, $\mathfrak{q}_i$, λ_i and μ_i. We have $\hat{\mathfrak{p}}_i = \mathrm{Ker}(\hat{\mu}_i)$; $\hat{\mathfrak{p}}_i$ is prime since $\hat{\mathcal{O}}_0$ is a regular local ring. Furthermore $\hat{\mathfrak{q}}_i$ is clearly the zero ideal of $\hat{\mathcal{O}}_i$. We apply ([1], Chap. IV, § 2, prop. 11) by putting (with the notations of that proposition) $A = \mathcal{O}_i$, $B = \hat{\mathcal{O}}_i$, $\mathfrak{p} = \mathfrak{p}_i$, $E = \mathcal{O}_i$ and $E' = \mathfrak{q}_i$; since $\hat{\mathfrak{p}}_i = \mathfrak{p}_i . \hat{\mathcal{O}}_i$ is prime, we have that $\hat{\mathfrak{q}}_i$ is $\hat{\mathfrak{p}}_i$-primary. Finally consider the ring R prorepresenting P, or more precisely P_ε, at the zero element ε of $P(\mathrm{Spec}(k))$; let $\beta_i = \beta(\alpha_i) : R \to \hat{\mathcal{O}}_i$ bet he surjective homomorphism from lemma I.4.3 corresponding with the functor morphism $(h_{G_i})_e \to P_\varepsilon$. Since $\hat{\lambda}_i : \hat{\mathcal{O}}_{i+1} \to \hat{\mathcal{O}}_i$ corresponds under the duality considered in I.4.1 with the functor morphism $(h_{G_i})_e \to (h_{G_{i+1}})_e$ obtained from $\varphi_i : G_i \to G_{i+1}$ and since we have a commutative diagram

$$\begin{array}{ccc} (h_{G_i})_e & \longrightarrow & (h_{G_{i+1}})_e \\ & \searrow \quad \swarrow & \\ & P_\varepsilon & \end{array}$$

we have also a commutative diagram

$$\begin{array}{ccc} \hat{\mathcal{O}}_i & \xleftarrow{\hat{\lambda}_i} & \hat{\mathcal{O}}_{i+1} \\ & {\scriptstyle\beta_i} \nwarrow \quad \nearrow {\scriptstyle\beta_{i+1}} & \\ & R & \end{array}$$

Let $\mathfrak{R}_i = \mathrm{Ker}(\beta_i)$, then $\mathfrak{R}_i = \beta_i^{-1}(\hat{\mathfrak{q}}_i)$ and $\mathfrak{R}_0 = \beta_0^{-1}(\hat{\mathfrak{q}}_0) = \beta_i^{-1}(\hat{\mathfrak{p}}_i)$, therefore $\mathfrak{R}_i$ *is* $\mathfrak{R}_0$*-primary*. From lemma I.7.4 follows that $\mathfrak{R}_0 = \mathfrak{r}(o)$ (the root of the zero ideal in R). Consider the sequence $\mathfrak{R}_0 \supseteq \mathfrak{R}_1 \supseteq \mathfrak{R}_2 \supseteq \ldots$; since every $\mathfrak{R}_i$ contains the $\mathfrak{R}_0$-primary ideal which enters in the primary decomposition of (o) in R, it follows that this sequence is finite, i.e. $\mathfrak{R}_i = \mathfrak{R}_{i_0}$ for $i >$ a certain i_0. Since the β_i are surjective by lemma I.4.3 this implies that the $\hat{\lambda}_i$, and hence the λ_i, are injective for $i > i_0$ and this completes the proof of lemma I.7.5 (1).

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(1) The referee suggested the following proof, which is more intelligible, but which uses more formal group theory:

Write $\psi_i = \varphi_{i-1} \circ \varphi_{i-2} \circ \ldots \circ \varphi_0 : G_0 \to G_i$, $\hat{G}_i = \mathrm{Spf}(\hat{\mathcal{O}}_{e,G_i})$, $\hat{P} = \mathrm{Spf}(R_\varepsilon)$ and let $\hat{\psi}_i : \hat{G}_0 \to \hat{G}_i$ and $\hat{\alpha}_i : \hat{G}_i \to \hat{P}$ be the homomorphisms of formal groups induced by ψ_i and α_i. Since the G_i have all the same subspace we have that $\mathrm{coker}(\psi_i) = G_i/G_0$ is a radical group with hyperalgebra A_i say. In order to prove the assertion it suffices to

Lemma **(I.7.7).** — *Let* (H_0, γ_0) *be as in lemma I.7.5 and let* R *be the ring prorepresenting* P *(or more precisely* P_ε*) at the unit element* ε *of* $P(\mathrm{Spec}(k))$. *Let* $\beta(\gamma_0) : R \to \hat{\mathcal{O}}_{e,H_0}$ *be the homomorphism from lemma I.4.3. Then* $\beta(\gamma_0)$ *is an isomorphism.*

Proof. — Since R is the local ring prorepresenting P at the unit element ε of $P(\mathrm{Spec}(k))$, the formal spectrum Spf(R) is a formal group (I.1.8). If the characteristic of the field k is zero then by a result of Cartier (cf. [2], Th. 26) formal groups and (hence) schemes of groups are reduced. Therefore $R = R_{red}$ and H_0 can be identified with the connected component G_0 of the unit element of G (from I.7.2); therefore $\hat{\mathcal{O}}_{e,H_0} = \hat{\mathcal{O}}_{e,G_0}$ and our lemma follows from lemma I.7.4.

(I.7.8) Assume therefore $\mathrm{Char}(k) = p \neq 0$. Let $\mathfrak{m}$ be the maximal ideal of R and let $\mathfrak{m}^{(p^n)}$ be the ideal generated by the elements x^{p^n} with $x \in \mathfrak{m}$ $(n = 1, 2, \ldots)$. Write $R_{(n)} = R/\mathfrak{m}^{(p^n)}$; let $\varphi_{(n)} : R \to R_{(n)}$ be the natural homomorphism. The formal spectrum $\mathrm{Spf}(R_{(n)})$ is the kernel of the homomorphism of formal groups $\mathrm{Spf}(R) \to \mathrm{Spf}(R)$ corresponding with the Frobenius map $F_{(n)} : R \to R$, defined by $F_{(n)}(x) = x^{p^n}$ for $x \in R$. Therefore $\mathrm{Spf}(R_{(n)})$ is a formal subgroup of Spf(R) and therefore a formal group itself; however since $R_{(n)}$ is a k-algebra which is a finite dimensional k-vector space, it follows immediately that $\mathrm{Spec}(R_{(n)})$ is a scheme of groups in the usual sense.

(I.7.9) Let τ be the canonical element (see I.2.3) of $P(\mathrm{Spec}(R))$ and $\tau_{(n)} = \tau \circ^a \varphi_{(n)} : \mathrm{Spec}(R_{(n)}) \to P$. We want to show that $\tau_{(n)}$ *is a homomorphism* (in the sense of I.5.1). In order to see this, let B be a local k-algebra of finite k-dimension (as k-vector space) and let $\mathfrak{n}$ be the maximal ideal of B. Let $P_\varepsilon(\mathrm{Spec}(B))$ be the subset of $P(\mathrm{Spec}(B))$ as usual (i.e., $\eta : \mathrm{Spec}(B) \to P$ is in $P_\varepsilon(\mathrm{Spec}(B))$ if and only if $\eta(\mathfrak{n}) = \varepsilon$; see I.1.5). Since (with the notations from I.1.4) $\mathrm{Hom}_c(R, B) = \mathrm{Hom}_k(R, B)$ we have a group isomorphism $\tau(\mathrm{Spec}(B)) : \mathrm{Hom}_k(R, B) \xrightarrow{\sim} P_\varepsilon(\mathrm{Spec}(B))$ (in fact it is precisely this bijective map which defines the group structure on Spf(R); see I.1.8). From the fact that $\mathrm{Spf}(R_{(n)})$ is a formal subgroup of Spf(R) and from the definition of $\tau_{(n)}$ it follows that $\tau_{(n)}(\mathrm{Spec}(B)) : \mathrm{Hom}_k(R_{(n)}, B) \to P_\varepsilon(\mathrm{Spec}(B))$ is a group homomorphism. We apply this remark to $B = R_{(n)} \otimes_k R_{(n)}$ $(i = 1, 2)$ and to the two ring homomorphisms $q_i : R_{(n)} \to R_{(n)} \otimes_k R_{(n)}$ $(i = 1, 2)$ defined by $q_1(r) = r \otimes 1$ and $q_2(r) = 1 \otimes r$. We obtain

$$\sigma_{(n)} \circ^a q_1 + \sigma_{(n)} \circ^a q_2 = \sigma_{(n)} \circ^a \mu_{(n)} \tag{1}$$

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show that the dimensions $[A_i : k]$ are bounded. However G_i/G_0 can be identified with $\mathrm{coker}(\hat{\psi}_i) = \hat{G}_i/\hat{G}_0$. On the other hand one has a commutative diagram of monomorphisms

$$\begin{array}{ccc} \hat{G}_0 & \xrightarrow{\hat{\alpha}_0} & \hat{P} \\ & {\scriptstyle \hat{\psi}_i}\searrow \quad \nearrow{\scriptstyle \hat{\alpha}_i} & \\ & \hat{G}_i & \end{array}$$

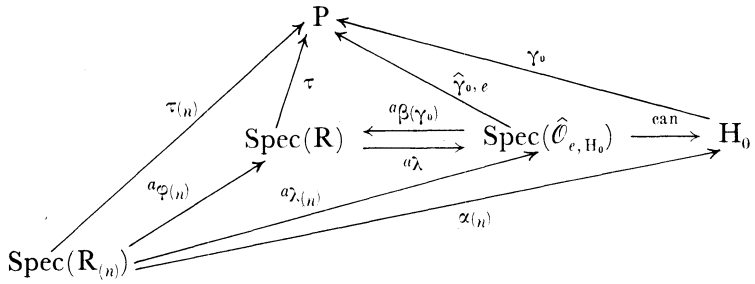
and therefore $\mathrm{coker}(\hat{\psi}_i) = \hat{G}_i/\hat{G}_0$ is a formal subgroup of $\mathrm{coker}(\hat{\alpha}_0) = \hat{P}/\hat{G}_0$. By lemma I.7.4, $\hat{P}/\hat{G}_0$ is radical with hyperalgebra A say. Therefore $[A_i : k] \leq [A : k]$.

where $\mu_{(n)} : R_{(n)} \to R_{(n)} \otimes_k R_{(n)}$ is the homomorphism corresponding with the group multiplication of $\mathrm{Spf}(R_{(n)})$. However the relation (1) implies (see footnote 6) that $\tau_{(n)}(T) : \mathrm{Hom}_k(T, \mathrm{Spec}(R_{(n)})) \to P(T)$ is a group homomorphism for all $T \in (\mathrm{Sch}/k)$; i.e. $\tau_{(n)}$ is a homomorphism in the sense of I.5.1.

(I.7.10) We can now complete the proof of lemma I.7.7. The

$$^a\tau_{(n)} : \mathrm{Spec}(R_{(n)}) \to P$$

are homomorphisms; therefore there are homomorphisms (of schemes of groups) $\alpha_{(n)} : \mathrm{Spec}(R_{(n)}) \to H_0$ such that $\tau_{(n)} = \gamma_0 \circ \alpha_{(n)}$. But since $R_{(n)}$ is a local ring and even a complete local ring, $\alpha_{(n)}$ can be factored through $\mathrm{Spec}(\hat{\mathcal{O}}_{e,H_0})$; therefore there exist k-homomorphisms (of rings) $\lambda_{(n)} : \hat{\mathcal{O}}_{e,H_0} \to R_{(n)}$ such that $(\hat{\gamma}_{0,e}) \circ {}^a\lambda_{(n)} = \tau_{(n)}$ $(n = 1, 2, \ldots)$ and clearly $\lambda_{(n)} = \varphi_{(nm)} \circ \lambda_{(m)}$ for $m \geqslant n$ (the $\varphi_{(nm)} : R_{(m)} \to R_{(n)}$ are the natural homomorphisms). Since $R = \varprojlim R_{(n)}$ there exists a continuous homomorphism $\lambda : \hat{\mathcal{O}}_{e,H_0} \to R$ such that $\lambda_{(n)} = \varphi_{(n)} \circ \lambda$; it follows easily from axiom P_2 that $(\hat{\gamma}_{0,e}) \circ {}^a\lambda = \tau$. On the other hand $\beta(\gamma_0) : R \to \hat{\mathcal{O}}_{e,H_0}$ is such that $\tau \circ {}^a\beta(\gamma_0) = \hat{\gamma}_{0,e}$ (see I.4.4). Therefore the continuous homomorphism $\lambda \circ \beta(\gamma_0) : R \to R$ is such that $\tau \circ {}^a(\lambda \circ \beta(\gamma_0)) = \tau$. However τ is the canonical element of $\mathrm{Spec}(R)$ and hence $\lambda \circ \beta(\gamma_0) = 1_R$; since $\beta(\gamma_0)$ is surjective (lemma I.4.3) it follows that both λ and $\beta(\gamma_0)$ are isomorphisms. (See diagram.)



(I.7.11) Let (H_0, γ_0) be as in lemma I.7.5. Since γ_0 is injective, we can identify $H_0(k) = \mathrm{Hom}_k(\mathrm{Spec}(k), H_0)$ by means of the map $\gamma_0(\mathrm{Spec}(k))$ with a subgroup of $P(\mathrm{Spec}(k))$. We consider (cf. I.5.5) the factor group $\mathfrak{h} = P(\mathrm{Spec}(k))/H_0(k)$; let a_σ be a set of representatives for $\mathfrak{h}$ in $P(\mathrm{Spec}(k))$ (we take as representative a_0 for the unit element in $\mathfrak{h}$ the unit element ε of $P(\mathrm{Spec}(k))$). For each $\sigma \in \mathfrak{h}$, let H_σ be another copy of H_0, and put $H = \coprod_{\sigma \in \mathfrak{h}} H_\sigma$. If $\pi : H_0 \to \mathrm{Spec}(k)$ is the structure morphism, then we define $\gamma_\sigma : H_\sigma \to P$ by $\gamma_\sigma = \gamma_0 + a_\sigma \circ \pi$ and $\gamma : H \to P$ is determined by the condition $\gamma / H_\sigma = \gamma_\sigma$ (axiom P_5). We note that $H(\mathrm{Spec}(k))$ can now be identified with $P(\mathrm{Spec}(k))$. Also we have, if G_0 is the connected component of the identity of the scheme of groups G from I.7.2, that $(H_0)_{\mathrm{red}}$ can be identified with G_0 and α_0 with γ / G_0 [1].

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[1] We do not bother to show that H is a scheme of groups; this will follow automatically.

We want to show that γ *is injective.* For this consider on $H\times_k H=\coprod_{\sigma,\tau\in\mathfrak{h}} H_\sigma\times_k H_\tau$ the element $\zeta=\gamma\circ p_1-\gamma\circ p_2 : H\times_k H\to P$ (where p_1 and p_2 are the projections on the first and second factor). We clearly have to show that the closed subscheme $N(\zeta)$ (from axiom P_6) coincides with the diagonal. If not then we can find a point $(t_1, t_2)\in N(\zeta)$ with t_1 and t_2 *closed points* on H and $t_1\neq t_2$. We get $\gamma(t_1)=\gamma(t_2)$ and it is easily seen that this is impossible *for closed points* (using the fact that γ_0 is injective and that the a_σ are representatives for $\mathfrak{h}$).

Lemma **(I.7.12).** — *The couple* (H, γ) *represents the functor* P.

Proof. — Let $T\in(Sch/k)$ and $\xi : T\to P$. *First we assume that* T *is connected, of finite type and that there exists a closed point* $t_0\in T$ *such that* $\xi(t_0)=0$. Let Z be the closed subscheme of $T\times_k H_0$ representing the functor $h_T\times_P h_{H_0}$ (axiom $P_6'=P_6$; the structure morphisms are $\xi : T\to P$ and $\gamma_0 : H_0\to P$). If q (resp. p) is the restriction of the projection $T\times_k H_0\to T$ (resp. $T\times_k H_0\to H_0$) to Z then *we want to show that q is an isomorphism*; because then $\varphi=p\circ q^{-1} : T\to H_0$ is such that $\gamma_0\circ\varphi=\xi$. The reduced prescheme Z_{red} is a subprescheme of $(T\times_k H_0)_{red}=T_{red}\times_k G_0$ (see I.7.11) and it is immediately seen that Z_{red} has the same subspace as the closed prescheme Z_1 on $T_{red}\times_k G_0$ which represents the functor $h_{T_{red}}\times_P h_{G_0}$ (the structure morphisms are $\xi/T_{red} : T_{red}\to P$ and $\alpha_0=\gamma_0/G_0 : G_0\to P$). However, since (G, α) represents the restriction of the functor to the category of preschemes which are reduced and a direct sum of preschemes of finite type over k, it follows that there exists $\psi : T_{red}\to G$ such that $\alpha\circ\psi=\xi/T_{red}$. Since T is connected and $\xi(t_0)=0$ it follows that ψ factors through G_0 and then clearly Z_1 is the graph of ψ; therefore Z_1 is isomorphic with T_{red} and hence reduced. Hence we have $Z_{red}=Z_1$ and $q_{red} : Z_{red}\to T_{red}$ is an isomorphism. Therefore by (*E.G.A.*, I, prop. 5.5.1, (vi) and II, cor. 5.4.6) q itself is proper and surjective; also we have that q is a monomorphism (this follows easily from the fact that γ_0 is injective). By lemma I.6.8 q is a closed immersion and is, as we have seen, also surjective. In order to conclude that q is an isomorphism it suffices therefore by (*S.G.A.*, I, th. 5.1) to show that q is " étale ". By (*S.G.A.*, I, prop. 4.5) it suffices to prove that q is étale in the closed points of Z.

(I.7.13) By (*E.G.A.*, II, prop. 1.6.4) q is affine and there is no loss in generality therefore if we take $T=\mathrm{Spec}(A)$ and $Z=\mathrm{Spec}(B)$. Let $z\in Z$ be a closed point and put $t=q(z)$; t is a closed point of T. Consider the local ring $A'=A_t$, let $T'=\mathrm{Spec}(A')$ and $Z'=T'\times_T Z=\mathrm{Spec}(B')$ with $B'=B\otimes_A A_t$. In order to prove that q is étale in z it clearly suffices to prove that $q'=1_{T'}\times_T q : Z'\to T'$ is étale in the (unique) point $z'\in Z'$ corresponding with z. We note that Z' on $T'\times_k H_0$ is clearly the subscheme which represents the functor $h_{T'}\times_P h_{H_0}$ where now the structure morphism $\xi' : T'\to P$ is the composition of the natural morphism $\mathrm{Spec}(A_t)\to\mathrm{Spec}(A)=T$ and $\xi : T\to P$. We make after the base extension $T'\to T$ still another base extension, namely $T''=\mathrm{Spec}(\hat{A}_t)\to T'$. This is a faithfully flat base extension and it suffices to show that $q'' : 1_{T''}\times_{T'} q' : Z''=T''\times_{T'} Z'\to T''$ is an isomorphism. Again we note that Z''

represents the functor $h_{T''} \times_P h_{H_0}$. Since q_{red} is an isomorphism there exists a closed point $x \in G_0$ such that $\alpha_0(x) = \xi(t)$. Let R (resp. R_1) be the ring prorepresenting P at the unit element (resp. at $\xi(t)$) of $P(\mathrm{Spec}(k))$; let τ (resp. τ_1) be the canonical element of $P(\mathrm{Spec}(R))$ (resp. of $P(\mathrm{Spec}(R_1))$). Since the functor P takes it values in the category of (abelian) groups we have an isomorphism between R and R_1 (resp. between $\hat{\mathcal{O}}_{e,H_0}$ and $\hat{\mathcal{O}}_{x,H_0}$) transforming the elements τ and τ_1 (resp. $\hat{\gamma}_{0,e}$ and $\hat{\gamma}_{0,x}$) into each other; therefore there is, by lemma I.7.7 and by Remark I.4.4, an isomorphism between R_1 and $\hat{\mathcal{O}}_{x,H_0}$ which transforms τ_1 and $\hat{\gamma}_{0,x}$ into each other. By lemma I.4.3 there is a homomorphism $\beta(\hat{\xi}_t) : R_1 \to \hat{\mathcal{O}}_{t,T} = \hat{A}_t$ which transforms τ_1 into $\hat{\xi}_t = \xi''$; there exists therefore a homomorphism $\lambda : \hat{\mathcal{O}}_{x,H_0} \to \hat{A}_t$ such that $\hat{\gamma}_{0,x} \circ^a \lambda = \xi''$, i.e., there exists a morphism $\varphi'' : T'' \to H_0$ such that $\xi'' = \gamma_0 \circ \varphi''$. Then it follows from the fact that γ_0 is injective that $\varphi'' \circ q'' = p''$ (where $p'' : Z'' \to H_0$ is the projection) and hence Z'' is the graph of φ'' and hence the projection $q'' : Z'' \to T''$ is an isomorphism. This shows that there exists a morphism $\varphi : T \to H$ such that $\xi = \gamma_0 \circ \varphi$.

(I.7.14) Next we assume that $T \in (\mathrm{Sch}/k)$ *is connected and of finite type* over k, but $\xi : T \to P$ is arbitrary. Let t_0 be a closed point of T and $\xi_1 = \xi - \xi(t_0)$. By what we have seen above there exists a morphism $\varphi_1 : T \to H_0$ such that $\gamma_0 \circ \varphi_1 = \xi_1$. Let $\xi(t_0) \bmod H_0(k) = \sigma$ (in the factor group $P(\mathrm{Spec}(k))/H_0(k)$), then $\xi(t_0) - a_\sigma$ (see I.7.11) corresponds with a closed point b on H_0. Consider $\varphi = t_b \circ \varphi_1 : T \to H_0$, where t_b is the translation on H_0 by the point b; it follows immediately from the fact that γ_0 is a homomorphism that $\gamma_0 \circ t_b \circ \varphi_1 = \xi - a_\sigma$. Since $H_\sigma = H_0$ we can interpret φ also as a morphism $T \to H_\sigma$ and $\gamma_\sigma \circ \varphi = \xi$. Therefore $\gamma(T) : \mathrm{Hom}_k(T, H) \to P(T)$ is clearly surjective (if T is connected and of finite type). Since γ is injective, the map $\gamma(T)$ is also injective and this completes the proof in this case.

(I.7.15) *Next let* T *be of finite type* over k, but otherwise arbitrary and $\xi : T \to P$. Let T_j $(j = 1, \ldots, n)$ be the connected components of T, then $T = \coprod T_j$; put $\xi_j = \xi/T_j$. There exists, as we have seen, a $\varphi_j : T_j \to H$ such that $\alpha \circ \varphi_j = \xi_j$. Let $\varphi : T \to H$ be defined by $\varphi/T_j = \varphi_j$; then we have $\alpha \circ \varphi = \xi$ by axiom P_5.

(I.7.16) *Next let* $T = \mathrm{Spec}(A)$ and $\xi : T \to P$. There exists by P_3 a subalgebra A' of A and an element $\xi' : \mathrm{Spec}(A') \to P$ such that A' is of finite type over k and such that $\xi = \xi' \circ^a i$ where $i : A' \to A$ is the injection. By I.7.15 there exists a morphism $\varphi' : \mathrm{Spec}(A') \to H$ such that $\gamma \circ \varphi' = \xi'$. If $\varphi = \varphi' \circ^a i$, then $\gamma \circ \varphi = \xi$, therefore $\gamma(\mathrm{Spec}(A)) : \mathrm{Hom}(\mathrm{Spec}(A), H) \to P(\mathrm{Spec}(A))$ is surjective and also injective because γ is injective.

(I.7.17) *Let finally* T *be arbitrary* and $\xi : T \to P$. Let $\{U_\alpha\}$ be an affine open covering of T and $\xi_\alpha = \xi/U_\alpha$. There exist $\varphi_\alpha : U_\alpha \to H$ such that $\gamma \circ \varphi_\alpha = \xi_\alpha$ (I.7.16), and since γ is injective we have $\varphi_\alpha/U_\alpha \cap U_\beta = \varphi_\beta/U_\alpha \cap U_\beta$. Hence there exists $\varphi : T \to H$

such that $\varphi/U_\alpha = \varphi_\alpha$. We have $\gamma \circ \varphi/U_\alpha = \xi_\alpha$, hence $\gamma \circ \varphi = \xi$ by axiom P_5. The uniqueness of φ follows from the fact that γ is injective. This completes the proof of lemma I.7.12 and *also the proof of proposition 3.*

(I.7.18) *Corollary of proposition 3. — Theorem 1 of section I.2 is true for functors* $P \in \mathbf{Hom}((\mathrm{Sch}/k)^0, \mathbf{Ab})$ *if the field k is algebraically closed.*

Proof. — By proposition 1 the functor P' on $(\mathrm{Ab}/k)_{\mathrm{red}}$ (see proposition 1 for notations) is representable by a couple (G, α) with an injective $\alpha : G \to P$. Consider the constant scheme of groups $\mathbf{Z}_k$ defined by the abelian group $\mathbf{Z}$. Then

$$\alpha(\mathbf{Z}_k) : \mathrm{Hom}_{(\mathrm{Ab}/k)_{\mathrm{red}}}(\mathbf{Z}_k, G) \to P'(\mathbf{Z}_k)$$

is bijective. However $\mathrm{Hom}_{(\mathrm{Ab}/k)_{\mathrm{red}}}(\mathbf{Z}_k, G)$ can be identified with $\mathrm{Hom}_k(\mathrm{Spec}(k), G)$ and (using P_5) $P'(\mathbf{Z}_k)$ with $P(\mathrm{Spec}(k))$, and by this identification $\alpha(\mathbf{Z}_k)$ corresponds with $\alpha(\mathrm{Spec}(k))$; from this it follows that the map $\alpha(\mathrm{Spec}(k)) : G(\mathrm{Spec}(k)) \to P(\mathrm{Spec}(k))$ is bijective. The corollary follows then immediately from proposition 2 and 3.

I.8. Completion of the proof of theorem 1 (the case of an arbitrary groundfield).

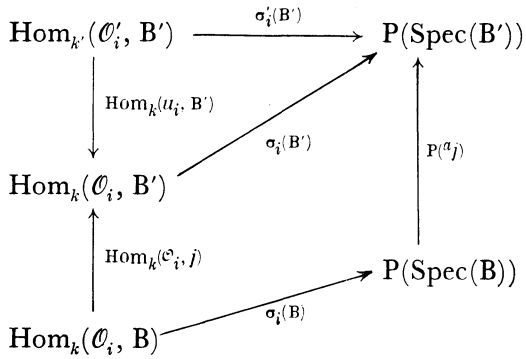
(I.8.1) We use the following notations in this section: k is the groundfield, k' is the algebraic closure of k, $S = \mathrm{Spec}(k)$, $S' = \mathrm{Spec}(k')$ and $S'' = S' \times_S S' = \mathrm{Spec}(k' \otimes_k k')$. We denote the natural homomorphism $S' \to S$ by f and $p_i : S'' \to S'$ $(i = 1, 2)$ is the projection on the i-th factor. Let $P \in \mathbf{Hom}((\mathrm{Sch}/k)^0, \mathbf{Ab})$. We denote the restriction of P to (Sch/k') by P' (i.e., put $P'(T') = P(T')$ for $T' \in (\mathrm{Sch}/k')$ where T' is considered as object of (Sch/k) and put $P'(u') = P(u')$ for every S'-morphism $u' : T'_1 \to T'_2$); similarly P'' is the restriction of P to (Sch/S'').

Lemma **(I.8.2)**. — *Let* $P \in \mathbf{Hom}((\mathrm{Sch}/k)^0, \mathbf{Ab})$ *satisfy* P_1 *and* P_3; *let* $\mathcal{O}$ *be the topological k-algebra which prorepresents the functor* P. *Then* P' *is strictly prorepresentable and is prorepresented by* $\mathcal{O}' = \mathcal{O} \hat{\otimes}_k k'$; *the local components of* $\mathcal{O}'$ *are noetherian rings.*

Proof. — Let $\{\mathcal{O}_i, \psi_{ji}, \sigma_i\}$ be the projective system which prorepresents P (see I.1.4) and $\mathcal{O} = \varprojlim \mathcal{O}_i$; then $\mathcal{O}' = \varprojlim \mathcal{O}'_i$ with $\mathcal{O}'_i = \mathcal{O}_i \otimes_k k'$. Clearly the $\mathcal{O}'_i$ are k'-algebras of finite k'-dimension; let $u_i : \mathcal{O}_i \to \mathcal{O}'_i$ be defined by $u_i(r) = r \otimes 1$ and put $\sigma'_i = \sigma_i \circ^a u_i$. Let B' be a k'-algebra of finite dimension. Consider the map $\sigma'_i(\mathrm{Spec}(B')) : \mathrm{Hom}_{k'}(\mathrm{Spec}(B'), \mathrm{Spec}(\mathcal{O}'_i)) \to P(\mathrm{Spec}(B'))$. In the following we identify $\mathrm{Hom}_{k'}(\mathrm{Spec}(B'), \mathrm{Spec}(\mathcal{O}'_i))$ with $\mathrm{Hom}_{k'}(\mathcal{O}'_i, B')$ and we write $\sigma'_i(B')$ instead of $\sigma'_i(\mathrm{Spec}(B'))$. We want to show that the maps $\sigma'_i(B') : \mathrm{Hom}_{k'}(\mathcal{O}'_i, B') \to P(\mathrm{Spec}(B'))$ induce a bijection

$$\varinjlim_i \mathrm{Hom}_{k'}(\mathcal{O}'_i, B') \xrightarrow{\sim} P(\mathrm{Spec}(B')).$$

Let B be a k-subalgebra of B′ which is of finite dimension as a k-vectorspace and let $j : B \to B'$ be the injection. Then we have the following commutative diagram



The $\mathcal{O}$ prorepresents P, therefore the $\sigma_i(B)$ induce a bijection (which we denote by $\sigma(B)$):

$$\sigma(B) : \varinjlim_i \mathrm{Hom}_k(\mathcal{O}_i, B) \xrightarrow{\sim} \mathrm{P}(\mathrm{Spec}(B))$$

therefore a bijection

$$\varinjlim_B \sigma(B) : \varinjlim_B \varinjlim_i \mathrm{Hom}_k(\mathcal{O}_i, B) \xrightarrow{\sim} \varinjlim_B \mathrm{P}(\mathrm{Spec}(B))$$

By P_3 we have $\varinjlim_B \mathrm{P}(\mathrm{Spec}(B)) = \mathrm{P}(\mathrm{Spec}(B'))$ and $\varinjlim_B \varinjlim_i \mathrm{Hom}_k(\mathcal{O}_i, B)$ can be identified with $\varinjlim_i \varinjlim_B \mathrm{Hom}_k(\mathcal{O}_i, B) = \varinjlim_i \mathrm{Hom}_k(\mathcal{O}_i, B')$, and this again is the same as $\varinjlim_i \mathrm{Hom}_{k'}(\mathcal{O}'_i, B')$; therefore we have an isomorphism $\varinjlim_i \mathrm{Hom}_{k'}(\mathcal{O}'_i, B') \xrightarrow{\sim} \mathrm{P}(\mathrm{Spec}(B'))$ and it is easily seen that this is just the map $\sigma'(B')$ (i.e., obtained from the $\sigma'_i(B')$).

The $\psi_{ji} : \mathcal{O}_i \to \mathcal{O}_j$ are surjective; therefore also the natural maps $\psi'_{ji} = \psi_{ji} \otimes 1_{k'}$ are surjective. Finally we want to show that the local components of $\mathcal{O}'$ are noetherian. Let $\mathcal{O} = \prod_\beta R_\beta$ where the R_β are the local components of $\mathcal{O}$ (see I.1.5); then $\mathcal{O}' \xrightarrow{\sim} \prod_\beta R'_\beta$ with $R'_\beta = R_\beta \hat{\otimes}_k k'$. Since R_β is noetherian, the R'_β is also noetherian (see [8], p. 64 c; if $\mathfrak{m}_\beta$ is the maximal ideal of R_β then $R_\beta/\mathfrak{m}_\beta$ is of finite dimension over k). The rings R'_β are not necessarily the local components of $\mathcal{O}'$ (see *loc. cit.* [8]); in fact the R'_β are only semi-local rings, but the local components are obtained from the R'_β by localization and therefore are also noetherian.

(I.8.3) Let P satisfy $P_1, P_2, \ldots, P_7$; then P′ satisfies the same axioms. For P_1 this was shown in lemma I.8.2; the other ones are straightforward. By the corollary to proposition 3 (see I.7.18), P′ is representable by a scheme of groups H′ which is locally of finite type over k'; let $\gamma' : H' \to P'$ be the canonical element.

(I.8.4) Let S be an arbitrary groundprescheme (in this number we do not stick to the conventions made in I.8.1), and let $f : S' \to S$ be a morphism and $S'' = S' \times_S S'$.

Let $P \in \mathbf{Hom}((\mathrm{Sch}/S)^0, \mathbf{Ens})$ and suppose that the restriction P' of P to (Sch/S') is representable by a couple (X', ρ'). Let $p_1, p_2 : S'' \to S'$ be the projections; put $X_i'' = X' \times_{S'} (S'', p_i)$ $(i = 1, 2)$ and let $q_i : X_i'' \to X'$ be the projections $(i = 1, 2)$; finally put $\rho_i'' = \rho' \circ q_i$. Then the following lemma holds:

Lemma (**I.8.5**). — *There exists an* S''*-isomorphism* $u : X_2'' \to X_1''$ *uniquely determined by the condition that* $\rho_2'' = \rho_1'' \circ u$. *This morphism is, for the* S'*-prescheme* X', *a descent data with respect to the morphism* $f : S' \to S$ ([5], I, A, def. 1.6). *If* $P \in \mathbf{Hom}((\mathrm{Sch}/S)^0, \mathbf{Ab})$ *then u is a group isomorphism.*

Proof. — The couple (X_1'', ρ_1'') (resp. (X_2'', ρ_2'')) represents the restriction P'' of P to (Sch/S''). Therefore there exist S''-morphisms $u : X_2'' \to X_1''$ and $v : X_1'' \to X_2''$ respectively determined by the conditions $\rho_2'' = \rho_1'' \circ u$ and $\rho_1'' = \rho_2'' \circ v$; clearly $v \circ u = 1_{X_1''}$ and $u \circ v = 1_{X_1''}$ and hence both are isomorphism. The verification that u is a descent data for X', with respect to $f : S' \to S$ is straightforward, also the remark in case $P \in \mathbf{Hom}((\mathrm{Sch}/S)^0, \mathbf{Ab})$.

Lemma (**I.8.6**). — *Let* H' *be a scheme of groups, locally of finite type over* $S' = \mathrm{Spec}(k')$. *Every descent data for the groupscheme* H', *with respect to* $f : S' \to S = \mathrm{Spec}(k)$ *is effective* ([5], I, A, def. 1.5). *Moreover, if* $H \in (\mathrm{Sch}/S)$ *is the prescheme obtained from such a descent data, then* H *is locally of finite type over* S.

Proof. — We use notations H_1'', H_2'', q_1, q_2 similarly as in I.8.4 and $u : H_2'' \to H_1''$ is a group isomorphism which is a descent data. Put $H'' = H_1''$ and let $\pi_1, \pi_2 : H'' \to H'$ be the equivalence relation on H' defined by $\pi_1 = q_1$, $\pi_2 = q_2 \circ u^{-1}$ (cf. *S.G.A.*, VIII, p. 25). We note that both π_1 and π_2 are quasi-compact (since they come, by base extension $H' \to S'$, from p_1 and $p_2 : S'' \to S'$). Let H_i' be the connected components of H'; then it follows in particular from the fact that π_1 is quasi-compact that $\pi_2(\pi_1^{-1}(x'))$, for every $x' \in H'$, is contained in a finite union of components of H'; let U' be the union of the components which intersect $\pi_2(\pi_1^{-1}(x'))$. U' is a S'-scheme of finite type (for H_0', the component of the identity, is of finite type over S' ([5], VI, p. 236-02) and the H_i' are obtained from H_0' by translation). The set U' is a saturated open set for the equivalence relation (we note that H_0' itself is saturated); by (*S.G.A.*, VIII, prop. 7.2) it suffices to show that the equivalence relation induced on this open set U' is effective. Finally we remark that every finite set of points on U' is contained in an affine open set, for this is true on H_0' (cf. [9], p. 59, example 2). Our lemma follows then from the following:

Lemma (**I.8.7**). — *Let* U' *be a prescheme of finite type over* k' *and let every finite set of points on* U' *be contained in an affine open set. Then every descent data on* U' *(with respect to the morphism* $f : \mathrm{Spec}(k') \to \mathrm{Spec}(k)$*) is effective. Furthermore, if* $U \in (\mathrm{Sch}/\mathrm{Spec}(k))$ *is obtained from this descent data, then* U *is of finite type over* S.

Proof. — It is well known (cf. *E.G.A.*, IV) that there exists an extension k_0 of finite degree over k and an $S_0 = \mathrm{Spec}(k_0)$-prescheme U_0 such that $U' \tilde{\to} U_0 \times_{S_0} S'$ and moreover

we can assume that U_0 is if finite type over S_0 and that the descent data on U' is obtained from a descent data on U_0 (with respect to the morphism $S_0 \to S$).

Next we want to show that every finite set F of points on U_0 is contained in an affine open set. Let $\pi : U' \to U_0$ be the projection, $\pi^{-1}(F)$ is a finite set and hence contained in an affine set V' of U'. There exists a finite extension k_1 of k_0 such that V' is of the form $V_1 \times_{k_1} k'$ where V_1 is an affine set on $U_1 = U_0 \times_{k_0} k_1$. Let $\pi_1 : U_1 \to U_0$ be the projection; then $\pi_1^{-1}(F) \subset V_1$. Similarly as in the proof of (*S.G.A.*, VIII, cor. 7.6), one can show that there is an affine neighbourhood of $\pi_1^{-1}(F)$ in V_1 which is saturated for the equivalence relation defined by π_1; this clearly proves the assertion that F is contained in an affine open set on U_0. The fact that the descent data on U_0 is effective follows now from (*S.G.A.*, VIII, prop. 7.6). Finally let $U_0 \tilde{\to} U \times_S S_0$, then by (*S.G.A.*, VIII, prop. 3.3) U is of finite type over S.

Lemma **(I.8.8).** — *The assumptions (and notations) are the same as in lemma I.8.5, but we suppose in addition that* $f : S' \to S$ *is faithfully flat and quasi-compact and that the descent data is effective. Let* X *be an* S*-prescheme obtained from this descent data; we identify* X′ *with* $X \times_S S'$. *Let* $g : X' \to X$ *be the projection. If* P *satisfies axiom* P_4, *then there exists a* $\rho : X \to P$ *such that* $\rho' = \rho \circ g$ *and* (X, ρ) *represents the functor* P.

Proof. — The morphism $g : X' \to X$ is faithfully flat and quasi-compact; consider the projections $\pi_1, \pi_2 : X' \times_X X' \to X'$. There is a canonical isomorphism $X' \times_X X' \tilde{\to} X \times_S S' \times_S S'$ and by means of this isomorphism we can identify π_i to $1_X \times p_i$ $(i = 1, 2)$. It follows by these identifications and by the construction of the descent data (see lemma I.8.5) that $\rho' \circ \pi_1 = \rho' \circ \pi_2$ and therefore there exists by axiom P_4 a $\rho : X \to P$ such that $\rho' = \rho \circ g$.

First we want to show that ρ is injective; we know that ρ' is injective (with respect to the category (Sch/S′)). Let $\varphi_1, \varphi_2 : T \to X$ (with $T \in (Sch/S)$) and let $\varphi_i' = \varphi_i \times_S 1_{S'} : T' = T \times_S S' \to X'$. We have commutative diagrams $(i = 1, 2)$

$$\begin{array}{ccc} T' & \xrightarrow{\varphi_i'} & X' \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\ T & \xrightarrow{\varphi_i} & X \end{array}$$

where h is the projection. If $\varphi_1 \neq \varphi_2$ then $\varphi_1' \neq \varphi_2'$ and hence

$$\rho \circ g \circ \varphi_1' = \rho' \circ \varphi_1' \neq \rho' \circ \varphi_2' = \rho \circ g \circ \varphi_2'$$

and therefore $\rho \circ \varphi_1 \neq \rho \circ \varphi_2$. Hence ρ is injective.

In order to complete the proof we must show that for every $\xi : T \to P$ there exists a $\varphi : T \to X$ such that $\xi = \rho \circ \varphi$. Again let $h : T' \to T$ and $\sigma_1, \sigma_2 : T' \times_T T' \to T'$ be the projections; put $\xi' = \xi \circ h$. Since (X', ρ') represents the restriction P′ of P

to (Sch/S') it follows that there exists a $\varphi' : T' \to X'$ such that $\xi' = \rho' \circ \varphi'$. Consider the diagram

$$\begin{array}{ccccc} T' \times_T T' & & & & \\ {\scriptstyle \sigma_1}\downarrow\downarrow{\scriptstyle \sigma_2} & & & & \\ T' & \xrightarrow{\varphi'} & X' & & \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} & & \\ T & \xrightarrow{?} & X & \xrightarrow{\rho} & P \end{array}$$

We have $\rho \circ g \circ \varphi' \circ \sigma_1 = \xi' \circ \sigma_1 = \xi \circ h \circ \sigma_1 = \xi \circ h \circ \sigma_2 = \xi' \circ \sigma_2 = \rho \circ g \circ \varphi' \circ \sigma_2$; hence, ρ being injective, $g \circ \varphi' \circ \sigma_1 = g \circ \varphi' \circ \sigma_2$. Therefore by (*S.G.A.*, VIII, prop. 5.1, *b*, cor. 4.3 and cor. 1.7) there exists a morphism $\varphi : T \to X$ such that $g \circ \varphi' = \varphi \circ h$. We have $\rho \circ \varphi \circ h = \rho \circ g \circ \varphi' = \xi'$; since also $\xi \circ h = \xi'$ we have by axiom P_4 (applied to the faithfully flat and quasi-compact morphism $h : T' \to T$) that $\rho \circ \varphi = \xi$.

We have proved now the following lemma (by combining lemma I.8.5, I.8.6 and I.8.8):

Lemma (**I.8.9**). — *Let* k'/k *be algebraic and* $P : (\mathrm{Sch}/k)^0 \to \mathbf{Ab}$ *a functor which satisfies* P_4 *and such that the restriction* P' *of* P *to* $(\mathrm{Sch}/k')^0$ *is representable by a scheme of groups, locally of finite type over* k'. *Then* P *itself is representable by a scheme of groups locally of finite type over* k.

(**I.8.10**) The proof of theorem 1 from section 2 is now completed by combining the result in I.8.3 with lemma I.8.9 applied to the algebraic closure k' of k.

I.9. A remark on the proof of the corollary (1.2.5).

(**I.9.1**) For the proof of the corollary we do not need the theory of generalized jacobians. The only place where this theory is used is in lemma I.6.3, which plays an important role in the proof of proposition 2 (see I.6.1). (Note: this is also the only place where P_7 is used). We shall outline a direct proof of preposition 2 under the assumptions of the corollary; because the entire section I.6 can be simplified then. We assume again that k is algebraically closed; let the functor P be a subfunctor of the functor $P' = \mathrm{Hom}_k(\ , G')$ where G' is a scheme of abelian groups, locally of finite type over k and assume that P has the properties mentioned in proposition 2. Clearly we have essentially to prove the following: if T is a reduced scheme of finite type and $\xi : T \to P$, then there exist $H \in (\mathrm{Ab}/k)_{\mathrm{red}}$, a homomorphism $\beta : H \to P$ and a morphism $\varphi : T \to H$ such that $\xi = \beta \circ \varphi$. For simplicity we assume (and this is clearly not a serious restriction) that T is connected and that there exists a closed point $t_0 \in T$ such that $\xi(t_0) = 0$.

(I.9.2) We use frequently the following lemma:

Lemma **(I.9.2)**. — *If* $f : T_1 \to T$ *is faithfully flat and quasi-compact and* $\alpha : T \to G'$ *is such that* $\alpha \circ f \in P(T_1)$ *(and not merely* $\in P'(T_1)$*) then* $\alpha \in P(T)$.

Proof. — Let $p_1, p_2 : T_1 \times_T T_1 \to T_1$ be the projections; we have $\alpha \circ f \circ p_1 = \alpha \circ f \circ p_2$ since $f \circ p_1 = f \circ p_2$. Hence by axiom P_4 there exists $\alpha_1 : T \to P$ such that $\alpha_1 \circ f = \alpha \circ f$ and then it follows by applying anew P_4, but this time to the functor P', that $\alpha = \alpha_1$.

(I.9.3) *Proof of proposition 2 in the case* P *is a subfunctor of* $\mathrm{Hom}_k(\ , G')$ *with* G' *a scheme of abelian groups, locally of finite type over k.*

Let T and $\xi : T \to P$ be as explained in I.9.1; we are looking for a suitable $H \in (\mathrm{Ab}/k)_{\mathrm{red}}$, a homomorphism $\beta : H \to P$ and $\varphi : T \to H$ such that $\xi = \beta \circ \varphi$. For H we take the smallest subscheme of groups in G'_{red} which contains the image $\xi(T)$ (note: $\xi(t_0) = e$). Let $\beta : H \to G'$ be the canonical immersion. We want to show that $\beta \in P(H)$; then clearly β is a homomorphism. Let T_n be the n-fold product of T over k and $\xi_n : T_n \to G'$ is the morphism defined (naively) by $\xi_n(t_1, \ldots, t_n) = \xi(t_1) + \ldots + \xi(t_n)$; then there is a factorization $\xi_n = \beta \circ \lambda_n$ with $\lambda_n : T_n \to H$. It is well known in the theory of group varieties that we can take n so large that λ_n is surjective (in fact we need only that $\lambda_n(T_n)$ contains an open dense set in H). By (*S.G.A.*, IV, cor. 6.11) there exists an open set U in H such that the restriction μ of λ_n to $\lambda_n^{-1}(U) = V$ is faithfully flat (and quasi-compact). By lemma I.9.2 it follows from the fact that $\beta \circ \lambda_n | V = \beta' \circ \mu$ (where $\beta' = \beta | U$) is in P(V) that β' is in P(U). Consider now the morphism $s : U \times_k U \xrightarrow{p} H \times_k H \xrightarrow{q} H \times_k H \xrightarrow{r} H$ where p is an open immersion, $q(a, b) = (a + b, b)$ and $r(a_1, b_1) = a_1$. The morphism r is obtained by base extension $H \to k$ from $H \to k$, hence is flat; q is an isomorphism and p an open immersion, hence s is flat and also (as is well known) surjective. Therefore s is faithfully flat (and quasi-compact). If $p_1, p_2 : U \times_H U \to U$ are the projections then we have $\beta \circ s = \beta' \circ p_1 + \beta' \circ p_2$; therefore $\beta \circ s \in P(U \times_H U)$ and hence (by lemma I.9.2) $\beta \in P(H)$ and this we wanted to show.

Finally we note that it follows from the definition of H that $\xi : T \to G'$ factors through H, i.e. there exists a morphism $\varphi : T \to H$ such that $\xi = \beta \circ \varphi$ and this finishes the proof.

PART II

THE PICARD SCHEME OF A PROPER SCHEME DEFINED OVER A FIELD

(II.1) In this section we use the following notations: k is an arbitrary field, $Z \in (\mathrm{Sch}/k)$, $\pi : Z \to \mathrm{Spec}(k)$ is the structure morphism. Let $T \in (\mathrm{Sch}/k)$; by

$$P(T) = \mathbf{Pic}_{Z/k}(T)$$

we denote the Picard functor defined in ([5], V, § 1); i.e. the functor obtained by " localizing " (by means of faithfully flat and quasi-compact morphisms) the functor (in T) $\mathrm{Pic}(Z \times_k T) = H^1(Z \times_k T, \mathcal{O}^*_{Z \times_k T})$. We want to show that this functor is representable under suitable conditions.

(II.2) First we make the following assumptions (until section II.15):

1) π is proper;
2) $\pi_*(\mathcal{O}_Z) = k$;
3) Z has a section $\sigma : \mathrm{Spec}(k) \to Z$.

(The assumptions 2) and 3) are auxiliary and will be eliminated later.)

Under these conditions an element $\xi : T \to P$ can be considered ([5], V, Remark 2.5) as an *invertible Module* on $Z \times_k T$ with a $\tau = (\sigma \times 1_T)$-*rigidification*; i.e. ξ is determined by a couple $(\mathscr{L}, \rho)$ where $\mathscr{L}$ is an invertible $\mathcal{O}_{Z \times_k T}$-Module and $\rho : \tau^*(\mathscr{L}) \tilde{\to} \mathcal{O}_T$ is an isomorphism. We shall say that two such couples $(\mathscr{L}, \rho)$ and $(\mathscr{L}', \rho')$ are isomorphic if there exists an isomorphism $\varphi : \mathscr{L} \tilde{\to} \mathscr{L}'$ such that $\rho' \circ \tau^*(\varphi) = \rho$ and $\xi : T \to P$ can be identified with an isomorphism class of such couples. We remark that an invertible $\mathcal{O}_{Z \times_k T}$-Module $\mathscr{L}$ determines already an element $\xi : T \to P$ ([5], I, B 4) and two isomorphic Modules $\mathscr{L}$ and $\mathscr{L}'$ determine the same ξ.

Lemma **(II.3)**. — *The functor* $\mathbf{Pic}_{Z/k}(T)$ *satisfies axiom* P_1.

This is ([5], II, C, prop. 3.1 and its corollary) [1].

Lemma **(II.4)**. — *The functor* $\mathbf{Pic}_{Z/k}(T)$ *satisfies axiom* P_2.

Proof. — Let A be a k-algebra which is a local, noetherian ring with maximal ideal $\mathfrak{m}$; suppose A is complete and separated with respect to the $\mathfrak{m}$-adic topology. Let $\varphi_n : A \to A_n = A/\mathfrak{m}^{n+1}$ and $\varphi_{mn} : A_n \to A_m$ $(n, m = 0, 1, \ldots; m \leqslant n)$ be the natural homo-

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[1] The corollary is based on ([5], II, prop. 5.1), the formulation of which is not correct. However in our case we need this proposition only if $\Lambda/\mathfrak{n}_\xi = K$ (with the notations of that proposition); in fact in our case Λ is the field k and it suffices that the restriction P' of P to $(\mathrm{Sch}/\bar{k})$ is prorepresentable, therefore we may assume $\Lambda = \bar{k} = K$.

morphisms. Write $X = Z \times_k \mathrm{Spec}(A)$, $X_n = Z \times_k (\mathrm{Spec}(A_n)$, $u_n = 1_Z \times {}^a\varphi_n : X_n \to X$ and $u_{nm} = 1_Z \times {}^a\varphi_{mn} : X_m \to X_n$. Furthermore the completion of X along X_0 is denoted by $\hat{X}$.

(II.5) In the following we consider systems $\{\mathscr{L}_n, \theta_{mn}\}$ where $\mathscr{L}_n$ is an $\mathscr{O}_{X_n}$-Module and $\theta_{mn} : u_{nm}^*(\mathscr{L}_n) \xrightarrow{\sim} \mathscr{L}_m$ is an $\mathscr{O}_{X_m}$-isomorphism $(n \geqslant m)$ such that $\theta_{kn} = \theta_{km} \circ u_{mk}^*(\theta_{mn})$ for $n \geqslant m \geqslant k$; such a system we call a *compatible system* of Modules on $\{X_n\}$. Two compatible systems $\{\mathscr{L}_n, \theta_{mn}\}$ and $\{\mathscr{L}'_n, \theta'_{mn}\}$ are said to be isomorphic if there are $\mathscr{O}_{X_n}$-isomorphisms $\lambda_n : \mathscr{L}_n \to \mathscr{L}'_n$ such that $\theta'_{mn} . u_{nm}^*(\lambda_n) = \lambda_m . \theta_{mn}$ for $n \geqslant m$. In the following we consider only systems with $\mathscr{L}_n$ which are *coherent* $\mathscr{O}_{X_n}$-Modules and the set of isomorphism classes of such compatible systems we denote by $\mathrm{Cl}(\{X_n\})$. We note that if $\xi_n \in \mathrm{P}(\mathrm{Spec}(A_n))$ $(n = 0, 1, \ldots)$ is such that $\mathrm{P}(u_{nm})(\xi_n) = \xi_m$ for $n \geqslant m$ and if ξ_n is determined by a couple $(\mathscr{L}_n, \rho_n)$ (see II, 2) then there exists an $\mathscr{O}_{X_n}$-isomorphism $\theta_{mn} : u_{nm}^*(\mathscr{L}_n) \xrightarrow{\sim} \mathscr{L}_m$ such that $({}^a\varphi_{mn})^*(\rho_n) = \rho_m \circ \tau_m^*(\theta_{mn})$ with $\tau_m = \sigma \times 1_{\mathrm{Spec}(A_m)}$; from the fact that a rigidified system $(\mathscr{L}_k, \rho_k)$ has no (non-trivial) automorphisms follows that

$$\theta_{kn} = \theta_{km} \circ u_{mk}^*(\theta_{mn})$$

for $n \geqslant m \geqslant k$. Hence the system $\{\mathscr{L}_n, \theta_{mn}\}$ is a compatible system and its class is clearly independent of the representatives $(\mathscr{L}_n, \rho_n)$; i.e. the system $\{\xi_n\}$ determines uniquely an element of $\mathrm{Cl}\{X_n\}$. Our lemma follows now from the following three statements.

a) Let $\mathrm{Cl}(\hat{X})$ denote the set of isomorphism classes of coherent $\mathscr{O}_{\hat{X}}$-Modules; then the natural map $\alpha : \mathrm{Cl}(\hat{X}) \to \mathrm{Cl}(\{X_n\})$ is bijective.

Proof. — The map α is defined as follows. Let $\mathscr{L}'$ be a representative of an element $\mathrm{Cl}(\mathscr{L}')$ of $\mathrm{Cl}(\hat{X})$ and let $v_n : X_n \to \hat{X}$ be the natural morphism of ringed spaces, then $\alpha(\mathrm{Cl}(\mathscr{L}')) = \mathrm{Cl}\{v_n^*(\mathscr{L}'), \theta_{mn}\}$ where θ_{mn} is the natural isomorphism between $u_{nm}^*(v_n^*(\mathscr{L}'))$ and $v_m^*(\mathscr{L}')$. The fact that α is bijective follows essentially from (*E.G.A.*, I, theorem 10.11.3 and cor. 10.11.4). (We note however that, if $(\mathscr{L}_n, \theta_{mn})$ is a compatible system of coherent $\mathscr{O}_{X_n}$-Modules then the condition $u_{nm}^*(\mathscr{L}_n) = \mathscr{L}_m$, as required in *E.G.A.*, I, 10.11.3, is not fulfilled in general. Therefore we must first proceed as in *E.G.A.*, I, 10.6.6: take $\mathscr{L}' = \varprojlim \mathscr{L}_n$, where the projective limit is taken in the category of sheaves on the space $X_0 = \hat{X}$; then $\mathscr{L}'$ has the structure of an $\mathscr{O}_{\hat{X}}$-Module. The fact that $\mathscr{L}'$ is coherent and that $\alpha(\mathrm{Cl}(\mathscr{L}')) = \mathrm{Cl}\{\mathscr{L}_n, \theta_{mn}\}$ follows from the assumption that the θ_{mn} are isomorphisms and from (*E.G.A.*, $\mathbf{0}_1$, prop. 7.2.9)).

b) Let $\mathrm{Cl}(X)$ denote the set of isomorphism classes of coherent $\mathscr{O}_X$-Modules; the natural map $\beta : \mathrm{Cl}(X) \to \mathrm{Cl}(\hat{X})$ (obtained from the functor $\mathscr{L} \rightsquigarrow \hat{\mathscr{L}}$, see *E.G.A.*, I, def. 10.8.4) is bijective.

Proof. — This follows from (*E.G.A.*, III, cor. 5.1.6).

c) Let $\mathscr{L}'$ be a coherent $\mathscr{O}_{\hat{X}}$-Module. Then $\mathscr{L}'$ is invertible if and only if $v_n^*(\mathscr{L}')$ is an invertible $\mathscr{O}_{X_n}$-Module for all n.

Furthermore, if $\mathscr{L}$ is a coherent $\mathscr{O}_X$-Module, then $\mathscr{L}$ is invertible if and only if $\hat{\mathscr{L}}$ is an invertible $\mathscr{O}_{\hat{X}}$-Module.

Proof. — If $\mathscr{L}'$ is invertible then it is clear that $v_n^*(\mathscr{L}')$ is invertible for every n. Conversely: suppose $v_n^*(\mathscr{L}')$ is invertible, or more generally, locally free for all n. Put $\mathscr{L}'_n = v_n^*(\mathscr{L}')$. Let U be an open formal affine set on $\hat{X}$ such that $v_0^*(\mathscr{L}')/U$ is free; in order to show that $\mathscr{L}'$ is locally free and of the same rank as $v_0^*(\mathscr{L}')$ it clearly suffices to show that the $\Gamma(U, \mathscr{L}'_n)$ are free $\Gamma(U, \mathscr{O}_{X_n})$-modules and that there exists a compatible system of bases for the $\Gamma(U, \mathscr{L}'_n)$. We know that $\Gamma(U, \mathscr{L}'_n)$ is a flat $\Gamma(U, \mathscr{O}_{X_n})$-module and that the kernel of $\Gamma(U, \mathscr{O}_{X_n}) \to \Gamma(U, \mathscr{O}_{X_{n-1}})$ is nilpotent; therefore if $\Gamma(U, \mathscr{L}'_{n-1})$ is a free $\Gamma(U, \mathscr{O}_{X_{n-1}})$-module, then $\Gamma(U, \mathscr{L}'_n)$ is free and a basis of $\Gamma(U, \mathscr{L}'_{n-1})$ can be lifted to $\Gamma(U, \mathscr{L}'_n)$ (see *S.G.A.*, IV, prop. 4.1 and its proof). This completes the proof that $\mathscr{L}'$ is invertible if all $\mathscr{L}'_n$ are invertible.

If $\mathscr{L}$ is invertible then it is clear that $\hat{\mathscr{L}}$ is invertible. Conversely if $\hat{\mathscr{L}}$ is invertible, consider $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{O}_X) \to \mathscr{O}_X$; using (*E.G.A.*, I, cor. 10.8.10) and (*E.G.A.*, III, cor. 5.1.3) we see that this is an isomorphism and we conclude by (*E.G.A.*, $\mathbf{0}_I$, 5.4.3) that $\mathscr{L}$ itself is invertible.

(II.6) The proof of lemma II.4 is clear now, for if $\{\xi_n\}$ with $\xi_n \in P(\mathrm{Spec}(A_n))$ is such that $P(u_{nm})(\xi_n) = \xi_m$ $(n \geq m)$ then $\{\xi_n\}$ determines, as we have seen, uniquely an element of $\mathrm{Cl}(\{X_n\})$ with invertible $\mathscr{L}_n$. This determines uniquely an element of $\mathrm{Cl}(\hat{X})$, hence of $\mathrm{Cl}(X)$, consisting of invertible $\mathscr{O}_X$-Modules and this determines uniquely an $\xi \in P(\mathrm{Spec}(A))$ by II.2; from the above descriptions of the map α and β it follows immediately that $P(u_n)(\xi) = \xi_n$ for all n.

Lemma **(II.7)**. — *The functor* $\mathbf{Pic}_{Z/k}(T)$ *satisfies axiom* P_3.

For the proof we refer to (*E.G.A.*, IV) (compare also with lemma I.3.2 where a proof of a statement or similar type is given).

Lemma **(II.8)**. — *The functor* $\mathbf{Pic}_{Z/k}(T)$ *satisfies axiom* P_4 *and* P_5.

This follows from the definition of the Picard functor.

Lemma **(II.9)**. — *Let* Z′ *and* Z *fulfill the assumptions 1), 2) and 3) from (II.2); assume furthermore that there exists a morphism* $f: Z' \to Z$ *such that* $\mathscr{O}_Z \xrightarrow{\sim} f_*(\mathscr{O}_{Z'})$. *Then the Picard functor* $P(T) = \mathbf{Pic}_{Z/k}(T)$ *is a subfunctor of* $P'(T) = \mathbf{Pic}_{Z'/k}(T)$.

Proof. — The assumption $\mathscr{O}_Z \xrightarrow{\sim} f_*(\mathscr{O}_{Z'})$ remains valid for every base extension $T \to \mathrm{Spec}(k)$ (*E.G.A.*, III, prop. 1.14.15). Write $X = Z \times_k T$, $X' = Z' \times_k T$ and $g = f_{(T)}$. If $\mathscr{L}$ is an invertible $\mathscr{O}_X$-Module, then $g^*(\mathscr{L})$ is an invertible $\mathscr{O}_{X'}$-Module and this defines (see (II.2)) a functor morphism $P(T) \to P'(T)$. In order to see that for every T this map $P(T) \to P'(T)$ is injective, we consider two rigidified, invertible $\mathscr{O}_X$-Modules $\mathscr{L}_1$ and $\mathscr{L}_2$; let $\mathscr{H} = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}_1, \mathscr{L}_2)$. If $\mathscr{L}'_i = g^*(\mathscr{L}_i)$ $(i = 1, 2)$, then it is immediately checked that $\mathscr{H}' = \mathscr{H}om_{\mathscr{O}_{X'}}(\mathscr{L}'_1, \mathscr{L}'_2) = g^*(\mathscr{H})$. We have to show that the natural map $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{L}_1, \mathscr{L}_2) = \Gamma(X, \mathscr{H}) \to \Gamma(X', \mathscr{H}') = \mathrm{Hom}_{\mathscr{O}_{X'}}(\mathscr{L}'_1, \mathscr{L}'_2)$ is bijective; but $\Gamma(X', \mathscr{H}') = \Gamma(X, g_*(\mathscr{H}'))$ and $g_*(\mathscr{H}') \xrightarrow{\sim} \mathscr{H} \otimes_{\mathscr{O}_X} g_*(\mathscr{O}_{X'})$; the lemma follows from the fact that $g_*(\mathscr{O}_{X'}) \xrightarrow{\sim} \mathscr{O}_X$.

Lemma **(II.10).** — *Let* Z_1 *and* Z *fulfill the assumptions 1), 2) and 3) from II.2; assume furthermore that there exists a morphism* $f: Z_1 \to Z$ *such that* Z_1 *is finite over* Z *and such that f is an effective epimorphism* ([5], III, p. 212-03). *Suppose the Picard functor* $P_1(Z)$ *of* Z_1 *is representable by a scheme of groups, locally of finite type over k. Then the Picard functor* P(T) *of* Z *is also representable by a scheme of groups, locally of finite type over k.*

Proof. — The functor P(T) satisfies P_1, P_2, P_3, P_4 and P_5; by the corollary of theorem 1 (I.2.5) it suffices to show that P can be embedded in a functor $Q: (Sch/k)^0 \to \mathbf{Ab}$ which is representable by a scheme of groups, locally of finite type over *k*.

Let $T \in (Sch/k)$, write $X = Z \times_k T$ and $X_1 = Z_1 \times_k T$ and consider the projections $p_1, p_2 : X_1 \times_X X_1 \to X_1$. We are going to define a suitable functor $Q \in \mathbf{Hom}((Sch/k)^0, \mathbf{Ab})$. Consider couples $(\mathscr{L}_1, \lambda)$ consisting of a rigidified invertible Module $\mathscr{L}_1$ on X_1 and an isomorphism $\lambda : p_1^*(\mathscr{L}_1) \tilde{\to} p_2^*(\mathscr{L}_1)$ on $X_1 \times_X X_1$; two such couples $(\mathscr{L}_1, \lambda)$ and $(\mathscr{L}_1', \lambda')$ are said to be isomorphic if there exists an isomorphism $\theta_1 : \mathscr{L}_1 \tilde{\to} \mathscr{L}_1'$ such that $\lambda' \circ p_1^*(\theta_1) = p_2^*(\theta_1) \circ \lambda$. Let Q(T) be the set of isomorphism classes of such couples; then clearly $Q \in \mathbf{Hom}((Sch/k)^0, \mathbf{Ab})$.

Let $\xi \in P(T)$; then (see (II.2)) ξ determines a rigidified invertible $\mathscr{O}_X$-Module $\mathscr{L}$, consider the couple $(g^*(\mathscr{L}), \lambda)$ where $g = f_{(T)}$ and λ is an isomorphism obtained by identifying $p_1^*(g^*(\mathscr{L}))$ and $p_2^*(g^*(\mathscr{L}))$ with $h^*(\mathscr{L})$ (where $h = g \circ p_1 = g \circ p_2$). The isomorphism class of the couple $(g^*(\mathscr{L}), \lambda)$ is an element of Q(T), depending only on ξ, and in this way we obtain clearly a functor morphism $\Phi : P \to Q$. Next we want to show that Φ is a monomorphism. If $\Phi(T)(\xi) = \Phi(T)(\xi')$ with $\xi : T \to P$ and $\xi' : T \to P$ and if ξ (resp. ξ') is representable by $\mathscr{L}$ (resp. $\mathscr{L}_1$) then there exists an isomorphism $\theta_1 : g^*(\mathscr{L}) = \mathscr{L}_1 \tilde{\to} g^*(\mathscr{L}') = \mathscr{L}_1'$ such that $\lambda' \circ p_1^*(\theta_1) = p_2^*(\theta_1) \circ \lambda$; but this means that after identification of $p_1^*(\mathscr{L}_1)$ and $p_2^*(\mathscr{L}_1)$ (resp. $p_1^*(\mathscr{L}_1')$ and $p_2^*(\mathscr{L}_1')$) with $h^*(\mathscr{L})$ (resp. with $h^*(\mathscr{L}')$) we have $p_1^*(\theta_1) = p_2^*(\theta_1)$. Consider the effective epimorphism $f: Z_1 \to Z$; according to ([5], I, p. 190-208) this is a descent morphism for the category of quasi-coherent, flat sheaves on Z and it remains so by a flat base extension $X \to Z$ (obtained from $T \to (Spec(k))$. Therefore there exists an $\mathscr{O}_X$-isomorphism $\theta : \mathscr{L} \tilde{\to} \mathscr{L}'$ such that $g^*(\theta) = \theta_1$. But then $\xi = \xi'$; i.e. Φ is a monomorphism.

Next we consider the functor morphism $\Psi : Q(T) \to P_1(T)$ defined in an obvious way. In order to complete the proof of lemma (II.10) we have to show that Q is representable by a scheme of groups, locally of finite type over *k*. Since P_1 is representable by such a scheme it suffices by ([6], IV, prop. 3.6 and remark 3.13) to show that $\psi : Q \to P_1$ is relatively representable by schemes which are locally of finite type over the groundscheme.

Write $Y = X_1 \times_Z X_1$; we note that $h : Y \to T$ is proper and flat (for $Z_1 \times_X Z_1 \to Spec(k)$ is proper by (*E.G.A.*, II, prop. 5.4.2 and cor. 5.4.3) and of course flat and *h* is obtained from this morphism by base extension $T \to Spec(k)$); let $\xi_1 : T \to P_1$ and take on X_1 a rigidified (with respect to some section), invertible $\mathscr{O}_{X_1}$-Module $\mathscr{L}_1$ which

determines ξ_1 and write $\mathscr{F}=p_1^*(\mathscr{L}_1)$ and $\mathscr{G}=p_2^*(\mathscr{L}_1)$. The proof of lemma (II.10) is completed by:

Lemma **(II.11)**. — *Let* $h : Y \to T$ *be a proper, flat morphism. Let* $\mathscr{F}$ *and* $\mathscr{G}$ *be invertible* $\mathscr{O}_Y$*-Modules. Suppose* $T' \in (Sch/T)$; *let* $\alpha : T' \to T$ *be the structure morphism; write* $Y' = Y \times_T T'$, $\mathscr{F}' = \alpha_{(Y)}^*(\mathscr{F})$, $\mathscr{G}' = \alpha_{(Y)}^*(\mathscr{G})$ *and let* $\mathrm{Isom}_{Y'}(\mathscr{F}', \mathscr{G}')$ *be the set of* $\mathscr{O}_{Y'}$*-isomorphisms from* $\mathscr{F}'$ *to* $\mathscr{G}'$. *Then the functor* $R : T' \rightsquigarrow \mathrm{Isom}_{Y'}(\mathscr{F}', \mathscr{G}')$ *on* (Sch/T) *is representable by an open set of a vectorbundle over* T *which is of finite type over* T.

First we prove:

Lemma **(II.12)**. — *The assumptions are the same as in lemma II.11. The functor* $R_1 : T' \rightsquigarrow \mathrm{Hom}_{Y'}(\mathscr{F}', \mathscr{G}')$ *is representable by a vector bundle over* T *which is of finite type over* T.

Proof. — Let $\mathscr{H}=\mathscr{H}om_{\mathscr{O}_Y}(\mathscr{F}, \mathscr{G})$ and $\mathscr{H}'=\mathscr{H}om_{\mathscr{O}_{Y'}}(\mathscr{F}', \mathscr{G}')$. Since $\mathscr{F}$ and $\mathscr{G}$ are invertible, we have that $\mathscr{H}$ is invertible, and clearly $\mathscr{H}'=\alpha_{(Y)}^*(\mathscr{H})$. Our problem therefore is to represent the functor (in T') $\Gamma(Y', \mathscr{H}')=\Gamma(T', g'_*(\mathscr{H}'))$ where $g'=g_{(T')}$ and where $\mathscr{H}$ is a coherent Y-flat Module. By (*E.G.A.*, III, § 7) there exists a coherent $\mathscr{O}_T$-Module $\mathscr{A}$ such that $g_*(\mathscr{H})=\mathscr{H}om_{\mathscr{O}_T}(\mathscr{A}, \mathscr{O}_T)$ and $\mathscr{A}$ commutes with base extension. Therefore we have to consider the functor $T' \rightsquigarrow \mathrm{Hom}_{\mathscr{O}_{T'}}(\mathscr{A}', \mathscr{O}_{T'})$ with $\mathscr{A}'=\alpha^*(\mathscr{A})$ and $\mathscr{A}$ is $\mathscr{O}_T$-coherent. However this functor is, as is well known (see (*E.G.A.*, II, 1.7.8) and [6], V, prop. 1.1), representable by a vector bundle over T which is of finite type over T.

(II.13) *The proof of lemma II.11.*

Consider the functor (mono)morphism $\rho : R \to R_1$; it suffices by ([6], IV, prop. 3.7) to show that ρ is relatively representable by open immersions. Let $\lambda' \in \mathrm{Hom}_{Y'}(\mathscr{F}', \mathscr{G}')$ be given; we have to consider on (Sch/T') the functor $R_2 : T'' \rightsquigarrow R_2(T'')$ where $R_2(T'')$ consists of one element if $(\beta_{(Y')})^*(\lambda')$ is an isomorphism ($\beta : T'' \to T'$ is the structure morphism and $Y'=Y \times_T T'$) and $R_2(T'')$ is empty if $(\beta_{(Y')})^*(\lambda')$ is not an isomorphism. Let (as above) $\mathscr{H}'=\mathscr{H}om_{\mathscr{O}_{Y'}}(\mathscr{F}', \mathscr{G}')$, then $\lambda' \in \Gamma(Y', \mathscr{H}')$. Consider the open set Y'_0 of Y' consisting of all points $y \in Y'$ such that $(\lambda')_y \notin \mathfrak{m}_y . \mathscr{H}'_y$ (where $\mathfrak{m}_y$ is the maximal ideal of $\mathscr{O}_{y, Y'}$). Consider the set $T'_0=T-g'(Y'-Y'_0)$, this set is open since $g' : Y' \to T'$ is proper. Clearly T'_0 represents the functor R_2 and this completes the proof of lemma (II.11).

Remark **(II.14)**. — The same argument works if $\mathscr{F}$ and $\mathscr{G}$ are locally free and of the same finite rank. Instead of $(\lambda')_y \notin \mathfrak{m}_y \mathscr{H}_y$ one has to require then that the " determinant " of λ' is a unit in $\mathscr{O}_{y, Y'}$.

(II.15) *Theorem 2.* — *Let* $\pi : Z \to \mathrm{Spec}(k)$ *be a proper morphism. Then the Picard functor* $\mathbf{Pic}_{Z/k}$ *is representable by a scheme of groups which is locally of finite type over k.*

Proof. — *a)* We first assume that *k* is *algebraically closed and* Z *integral.* Then $\pi_*(\mathscr{O}_Z)=k$ (*E.G.A.*, II, cor. 7.3.11) and clearly Z has a rational point. Therefore we can apply to Z lemmas II.3, 4, 7 and 8. According to the lemma of Chow (*E.G.A.*,

Th. 5.6.1 and cor. 5.6.2) there exists a scheme Z′ and a surjective morphism $f: Z' \to Z$ such that f and $\pi' = \pi \circ f$ are projective. Consider the coherent $\mathcal{O}_Z$-Algebra $f_*(\mathcal{O}_{Z'})$; let Z_1 be the finite Z-scheme such that $\mathcal{A}(Z_1) \tilde{\to} f_*(\mathcal{O}_{Z'})$ (see *E.G.A.*, III, Th. 4.3.1); then f factors as $Z' \xrightarrow{g} Z_1 \xrightarrow{h} Z$ and $\mathcal{O}_{Z_1} \tilde{\to} g_*(\mathcal{O}_{Z'})$. By ([5], V, cor. 6.6) the Picard scheme of Z′ exists (and is locally of finite type) hence by lemma II.9 and the corollary in I.2.5 the same is true for the Picard scheme of Z_1. Next consider $h: Z_1 \to Z$; this morphism is surjective and finite. Since Z is integral h gives an injection in the corresponding sheaves of rings and hence h is an epimorphism. By a result of ([5], I, p. 190-08) h can be factored into a finite product of effective epimorphisms (see [5], III, p. 212-03 below). It follows then by lemma II.10 that the Picard functor of Z is representable by a scheme of groups, locally of finite type over k.

b) Assume k algebraically closed and Z *reduced.* The theorem follows then from *a)* above and from ([5], V, cor. 6.4).

c) Assume *k algebraically closed* (and no condition on Z). The theorem follows from *b)* and from Oort's theorem ([7], 8).

d) General case. The theorem follows from lemma I.8.9 applied to the algebraic closure k' of k after using lemma II.8 and the result in *c)* above.

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