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SOME REMARKS ON THE OPTIONAL DECOMPOSITION THEOREM

C. Stricker and J.A. Yan

Summary : Let S be a vector-valued semimartingale and $\mathcal{Z}(S)$ the set of all strictly positive local martingales Z with $Z_0 = 1$ such that ZS is a local martingale. Assume V (resp. U) is a nonnegative process such that for each $Z \in \mathcal{Z}(S)$ ZV is a supermartingale (resp. ZU is a local submartingale with $\sup_{Z\in\mathcal{Z}(S),\tau\in\mathcal{T}^f} E(Z_\tau U_\tau) < +\infty$ where $\mathcal{T}^f$ denotes the set of all finite stopping times). Then V (resp. U) admits a decomposition $V = V_0 + \phi \cdot S - C$ (resp. $U = U_0 + \psi \cdot S + A$) where C and A are adapted increasing processes with $C_0 = A_0 = 0$. The first result is a slight generalization of the optional decomposition theorem (see [2,4,7]) and the second one is new. As an application to mathematical finance, if S is interpreted as the discounted price process of the stocks, we show $\mathcal{Z}(S)$ contains exactly one element iff the market is complete.

1. Introduction and motivations.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a usual stochastic basis. For simplicity we assume $\mathcal{F}_0$ is *trivial*. Consider a model of a security market which consists of $d+1$ assets : one bond and d stocks. We choose the bond as a numéraire and denote by $S = (S^1, \ldots, S^d)$ the discounted price process of the stocks. We fix a time horizon $[0, T]$. Let B be a discounted European contingent claim (i.e. a nonnegative $\mathcal{F}_T$-measurable r.v.). Assume that there exists at least one probability measure Q, equivalent to P such that S is a local martingale under Q. We denote by $\mathcal{P}(S)$ the set of all such probability measures Q. Put

$$V_t := \text{ess sup}_{Q\in\mathcal{P}(S)} E^Q[B|\mathcal{F}_t], \qquad 0 \le t \le T. \tag{1.1}$$

If $\sup_{Q\in\mathcal{P}(S)} E^Q[B] < +\infty$, then (V_t) is a supermartingale under each $Q \in \mathcal{P}(S)$. When S is a diffusion process, El Karoui and Quenez [2] proved that V admits a decomposition of the form

$$V_t = V_0 + (\phi \cdot S)_t - C_t, \qquad 0 \le t \le T, \tag{1.2}$$

where ϕ is a vector-valued predictable process, integrable w.r.t. S and C is an adapted increasing process with $C_0 = 0$. Notice that since $V_0 + \phi \cdot S$ is nonnegative, $\phi \cdot S$ is also a local martingale under $Q \in \mathcal{P}(S)$ (see Emery [3] and Ansel/Stricker [1]). The process V is called the value process associated to the problem of hedging the contingent claim B. Here the financial meaning of the processes ϕ and C is clear : If the "option-writer"(i.e. the seller of the contingent claim B) invests the initial capital V_0 in the market and uses the hedging strategy ϕ, he can obtain a cumulative profit

called the cumulated consumptions C during the time interval $[0,T]$ and replicate the contigent claim B at time T. V_0 is called the "selling price" of the contingent claim B, because with this price as initial capital one can hedge the contingent claim without risk. Inspired by El Karoui/Quenez [2], Kramkov [7] showed that a decomposition of the form (1.2) is valid in a more general situation, i.e. (S_t) is a locally bounded process and (V_t) is a nonnegative process such that for each $Q \in \mathcal{P}(S)$, (V_t) is a Q-supermartingale. This unpleasant "local boundedness" condition has been removed in the recent paper of Föllmer/Kabanov [4] by means of the "Lagrange Multipliers" method. The following theorem is the main result of their paper :

Theorem 1.1. *Let S be a vector-valued process defined on $(\Omega,(\mathcal{F}_t),\mathcal{F},P)$. Denote by $\mathcal{P}(S)$ the set of all probability measures $Q \sim P$ such that S is a Q-local martingale. Suppose that $\mathcal{P}(S) \neq \emptyset$ and V is a nonnegative process. Then the following statements are equivalent :*
i) V is a Q-supermartingale for each $Q \in \mathcal{P}(S)$.
ii) There exist a predictable process ϕ, integrable w.r.t. S and an adapted increasing process C with $C_0 = 0$ such that $V = V_0 + \phi \cdot S - C$ and $E^Q(C_\infty) < +\infty$ for each $Q \in \mathcal{P}(S)$.

This theorem is an important contribution both to the theory of semimartingales and to mathematical finance. The first aim of this paper is to weaken the assumption $\mathcal{P}(S) \neq \emptyset$ in Theorem 1.1 . It turns out that if there exists at least one strictly positive local martingale Z with $Z_0 = 1$ such that ZS is a local martingale (Z is called *a strict martingale density for* S), then any nonnegative process V satisfying the property that ZV is a supermartingale for each strict martingale density Z for S admits a decomposition of the form (1.2)(see Theorem 2.1 below). When no strict martingale density for S exists, we give a characterization of those stopping times τ such that S^τ has a strict martingale density (see Theorem 2.2 below). The second aim of our paper is to investigate the submartingale case. From the option-buyer's point of view the following process

$$U_t := \text{ess inf}_{Q \in \mathcal{P}(S)} E^Q[B|\mathcal{F}_t], \qquad 0 \leq t \leq T$$

should be the value process associated to the problem of hedging the contingent claim B. Observe that (U_t) is a submartingale under each $Q \in \mathcal{P}(S)$. A very natural question is the following : does U admit a decomposition of the form

$$U = U_0 + (\psi \cdot S)_t + A_t, \qquad 0 \leq t \leq T, \tag{1.3}$$

where (A_t) is an adapted increasing process with $A_0 = 0$. It turns out that in the general setting of Theorem 1.1 a necessary and sufficient condition for a process U to admit a decomposition of form (1.3) is available. We shall prove this result in the setting of Theorem 1.1 as well as in our generalized setting (see Theorem 2.3 and 2.4 below). The decomposition (1.3) has the following financial meaning. In contrast to the selling price V_0, U_0 is called the purchase price of the contingent claim B, because with this price as the initial capital one should save the amount A during the time interval $[0,T]$ in order to hedge the contingent claim B at time T. So this price U_0 is favourable to the option buyer.

As an application of the above two results we give a characterization of the replicability of a contingent claim which generalizes a previous result of Ansel/Stricker [1] and Jacka [5] .

2. Main results.

Throughout this paper we consider a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ which satisfies the usual conditions. We denote by $\mathcal{M}_{loc}$ the set of all local martingales. A strictly positive local martingale Z with $Z_0 = 1$ is called a *strict martingale density* for a vector-valued process S if $ZS \in \mathcal{M}_{loc}$. We denote by $\mathcal{Z}(S)$ the set of all strict martingale densities for S, and by $\mathcal{P}(S)$ the set of all laws $Q \sim P$ such that S is a local martingale under Q. In the sequel we also denote by $\mathcal{P}(S)$ the set of density processes $M_t := E(\frac{dQ}{dP}|\mathcal{F}_t)$ associated to $Q \in \mathcal{P}(S)$. For a semimartingale X we denote by $\mathcal{E}(X)$ the Doléans-Dade exponential of X, i.e. the unique solution Y of the stochastic differential equation $dY = Y_- dX$ and $Y_0 = 1$. The following three lemmas that are probably known, are the key for the proof of our main results. For the sake of completeness we shall give the proofs.

Lemma 2.1. *Let V be an optional process, Y a local martingale and T a stopping time. If $Y^T V^T$ is a local supermartingale, then YV^T is also a local supermartingale.*

Proof. Notice that

$$YV^T = (Y - Y^T)V^T + Y^T V^T = (V_T 1_{]\!]T,\infty[\![}) \cdot Y + Y^T V^T.$$

Since Y is a local martingale and $V_T 1_{]\!]T,\infty[\![}$ is a locally bounded predictable process, $(V_T 1_{]\!]T,\infty[\![}) \cdot Y$ is a local martingale. Hence YV^T is a local supermartingale.

Lemma 2.2. *Let Z and M be two strictly positive uniformly integrable martingales and T be a stopping time. Assume $Z = Z^T$ and $M_T > 0$. Put $Z' := ZM(M^T)^{-1}$. Then Z' is a strictly positive uniformly integrable martingale.*

Proof. First of all Z' is a local martingale because $Z = Z^T$. Since Z' is positive, it is a supermartingale. Thus it remains to prove that $E(Z'_\infty) = E(Z'_0)$. We have $M_T = E[M_\infty|\mathcal{F}_T] = E[M_T M_\infty M_T^{-1}|\mathcal{F}_T] = M_T E[M_\infty M_T^{-1}|\mathcal{F}_T]$. Thus $E[M_\infty M_T^{-1}|\mathcal{F}_T] = 1$. Now $E[Z'_\infty] = E(Z_T E[M_\infty M_T^{-1}|\mathcal{F}_T]) = E(Z_T) = E(Z_0) = E(Z'_0)$.

Lemma 2.3. *Let S be a vector-valued process and T, T_1, T_2 be stopping times.*
i) We have $\mathcal{Z}(S) \subset \mathcal{Z}(S^T)$.
ii) If $Z' \in \mathcal{Z}(S^{T_1})$ and $Z'' \in \mathcal{Z}(S^{T_2})$, then $Z := (Z')^{T_1}(Z'')^{T_2}[(Z'')^{T_1 \wedge T_2})]^{-1}$ belongs to $\mathcal{Z}(S^{T_1 \vee T_2})$.
iii) If $\mathcal{P}(S^{T_1}) \neq \emptyset$ and $\mathcal{P}(S^{T_2}) \neq \emptyset$, then $\mathcal{P}(S^{T_1 \vee T_2}) \neq \emptyset$.

Proof. We only need to prove the lemma for the case when S is a real-valued process. Since $ZS^T = (Z - Z^T)S^T + (ZS)^T = (S_T 1_{]\!]T,+\infty[\![}) \cdot Z + (ZS)^T$, we conclude that $\mathcal{Z}(S) \subset \mathcal{Z}(S^T)$. Now we are going to prove the second assertion. We have $Z(S^{T_1 \vee T_2}) = (ZS)^{T_1 \vee T_2} = (ZS)^{T_2} - (ZS)^{T_1 \vee T_2} + (ZS)^{T_1} = \frac{Z'_{T_1}}{Z''_{T_1}} 1_{T_1 < +\infty}((Z''S)^{T_2} -$

$(Z''S)^{T_1\vee T_2}) + (Z'S)^{T_1} = \left(\frac{Z'_{T_1\wedge T_2}}{Z''_{T_1\wedge T_2}} 1_{]T_1\wedge T_2, T_2]}\right) \cdot (Z''S)^{T_2} + (Z'S)^{T_1}$. The last assertion follows from ii) and Lemma 2.2 applied to $(Z')^{T_1}$ and $(Z'')^{T_2}$ with $T = T_1$.

Definition 2.1. *A subset $\mathcal{H}$ of $\mathcal{Z}(S)$ is called dense in $\mathcal{Z}(S)$ if for each $Z \in \mathcal{Z}(S)$ there exists a sequence $Z^n \in \mathcal{H}$ such that $\forall t \geq 0$ $Z^n_t \longrightarrow Z_t$ a.e.*

If $\mathcal{P}(S) \neq \emptyset$, then $\mathcal{P}(S)$ is dense in $\mathcal{Z}(S)$. Indeed, let $Z \in \mathcal{Z}(S)$ and (T_n) be an increasing sequence of finite stopping times converging to $+\infty$ such that Z^{T_n} is a uniformly integrable martingale. Let $Q \in \mathcal{P}(S)$, $M := E(\frac{dQ}{dP}|\mathcal{F}_t)$ and $Z^{(n)} := Z^{T_n} M(M^{T_n})^{-1}$. Then by Lemma 2.2 and 2.3 $Z^{(n)}$ is a uniformly integrable martingale which belongs to $\mathcal{P}(S)$. Moreover $\forall t \geq 0$ $Z^{(n)}_t$ converges to Z_t. Thus the next theorem is a generalization of Theorem 1.1.

Theorem 2.1. *Let S be a vector-valued semimartingale such that $\mathcal{Z}(S) \neq \emptyset$, $\mathcal{H}$ a dense subset of $\mathcal{Z}(S)$, $\mathcal{T}$ the set of all stopping times and V a nonnegative process . The following statements are equivalent :*
i) For each $Z \in \mathcal{H}$, ZV is a supermartingale.
ii) For each $Z \in \mathcal{Z}(S)$, ZV is a supermartingale.
iii) V admits a decomposition of the form :

$$V = V_0 + \phi \cdot S - C, \tag{2.1}$$

such that ϕ is a predictable process integrable w.r.t. S, $Z(\phi \cdot S)$ a local martingale for each $Z \in \mathcal{Z}(S)$, C an adapted increasing process, $C_0 = 0$ and $\forall Z \in \mathcal{Z}(S)$, $\forall \tau \in \mathcal{T}$ $E(Z_\tau C_\tau) < +\infty$.
Moreover, if i), ii) or iii) holds, then $\sup_{Z\in\mathcal{Z}(S),\tau\in\mathcal{T}} E(Z_\tau C_\tau) \leq V_0$.

Proof. We first prove i) $\Longrightarrow$ii). Let $Z \in \mathcal{Z}(S)$. Then there exists a sequence $Z^n \in \mathcal{H}$ such that $\forall t \geq 0$ $Z^n_t \longrightarrow Z_t$ a.e.. Since $Z^n V$ is a supermartingale, for $s \leq t$ we have $E(Z^n_t V_t|\mathcal{F}_t) \leq Z^n_s V_s$. According to Fatou's lemma $E(Z_t V_t|\mathcal{F}_t) \leq Z_s V_s$, i.e. ZV is a supermartingale.
Now we prove ii) $\Longrightarrow$ iii). We take an arbitrary but fixed element $M \in \mathcal{Z}(S)$ and take an increasing sequence (T_n) of finite stopping times with $T_n \longrightarrow +\infty$ such that each M^{T_n} is a uniformly integrable martingale. We shall show that each pair (S^{T_n}, V^{T_n}) satisfies the assumption of Theorem 1.1. First of all, $\mathcal{P}(S^{T_n}) \neq \emptyset$, because $M^{T_n} \in \mathcal{P}(S^{T_n})$. Now fix an n and let $Q \in \mathcal{P}(S^{T_n})$. Put $Y_t := E(\frac{dQ}{dP}|\mathcal{F}_t)$. Then $Y \in \mathcal{Z}(S^{T_n})$. Set $Y' := Y^{T_n} M(M^{T_n})^{-1}$. By Lemma 2.3 $Y' \in \mathcal{Z}(S)$. Thus $Y'V$ is a supermartingale under P. Since $(Y')^{T_n} = Y^{T_n}$, we know that $(YV)^{T_n}$ is also a supermartingale under P. According to Lemma 2.1 and taking in account the fact that YV^{T_n} is nonnegative with $Y_0 V_0^{T_n} = V_0$ being integrable, YV^{T_n} is a supermartingale under P. This is equivalent to saying that V^{T_n} is a supermartingale under Q. Therefore we can apply Theorem 1.1 to get the following decomposition of V :

$$V^{T_n} = V_0 + \phi^{(n)} \cdot S^{T_n} - C^{(n)},$$

where $\phi^{(n)}$ is a vector-valued predictable process which is integrable w.r.t. S^{T_n} and $C^{(n)}$ is an adapted increasing process with $(C^{(n)})^{T_n} = C^{(n)}$. Obviously we can assume

$\phi^{(n)}\mathbf{1}_{]T_n,+\infty[} = 0$. Put

$$\phi := \phi^{(1)} + \sum_{k=1}^{\infty}(\phi^{(k+1)} - \phi^{(k)}) \text{ and } C := C^{(1)} + \sum_{k=1}^{\infty}(C^{(k+1)} - C^{(k)}).$$

Then we obtain a decomposition of the form (2.1). Remark that for $n \geq k$, we have $\phi^{(k)} \cdot S^{T_n} = \phi^{(k)} \cdot S^{T_k}$. So it is easy to see that we have

$$(\phi \cdot S)^{T_n} = \phi^{(n)} \cdot S^{T_n}.$$

Next we show that for each $Z \in \mathcal{Z}(S)$, $Z(\phi \cdot S)$ is a local martingale. Let $Z \in \mathcal{Z}(S)$. Take an increasing sequence T_n of finite stopping times with $T_n \longrightarrow +\infty$ such that each Z^{T_n} is a uniformly integrable martingale. Now S^{T_n} is a local martingale under the law $Q^n := Z_{T_n}P$, $V_0 + \phi \cdot S \geq 0$ and $(\phi \cdot S)^{T_n} = \phi \cdot S^{T_n}$. Therefore $(\phi \cdot S)^{T_n}$ is a Q^n local martingale (see Ansel/Stricker [1] Corollaire 3.5). Hence $(Z(\phi \cdot S))^{T_n}$ is a P-local martingale for each T_n and therefore $Z(\phi \cdot S)$ is a P-local martingale. Moreover since $V_0 + \phi \cdot S$ is nonnegative, $Z(\phi \cdot S + V_0)$ is also a P-supermartingale. So $\sup_{Z \in \mathcal{Z}(S), \tau \in CT} E(Z_\tau C_\tau) \leq V_0$.

It remains to prove iii) $\Longrightarrow$ i). Let $Z \in \mathcal{Z}(S)$. Since for each stopping time τ $E(Z_\tau C_\tau) < +\infty$, the process ZC is locally integrable and $ZC = C_- \cdot Z + Z \cdot C$ is a submartingale. Therefore the assumption that $Z(\phi \cdot S)$ is a P local martingale implies ZV is a P-supermartingale.

Next we give a straightforward application of the previous theorem to mathematical finance.

Corollary 2.1. *Let S be a vector-valued process such that $\mathcal{Z}(S) \neq \emptyset$. Then $\mathcal{Z}(S)$ contains only one element iff there exists $Z \in \mathcal{Z}(S)$ such that for any finite stopping time R, each bounded $\mathcal{F}_R$-measurable random variable ξ admits a representation of the form*

$$\xi = x + \int_0^R \phi_s dS_s \tag{2.2}$$

where ϕ is a vector-valued predictable process which is integrable w.r.t S and $Z(x+\phi \cdot S)$ is a uniformly integrable martingale.

Proof. Assume $\mathcal{Z}(S)$ contains only one element Z. We can and will assume ξ is nonnegative. Put

$$V_t := Z_t^{-1} E(Z_R \xi | \mathcal{F}_t),\ t \geq 0.$$

Then ZV is a nonnegative martingale. By Theorem 2.1 V admits a decomposition of the form (2.1) with $C = 0$:

$$V_t = V_0 + (\phi \cdot S)_t.$$

Since $V_R = \xi$ we get (2.2).

The converse is straightforward. Let Z' be another strict martingale density for S and τ a stopping time such that Z^τ and $(Z')^\tau$ are uniformly integrable martingales. Then it is well-known (see for instance Corollary 11.4 page 340 of [6]) that (2.2) implies $Z^\tau = (Z')^\tau$. Therefore we get $Z = Z'$ and the proof is completed.

Now, given a RCLL adapted vector-valued process S, we will investigate those stopping times R for which $\mathcal{Z}(S^R) \neq \emptyset$. Recall that a subset B of $\mathbb{R}^+ \times \Omega$ is said of type $[\![0, \cdot\|$ if there exists a nonnegative r.v. R such that each section $B(\omega)$ is not empty and equal to $[0, R(\omega)]$ or $[0, R(\omega))$. According to Lemma 5.2 of Jacod [6], a set B of type $[\![0, \cdot\|$ is predictable iff there exists an increasing sequence (T_n) of stopping times such that $T_n \to R$ and $B = \cup_{n=1}^{\infty} [\![0, T_n]\!]$.

Theorem 2.2. *Let S be a vector-valued RCLL adapted process. Then there exists a unique (up to an evanescent set) predictable set B of type $[\![0, \cdot\|$ such that for each stopping time τ, $[\![0, \tau]\!] \subset B$ iff $\mathcal{Z}(S^\tau) \neq \emptyset$. Moreover there exists an increasing sequence of stopping times (T_n) such that $\mathcal{Z}(S^{T_n}) \neq \emptyset$ and $B = \cup_n [\![0, T_n]\!]$.*

Proof. Denote by $\mathcal{T}$ the set of all stopping times. Put $\mathcal{C} := \{\tau \in \mathcal{T} : \mathcal{Z}(S^\tau) \neq \emptyset\}$. Then $\tau \equiv 0$ belongs to $\mathcal{C}$. Set $R := \text{ess sup } \mathcal{C}$ and $\mathcal{C}_1 := \{\tau \in \mathcal{C} : \tau \equiv 0 \text{ or } P(\tau = R) > 0\}$. By lemma 2.3, if T_1 and T_2 belong to $\mathcal{C}$, then $T_1 \vee T_2 \in \mathcal{C}$. Consequently, there exists an increasing sequence (R^n) of elements of $\mathcal{C}$ such that $R^n \uparrow R$. If T_1 and T_2 belong to $\mathcal{C}_1$, it is easy to check that $T_1 \vee T_2 \in \mathcal{C}_1$. Hence there exists an increasing sequence (U_n) of elements of $\mathcal{C}_1$ such that $U^n \uparrow \text{ess sup } \mathcal{C}_1$. We put

$$T_n := R^n \vee U^n \quad \text{and} \quad B := \cup_n [\![0, T^n]\!].$$

Then B is predictable. It is obvious that

$$\tau \in \mathcal{C} \Longrightarrow [\![0, \tau]\!] \subset B.$$

Now we are going to show the converse. Assume τ is a stopping time such that $[\![0, \tau]\!] \subset B$. Put

$$R_n := T^n_{[T^n < \tau]}.$$

It is easy to see that $R_n \uparrow +\infty$ a.s. and $R_n \wedge \tau = T^n \wedge \tau$. Let $V^{(n)} \in \mathcal{Z}(S^{T_n})$. We can assume $(V^{(n)})^{T_n} = V^{(n)}$. Now if we successively apply Lemma 2.3 to $Z^{(n)}$ and $V^{(n+1)}$, we can construct a sequence $Z^{(n)} \in \mathcal{Z}(S^{T_n})$ such that $(Z^{(n+1)})^{T_n} = Z^{(n)}$. Put $Z_t := Z_t^{(n)}$ for $t \in [0, T_n]$. Since $(Z^\tau)^{R_n} = Z^{T_n \wedge \tau} = (Z^{(n)})^\tau$ and $((ZS)^\tau)^{R_n} = (Z^{(n)} S^{T_n})^\tau$ are local martingales, Z^τ and $(ZS)^\tau$ are local martingales too. Hence $\mathcal{Z}(S^\tau) \neq \emptyset$ and the proof of Theorem 2.2 is complete.

The next theorem is a counterpart of Theorem 1.1 for the local submartingale case.

Theorem 2.3. *Let S be a vector-valued semimartingale such that $\mathcal{P}(S) \neq \emptyset$ and U be a nonnegative process. The following statements are equivalent :*
i) U admits a decomposition of the form

$$U = U_0 + \psi \cdot S + A, \tag{2.3}$$

where ψ is a predictable process, integrable w.r.t. S, such that $\psi \cdot S$ is a local martingale under each $Q \in \mathcal{P}(S)$, and A is an adapted increasing process with $A_0 = 0$ and $\sup_{Q \in \mathcal{P}(S)} E^Q(A_\infty)) < +\infty$.
ii) U is a local submartingale under each law $Q \in \mathcal{P}(S)$ and $\sup_{Q \in \mathcal{P}(S),\ \tau \in \mathcal{T}^f} E^Q(U_\tau) < +\infty$ where $\mathcal{T}^f$ is the set of all finite stopping times.

Proof. We first prove i) $\Longrightarrow$ ii). Since $\psi \cdot S \geq -A$ and $E^Q(A_\infty)) < +\infty$, $\psi \cdot S$ is a supermartingale under each $Q \in \mathcal{P}(S)$ and $\sup_{Q \in \mathcal{P}(S),\ \tau \in \mathcal{T}^f} E^Q((\psi \cdot S)_\tau) < +\infty$. Therefore we get $\sup_{Q \in \mathcal{P}(S),\ \tau \in \mathcal{T}^f} E^Q(U_\tau) < +\infty$.

Now we prove ii) $\Longrightarrow$ i). Denote by $\mathcal{T}_t^f$ the set of all stopping times taking values in $[t, +\infty)$. Put

$$V_t := \text{ ess sup }_{Q \in \mathcal{P}(S), \tau \in \mathcal{T}_t^f} E^Q(U_\tau | \mathcal{F}_t).$$

Kramkov showed in [7] (Proposition 4.3) that (V_t) is a Q-supermartingale for each $Q \in \mathcal{P}(S)$. According to Theorem 1.1 V admits a decomposition of the form (2.1)

$$V_t = V_0 + \phi^{(1)} \cdot S - C^{(1)}.$$

Put $W := V_0 + \phi^{(1)} \cdot S - U = V - U + C^{(1)}$. Then under each $Q \in \mathcal{P}(S)$ W is a nonnegative supermartingale. Thus, according to Theorem 1.1 , W admits a decomposition of the form (2.1)

$$W = W_0 + \phi^{(2)} \cdot S - C^{(2)}.$$

Put

$$\psi := \phi^{(1)} - \phi^{(2)},\ A := C^{(2)}.$$

We get a decomposition of the form (2.3). Since $A_\infty = C^{(2)}$, $\sup_{Q \in \mathcal{P}(S)} E^Q(A_\infty)) < +\infty$.

The following lemma is a slight generalization of a result due to Kramkov (see [7] Proposition 4.3).

Lemma 2.4. *Let S be a vector-valued process with $\mathcal{Z}(S) \neq \emptyset$. Let $f := (f_t)_{t \geq 0}$ be a nonnegative adapted RCLL process such that*

$$(*) \qquad a := \sup_{Z \in \mathcal{Z}(S), \tau \in \mathcal{T}^f} E(Z_\tau f_\tau) < +\infty.$$

Then there exists an adapted RCLL process V such that V dominates f and for each $Z \in \mathcal{Z}(S)$ ZV is a supermartingale.

Proof. First of all, we are going to show that for any stopping time T such that $\mathcal{P}(S^T) \neq \emptyset$ we have

$$\sup_{Q \in \mathcal{P}(S^T), \tau \in \mathcal{T}^f} E^Q(f_{T \wedge \tau}) < +\infty.$$

In fact, let $Q \in \mathcal{P}(S^T)$ and M be the associated density process w.r.t. P. For $\tau \in \mathcal{T}^f$ $M^{T \wedge \tau} \in \mathcal{Z}(S^{T \wedge \tau})$. By Lemma 2.3 there exists a $Z \in \mathcal{Z}(S)$ such that $Z^{T \wedge \tau} = M^{T \wedge \tau}$. Thus we have

$$E^Q(f_{T \wedge \tau}) = E(M_\infty f_{T \wedge \tau}) = E(M_{T \wedge \tau} f_{T \wedge \tau}) = E(Z_{T \wedge \tau} f_{T \wedge \tau}) \leq a.$$

Now let $T_n \in \mathcal{T}^f$ be such that $T_n \uparrow +\infty$ and for each n $\mathcal{P}(S^{T_n}) \neq \emptyset$. By Proposition 4.3 in [7], for each n there exists an adapted RCLL process $V^{(n)}$ such that

$$V_t^{(n)} = \text{ ess sup }_{Q \in \mathcal{P}(S^{T_n}), \tau \in \mathcal{T}_t^f} E^Q(f_{T_n \wedge \tau} | \mathcal{F}_t).$$

$V^{(n)}$ is a Q supermartingale for each $Q \in \mathcal{P}(S^{T_n})$. We claim that this latter property implies that for each $Z \in \mathcal{Z}(S^{T_n})$ $ZV^{(n)}$ is a supermartingale. In fact, take an increasing sequence (τ_m) of finite stopping times such that each Z^{τ_m} is a uniformly integrable martingale. Now fix m. According to Lemma 2.2, we can construct a uniformly integrable martingale M such that $M \in \mathcal{Z}(S^{T_n})$ and $M^{\tau_m} = Z^{\tau_m}$. Since $MV^{(n)}$ is a supermartingale, $(ZV^{(n)})^{\tau_m} = (MV^{(n)})^{\tau_m}$ is also a supermartingale. Consequently, $ZV^{(n)}$ is a supermartingale. In particular for each $Z \in \mathcal{Z}(S)$ $ZV^{(n)}$ is a supermartingale because $\mathcal{Z}(S) \subset \mathcal{Z}(S^{T_n})$. Moreover, we have for each $\tau \in \mathcal{T}^f$

$$E(Z_\tau V_\tau^{(n)}) \leq V_0^{(n)} = \sup_{Q \in \mathcal{P}(S^{T_n}), \tau \in \mathcal{T}^f} E^Q(f_{T_n \wedge \tau}) \leq a.$$

Put $V_t := \sup_n V_t^{(n)}$. By Lemma 2.2 and 2.3, for any $Q \in \mathcal{P}(S^{T_n})$ we can construct a $Q' \in \mathcal{P}(S^{T_n+1})$ such that $Q'_{|\mathcal{F}_{T_n}} = Q_{|\mathcal{F}_{T_n}}$.Therefore $V_t^{(n)} \uparrow V_t$ and $(Z_t V_t)$ is a supermartingale. Now we are going to prove that V is a RCLL process. Let $\tau_n \in \mathcal{T}^f$, $\tau_n \downarrow \tau$. We have $\lim_{n\to\infty} E(Z_{\tau_n} V_{\tau_n}) = \lim_{n\to\infty} \lim_{m\to\infty} E(Z_{\tau_n} V_{\tau_n}^{(m)}) = \lim_{m\to\infty} \lim_{n\to\infty} E(Z_{\tau_n} V_{\tau_n}^{(m)})$ $= \lim_{m\to\infty} E(Z_\tau V_\tau^{(m)}) = E(Z_\tau V_\tau)$. Here $E(Z_{\tau_n} V_{\tau_n}^{(m)})$ increases both in m and in n, so the interchange of the limits is allowed. Since ZV is an optional process, we conclude that ZV is a RCLL supermartingale. Thus V is an adapted RCLL process. Obviously V dominates f.

Remark 1. If $\mathcal{P}(S) \neq \emptyset$ and $b := \sup_{Q \in \mathcal{P}(S), \tau \in \mathcal{T}^f} E^Q(f_\tau) < +\infty$, then (*) holds. In fact, let $Z \in \mathcal{Z}(S)$ and $\tau \in \mathcal{T}^f$. Take an increasing sequence of finite stopping times T_n converging to $+\infty$ such that Z^{T_n} is a uniformly integrable martingale. By Lemma 2.2 we can construct a uniformly integrable martingale M such that $M \in \mathcal{P}(S)$ and $M^{T_n \wedge \tau} = Z^{T_n \wedge \tau}$. Thus we have

$$E(Z_{T_n \wedge \tau} f_{T_n \wedge \tau}) = E(M_{T_n \wedge \tau} f_{T_n \wedge \tau}) = E(M_\infty f_{T_n \wedge \tau}) \leq \sup_{Q \in \mathcal{P}(S), \tau \in \mathcal{T}_0^f} E^Q(f_\tau).$$

By Fatou's lemma, we get $E(Z_\tau f_\tau) \leq b$. Thus, Lemma 2.4 extends Proposition 4.3 of [7].

Remark 2. It is easy to prove that the process V constructed in the proof of Lemma 2.4 is the smallest process dominating f such that ZV is a supermartingale for each $Z \in \mathcal{Z}(S)$ and, since $\cup_n \mathcal{P}(S^{T_n})$ is dense in $\mathcal{Z}(S)$, we have

$$V_t = \text{ ess sup }_{Z \in \mathcal{Z}(S), \tau \in \mathcal{T}_t^f} (E(Z_\tau f_\tau | \mathcal{F}_t) Z_t^{-1}).$$

The next theorem is a counterpart of Theorem 2.1 for the local submartingale case.

Theorem 2.4. *Let S be a vector-valued process such that $\mathcal{Z}(S) \neq \emptyset$ and U be a nonnegative process. The following statements are equivalent :*
i) U admits a decomposition of the form :

(2.4), $$U = U_0 + \psi \cdot S + A,$$

where ψ is a predictable process, integrable w.r.t. S, such that $Z(\psi \cdot S)$ is a local martingale for each $Z \in \mathcal{Z}(S)$, A is an adapted increasing process with $A_0 = 0$ and $\sup_{Z \in \mathcal{Z}(S), \tau \in \mathcal{T}^f} E(Z_\tau A_\tau) < +\infty$.

ii) For each $Z \in \mathcal{Z}(S)$, ZU is a local submartingale and $\sup_{Z\in\mathcal{Z}(S),\tau\in\mathcal{T}^f} E(Z_\tau U_\tau) < +\infty$.
iii) For each $Z \in \mathcal{Z}(S)$, ZU is a local submartingale and there exists a process V dominating U such that ZV is a supermartingale for each $Z \in \mathcal{Z}(S)$.

Proof. First we are going to prove i) $\Longrightarrow$ ii). Let $Z \in \mathcal{Z}(S)$, $\tau \in \mathcal{T}^f$ and (τ_n) be an increasing sequence of finite stopping times converging to $+\infty$ such that Z^{T_n} and $[Z(\psi \cdot S)]^{T_n}$ are uniformly integrable martingales. By (2.4) we have

$$E(Z_{\tau\wedge T_n} U_{\tau\wedge T_n}) = U_0 + E(Z_{\tau\wedge T_n} A_{\tau\wedge T_n}) \leq U_0 + \sup_{Z\in\mathcal{Z}(S),\tau\in\mathcal{T}^f} E(Z_\tau A_\tau).$$

Let $n \longrightarrow +\infty$ we get $E(Z_\tau U_\tau) \leq U_0 + \sup_{Z\in\mathcal{Z}(S),\tau\in\mathcal{T}^f} E(Z_\tau A_\tau) < +\infty$.
ii) $\Longrightarrow$ iii) is a consequence of Lemma 2.4.
Next we are going to prove iii) $\Longrightarrow$ i). Assume iii) holds. By Theorem 2.1 V admits a decomposition of the form

$$V = V_0 + \phi_0^{(1)} \cdot S - C^{(1)}.$$

Put $W := V_0 + \phi_0^{(1)} \cdot S - U = V - U + C^{(1)}$. Then ZW is a nonnegative supermartingale. Again by Theorem 2.1 W admits a decomposition of the form

$$W = U_0 + \phi^{(2)} \cdot S - C^{(2)}.$$

By putting $\psi := \phi^{(1)} - \phi^{(2)}, A := C^{(2)}$, we get (2.4). It remains to show that $\sup_{Z\in\mathcal{Z}(S),\tau\in\mathcal{T}^f} E(Z_\tau A_\tau) < +\infty$. Since V dominates U, we have

$$\sup_{Z\in\mathcal{Z}(S),\tau\in\mathcal{T}^f} E(Z_\tau U_\tau) \leq \sup_{Z\in\mathcal{Z}(S),\tau\in\mathcal{T}^f} E(Z_\tau U_\tau) \leq V_0 < +\infty$$

By a same argument as in the proof of i) $\Longrightarrow$ ii) we conclude that

$$\sup_{Z\in\mathcal{Z}(S),\tau\in\mathcal{T}^f} E(Z_\tau A_\tau) \leq V_0 - U_0.$$

3. Application to Mathematical Finance.

Now we fix a time horizon T. By stopping at T, all results of section 2 are applicable to the present case. As mentioned in section 1, the vector-valued semimartingale S can be interpreted as the discounted price process of stocks in a security market. We suppose that $\mathcal{Z}(S) \neq \emptyset$, but we don't assume $\mathcal{P}(S) \neq \emptyset$. In general, the market is incomplete. So there exist contingent claims which are not replicable. Here, by a contingent claim, we mean a nonnegative $\mathcal{F}_T$-random variable. The contingent claim B is said to be *replicable* if there exist $x \geq 0$, a vector-valued predictable process ϕ integrable w.r.t. S and $Z \in \mathcal{Z}(S)$ such that $Z(x + \phi \cdot S)$ is a martingale and $B = x + (\phi \cdot S)_T$. In that case $\phi \cdot S$ is uniquely defined. In the sequel we consider only those contingent claims B which satisfy the following condition :

$$\sup_{Z\in\mathcal{Z}(S)} E(Z_T B) < \infty.$$

Such a contingent claim will be called *tradable.* Of course, each bounded contingent claim is tradable. If B is a tradable claim, we put :

$$V_t := \text{ess sup}_{Z\in\mathcal{Z}(S)}(E[Z_T B|\mathcal{F}_t]Z_t^{-1}),\ 0 \le t \le T. \tag{3.1}$$

$$U_t := \text{ess inf}_{Z\in\mathcal{Z}(S)}(E[Z_T B|\mathcal{F}_t]Z_t^{-1}),\ 0 \le t \le T. \tag{3.2}$$

The following theorem, an immediate consequence of Theorem 2.1 and 2.4, shows that (V_t) and (U_t) can be interpreted as the value processes associated to the problem of hedging the contingent claim B for option-writer and option-buyer respectively.

Theorem 3.1. *Assume $\mathcal{Z}(S) \neq \emptyset$. Then (V_t) and (U_t) admit the following decompositions :*

$$V_t = V_0 + (\phi \cdot S)_t - C_t,\ 0 \le t \le T, \tag{3.3}$$

$$U_t = U_0 + (\psi \cdot S)_t + A_t,\ 0 \le t \le T, \tag{3.4}$$

where ϕ and ψ are predictable vector-valued processes, integrable w.r.t. S and C and A are adapted, increasing processes with $C_0 = A_0 = 0$, such that for each $Z \in \mathcal{Z}(S)$, $Z(\phi \cdot S)$ and $Z(\psi \cdot S)$ are local martingales.

If $\mathcal{P}(S) \neq \emptyset$, it is easy to see that we have

$$V_t := \text{ess sup}_{Q\in\mathcal{P}(S)}E^Q[B|\mathcal{F}_t],\ 0 \le t \le T, \tag{3.3}'$$

$$U_t := \text{ess inf}_{Q\in\mathcal{P}(S)}E^Q[B|\mathcal{F}_t],\ 0 \le t \le T. \tag{3.4}'$$

In this case, Ansel/Stricker [1] proved that B is replicable iff there exists a $Q \in \mathcal{P}(S)$ such that $V_0 = E^Q(B)$. The following theorem extends this result to the general case.

Theorem 3.2. *Assume $\mathcal{Z}(S) \neq \emptyset$. Then the following statements are equivalent :*
i) B is replicable.
ii) $\exists Z' \in \mathcal{Z}(S)$ such that $E(Z'_T B) = \sup_{Z\in\mathcal{Z}(S)} E(Z_T B)$.
iii) $\exists Z' \in \mathcal{Z}(S)$ such that $Z'V$ is a martingale.

Proof. i) $\Longrightarrow$ ii). Asume $B = x + (\phi \cdot S)_T$ and there is a $Z' \in \mathcal{Z}(S)$ such that $Z'(x + \phi \cdot S)$ is a martingale. Since $x + \phi \cdot S$ is nonnegative, by Corollary 3.5 in Ansel/Stricker [1], $\forall Z \in \mathcal{Z}(S)$ $Z(x + \phi \cdot S)$ is a nonnegative local martingale. Thus, it is a supermartingale and we have $E(Z_T B) \le E(Z_0 x) = x = E(Z'_T B)$.
ii) $\Longrightarrow$ iii). Assume ii) holds. Since $Z'V$ is a supermartingale and $E(Z'_T V_T) = EZ'_T B) = \sup_{Z\in\mathcal{Z}(S)} E(Z_T B) = V_0 = E(Z'_T V_T)$, $Z'V$ is a martingale.
iii) $\Longrightarrow$ i). Assume $Z'V$ is a martigale. By (3.3) we must have $C_t = C_0 = 0$. So i) holds.

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Note added in proofs : When this work was completed, we learned that Föllmer and Kabanov modified their proof of Theorem 1.1 in order to remove the assumption that V is nonnegative. So Theorem 1.1 can also be stated for submartingales. Recently Delbaen and Schachermayer provided the proof of this general result using completely different ideas.