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The Brownian Burglar: conditioning Brownian motion by its local time process

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Imagine a Brownian crook who spent a month in a large metropolis. The number of nights he spent in hotels A,B,C...etc. is known; but not the order, nor his itinerary. So the only information the police has is total hotel bills.....

Let $(W_t; t \geq 0)$ be reflecting Brownian motion issuing from zero, and let $l(t,y)$, for $y \in \mathbb{R}^+$ and $t \geq 0$, denote the local time that W has accrued at level y by time t. Throughout this paper our normalisation of local time is such that it is an occupation density with respect to Lebesgue measure. Let T_1 be the first time t such that $W_t = 1$. The celebrated Ray-Knight theorem describes the law of $(l(T_1, 1-y); 0 \leq y \leq 1)$ as being that of a diffusion; specifically a squared Bessel process of dimension two, started from zero. The question now naturally arises of obtaining some description of W conditional on these local times.

To begin one may look for functionals of W for which we can describe the conditional law. Such a functional is the process $(l(T_a, y); 0 \leq y \leq a)$ for some $a \in (0,1)$. Specifically we find that

$$\frac{l(T_a, y)}{l(T_1, y)} = Y^{2,0}_{\int_y^a \frac{dz}{l(T_1,z)}}, \tag{0.1}$$

where $Y^{2,0}$ is a diffusion, independent of $(l(T_1, 1-y); 0 \leq y \leq 1)$, and with generator $2y(1-y)D^2 + 2(1-y)D$. It belongs to a class of diffusions on $[0,1]$ known as Jacobi diffusions. We were then motivated to try and obtain a process $\hat{W}$, which we call the burglar, whose local times would give rise to such diffusions. Define $\hat{W}$ via the space-time change

$$\theta(W_t) = \hat{W}_{A_t}, \tag{0.2}$$

for $0 \leq t < T_1$, where

$$A_t = \int_0^t \frac{ds}{\left(l(T_1, W_s)\right)^2} \qquad \text{and} \qquad \theta(y) = \int_0^y \frac{dz}{l(T_1, z)}.$$

With probability one, the function θ maps $[0,1)$ onto $\mathbb{R}^+$, and A maps $[0, T_1)$ onto $[0, \infty)$; both being continuous and strictly increasing. The main result of this paper is to show that the burglar, so defined, is independent of the local times accrued by W at time T_1.

Theorem 1. *The reflecting Brownian motion* $(W_t; 0 \leq t < T_1)$ *admits the representation* (0.2) *in terms of its local times* $(l(T_1, y); 0 \leq y \leq 1)$ *and an independent burglar* $(\hat{W}_u; 0 \leq u < \infty)$.

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Although we leave to a future article a thorough study of the law of the burglar, including Markovian properties and martingale characterisations, already (0.1) may be interpreted as a Ray-Knight theorem for the local times of $\hat{W}$. In fact, we now present burglar variants of the two classical Ray-Knight theorems for Brownian local times. We first observe that the burglar $\hat{W}$ possesses a jointly continuous local time process $\big(\rho(t,y); t \geq 0, y \geq 0\big)$. Recall the definitions of θ and A.

Lemma 2. *Define, for* $0 \leq t < T_1$, *and* $0 \leq y < 1$,

$$\rho\big(A_t, \theta(y)\big) = \frac{l(t,y)}{l(T_1,y)},$$

then ρ *is a jointly continuous local time process for the burglar, in that for any positive measurable function* f *on* $\mathbb{R}^+$,

$$\int_0^t f(\hat{W}_s)\, ds = \int_0^\infty f(y)\rho(t,y)\, dy.$$

Theorem 3. *The local times* $\big(\rho(t,y); y \in \mathbb{R}^+, t \in \mathbb{R}^+\big)$ *of a Brownian burglar* $\hat{W}$ *admit the following descriptions.*

For $a > 0$ *let* $\hat{T}_a = \inf\{t : \hat{W}_t = a\}$. *Then we have*

$$\big(\rho(\hat{T}_a, a-y); 0 \leq y \leq a\big) \overset{law}{=} \big(Y_y^{2,0}; 0 \leq y \leq a\big),$$

where $Y_0^{2,0} = 0$.

For $0 \leq s \leq 1$ *let* $\hat{\tau}_s = \inf\{u : \rho(u,0) \geq s\}$. *Then we have*

$$\big(\rho(\hat{\tau}_s, y); y \geq 0\big) \overset{law}{=} \big(Y_y^{0,2}; y \geq 0\big),$$

where $Y_0^{0,2} = s$.

In the above $Y^{0,2}$ denotes a Jacobi diffusion with generator $2y(1-y)D^2 - 2yD$. Our paper is organised as follows.

- Section 1 contains a discussion showing how a group action on a probability space can induce a factorisation. This is illustrated with reference to the standard Brownian bridge and the skew-product representation of planar Brownian motion.
- In Section 2 we prove the independence of the burglar and $\big(l(T_1,y); 0 \leq y \leq 1\big)$ as an application of the general method presented in the previous section.
- Section 3 contains a proof of the Ray-Knight theorems for the burglar (Theorem 3 above).
- In Section 4 we give an application of the burglar to the problem of describing W conditional on $\big(l(\tau_1,y); y \in \mathbb{R}^+\big)$, where $\tau_1 = \inf\{t : l(t,0) = 1\}$. In order to do this we must decompose the path of W at its maximum. The result of this section can be seen as describing a contour process for the Fleming-Viot process, and should be compared with the skew-decomposition of super-Brownian motion in terms of its total mass process and an independent Fleming-Viot process achieved by March and Etheridge [7]; see also Dawson [5].

1 Group actions and factorisations

Suppose that $(\Omega, \mathcal{F}, \mu)$ is a probability space and that $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$. It is well known (and the cause of much grief!) that, in general, there are many different independent complements to $\mathcal{G}$. That is sub-σ-algebras $\mathcal{H}$ such that $\mathcal{F} = \mathcal{G} \vee \mathcal{H}$ and such that $\mathcal{G}$ and $\mathcal{H}$ are independent. However it is usual to have some additional structure on Ω which allows one to single out some distinguished complement in a natural manner. Here we will be concerned with cases in which this additional structure arises from the action of a group G on Ω.

We suppose that we have a second probability space $(E, \mathcal{E}, \nu)$ on which there is a G-action also defined; and a measurable map $\phi : \Omega \mapsto E$ so that ν is the image of μ under ϕ, and so that ϕ is a homomorphism of G-spaces, that is,

$$\phi(g\omega) = g\phi(\omega), \tag{1.1}$$

for all $g \in G$ and $\omega \in \Omega$. We assume that the measures μ and ν are quasi-invariant under the action of G on Ω and E, and denote their images under the transformation associated with an element $g \in G$ by μ^g and ν^g respectively. In the dynamical systems literature the space E is known as a *factor* of Ω, see, for example, Cornfeld, Fomin and Sinai [4].

We are interested in conditions under which the space $(\Omega, \mathcal{F}, \mu)$ can be identified with the product of $(E, \mathcal{E}, \nu)$ and some complementary space. To develop this fully would involve us in much measure theoretic detail, as described by Rohlin [16]. This would be, for our purpose, unnecessary, and we can be satisfied with the following elementary lemma. Recall that the action of G on E is said to be ergodic if every $\mathcal{E}$-measurable, G-invariant function on E is constant modulo a set of measure zero.

Lemma 4. *Consider a probability space $(\Omega, \mathcal{F}, \mu)$ upon which a group G acts. Let ϕ be a homomorphism between this space and a factor $(E, \mathcal{E}, \nu)$. Let $\mathcal{G} = \sigma(\phi)$ and $\mathcal{H}$ be the σ-algebra of G-invariant subsets of Ω, thus*

$$\mathcal{H} = \{H \in \mathcal{F} : \mu(H \triangle gH) = 0 \text{ for all } g \in G\}.$$

Then if the Radon-Nikodým densities $d\mu^g/d\mu$ are $\mathcal{G}$-measurable for all $g \in G$, and if the action of G on E is ergodic, the σ-algebras $\mathcal{G}$ and $\mathcal{H}$ are independent.

Proof. Let $H \in \mathcal{H}$ and consider the conditional expectation $\mu[\mathcal{I}_H|\mathcal{G}]$. For any $g \in G$, we find that for μ-almost all $\omega \in \Omega$,

$$\mu[\mathcal{I}_H|\mathcal{G}](\omega) = \mu^g[\mathcal{I}_{gH}|g\mathcal{G}](g\omega) = \mu^g[\mathcal{I}_H|\mathcal{G}](g\omega).$$

But, since $d\mu^g/d\mu$ is $\mathcal{G}$-measurable, we have

$$\mu^g[\mathcal{I}_H|\mathcal{G}] = \mu[\mathcal{I}_H|\mathcal{G}].$$

Thus $\mu[\mathcal{I}_H|\mathcal{G}](\omega) = \mu[\mathcal{I}_H|\mathcal{G}](g\omega)$ for μ-almost all ω. Now the ergodicity of the G-action implies that $\mu[\mathcal{I}_H|\mathcal{G}]$ is equal to some constant μ-almost always, and thus the σ-algebras $\mathcal{G}$ and $\mathcal{H}$ are independent. □

This lemma says nothing to guarantee that $\mathcal{F} = \mathcal{G} \vee \mathcal{H}$; indeed in general this is manifestly false. However in the examples we consider it will be evident that $\mathcal{G}$ and $\mathcal{H}$ do generate everything.

We now illustrate the above discussion with some concrete examples. It is important to stress that there are alternative treatments of these than the one we will describe. However we hope that our rather unusual approach will pay dividends in demonstrating that the Burglar is constructed in a very natural way.

Our first example is that of the Brownian bridge. Let $(X_t; 0 \leq t \leq 1)$ be a Brownian motion issuing from zero. We wish to find an independent complement to X_1. We take $(\Omega, \mathcal{F}, \mu)$ to be Wiener space and consider $(X_t; 0 \leq t \leq 1)$ to be the co-ordinate projection maps on Ω. Introduce the action of the group $G \equiv (\mathbb{R}, +)$ on Ω by defining, for any $a \in G$ and $\omega \in \Omega$,

$$X_t(a\omega) = X_t(\omega) + at, \tag{1.2}$$

for $0 \leq t \leq 1$. Of course the measure μ^a is the law of Brownian motion with drift a, and the absolute continuity relation,

$$\frac{d\mu^a}{d\mu} = \exp\{aX_1 - \tfrac{1}{2}a^2\}, \tag{1.3}$$

holds. For the factor we take the map $\phi \equiv X_1$, and the probability space $(E, \mathcal{E}, \nu)$ is just the real line equipped with its Borel σ-algebra and standard Gaussian measure, with G acting by translation. Now consider the bridge $(\hat{X}_t; 0 \leq t \leq 1)$, defined by

$$\hat{X}_t(\omega) = X_t(\omega) - X_1(\omega)t, \tag{1.4}$$

for $0 \leq t \leq 1$. Note $\hat{X}$ is invariant under the action of G:

$$\hat{X}_t(a\omega) = \hat{X}_t(\omega), \tag{1.5}$$

for all $\omega \in \Omega$ and $a \in G$. Thus, since G acts transitively on E, and the Radon-Nikodým derivatives (1.3) are $\sigma(\phi)$-measurable, Lemma 4 is applicable and the bridge $\hat{X}$ is independent of X_1. Bridges of the gamma process, see Vershik and Yor [17], may be treated in exactly the same manner.

Our second example, slightly more involved, is that of the decomposition of planar Brownian motion into its radial and angular parts. However before we present this, let us introduce a group and a probability space that will play a central role both in this example and in the construction of the burglar. Let G be the group of increasing C^2-diffeomorphisms of $[0,1]$. Let $(E, \mathcal{E}, \nu)$ be the space of continuous paths indexed by $[0,1]$, together with the usual Borel σ-algebra, and ν the law of the squared Bessel process of dimension two starting from zero. Let $(Z_t; 0 \leq t \leq 1)$ be the co-ordinate projection maps on E. We can define an action of G on E as follows. For any $g \in G$ define for each $\omega \in E$ its image $g\omega$ under the action of g via,

$$Z_{g(t)}(g\omega) = g'(t)Z_t(\omega), \tag{1.6}$$

for all $0 \leq t \leq 1$. Under ν^g, $(Z_t; 0 \leq t \leq 1)$ is a time-inhomogeneous diffusion with generator

$$2z\frac{d^2}{dz^2} + \big(2F_g(t)z + 2\big)\frac{d}{dz}, \tag{1.7}$$

where $F_g(t) = -h''(t)/2h'(t)$, if $h = g^{-1}$. Such diffusions are considered by Pitman and Yor [13].

Lemma 5. *The action of G on $(E, \mathcal{E}, \nu)$ is ergodic.*

Proof. The law ν^g is absolutely continuous with respect to ν with Radon-Nikodým derivative

$$\frac{d\nu^g}{d\nu} = \exp\left\{\int_0^1 F_g(t)dM_t - \tfrac{1}{2}\int_0^1 F_g^2(t)Z_t dt\right\},$$

where $2M_t = Z_t - 2t$. Let Ξ be the collection of random variables which are proportional to $d\nu^g/d\nu$ for some $g \in G$, and lie in $\mathcal{L}^2(\nu)$. We will show that Ξ is total in $\mathcal{L}^2(\nu)$. The ergodicity of the G-action follows easily from this, for if Φ is a bounded random variable that is invariant under the action of G, then $\Phi - \int_E \Phi d\nu$ is orthogonal to each member of Ξ, and thus almost surely zero.

Suppose that F_g is continuously differentiable, then

$$F_g(1)Z_1 = \int_0^1 F_g(t)dZ_t + \int_0^1 Z_t dF_g(t),$$

and we may write

$$\frac{d\nu^g}{d\nu} = \exp\left\{\tfrac{1}{2}F_g(1)Z_1 - \int_0^1 F_g(t)dt - \tfrac{1}{2}\int_0^1 Z_t\left[dF_g(t) + F_g^2(t)dt\right]\right\}.$$

So the $\mathcal{L}^2$-closure of Ξ contains every random variable of the form

$$\exp\left\{-\tfrac{1}{2}\int_0^1 Z_t\alpha(dt)\right\},$$

where α is a positive measure on $[0,1]$; and the collection of all such random variables is closed under multiplication and generates $\mathcal{E}$. Thus if Φ is orthogonal to Ξ, on application of the monotone class lemma, we may deduce that Φ is orthogonal to all of $\mathcal{L}^2(\nu)$ and must be zero. □

Returning to the planar Brownian motion let $(X_t; t \in [0,1])$ be the co-ordinate projection maps on Ω, the space of all continuous paths in the plane $\mathbb{R}^2$, issuing from the origin, indexed by time in $[0,1]$. We may take μ to be the law of planar Brownian motion issuing from zero. We define the action of G on Ω as follows. For $g \in G$ the image of $\omega \in \Omega$ under the action of g satisfies

$$X_{g(t)}(g\omega) = \sqrt{g'(t)}X_t(\omega), \tag{1.8}$$

for all $t \in [0,1]$. Under μ^g the process X has a radial drift. In fact it satisfies

$$X_t = \beta_t + \int_0^t F_g(s)X_s\,ds, \tag{1.9}$$

where β, under μ^g, is a planar Brownian motion. The measure μ^g is absolutely continuous with respect to μ with the Radon-Nikodým derivative being

$$\frac{d\mu^g}{d\mu} = \exp\left\{\int_0^1 F_g(t)dM_t - \tfrac{1}{2}\int_0^1 F_g^2(t)|X_t|^2 dt\right\}, \tag{1.10}$$

where $2M_t = |X_t|^2 - 2t$.

The factor space is to be the radial part of X_t. Specifically define the operator $\phi : \Omega \mapsto E$ by,

$$Z_t(\phi\omega) = |X_t(\omega)|^2, \tag{1.11}$$

for all $t \in [0,1]$. It is well known that the measure ν, the law of the squared Bessel process of dimension two starting from zero, is the image of μ under ϕ. Observe that the Radon-Nikodým derivative (1.10) is $\sigma(\phi)$-measurable. Now consider the angular part of X appropriately time-changed:

$$\hat{X}_{\int_t^1 \frac{dv}{|X_v|^2}} = \frac{X_t}{|X_t|}. \tag{1.12}$$

With probability one, this defines a process $\left(\hat{X}_u; 0 \leq u < \infty\right)$, which is, in fact, a Brownian motion on the unit circle. It is easy to check that $\hat{X}$ is invariant under the action of G on Ω, and consequently we deduce, with the aid of Lemma 4, that it is independent of the radial process.

Define $E_0 \subset E$ by,

$$E_0 = \left\{ \eta \in E : Z_t(\eta) > 0 \text{ for } 0 < t \leq 1 \quad \text{and} \quad \int_{0+}^1 \frac{dt}{Z_t(\eta)} = \infty \right\}. \tag{1.13}$$

Observe that $\nu(E_0) = 1$. For $\eta \in E_0$ define a process $\left(\hat{X}_t^\eta; 0 \leq t \leq 1\right)$ on the probability space $\left(\Omega, \mathcal{F}, \mu\right)$, via,

$$\hat{X}_t^\eta(\omega) = Z_t(\eta)\hat{X}_{\int_t^1 \frac{dv}{Z_v(\eta)}}(\omega). \tag{1.14}$$

$\hat{X}^\eta$ is a process with radial part $Z_t(\eta)$ and (time-changed) angular part $\hat{X}_u$. Its law μ_η is supported on the fibre $\phi^{-1}(\eta) \subset \Omega$. The family of laws $\left(\mu_\eta; \eta \in E_0\right)$ form a regular probability distribution for μ given ϕ, see Parthasarathy [12].

2 The Burglar

We shall now apply the technique we have demonstrated in the previous section to our original problem of conditioning with respect to local times. For presentational reasons it is convenient to reverse the rôles that zero and one take in the introduction. Thus our Brownian motion is reflected down from level one and stopped on first reaching level zero.

We take Ω to be the space of continuous paths taking values in the interval $[0,1]$, starting from 1, and stopped at $T_0(\omega)$ which is the first time the path reaches 0. The σ-algebra $\mathcal{F}$ is the Borel σ-algebra generated by the uniform topology, and the measure μ will be the law of Brownian motion on $[0,1]$ with reflection at the boundaries, stopped on hitting level 0. Denote the co-ordinate projections by $\left(X_t; 0 \leq t \leq T_0\right)$. A path admits a bicontinuous local time $\left(l(t,y); 0 \leq t \leq T_0, 0 \leq y \leq 1\right)$ with probability one, and we extend l to the whole of Ω, defining it to be identically zero otherwise. We will be concerned with exactly the same group G as that featured in the second example

of the preceding section, but this time the group action is defined very differently. For $g \in G$ define the image of $\omega \in \Omega$ under the action of g via,

$$g\big(X_t(\omega)\big) = X_{H_t}(g\omega), \tag{2.1}$$

where

$$H_t = \int_0^t g'\big(X_s(\omega)\big)^2 ds. \tag{2.2}$$

It is simple to check that this does indeed define a G-action. Under μ^g the co-ordinate process satisfies

$$X_t = \beta_t - \tfrac{1}{2} l(t,1) + \int_0^t F_g(X_s) ds, \tag{2.3}$$

where β is a μ^g-Brownian motion. Thus μ^g is absolutely continuous with respect to μ and the Radon-Nikodým derivative is given by,

$$\frac{d\mu^g}{d\mu} = \exp\left\{ \int_0^{T_0} F_g(X_t) dB_t - \tfrac{1}{2} \int_0^{T_0} F_g^2(X_t) dt \right\}, \tag{2.4}$$

where $B_t = X_t + \frac{1}{2} l(t,1)$.

We retain the space $(E, \mathcal{E}, \nu)$ as before; but $\phi : \Omega \mapsto E$ now takes the form:

$$Z_y(\phi\omega) = l(T_0, y)(\omega), \tag{2.5}$$

for $y \in [0,1]$. The Ray-Knight theorem states that ν is indeed the image of μ under ϕ. That ϕ is a homomorphism of G-spaces follows immediately from the following lemma.

Lemma 6. *If the path ω admits a bicontinuous local time $\big(l(t,y); 0 \le t \le T_0, 0 \le y \le 1\big)$, then the path $g\omega$ defined by equation (2.1) does also, and denoting this by $\big(l^g(t,y); 0 \le t \le T_0^g, 0 \le y \le 1\big)$, the two are related by,*

$$l^g\big(H_t, g(y)\big) = g'(y) l(t,y),$$

for all $y \in [0,1]$ and $0 \le t \le T_0$.

Proof. For any bounded, measurable test function f we have, for $0 \le t \le T_0$,

$$\begin{aligned} \int_0^{H_t} f\big(X_v(g\omega)\big) dv = \int_0^t f\big(X_{H_s}(g\omega)\big) dH_s &= \int_0^t f\big(g\big(X_s(\omega)\big)\big) g'\big(X_s(\omega)\big)^2 ds \\ &= \int_0^1 f\big(g(y)\big) g'(y)^2 l(t,y) dy. \end{aligned}$$

Thus, defining $l^g\big(H_t, g(y)\big) = g'(y) l(t,y)$, we find, substituting $u = H_t$ and $z = g(y)$,

$$\int_0^u f\big(X_v(g\omega)\big) dv = \int_0^1 f(z) l^g(u,z) dz,$$

and so l^g forms a bicontinuous local time for $g\omega$. □

The arguments of Bouleau and Yor, [3], show that,

$$-\int_0^1 F_g(y)dy = \int_0^{T_0} F_g(X_t)dB_t - \tfrac{1}{2}\int_0^1 F_g(y)d_y l(T_0,y). \tag{2.6}$$

Using this, the Radon-Nikodým derivative (2.4) may be re-written in the form:

$$\frac{d\mu^g}{d\mu} = \exp\left\{\int_0^1 F_g(y)dM_y - \tfrac{1}{2}\int_0^1 F_g^2(y)l(T_0,y)dy\right\}, \tag{2.7}$$

where $2M_y = l(T_0,y) - 2y$, and is now evidently $\sigma(\phi)$-measurable. Define $\hat{X}$ via the space-time change

$$\theta(X_t) = \hat{X}_{A_t}, \tag{2.8}$$

for $0 \le t < T_0$, where

$$A_t = \int_0^t \frac{ds}{\big(l(T_0,X_s)\big)^2} \qquad \text{and} \qquad \theta(y) = \int_y^1 \frac{dz}{l(T_0,z)}. \tag{2.9}$$

With probability one, $\big(\hat{X}_u; 0 \le u < \infty\big)$ is well-defined, and satisfies $\lim_{u\to\infty}\hat{X}_u = \infty$. The process $\big(\hat{X}_u; u \ge 0\big)$, or more generally any process with the same law, will be called a Brownian burglar. The following invariance property is the key to proving the independence claimed in Theorem 1.

Lemma 7. *The burglar $\hat{X}$ defined by equations* (2.8) *and* (2.9) *is invariant under the action of G on Ω,*

Proof. Fix $g \in G$. Write A^g and θ^g for the functions A and θ translated by the action of g; that is

$$A_t^g(\omega) = A_t(g\omega) \qquad \text{and} \qquad \theta^g(y)(\omega) = \theta(y)(g\omega).$$

Now we make repeated use of the previous lemma. For any $y \in (0,1]$,

$$\theta^g\big(g(y)\big) = \int_{g(y)}^1 \frac{dz}{l^g(T_0^g,z)} = \int_y^1 \frac{g'(x)dx}{l^g\big(T_0^g,g(x)\big)} = \int_y^1 \frac{dx}{l(T_0,x)} = \theta(y).$$

Similarly we find,

$$\begin{aligned} A^g_{H_t} = \int_0^{H_t} \frac{ds}{l^g\big(T_0^g,X_s(g\omega)\big)^2} &= \int_0^t \frac{dH_u}{l^g\big(T_0^g,X_{H_u}(g\omega)\big)^2} \\ &= \int_0^t \frac{g'\big(X_u(\omega)\big)^2 du}{l^g\big(T_0^g,g(X_u(\omega))\big)^2} = \int_0^t \frac{du}{l\big(T_0,X_u(\omega)\big)^2} = A_t. \end{aligned}$$

Now consider the definition of the burglar,

$$\theta\big(X_t(\omega)\big) = \hat{X}_{A_t}(\omega),$$

and the same relationship at $g\omega$,

$$\theta^g\big(X_t(g\omega)\big) = \hat{X}_{A_t^g}(g\omega).$$

Replacing t by H_t, this latter equation becomes,

$$\theta^g\big(X_{H_t}(g\omega)\big) = \hat{X}_{A^g_{H_t}}(g\omega),$$

and now, using equation (2.1) and the relationships that we have just derived, we obtain,

$$\theta\big(X_t(\omega)\big) = \hat{X}_{A_t}(g\omega).$$

But comparing this with the original equation defining the burglar, we deduce,

$$\hat{X}_u(g\omega) = \hat{X}_u(\omega),$$

for all u, and the invariance is proven. □

In order to prove Theorem 1 we apply Lemma 4, having now confirmed that its premises hold.

Recall the definition of E_0, made in the previous section. For $\eta \in E_0$, let $k : \mathbb{R}^+ \mapsto (0,1]$ be the function

$$k(a) = \sup\left\{y : \int_y^1 \frac{dz}{\eta(z)} \geq a\right\}, \tag{2.10}$$

and then define a process $\hat{X}^\eta$ on $\big(\Omega, \mathcal{F}, \mu\big)$ via the space-time change,

$$k(\hat{X}_t) = \hat{X}^\eta_{K_t}, \tag{2.11}$$

where

$$K_t = \int_0^t k'(\hat{X}_s)^2 ds.$$

The process $\big(\hat{X}^\eta_t; 0 \leq t < T_0\big)$ can be thought of as X conditioned on $l(T_0, \cdot) = \eta(\cdot)$. In fact, if we denote the law of $\hat{X}^\eta$, which is supported on $\phi^{-1}(\eta) \subset \Omega$, by μ_η, then the family $\big(\mu_\eta : \eta \in E_0\big)$ form a regular probability distribution for μ given ϕ.

3 Some Ray-Knight Theorems

We denote by $Y^{d,d'}$ a diffusion on $[0,1]$ with infinitesimal generator

$$2y(1-y)D^2 + \big(d - (d+d')y\big)D. \tag{3.1}$$

These diffusions, called Jacobi diffusions with dimensions d and d', have been well studied, particularly in relations to models in genetics, see for example Ethier-Kurtz [8], Karlin-Taylor [9] and Kimura [10], or more recently in financial models, see Delbaen-Shirakawa [6]. For other studies and motivations, including hypercontractivity, see Bakry [1] and Mazet [11]. Some further results are given by the authors of this paper in [18], where the Jacobi diffusions are introduced via the following proposition.

Proposition 8. *Let* $(Z_t; t \geq 0)$ *and* $(Z'_t; t \geq 0)$ *be two independent squared Bessel processes of dimensions* d *and* d' *starting from* z *and* z' *respectively, with* $z + z' > 0$, *and let* $T = \inf\{u : Z_u + Z'_u = 0\}$. *Then there exists a Markov process* $(Y_u^{d,d'} : u \geq 0)$, *a diffusion on* $[0,1]$ *with infinitesimal generator given by* (3.1) *such that*

$$\frac{Z_t}{Z_t + Z'_t} = Y^{d,d'}_{\left(\int_0^t \frac{ds}{Z_s + Z'_s}\right)}, \qquad \text{for } 0 \leq t < T,$$

with $Y^{d,d'}$ *being independent of* $(Z_t + Z'_t; t \geq 0)$.

The above skew-product decomposition also holds when Z *and* Z' *are replaced by processes* $\tilde{Z}$ *and* $\tilde{Z}'$ *obtained from* Z *and* Z' *via,*

$$\tilde{Z}_t = \frac{1}{u'(t)} Z_{u(t)} \qquad \text{and} \qquad \tilde{Z}'_t = \frac{1}{u'(t)} Z'_{u(t)},$$

with $u : [0,\infty) \mapsto [0,\infty)$ *a strictly increasing,* C^1*-function, and* $u(0) = 0$.

Proof. We give a proof based on the stochastic calculus. The squared Bessel processes Z and Z' satisfy

$$Z_t = z + \int_0^t 2\sqrt{Z_s} d\beta_s + dt,$$
$$Z'_t = z' + \int_0^t 2\sqrt{Z'_s} d\beta'_s + d't,$$

where β and β' are independent Brownian motions. Now we sum these two expressions and use 'Pythagoras':

$$Z_t + Z'_t = z + z' + \int_0^t 2\sqrt{Z_s + Z'_s} d\gamma_s + (d + d')t,$$

where γ is the Brownian motion:

$$\gamma_t = \int_0^t \frac{\sqrt{Z_s} d\beta_s + \sqrt{Z'_s} d\beta'_s}{\sqrt{Z_s + Z'_s}},$$

defined up to the time T. Now one introduces,

$$\xi_t = \frac{Z_t}{Z_t + Z'_t},$$

for $0 \leq t < T$. We deduce with the aid of Itô's formula that

$$\xi_t = \xi_0 + 2\int_0^t \sqrt{\xi_s(1-\xi_s)} \frac{\sqrt{1-\xi_s} d\beta_s - \sqrt{\xi_s} d\beta'_s}{\sqrt{Z_s + Z'_s}} + \int_0^t \left(d - (d+d')\xi_s\right) \frac{ds}{Z_s + Z'_s}.$$

The process $Y^{d,d'}$ is obtained as a time-change of ξ, thus $Y_u^{d,d'} = \xi_{\alpha_u}$ where $\alpha_u = \inf\{t : \int_0^t ds/(Z_s + Z'_s) \geq u\}$. Applying this time-change to the above equation we obtain

$$Y_u^{d,d'} = Y_0^{d,d'} + 2\int_0^u \sqrt{Y_v^{d,d'}(1 - Y_v^{d,d'})} d\hat{\beta}_v + \int_0^u \left(d - (d+d')Y_v^{d,d'}\right) dv,$$

where $\hat{\beta}$ is the time-change of

$$\int_0^t \frac{\sqrt{1-\xi_s}d\beta_s - \sqrt{\xi_s}d\beta'_s}{\sqrt{Z_s + Z'_s}}.$$

Observe that this martingale is orthogonal to γ, and thus we deduce using Knight's Theorem on continuous orthogonal martingales that the Brownian motions $\hat{\beta}$ and γ are independent. Now $Y^{d,d'}$ is adapted to the filtration of $\hat{\beta}$, and consequently independent of $(Z_t + Z'_t; t \geq 0)$.

The extension to $\tilde{Z}$ and $\tilde{Z}'$ follows immediately on making the deterministic time change $t \mapsto u(t)$. □

Before proceeding to the proofs of the Ray-Knight theorems for the burglar, we must prove Lemma 2 which confirms that the burglar possesses local times.

Proof of Lemma 2. This is really the same argument as for Lemma 6, but it bears repeating. For any bounded and compactly supported, measurable test function f on $[0,1)$, we have,

$$\int_0^{A_t} f(\hat{W}_v)dv = \int_0^t f(\hat{W}_{A_s})dA_s = \int_0^t f(\theta(W_s))\frac{ds}{l(T_1, W_s)^2}$$
$$= \int_0^1 f(\theta(y))\frac{l(t,y)}{l(T_1,y)^2}dy.$$

Consequently, if ρ is defined by,

$$\rho(A_t, \theta(y)) = \frac{l(t,y)}{l(T_1,y)},$$

for $y \in [0,1)$ and $0 \leq t < T_1$, then, substituting $z = \theta(y)$ and $u = A_t$, we obtain,

$$\int_0^u f(\hat{W}_v)dv = \int_0^\infty f(z)\rho(u,z)dz.$$

□

Proof of first part of Theorem 3. Fix $x \in (0,1)$. Consider the local times of the reflecting Brownian motion $(W_t; 0 \leq t \leq T_1)$, stopped when it first hits level 1, split into a contribution from before time $T_x = \inf\{u : W_u = x\}$ and a contribution from between times T_x and T_1. It follows from the Ray-Knight theorems for Brownian motion, and arguments familiar in excursion theory, that

$$(l(T_x,y), l(T_1,y) - l(T_x,y); 0 \leq y \leq x) \stackrel{law}{=} (Z^2(x-y), Z^0(x-y); 0 \leq y \leq x),$$

where Z^2 is a squared Bessel process of dimension two starting from zero and Z^0 is an independent squared Bessel process of dimension zero starting from $l(T_1,x)$. By Lemma 2, proved above,

$$\rho(\hat{T}_{\theta(x)}, \theta(y)) = \frac{l(T_x,y)}{l(T_1,y)},$$

and so we deduce from Proposition 8 that

$$\big(\rho(\hat{T}_{\theta(x)}, \theta(x) - y); 0 \leq y \leq \theta(x)\big) \overset{law}{=} \big(Y_y^{2,0}; 0 \leq y \leq \theta(x)\big),$$

where $Y_0^{2,0} = 0$. Now ρ is independent of $\theta(x)$, since the burglar is independent of $\big(l(T_1, y); y \in \mathbb{R}^+\big)$, whence for each fixed a we must have,

$$\big(\rho(\hat{T}_a, a - y); 0 \leq y \leq a\big) \overset{law}{=} \big(Y_y^{2,0}; 0 \leq y \leq a\big),$$

as desired. □

Proof of second part of Theorem 3. Fix $s \in [0,1]$. Let $\sigma = l(T_1, 0)$, and then $\tau_{s\sigma} = \inf\{t : l(t,0) \geq sl(T_1,0)\}$. This time we split the local times that W has attained by time T_1 into a contribution from before time $\tau_{s\sigma}$, and a contribution from between times $\tau_{s\sigma}$ and T_1. We find that,

$$\big(l(\tau_{s\sigma}, y), l(T_1, y) - l(\tau_{s\sigma}, y); 0 \leq y \leq 1\big) \overset{law}{=} \big(Z^0_{s\sigma \to 0}(y), Z^2_{(1-s)\sigma \to 0}(y); 0 \leq y \leq 1\big),$$

where $Z^d_{x \to 0}$ denotes the bridge of the squared Bessel process of dimension d, from x to 0. The two bridges appearing in the above equation are taken to be independent. By virtue of Lemma 2 we have, since $A_{\tau_{s\sigma}} = \hat{\tau}_s$,

$$\rho(\hat{\tau}_s, \theta(x)) = \frac{l(\tau_{s\sigma}, x)}{l(T_1, x)},$$

and since, see Revuz and Yor, [15],

$$\big(Z^d_{x \to 0}(t); 0 \leq t \leq 1\big) \overset{law}{=} \big((1-t)^2 Z^d_{t/(1-t)}; 0 \leq t \leq 1\big)$$

we may apply Proposition 8 to obtain the result. □

4 Stopping at τ_1

In this section we describe $(W_t; 0 \leq t \leq \tau_1)$ conditional on $\big(l(\tau_1, y); y \in \mathbb{R}^+\big)$ where, as usual, $\tau_1 = \inf\{u : l(u,0) \geq 1\}$.

Theorem 9. *We consider a reflecting Brownian motion,* $\big(W_t; 0 \leq t \leq \tau_1\big)$, *with its maximum* $M = \sup_{0 \leq t \leq \tau_1} W_t$ *attained at time* T_M. *Then, there exists a Jacobi diffusion,* $Y^{2,2}$, *independent of* $\big(l(\tau_1, y); y \in \mathbb{R}^+\big)$, *such that, for* $0 \leq y < M$,

$$\frac{l(T_M, y)}{l(\tau_1, y)} = Y^{2,2}_{\left(\int_0^y \frac{dz}{l(\tau_1, z)}\right)},$$

with $Y_0^{2,2} = l(T_M, 0)/l(\tau_1, 0)$ *having uniform distribution.*

We define a process $\big(\hat{W}^{(1)}_u; u \geq 0\big)$ *by*

$$\int_0^{W_t} \frac{dy}{l(T_M, y)} = \hat{W}^{(1)}_{\left(\int_0^t \frac{ds}{(l(T_M, W_s))^2}\right)},$$

for $0 \le t < T_M$, *and a process* $\left(\hat{W}_u^{(2)}; u \ge 0\right)$ *by*

$$\int_0^{W_t} \frac{dy}{l(\tau_1, y) - l(T_M, y)} = \hat{W}^{(2)}_{\left(\int_{\tau_1 - t}^{\tau_1} \frac{ds}{(l(\tau_1, W_s) - l(T_M, W_s))^2}\right)},$$

for $T_M < t \le \tau_1$. *Then* $\hat{W}^{(1)}$ *and* $\hat{W}^{(2)}$ *both have the law of the Brownian burglar. The four processes* $\hat{W}^{(1)}$, $\hat{W}^{(2)}$, $Y^{2,2}$ *and* $l(\tau_1, \cdot)$ *are independent, and from them we can reconstruct* $\left(W_t; 0 \le t \le \tau_1\right)$.

We use the following lemma which is a combination of the agreement formula of Pitman-Yor [14] and the relationship between the bridge and the pseudo-bridge given by Biane-Le Gall-Yor [2].

Lemma 10. *Let* $R^{(1)}$ *and* $R^{(2)}$ *be two independent* $BES(1)$ *processes starting from* 0, *and let* $T^{(1)}$ *and* $T^{(2)}$ *be their respective hitting times of level* 1. *Define* $R^{(+)}$ *by connecting the paths of* $R^{(1)}$ *and* $R^{(2)}$ *back to back:*

$$R_t^{(+)} = \begin{cases} R_t^{(1)} & \text{if } t \le T^{(1)} \\ R^{(2)}_{T^{(1)} + T^{(2)} - t} & \text{if } T^{(1)} \le t \le T^{(1)} + T^{(2)}. \end{cases}$$

Now finally let R *be obtained by scaling* $R^{(+)}$ *so as to normalise its local time:*

$$R_t = \frac{1}{l^{(1)} + l^{(2)}} R^{(+)}\left((l^{(1)} + l^{(2)})^2 t\right), \qquad \text{for } t < \frac{T^{(1)} + T^{(2)}}{\left(l^{(1)} + l^{(2)}\right)^2},$$

where $l^{(1)}$ *is the local time at level* 0 *that* $R^{(1)}$ *has accrued by time* $T^{(1)}$ *and* $l^{(2)}$ *is similarly defined. Then the law of* R *is equivalent to the law of the reflecting Brownian motion* W *run until its local time at level* 0 *first reaches* 1, *and for any suitable path-functional* F

$$\mathbb{E}\left[F(R)\right] = \mathbb{E}\left[\frac{1}{2M} F(W)\right],$$

where $M = \sup_{0 \le t \le \tau_1} W_t$.

We will be satisfied with sketching the proof of this lemma. The above mentioned references give some more detail. Let L^W be the local time at zero that W has accrued when it attains its maximum M^W. Begin by observing that,

$$\mathbb{P}\left(M^W \in dm, L^W \in dl\right) = \frac{e^{-1/2m}}{2m^2}\, dm dl. \tag{4.1}$$

Using the law of $l^{(1)}$ and $l^{(2)}$, a simple calculation confirms that,

$$\mathbb{P}\left(M^R \in dm, L^R \in dl\right) = \frac{e^{-1/2m}}{4m^3}\, dm dl, \tag{4.2}$$

where L^R and M^R have the obvious meaning. Thus the conclusion of the lemma holds for F depending only on the maximum level attained and the local time at zero when this occurs. In order to lift the result to an equality of laws on path space, we condition on these two quantities. We can then easily confirm, using Brownian scaling and Williams' description of the Itô excursion measure, that the excursions from zero have identical conditional law under the two regimes.

Proof of Theorem 9. Consider the construction of the preceding lemma. Define burglars $\hat{R}^{(1)}$ and $\hat{R}^{(2)}$ from the processes $R^{(1)}$ and $R^{(2)}$ in the usual manner. Define $\tilde{Y}^{2,2}$ via,

$$\frac{l^{(1)}(y)}{l^{(1)}(y)+l^{(2)}(y)} = Y^{2,2}_{\int_0^y \frac{dx}{l^{(1)}(x)+l^{(2)}(x)}}, \tag{4.3}$$

where $l^{(1)}(y)$ is the local time at level y accrued by $R^{(1)}$ before $T^{(1)}$ and $l^{(2)}(y)$ is similarly defined. As is used in the proof of Theorem 3 we have

$$\left(l^{(1)}(y); 0 \le y \le 1\right) \stackrel{law}{=} \left(Z^2(1-y); 0 \le y \le 1\right) \stackrel{law}{=} \left(Z^2_{l^{(1)}\to 0}(y); 0 \le y \le 1\right),$$

and $Y^{2,2}$ is a Jacobi diffusion by virtue of Proposition 8. The four processes $\hat{R}^{(1)}$, $\hat{R}^{(2)}$, $Y^{2,2}$ and $l(\tau_1,\cdot)$ must be independent as a consequence of the independence of the two $BES(1)$ processes, and the results of Theorem 1 and Proposition 8.

Now let the reflecting Brownian motion W be obtained by completing the construction of R, and then making the appropriate change of measure. It is simple to check that $\hat{R}^{(1)} = \hat{W}^{(1)}$ and $\hat{R}^{(2)} = \hat{W}^{(2)}$. The process $Y^{2,2}$ just defined by (4.3) is also identical to that defined in the statement of the theorem. Since the change of measure we have made affects only the marginal law of $l(\tau_1,\cdot)$ the theorem is proved. □

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