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THE LEVEL SETS OF ITERATED BROWNIAN MOTION

BY

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ABSTRACT. We show that the Hausdorff dimension of every level set of iterated Brownian motion is equal to 3/4.

§1. Introduction and Main Result. Suppose $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space, rich enough to carry three independent Brownian motions, X^+, X^- and Y, all starting from the origin. Iterated Brownian motion (IBM) is the process defined by $Z(t) = X\big(Y(t)\big)$, where $X(t) = X^+(t)1_{[0,\infty)}(t) + X^-(-t)1_{(-\infty,0)}(t)$. The probabilistic and analytical properties of IBM and related processes have been the subject of recent vigorous investigations; see BERTOIN [B], BURDZY [B1,B2], CSÁKI ET AL. [CsCsFR1,CsCsFR2], DEHEUVELS AND MASON [DM], FUNAKI [F], HU ET AL. [HPS], HU AND SHI [HS], KHOSHNEVISAN AND LEWIS [KL1,KL2] and SHI [S] together with their combined references. Define the set–valued x–level set process, $\mathcal{L}_x(t)$, by

$$\mathcal{L}_x(t) = \big\{0 \le s \le t : \ Z(s) = x\big\}, \qquad \text{for all } x \in \mathbf{R}^1. \tag{1.1}$$

The main result of this paper is the following analogue of Paul Lévy's well–known result for Brownian motion (see ITÔ AND MCKEAN [IM] and ADLER [A]):

(1.2) Theorem. *Let* $\dim_H$ *denote Hausdorff dimension. Then, outside a single null set,*

$$\dim_H \big(\mathcal{L}_x(t)\big) = \tfrac{3}{4},$$

simultaneously for all $t \ge 0$ *and all* x *in the interior of* $Z([0,t])$.

Here and throughout, if $f : \mathbf{R}^1 \mapsto \mathbf{R}^1$ is Borel measurable and $A \subset \mathbf{R}^1$ is measurable, then $f(A) = \{y : y = f(x) \text{ for some } x \in A\}$.

The proof of Theorem (1.2) uses a capacity argument due to Frostman (see ADLER [A]) and relies on the following which has been discovered independently and at the same time by CSÁKI ET AL. [CsCsFR2]:

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(1.3) Proposition. *There exists an almost surely jointly continuous family of "local times", $\{\ell_t^a; t \geq 0, a \in \mathbf{R}^1\}$, such that for all Borel measurable integrable functions, $f : \mathbf{R}^1 \mapsto \mathbf{R}^1$ and all $t \geq 0$,*

$$\int_0^t f\big(Z(s)\big)ds = \int_{-\infty}^{\infty} f(a)\ell_t^a da.$$

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§2. Local Times. If B is any Brownian motion, its process of local times will be denoted by $L_t^a(B)$. These satisfy the following occupation density formula: for any Borel measurable $f : \mathbf{R}^1 \mapsto \mathbf{R}^1$ and all $t \geq 0$,

$$\int_0^t f\big(B(r)\big)dr = \int_{-\infty}^{\infty} f(a)L_t^a(B)da. \tag{2.1}$$

For a stochastic calculus description as well as many deep properties of local times see REVUZ AND YOR [RY]. Proposition (1.3) is a consequence of the following real variable fact:

(2.2) Proposition. *Let $K(b, da)$ be the kernel defined by*

$$K(b, da) = L_{da}^b(X^+) + L_{da}^b(X^-).$$

Then the local times ℓ are given by

$$\ell_t^b = \int_{Y\left([0,t]\right)} L_t^a(Y)K(b, da).$$

Proof. Let $f : \mathbf{R}^1 \mapsto \mathbf{R}^1$ be as in Proposition (1.3). Viewing $f(Z)$ as $(f \circ X)(Y)$, we see from (2.1) that for all $t \geq 0$, a.s.,

$$\begin{aligned} \int_0^t f\big(Z(s)\big)ds &= \int_{-\infty}^{\infty}(f \circ X)(a)L_t^a(Y)da \\ &= \int_0^{\infty}(f \circ X^+)(a)L_t^a(Y)da + \int_{-\infty}^0(f \circ X^-)(a)L_t^a(Y)da. \end{aligned} \tag{2.3}$$

By (2.1) and a monotone class argument, for any jointly measurable $F : \mathbf{R}^1 \times \mathbf{R}_+^1 \mapsto \mathbf{R}^1$,

$$\int_0^{\infty} F\big(X^{\pm}(s), s\big)ds = \int_{-\infty}^{\infty}\int_0^{\infty} F(a, t)L_{dt}^a(X^{\pm})da.$$

Applying (2.3),

$$\begin{aligned} \int_0^t f\big(Z(s)\big)ds = &\int_{-\infty}^{\infty} f(b)db \int_0^{\infty} L_t^a(Y)L_{da}^b(X^+) \\ &+ \int_{-\infty}^{\infty} f(b)db \int_0^{\infty} L_t^{-a}(Y)L_{da}^b(X^-). \end{aligned}$$

Since $a \mapsto L_t^a(Y)$ is a.s. supported on $Y([0,t])$,

$$\int_0^t f\big(Z(s)\big)ds = \int_{-\infty}^{\infty} f(b)db \int\limits_{\mathbf{R}^1_+ \cap Y([0,t])} L_t^a(Y) L_{da}^b(X^+) + \int_{-\infty}^{\infty} f(b)db \int\limits_{\mathbf{R}^1_+ \cap Y([0,t])} L_t^a(Y) L_{da}^b(X^-).$$

The proposition follows from a change of variables. The joint continuity of ℓ_t^a follows from that of $L_t^a(B)$ for any Brownian motion, B; see REVUZ AND YOR [RY]. ///

(2.4) Proposition. *For any $T > 0$, almost surely,*

$$\limsup_{\varepsilon \to 0} \sup_{a \in \mathbf{R}^1} \sup_{0 \le t \le T} \frac{\ell_{t+\varepsilon}^a - \ell_t^a}{\varepsilon^{3/4} \big(\ln(1/\varepsilon)\big)^{5/4}} \le 2^{3/2}, \qquad \text{a.s.}.$$

With more work, one can improve the upper bound of $2^{3/2}$ in the above. However, we do not even know whether the power of the logarithm is the correct one. Therefore, we will be satisfied with our simple proof of the upper bound.

Proof. By Proposition (2.2),

$$\begin{aligned} \ell_{t+\varepsilon}^a - \ell_t^a &= \int\limits_{Y([0,t])} \big(L_{t+\varepsilon}^r(Y) - L_t^r(Y)\big) K(a, dr) \\ &\quad + \int\limits_{Y([0,t+\varepsilon]) \setminus Y([0,t])} L_{t+\varepsilon}^r(Y) K(a, dr) \\ &= \mathrm{I}_{(2.5)} + \mathrm{II}_{(2.5)}. \end{aligned} \tag{2.5}$$

For all $r \in Y([0,t]) \setminus Y([t, t+\varepsilon])$, $L_{t+\varepsilon}^r(Y) = L_t^r(Y)$. Hence, as $\varepsilon \to 0^+$,

$$\begin{aligned} \mathrm{I}_{(2.5)} &= \int\limits_{Y([t,t+\varepsilon]) \cap Y([0,t])} \big(L_{t+\varepsilon}^r(Y) - L_t^r(Y)\big) K(a, dr) \\ &\le \sup_r \big(L_{t+\varepsilon}^r(Y) - L_t^r(Y)\big) K\big(a, Y([t, t+\varepsilon])\big) \\ &\le (1 + o(1)) \sqrt{2\varepsilon \ln(1/\varepsilon)}\, K\big(a, Y([t, t+\varepsilon])\big), \end{aligned}$$

uniformly over all $a \in \mathbf{R}^1$ and all $0 \le t \le T$. We have used the uniform modulus of continuity of local times in time; see Lemma 5(c) of PERKINS [P]. By Lévy's modulus of continuity ([RY]), as $\varepsilon \to 0^+$,

$$\big|Y([t, t+\varepsilon])\big| \le (1 + o(1)) \sqrt{2\varepsilon \ln(1/\varepsilon)},$$

uniformly over all $0 \le t \le T$. Hence, by MCKEAN [Mc] and the independence of X^+ and X^-, as $\varepsilon \to 0^+$, uniformly over all $a \in \mathbf{R}^1$ and $0 \le t \le T$,

$$\begin{aligned} K\big(a, Y([t, t+\varepsilon])\big) &\le (1 + o(1)) \sqrt{2\sqrt{2\varepsilon \ln(1/\varepsilon)} \big| \ln \sqrt{2\varepsilon \ln(1/\varepsilon)} \big|} \\ &= (1 + o(1)) 2^{1/4} \varepsilon^{1/4} \big(\ln(1/\varepsilon)\big)^{3/4}. \end{aligned}$$

This implies that as $\varepsilon \to 0^+$,

$$\text{(2.6)} \qquad \mathrm{I}_{(2.5)} \le (1+o(1))2^{3/4}\varepsilon^{3/4}\big(\ln(1/\varepsilon)\big)^{5/4}.$$

The bound for $\mathrm{II}_{(2.5)}$ is very similar. Note that for all $r \in Y([0,t+\varepsilon]) \setminus Y([0,t])$, $L_t^r(Y) = 0$. Hence, making similar arguments as above, we see that as $\varepsilon \to 0^+$, uniformly over all $a \in \mathbb{R}^1$ and $0 \le t \le T$,

$$\begin{aligned}
\mathrm{II}_{(2.5)} &= \int\limits_{Y([0,t+\varepsilon])\setminus Y([0,t])} \big(L_{t+\varepsilon}^r(Y) - L_t^r(Y)\big)K(a,dr) \\
&\le \sup_r \big(L_{t+\varepsilon}^r(Y) - L_t^r(Y)\big)K\big(a, Y([t,t+\varepsilon])\big) \\
&\le (1+o(1))\sqrt{2\varepsilon\ln(1/\varepsilon)}\sqrt{2|Y([t,t+\varepsilon])|\cdot|\ln|Y([t,t+\varepsilon])||} \\
&= (1+o(1))2^{3/4}\varepsilon^{3/4}\big(\ln(1/\varepsilon)\big)^{5/4}.
\end{aligned}$$

Together with (2.5) and (2.6), the above implies the result. ///

§3. The proof of Theorem (1.2). Once there is a modulus of continuity of local times (in t), we proceed by Frostman's capacity method as outlined in ADLER [A], for example. Recall from KHOSHNEVISAN AND LEWIS [KL] that for any $T > 0$, almost surely,

$$\text{(3.1)} \qquad \lim_{\varepsilon\to 0}\sup_{0\le t\le T}\frac{|Z(t+\varepsilon)-Z(t)|}{\varepsilon^{1/4}\big(\ln(1/\varepsilon)\big)^{3/4}} = 1, \qquad \text{a.s. .}$$

In particular, we see that Z is Hölder continuous of order $\gamma < 1/4$. By Proposition (1.3) and Lemma 7 of [A], simultaneously over all $x \in \mathbb{R}^1$ $\dim_H \mathcal{L}_x(t) \le 3/4$. Moreover, by Proposition (2.4), $t \mapsto \ell_t^a$ is Hölder continuous of order $\gamma < 3/4$, uniformly in $a \in \mathbb{R}^1$. By Frostman's lemma, (see the proof of Lemma 6 of [A]), simultaneously over all x in the interior of $Z([0,t])$, $\dim_H \mathcal{L}_x(t) \ge 3/4$. This proves the result. ///

We conclude this section with some open problem.

Problem 1. Define $Z^+(t) = \sup_{0\le s\le t} Z(s)$. BERTOIN [B] proves that for all $T > 0$, almost surely,

$$\lim_{\varepsilon\to 0}\sup_{0\le t\le T}\frac{|Z^+(t+\varepsilon)-Z^+(t)|}{\varepsilon^{1/4}\big(\ln(1/\varepsilon)\big)^{3/4}} = \frac{1}{2^{1/4}\cdot 3^{3/4}}.$$

In light of (3.1), this says that Z^+ is smoother than Z. Is there a probabilistic explanation for this, in terms of (say) path decompositions?

Problem 2. Define

$$\mathcal{S}^+(t) \triangleq \big\{t \in [0,1] : Z(t) > Z(s) \text{ for all } s < t\big\}.$$

According to BERTOIN [B], the Hausdorff dimension of $\mathcal{S}^+(t)$ is almost surely $1/4$. This is in sharp contrast with Theorem (1.2) and the analogous result for Brownian motion which is a consequence of Lévy's characterization of Brownian motion. Is there a probabilistic explanation for this apparent difference?

Problem 3. By being more careful, it is possible to show that φ–Hausdorff measure of $\mathcal{L}_0(t)$ is a.s. (strictly) positive, if $\varphi(\varepsilon) = \varepsilon^{3/4}\big(\ln(1/\varepsilon)\big)^{5/4}$. Is this sharp? For the corresponding problem for $\mathcal{S}^+(t)$, see BERTOIN [B].

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